PHOTOPRODUCTION OF π-MESONS

F. A. BERENDS

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PROEFSCHRIFT

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The subject of this dissertation is to calculate the cross-section for the production of a π-meson from the collision of a photon and a nucleon ($\gamma + N \rightarrow \pi + N$). In this problem the electromagnetic interaction is treated in first order. As is well-known, the strong interactions cannot be approximated by using a perturbation treatment. Accordingly, it is necessary to deal with matrix elements of the electromagnetic current of the strongly interacting particles containing the dynamics of the full strong interactions. This is shown in detail in Chapter I, where also some necessary kinematics is given.

To overcome the problem of the strong interactions, the assumption is made that the matrix elements as a function of energy and momentum transfer have certain analytic properties for complex values of these variables. This is expressed in the form of dispersion relations for the relevant transition amplitudes. The explicit details of this procedure are shown in Chapter II, where unitarity of the S-matrix is used to find the singularities of the amplitudes. Although simple in principle, there are several complications to take care of, as can be seen in Chapter II. The appropriate way to bring the theory in a calculable form is to expand the matrix elements in terms of multipole amplitudes and to write down dispersion relations for the latter. This is done in Chapter III, where also the unitarity condition for multipole amplitudes is considered.

In Chapter IV, it is pointed out that up to a certain energy of the process (namely 500 MeV photon laboratory energy), the dispersion relations given in the previous chapter can be solved. In this procedure the essential step is that one relates the amplitudes for photoproduction to the amplitudes of pion-nucleon scattering through the unitarity condition. Inserting experimental results for π - N scattering, it is possible to solve the dispersion relations for the photoproduction amplitudes. The details of the method and the results can be found in this chapter.

Finally four appendices are included. To avoid any misunderstanding about the conventions used, appendix A gives the definitions for the fields, Dirac matrices, $\mathcal{C}$-, $\mathcal{P}$- and $\mathcal{T}$-operations, whereas appendix B gives the isospin conventions. The connection of the multipoles and the helicity formalism is derived in appendix C, while in appendix D all sorts of cross-section and polarization formulae for photoproduction are compiled.

Because of the close analogy between photoproduction ($\gamma + N \rightarrow \pi + N$) and electroproduction ($e + N \rightarrow e + \pi + N$) of a pion, the dispersion theory for the latter is discussed simultaneously with the former. Due to a lack of information on certain quantities (form factors) in the dispersion relations for electroproduction, a numerical evaluation as for photoproduction cannot be given yet.

Although the dispersion theory can lead to predictions only in a limited region, the theory is still of use outside this region. Dispersion relations can be used as a constraint on multipole fits to experimental data. The experimental data in itself are too scanty to give unique multipole fits. Dispersion relations can help to remedy this problem, as has been done for the pion-nucleon phase shift analysis. Thus it is hoped that multipoles up to the energy of 1 GeV can be obtained. The multipoles calculated in ChapterIV are of great use as a starting point for such fitting procedures.

Knowledge of photoproduction is of importance for several related questions as there are:

a Nucleon isobars

Knowledge of photoproduction multipoles up to 1 GeV will give additional information on recently discovered pion-nucleon resonances. It may be possible that some of these show up more clearly in the multipoles than in the pion-nucleon phase shifts. Then information on the position and electromagnetic couplings may be obtained.

b Quark model

The quark model gives certain predictions about relations of multipoles[1] and, of course, of the classification of resonances[2]. From the information of photoproduction these statements can be checked.

c Sum rules

Recently a great number of sum rules have been obtained. Some of them seem to follow from rather reliable assumptions as e.g. the Cabibbo-Radicati[3] and Drell-Hearn[4] sum rules. Others use more dubious assumptions. Many of them can be calculated to a good degree of accuracy, when photoproduction multipoles are known.

d Electroproduction

Electroproduction can be used to get extra information on electromagnetic form factors. In fact, for π^+ - production from protons the pion electromagnetic form factor is unknown and the neutron form factor is only known from experiments on deuterium. For π^o-production only the known proton electromagnetic form factor occurs.

e Compton scattering

The unitarity condition relates Compton scattering to the photoproduction multipoles. So in good approximation the imaginary parts of the Compton amplitudes can be calculated. The s- and u-channel poles are also known. Of the t-channel contributions only the π^o-exchange is known, the η- and two-pion-exchanges being the unknowns. Then a one or two parameter theory for Compton scattering results, depending on the importance of the two pion contribution.

f Vector meson coupling constants

Some idea of these coupling constants can be obtained by using dispersion relation techniques, as was done for the $\pi - \pi$ interaction[5]. One then needs a good knowledge of the multipoles. It is interesting to get information on the coupling constants $g_{\rho\pi\gamma}$, $g_{\omega NN}$ and $g_{\rho NN}$ because of certain SU(3) and SU(6) predictions.

CHAPTER I

KINEMATICAL AND DYNAMICAL CONCEPTS

SUMMARY

In section 1 the problem of this thesis is stated and some historical remarks are made. The kinematics and the notation are given in section 2. In section 3 the connection between the S-matrix for photo- and electroproduction and the electromagnetic current of the strongly interacting particles is derived.

1 STATEMENT OF THE PROBLEM AND SOME HISTORICAL REMARKS

For the theoretical description of photo- or electroproduction of pions, both the electromagnetic and strong interactions have to be taken into account. As there exists no theory for the latter, the difficult part in the theory of photo- and electroproduction comes from the strong interactions. In the last ten years two approaches have been followed to overcome this problem.

One approach is to use dispersion relations, which makes it possible to relate photo- and electroproduction to pion-nucleon scattering. Then predictions for the former processes become possible, although the energy region, where the theory can be applied is limited. Chew, Goldberger, Low and Nambu[6] (henceforth referred to as C.G.L.N.) were the first to obtain qualitative agreement. Hereafter the theory was improved by various authors (see section 15 for more details). However, still many approximations were made and the solution methods were developed considerably. It is the purpose of this investigation to use the theory as far as possible. The theory of the multipole dispersion relations, therefore, must be developed much further than has been done. Then it becomes possible to make fewer approximations than before. Using the so-called conformal mapping technique, the dispersion relations can be solved into more detail than was previously done. Also estimates can be made of the errors on the solutions arising from errors on the pion-nucleon phase shifts. The more detailed knowledge of these phase shifts nowadays, of course, makes a more detailed treatment of photoproduction feasible.

The other approach to describe photo- and electroproduction is by using a model, the so-called isobar model. This approach, which is not followed here, wants to give a simple picture of the process in terms of a sum of generalized Born approximations. In the region below 1 GeV photon laboratory energy, the assumption is made that the Born approximations of nucleon, pion and isobar exchanges dominate the process. As the coupling constants are unknown, one has a many parameter theory. Gourdin and Salin[7] obtained in this way a twelve parameter theory, which gave a good fit to the experiments. This model is less satisfactory than the dispersion treatment, which is parameter free. The reason for this is the large number of parameters and the neglect of terms, which are known from the dispersion theory to be important.

2 KINEMATICS AND NOTATION

The kinematics will be given for electroproduction, the photoproduction formulae being obtained by putting the "photon mass" ($-K^2$) zero.

Let K_1 and K_2 be the four-momenta of the initial and final electron, P_1 and P_2 the four-momenta of the initial and final nucleon, and Q the four-momentum of the created pion. Using the approximation in which only one photon is exchanged (see fig. 1), the four-momentum of the virtual photon is given by

$$K = K_1 - K_2 . \qquad (2.1)$$

The symbol K is also used for the four-momentum of the real photon in the case of photoproduction. In the latter case the lepton vertex in fig. 1 is left out.
Then

$$K + P_1 = Q + P_2 , \qquad (2.2)$$

and

$$K_1^2 = K_2^2 = -m_e^2 \quad , \quad P_1^2 = P_2^2 = -m^2 \quad , Q^2 = -\mu^2 \quad .$$

This is illustrated in fig. 1.

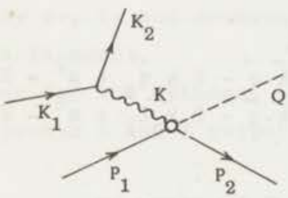

fig. 1

As Lorentz invariant kinematical variables, the usual Mandelstam kinematical variables are chosen

$$\begin{aligned} s &= -(K + P_1)^2 , \\ t &= -(K - Q)^2 , \\ u &= -(K - P_2)^2 , \end{aligned} \qquad (2.3)$$

and they satisfy

$$s + t + u = 2m^2 + \mu^2 - K^2 . \qquad (2.4)$$

For the dispersion treatment, it is advantageous to use the pion-nucleon centre-of-mass system. Let W be the total energy in this system and define

$$
\begin{aligned}
K &= (\vec{k}, ik_o) \quad , \\
P_1 &= (-\vec{k}, iE_1) \quad , \\
Q &= (\vec{q}, iq_o) \quad , \\
P_2 &= (-\vec{q}, iE_2) \quad ,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
k &= |\vec{k}| \quad , \qquad \hat{k} = \frac{\vec{k}}{k} \quad , \\
q &= |\vec{q}| \quad , \qquad \hat{q} = \frac{\vec{q}}{q} \quad .
\end{aligned}
\tag{2.6}
$$

In the pion-nucleon centre-of-mass system, the Mandelstam variables take the form

$$
\begin{aligned}
s &= W^2, \\
t &= 2\,\vec{k}.\vec{q} - 2\,k_o q_o + \mu^2 - K^2 \quad , \\
u &= -2\vec{k}.\vec{q} - 2\,k_o E_2 + m^2 - K^2 \quad ,
\end{aligned}
\tag{2.7}
$$

and the scattering angle θ is given by

$$
\cos\theta = \frac{\vec{k}.\vec{q}}{kq} = \hat{k}.\hat{q} = x \quad . \tag{2.8}
$$

The electron four-momenta have components

$$
\begin{aligned}
K_1 &= (\vec{k}_1, ik_{10}) \quad , \\
K_2 &= (\vec{k}_2, ik_{20}) \quad .
\end{aligned}
\tag{2.9}
$$

The virtual photon has a spacelike four-momentum K and therefore a negative squared "mass" of $-K^2$.

Some useful kinematical relations, which express the energies in the centre-of-mass system in terms of W and K^2 are

$$
k_o = \frac{W^2 - K^2 - m^2}{2W} \quad ,
$$

$$E_1 = \frac{W^2 + m^2 + K^2}{2W} ,$$

$$E_2 = \frac{W^2 - \mu^2 + m^2}{2W} ,$$

$$q_o = \frac{W^2 + \mu^2 - m^2}{2W} , \qquad (2.10)$$

$$E_1 \pm m = \frac{(W \pm m)^2 + K^2}{2W} ,$$

$$E_2 \pm m = \frac{(W \pm m)^2 - \mu^2}{2W} . \qquad (2.10)$$

One often denotes the reactions $\gamma + N_1 \rightarrow \pi + N_2$, $\gamma + \pi \rightarrow \bar{N}_1 + N_2$ and $\gamma + \bar{N}_2 \rightarrow \pi + \bar{N}_1$ by s-, t- and u-channel respectively, also when a virtual photon is meant.

The conventions for the spinors, γ -matrices and isospin are summarized in Appendix A and B respectively.

3 S-MATRIX FOR PHOTO- AND ELECTROPRODUCTION

For finding the cross-sections for photo- and electroproduction, it is necessary to consider first the S-matrix for the processes. The connection of the S-matrix for photo- and electroproduction with the electromagnetic current will be shown in the following. The result is valid only for the lowest possible order of electromagnetic interaction. This means first order for photoproduction and second order for electroproduction. Several ways can be followed for the derivation. One method uses the interaction picture for the total interaction. Another method uses the asymptotic in- and out-states [8]. The first method will be used here.

The total interaction described by a strong and an electromagnetic interaction hamiltonian

$$H_i = H_s + H_{e.m.} , \qquad (3.1)$$

leads to the following Dyson series for the S-operator

$$S_I = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dx_1 \ldots dx_n T\Big(H_i(x_1)H_i(x_2)..H_i(x_n)\Big). \tag{3.2}$$

This operator taken between bare states gives the S-matrix. In the cases at hand, it is sufficient to consider only the terms of first and second order in the electromagnetic interaction, which reduces eq.(3.2) to

$$S_I = S_s + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} n \int_{-\infty}^{+\infty} dx\, dx_2 \ldots dx_n\, T\Big(H_{e.m}(x)H_s(x_2)..H_s(x_n)\Big) +$$

$$\sum_{n=2}^{\infty} \frac{(-i)^n}{n!} \binom{n}{2} \int_{-\infty}^{+\infty} dx\, dy\, dx_3 \ldots dx_n\, T\Big(H_{e.m}(x)H_{e.m}(y)H_s(x_3)\ldots H_s(x_n)\Big) =$$

$$S_s - i\int_{-\infty}^{+\infty} dx\, T\Big(H_{e.m}(x)S_s\Big) - \frac{1}{2!}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, T\Big(H_{e.m}(x)H_{e.m}(y)S_s\Big), \tag{3.3}$$

where S_s is the strong interaction S-operator, which is given by eq.(3.2) when H_i is replaced by H_s. S_s is also often denoted by $U(+\infty,-\infty)$, i.e. the unitary operator $U(t_2,t_1)$, which describes the motion of the system from t_1 to t_2 is taken for $t_1 = -\infty$ to $t_2 = +\infty$. In terms of this operator the in- and out-states for the strong interactions are defined

$$U(0,\mp\infty)|\alpha\rangle = |\alpha\rangle_{in}, |\alpha\rangle_{out}. \tag{3.4}$$

Also the connection of an operator $O^H(t)$ in the Heisenberg picture for the strong interactions and the operator $O^I(t)$ in the interaction picture is given by $U(t_1,t_2)$

$$O^H(t) = U(0,t)O^I(t)U(t,0). \tag{3.5}$$

Use of eqs.(3.3), (3.4) and (3.5) gives for photoproduction

$$\langle\pi N|S_I|\gamma N\rangle = -i\int_{-\infty}^{+\infty} dx\, {}_{out}\langle\pi N|H^H_{e.m}(x)|\gamma N\rangle_{in}. \tag{3.6}$$

The interaction is of the form

$$H^{H}_{e.m}(x) = A_\mu(x)\ J^{H}_\mu(x). \qquad (3.7)$$

The coupling constant e is absorbed in the electromagnetic current, which in lowest order in e depends on leptons and hadrons. As in- and out-states in eq.(3.6) depend only on strong interactions, we have for a photon of momentum $\vec{k}$ and polarization λ (see appendix A)

$$|\gamma N\rangle_{in} = a^{\lambda *}_{\vec{k}}|N\rangle_{in}. \qquad (3.8)$$

Thus one obtains for photoproduction

$$\langle \pi N|S_I|\gamma N\rangle = -i\frac{\varepsilon^{\lambda}_{\mu}}{(2\pi)^{3/2}(2k_0)^{1/2}}(2\pi)^4\delta(P_1+K-P_2-Q)\ x$$

$$_{out}\langle \pi N|J^{H}_{\mu}(0)|N\rangle_{in} = -i\left(\frac{m^2}{4k_0q_0E_1E_2}\right)^{\frac{1}{2}}\frac{1}{(2\pi)^2}\delta(P_1+K-P_2-Q)T_{fi}, \qquad (3.9)$$

where T_{fi} is the T-matrix for photoproduction.

For electroproduction eqs.(3.3), (3.4), (3.5) and (3.7) are used. The time ordered product is changed into the normal product by the Wick reordering theorem

$$_{out}\langle e^-\pi N|T(A_\mu(x)\ J^{H}_{\mu}(x)\ A_\nu(y)\ J^{H}_{\nu}(y))|e^-N\rangle_{in} =$$

$$_{out}\langle e^-\pi N|N(J^{H}_{\mu}(x)\ J^{H}_{\nu}(y))A^{\cdot}_{\mu}(x)\ A^{\cdot}_{\nu}(y)|e^-N\rangle_{in},$$

where the photon fields are contracted. Then the result is

$$\langle e^-\pi N|S_I|e^-N\rangle = -\frac{1}{2!}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\ A^{\cdot}_{\mu}(x)\ A^{\cdot}_{\nu}(y) \quad x$$

$$_{out}\langle e^-\pi N|N(J^{H}_{\mu}(x)\ J^{H}_{\nu}(y))|e^-N\rangle_{in} =$$

$$-\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\langle e^-|J^{H}_{\mu}(x)|e^-\rangle A^{\cdot}_{\mu}(x)\ A^{\cdot}_{\nu}(y)\ {}_{out}\langle\pi N|J^{H}_{\nu}(y)|N\rangle_{in} =$$

$$-\left(\frac{m_e^2}{k_{10}k_{20}}\right)^{\frac{1}{2}} 2\pi\, \delta(K_1+P_1-K_2-Q-P_2) \frac{e\, \bar{u}(\vec{K}_1)\gamma_\mu u(\vec{K}_2)}{K^2} \; {}_{out}\langle \pi N | J_\mu^H(0) | N \rangle_{in} . \tag{3.10}$$

$$\langle e^- \pi N | S_I | e^- N \rangle = -\left(\frac{m_e^2}{k_{10}k_{20}}\right)^{\frac{1}{2}} \frac{1}{(2\pi)^{7/2}} \left(\frac{m^2}{2E_1E_2q_0}\right)^{\frac{1}{2}} \delta(K_1+P_1-K_2-Q-P_2) T_{fi} . \tag{3.11}$$

Thus in both cases T_{fi} and $\varepsilon_\mu \; {}_{out}\langle \pi N | J_\mu^H(0) | N \rangle_{in}$ are proportional with the same kinematical factor $\left(\frac{2q_0E_1E_2}{m^2}\right)^{\frac{1}{2}} (2\pi)^{9/2}$. It is useful to define ${}_{out}\langle \pi N | J_\mu^H(0) | N \rangle_{in}$ such that

$$T_{fi} = \varepsilon_\mu \; {}_{out}\langle \pi N | J_\mu^H(0) | N \rangle_{in} = \langle f | T | i \rangle . \tag{3.12}$$

For photoproduction one has the polarization vector

$$\varepsilon_\mu = \varepsilon_\mu^\lambda , \tag{3.13}$$

whereas for electroproduction one has the lepton vertex

$$\varepsilon_\mu = e \frac{\bar{u}(k_2)\gamma_\mu \; u(k_1)}{K^2} . \tag{3.14}$$

Once the T-matrix is known, the cross-sections can be calculated. The invariant cross-section formulae are given by

$$d^6\sigma = \frac{1}{(2\pi)^2} \delta(K+P_1-Q-P_2) \frac{m^2}{4(K.P_1)} |T_{fi}|^2 \frac{d\vec{q}}{q_0} \frac{d\vec{p}_2}{E_2} , \tag{3.15}$$

for photoproduction and by

$$d^9\sigma = \frac{1}{(2\pi)^5} \delta(K_1+P_1-K_2-Q-P_2) \frac{m^2 m_e^2}{2} \frac{1}{[(K_1.P_1)^2 - m^2 m_e^2]^{\frac{1}{2}}} |T_{fi}|^2 \frac{d\vec{p}_2}{E_2} \frac{d\vec{k}_2}{k_{20}} \frac{d\vec{q}}{q_0} , \tag{3.16}$$

for electroproduction.

Later on, the differential cross-section for photoproduc- in the centre-of-mass system will be needed

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left(\frac{m}{4\pi W}\right)^2 |T_{fi}|^2 . \tag{3.17}$$

CHAPTER II

DISPERSION RELATIONS

SUMMARY

In this chapter fixed-t dispersion relations for photo- and electroproduction will be given (section 9). Although these dispersion relations are an assumption, some discussion is given about the connection with the Mandelstam representation. In the first place amplitudes are introduced, which do not have kinematical singularities. It turns out that these amplitudes are not independent, which is due to current conservation. The reduction to an independent set of amplitudes entails the introduction of kinematical singularities (section 4). The role of C-invariance and the relation of the amplitudes of one channel with those of an other channel is discussed in section 5. Then it is shown in section 6 which singularities must be present because of unitarity. The poles in the amplitudes, which do not have kinematical singularities are given in section 7. In section 8 the compatibility of the Mandelstam representation and current conservation is discussed. After assuming a Mandelstam representation for the amplitudes free of kinematical singularities, fixed-t dispersion relations can be derived for the independent set of amplitudes. These are given in compact notation in section 9. In section 10 some comments are made in regard to the arguments given in the literature connected with these questions.

4 THE INVARIANT AMPLITUDES

In this section the invariant amplitudes are introduced, for which the dispersion relations will be assumed to be valid. First the decomposition of T_{fi} in isospace is discussed and then the spin and momentum dependence is dealt with.

A Isospin dependence

The electromagnetic current consists of an isoscalar and isovector part, just as the integrated fourth component -the charge Q- is decomposed according to the Gell-Mann-Nishijima rule

$$Q = \tfrac{1}{2}Y + I_3, \tag{4.1}$$

where Y and I_3 are the hypercharge and third component of the isospin respectively. Thus one writes

$$T_{fi} = {}_{out}\langle\pi N|\varepsilon_\mu J_\mu|N\rangle_{in} = {}_{out}\langle\pi N|\varepsilon_\mu J_\mu^S + \varepsilon_\mu J_\mu^V|N\rangle_{in} \tag{4.2}$$

The first part conserves isospin. Therefore an invariant must be constructed from the available isospinors of the nucleons and the isovector of the pion (see appendix B). The only possible form is

$${}_{out}\langle\pi N|\varepsilon_\mu J^S|N\rangle_{in} = A^0 V_\alpha^\dagger \chi^\dagger \tau_\alpha \chi, \tag{4.3}$$

where A^0 still depends on the kinematical variables and where

$$\chi(p)=\begin{pmatrix}1\\0\end{pmatrix}, \chi(n)=\begin{pmatrix}0\\1\end{pmatrix}, \quad V_\alpha(\pi^\pm)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm i\\0\end{pmatrix}, \quad V_\alpha(\pi^0)=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{4.4}$$

The second part behaves like the third component of an isovector. Such a vector can be constructed in two ways, giving rise to two amplitudes A^+ and A^-

$${}_{out}\langle\pi N|\varepsilon_\mu J_\mu^V|N\rangle_{in} = V_\alpha^\dagger \chi^\dagger (A^+\tfrac{1}{2}\{\tau_\alpha,\tau_3\} + A^-\tfrac{1}{2}[\tau_\alpha,\tau_3])\chi \tag{4.5}$$

$$= V_\alpha^\dagger \chi^\dagger (A^+\delta_{\alpha 3} + A^-\tfrac{1}{2}[\tau_\alpha,\tau_3])\chi. \tag{4.6}$$

With the help of eq.(4.4) the relation of the amplitudes to the various physical processes follows

$$\begin{aligned}
\langle n\pi^+|T|\gamma p\rangle &= \sqrt{2}(A^- + A^0), \\
\langle p\pi^0|T|\gamma p\rangle &= A^+ + A^0, \\
\langle p\pi^-|T|\gamma n\rangle &= \sqrt{2}(-A^- + A^0), \\
\langle p\pi^0|T|\gamma n\rangle &= -A^0 + A^+.
\end{aligned} \tag{4.7}$$

The initial state always has isospin $\frac{1}{2}$, the final state has isospin $\frac{1}{2}$ or 3/2. For the isovector current both final states are possible. The corresponding invariants A^1 and A^3 respectively are related to A^+ and A^- as follows

$$\begin{aligned}
A^+ &= \frac{2A^3 + A^1}{3}, \\
A^- &= \frac{A^1 - A^3}{3}.
\end{aligned} \tag{4.8}$$

One may notice that the decomposition of the vector part is also of importance for neutrinoproduction of pions. The axial and vector currents in this process are both assumed to have an isovector character. The $\Delta Q = \pm 1$ currents are obtained by replacing τ_3 in eq.(4.5) by $\tau_\pm = \tau_1 \pm i\tau_2$.

B S p i n a n d m o m e n t u m d e p e n d e n c e

In order to apply dispersion relation techniques amplitudes $B(s,t,u,K^2)$ are introduced, which depend on the kinematical invariants and describe the physical processes. In the applications K^2 is fixed and two variables out of s, t, u can be varied (cf. eq.(2.4)).

As

$$T_{fi} = \varepsilon_\mu \, {}_{out}\langle \pi N | J_\mu | N \rangle_{in},$$

the most general structure of the four-vector ${}_{out}\langle \pi N | J_\mu | N \rangle_{in}$ must be found. One constructs a quantity which, when sandwiched between $\bar{u}(\vec{p}_2)$ and $u(\vec{p}_1)$ behaves like a vector. Because of momentum conservation, it is sufficient for this construction to use only the vectors K_μ, Q_μ and $P_\mu = \frac{1}{2}(P_1 + P_2)_\mu$.

Scalars constructed from these vectors are absorbed in the amplitudes B. In addition, scalars constructed from γ-matrices and from these vectors, can be reduced to expressions without γ-matrices or can be reduced to the expressions $(\gamma.K)$, $(\gamma.P_1)$ and $(\gamma.P_2)$. The last two can be eliminated by the Dirac equation. Thus the general form of the current J_μ is *

$$J_\mu = \sum_{i=1}^{8} B_i(s,\ t,\ u,\ K^2)N_\mu^i, \qquad (4.9)$$

where

$$\begin{aligned}
N_\mu^1 &= i\gamma_5\gamma_\mu(\gamma.K),\\
N_\mu^2 &= 2i\gamma_5 P_\mu,\\
N_\mu^3 &= 2i\gamma_5 Q_\mu,\\
N_\mu^4 &= 2i\gamma_5 K_\mu,\\
N_\mu^5 &= \gamma_5\gamma_\mu,\\
N_\mu^6 &= \gamma_5(\gamma.K)P_\mu,\\
N_\mu^7 &= \gamma_5(\gamma.K)K_\mu,\\
N_\mu^8 &= \gamma_5(\gamma.K)Q_\mu.
\end{aligned} \qquad (4.10)$$

There is also a similar set as (4.10) without a γ_5, but this set is rejected by parity conservation. Furthermore, factors of i are introduced in such a way that the definitions of real and imaginary part of the amplitudes B are uniform. In fact, the convention of ref. 6 is followed. This amounts to the following: the imaginary part of the matrixelement corresponds to the real part of the amplitudes B in eq.(4.9).

Current conservation imposes restrictions on eq.(4.9), which reduces the number of invariants. One may notice however that there is a case in which the full current (4.9) should be used. This is the case for the description of the axial current

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* We use in the following J_μ both for the operator (4.9) and for its matrixelement $\bar{u}\ J_\mu\ u$.

in neutrinoproduction of pions, where the γ_5's should be omitted.

In case of a conserved current

$$K_\mu \; J_\mu = 0, \tag{4.11}$$

or

$$\frac{K^2}{2} B_1 + P.K \; B_2 + Q.K \; B_3 + K^2 \; B_4 = 0,$$
$$B_5 + P.K \; B_6 + K^2 \; B_7 + Q.K \; B_8 = 0, \tag{4.12}$$

two amplitudes can be eliminated. A choice is made, which is possible for both photo- and electroproduction and which avoids unnecessary kinematical singularities. Eliminating B_3 and B_5 one obtains

$$\begin{aligned} J_\mu = {} & B_1\left(N^1_\mu - \frac{K^2}{2(Q.K)} N^3_\mu\right) + B_2\left(N^2_\mu - \frac{P.K}{Q.K} N^3_\mu\right) + \\ & B_4\left(N^4_\mu - \frac{K^2}{Q.K} N^3_\mu\right) + B_6\left(N^6_\mu - P.K \; N^5_\mu\right) + \\ & B_7\left(N^7_\mu - K^2 \; N^5_\mu\right) + B_8\left(N^8_\mu - Q.K \; N^5_\mu\right). \end{aligned} \tag{4.13}$$

Defining

$$F_{\mu\nu} = \varepsilon_\mu K_\nu - \varepsilon_\nu K_\mu , \tag{4.14}$$

one can write

$$\begin{aligned} \varepsilon_\mu J_\mu = {} & B_1\left(\frac{i}{2}\gamma_5\gamma_\mu\gamma_\nu F_{\mu\nu} + i\gamma_5 \frac{K_\mu \; Q_\nu}{Q.K} F_{\mu\nu}\right) \\ & + B_2 \; 2i\gamma_5 \frac{P_\mu \; Q_\nu}{Q.K} F_{\mu\nu} + B_4 \; 2i\gamma_5 \frac{K_\mu \; Q_\nu}{Q.K} F_{\mu\nu} \\ & - B_6\gamma_5\gamma_\mu P_\nu F_{\mu\nu} + B_7\gamma_5 K_\mu\gamma_\nu F_{\mu\nu} - B_8\gamma_5\gamma_\mu Q_\nu F_{\mu\nu}. \end{aligned} \tag{4.15}$$

A set of explicitly "gauge invariant" matrices can be introduced by the definitions

$$
\begin{aligned}
M_1 &= \tfrac{1}{2} i \gamma_5 \gamma_\mu \gamma_\nu F_{\mu\nu}, \\
M_2 &= 2 i \gamma_5 P_\mu (Q - \tfrac{1}{2}K)_\nu F_{\mu\nu}, \\
M_3 &= \gamma_5 \gamma_\mu Q_\nu F_{\mu\nu}, \\
M_4 &= 2\gamma_5 \gamma_\mu P_\nu F_{\mu\nu} - 2m\, M_1, \\
M_5 &= i\gamma_5 K_\mu Q_\nu F_{\mu\nu}, \\
M_6 &= \gamma_5 K_\mu \gamma_\nu F_{\mu\nu},
\end{aligned}
\tag{4.16}
$$

which is the set used by Dennery [9]. Another set appearing in the literature [10)*] replaces the above M_2 by the quantity

$$
M_2' = 2i\gamma_5 P_\mu Q_\nu F_{\mu\nu} = \frac{2Q.K}{t-\mu^2} M_2 - \frac{2P.K}{t-\mu^2} M_5. \tag{4.17}
$$

Both sets reduce for photoproduction, where M_2 equals M_2', M_5 and M_6 are zero, to the set of C.G.L.N. [6].

In this way eq.(4.15) becomes

$$
\varepsilon_\mu J_\mu = \sum_{i=1}^{6} A_i(s, t, u, K^2) M_i, \tag{4.18}
$$

where

$$
\begin{aligned}
A_1 &= B_1 - m\, B_6, \\
A_2 &= \frac{2B_2}{t-\mu^2}, \quad A_2' = \frac{B_2}{Q.K}, \\
A_3 &= - B_8, \\
A_4 &= - \frac{B_6}{2}, \\
A_5 &= \frac{1}{Q.K}\left(B_1 + 2B_4 - \frac{2P.K}{t-\mu^2} B_2\right), \quad A_5' = \frac{B_1+2B_4}{Q.K}, \\
A_6 &= B_7.
\end{aligned}
\tag{4.19}
$$

The primed quantities belong to the set M_2'. Eq.(4.18) gives the general expression for a conserved current. So also in the case of neutrinoproduction the vector part contracted with the lepton

---
* Referred to as F.N.W. in the following.

vertex ε_μ has the form (4.18). In principle the kinematical singularities are cancelled in physical quantities. Consider e.g. the set of F.N.W. Kinematical singularities arise in A_2' and A_5' for $Q.K = 0$. In the current J_μ they cancel because at $Q.K = 0$ from eq.(4.12) follows

$$\frac{K^2}{2} (B_1 + 2B_4) + P.K\, B_2 = 0 \qquad (4.20)$$

So $A_2' M_2' + A_5' M_5'$ is regular at $Q.K = 0$.

Nevertheless in numerical calculations one will not obtain eq.(4.20) or equivalently

$$\frac{K^2}{2} A_5' = - P.K\, A_2', \qquad (4.21)$$

at $Q.K = 0$. The dispersion relations for fixed t and K^2 are the same for A_2' and A_5' at this value (see section 9).

So in numerical calculations difficulties may arise because the computed quantities do not obey eq.(4.21) exactly and the value $Q.K = 0$ is in the physical region. Therefore it is preferable to use the set (4.16) for which A_2 and A_5 have the kinematical singularities $t = \mu^2$ outside the physical region.

5 $\mathcal{C}$-INVARIANCE AND CROSSING RELATIONS

Assuming $\mathcal{C}_{st}$-invariance for the strong interactions, one can classify other interactions according to their behaviour under the strong interaction particle anti-particle operator $\mathcal{C}_{st}$. In particular, as $\mathcal{C}_{st}^2 = 1$, the T-matrix can be split into a $\mathcal{C}_{st}$-even and a $\mathcal{C}_{st}$-odd part

$$T_1 = T + \mathcal{C}_{st} T\, \mathcal{C}_{st}^{-1}, \qquad (5.1)$$

$$T_2 = T - \mathcal{C}_{st} T\, \mathcal{C}_{st}^{-1}, \qquad (5.2)$$

$$T = \tfrac{1}{2} (T_1 + T_2). \qquad (5.3)$$

For photo- and electroproduction this division can be described by a corresponding division of the electromagnetic current [11] of the hadrons

$$J_\mu = J_{1\mu} + J_{2\mu}, \qquad (5.4)$$

$$\mathcal{C}_{st}\, J_{1\mu}\, \mathcal{C}^{-1}_{st} = -J_{1\mu}, \qquad (5.5)$$

$$\mathcal{C}_{st}\, J_{2\mu}\, \mathcal{C}^{-1}_{st} = J_{2\mu}, \qquad (5.6)$$

while A_μ, the electromagnetic field, obeys

$$\mathcal{C}_{st}\, A_\mu\, \mathcal{C}^{-1}_{st} = -A_\mu. \qquad (5.7)$$

As the hadronic electromagnetic current is conserved by the strong interactions and as $\mathcal{C}_{st}$ is conserved by the strong interactions, both $J_{1\mu}$ and $J_{2\mu}$ are conserved by these interactions. So for both currents one can introduce invariant amplitudes A_i as coefficient of M_i. Denoting the amplitudes for $J_{2\mu}$ by $A_7, \ldots, A_{12}$, the total current is written as

$$_{out}\langle \pi N | \varepsilon_\mu\, J_\mu | N \rangle_{in} = \bar{u}(\vec{p}_2)[\sum_{i=1}^{12} A_i(s,\, t,\, u,\, K^2) M_i] u(\vec{p}_1), \qquad (5.8)$$

where $M_{i+6} = M_i$ for $i = 1, \ldots 6$.

The basic assumption now is that the amplitudes A_i (s, t, u, K^2) are for fixed K^2 meromorphic functions in two variables e.g. s and t (s, t and u are related by eq.(2.4)). Furthermore, these amplitudes are assumed to be the physical amplitudes for the s-, t- and u-channel, when the variables s and t are taken at physical values for these channels. So by analytic continuation in unphysical regions one obtains from the s-channel amplitudes the u- and t-channel amplitudes.

Thus the T-matrix elements for the three channels are related to the amplitudes (suppressing K^2 dependence) by the so-called substitution rule

s-channel:

$$\langle N(P_2), \pi^\alpha(Q) | T | N(P_1), \gamma(K) \rangle = \bar{u}(\vec{p}_2)[\sum_{i=1}^{12} A_i(s,t,u) M_i] u(\vec{p}_1), \qquad (5.9)$$

t-channel:

$$\langle N(P_2),\bar{N}(-P_1)|T|\pi^\alpha(-Q),\gamma(K)\rangle = \bar{u}(\vec{p}_2)[\sum_i^{12} A_i(s,t,u)M_i]v(-\vec{p}_1), \tag{5.10}$$

u-channel:

$$\langle \bar{N}(-P_1),\pi^\alpha(Q)|T|\bar{N}(-P_2),\gamma(K)\rangle = -\bar{v}(-\vec{p}_2)[\sum_{i=1}^{12} A_i(s,t,u)M_i]v(-\vec{p}_1) \tag{5.11}$$

Here π^α means "1", "2" or "3" meson. In the right hand sides the isospin dependence is absorbed in A_i. Physical matrix elements are obtained by taking the momenta (K, P_1, P_2, Q), $(K, -Q, P_2, -P_1)$ or $(K, -P_2, Q, -P_1)$ at physical values. For electroproduction $\gamma(K)$ in the initial state is replaced by $e(K_1)$ and $e(K_2)$ is added in the final state. The -sign in (5.11) is due to the anticommutation rules of the nucleon fields. For the $\mathcal{C}_{st}$-even and -odd amplitudes one can derive crossing symmetries. For instance from eqs. (5.1), (5.7) and

$$\mathcal{C}_{st}\pi^\alpha\,\mathcal{C}_{st}^{-1} = -(-1)^\alpha\pi^\alpha, \tag{5.12}$$

follows

$$\langle N(P_2),\pi^\alpha(Q)|T_1|N(P_1),\gamma(K)\rangle = (-1)^\alpha\langle \bar{N}(P_2),\pi^\alpha(Q)|T_1|\bar{N}(P_1),\gamma(K)\rangle. \tag{5.13}$$

Combined with the substitution rule, eqs.(5.9) and (5.11), this gives

$$\chi^+(2)\bar{u}(\vec{p}_2)[\sum_{i=1}^{6}(A_i^0(s,t,u)\tau_\alpha+A_i^+(s,t,u)\delta_{\alpha 3}+A_i^-(s,t,u)\tfrac{1}{2}[\tau_\alpha,\tau_3])\times$$

$$M_i(P,K,Q)]u(\vec{p}_1)\chi(1) = -(-1)^\alpha\chi^+(1)\bar{v}(\vec{p}_1)[\sum_{i=1}^{6}(A_i^0(u,t,s)\tau_\alpha+$$

$$A_i^+(u,t,s)\delta_{\alpha 3}+A_i^-(u,t,s)\tfrac{1}{2}[\tau_\alpha,\tau_3])M_i(-P,K,Q)]v(\vec{p}_2)\chi(2). \tag{5.14}$$

Use of the C-matrix (see appendix A) gives (~ stands for transposition)

$$v = C\,\tilde{\bar{u}}, \quad \bar{v} = -\tilde{u}\,C^{-1}. \tag{5.15}$$

The fact that the isospinors are real gives

$$\chi^\dagger(1)[A^0\tau_\alpha + A^+\delta_{\alpha 3} + A^-\tfrac{1}{2}[\tau_\alpha,\tau_3]]\chi(2) =$$

$$-(-1)^\alpha\chi^\dagger(2)[A^0\tau_\alpha + A^+\delta_{\alpha 3} - A^-\tfrac{1}{2}[\tau_\alpha,\tau_3]]\chi(1). \qquad (5.16)$$

Combining eqs.(5.14), (5.15) and (5.16) yields the crossing symmetry properties, that $A^{0,+}_{1,2,4}$ and $A^-_{3,5,6}$ are even under s and u interchange and $A^{0,+}_{3,5,6}$ and $A^-_{1,2,4}$ are odd under s and u interchange. Using the vector notation $\tilde{A}$ for the vector with components A_1, .. A_6 this crossing symmetry is expressed by

$$\tilde{A}\ (s,t,u) = [\xi]\ \tilde{A}\ (u,t,s), \qquad (5.17)$$

where

$$[\xi] = \xi \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}$$

where ξ = +1 for 0,+ isospin index and ξ = -1 for - isospin index. For the C_{st}-odd amplitudes the properties are just opposite, so $-\xi$ must be used instead of ξ.

6 UNITARITY AND THE SINGULARITIES

In this section a sketch is given of how the singularities of the amplitudes A_i are found. For a more detailed account of the principles involved, see the literature [12].

From the unitarity of the S-matrix

$$S^\dagger S = 1, \qquad (6.1)$$

and the definition of the T-matrix by

$$S = 1 + i\,T, \qquad (6.2)$$

it follows that

$$-i(T - T^{\dagger}) = T^{\dagger}T. \qquad (6.3)$$

By the definition (6.2) some numerical factors as well as the δ-function for the energy momentum conservation are absorbed in T. For the matrix elements this yields

$$\langle f|T|i\rangle - \langle f|T^{\dagger}|i\rangle = i\sum_n \langle f|T^{\dagger}|n\rangle\langle n|T|i\rangle, \qquad (6.4)$$

where the summation runs over all (possible) physical intermediate states. Olive [13)] has shown that when $\langle f|T|i\rangle$ is the boundary value of an analytic function for real values of the relevant physical variables, the opposite boundary value is given by $\langle f|T^{\dagger}|i\rangle$. In other words, for values of the variables for which the right hand side of eq.(6.4) vanishes there is no discontinuity in $\langle f|T|i\rangle$. When the right hand side of eq.(6.4) does not vanish, there is a discontinuity in $\langle f|T|i\rangle$, which is given by $\sum_n \langle f|T^{\dagger}|n\rangle\langle n|T|i\rangle$.

In the following, the initial state is a $|\gamma N\rangle$ state and the final state a $|\pi N\rangle$ state. When eq. (6.4) is considered in lowest order in the electromagnetic interaction, the intermediate states $|n\rangle$ will contain only strongly interacting particles, in particular no photons. This means that $\langle f|T^{\dagger}|n\rangle$ is a strong interaction matrix element. Thus the discontinuity of the photoproduction T-matrix is related to strong interactions. Explicit use of this will be made in section 14.

The T-matrix for photoproduction can again be decomposed in a $\mathcal{C}_{st}$-even and $\mathcal{C}_{st}$-odd part (see eqs.(5.1) and (5.2)). Because $\langle \pi N|T^{\dagger}|n\rangle$ is a $\mathcal{C}_{st}$-even function two equations are obtained from eq.(6.4)

$$\langle \pi N|T_1|\gamma N\rangle - \langle \pi N|T_1^{\dagger}|\gamma N\rangle = i\sum_n \langle \pi N|T^{\dagger}|n\rangle\langle n|T_1|\gamma N\rangle, \qquad (6.5)$$

$$\langle \pi N|T_2|\gamma N\rangle - \langle \pi N|T_2^{\dagger}|\gamma N\rangle = i\sum_n \langle \pi N|T^{\dagger}|n\rangle\langle n|T_2|\gamma N\rangle. \qquad (6.6)$$

These equations then give the discontinuities of the T_1 and T_2

parts. As $\mathcal{C}_{st}\mathcal{P}_{st}\mathcal{T}_{st}$ invariance is assumed and as $\mathcal{P}_{st}$ is conserved, both in strong and electromagnetic interactions, T_1 and T_2 are also $\mathcal{T}_{st}$-even and -odd functions. Thinking of $|\pi N\rangle$ and $|\gamma N\rangle$ states as states with definite angular momentum one can apply the $\mathcal{T}$-operation character of eqs.(6.5) and (6.6) to give

$$2\,\mathrm{Im}\langle\pi N|T_1|\gamma N\rangle = \sum_n \langle\pi N|T^\dagger|n\rangle\langle n|T_1|\gamma N\rangle, \tag{6.7}$$

$$-2i\,\mathrm{Re}\langle\pi N|T_2|\gamma N\rangle = \sum_n \langle\pi N|T^\dagger|n\rangle\langle n|T_2|\gamma N\rangle. \tag{6.8}$$

This implies that, for values of s below the value for which the sum over intermediate states contributes, T_1 is a real function and T_2 an imaginary function.

The invariant amplitudes A_i (or B_i) in terms of which T is expressed in eqs.(3.12) and (5.8) obey similar rules. Assume for instance that $\mathcal{T}$-invariance holds i.e. $T_2 = 0$. Then the processes $\gamma N \to \pi N$ and $\pi N \to \gamma N$ are related by

$$_{in}\langle\pi N|T_1|\gamma N\rangle_{in} = {}_{out}\langle\gamma' N'|T_1|\pi' N'\rangle_{out} = {}_{in}\langle\gamma' N'|T_1|\pi' N'\rangle_{in}, \tag{6.9}$$

where the prime denotes the $\mathcal{T}$-reflected state e.g.

$$\varepsilon'_\mu = -\zeta_\mu \varepsilon_\mu, \qquad P'_{1,2} = -\zeta_\mu P_{1,2}, \tag{6.10}$$

etc., with

$$\zeta_\mu = +1 \text{ for } \mu = 1,2,3, = -1 \text{ for } \mu = 4.$$

A priori $\pi N \to \gamma N$ is described by other amplitudes D_i than $\gamma N \to \pi N$

$$\langle\pi N|T_1|\gamma N\rangle = \bar{u}(\vec{p}_2)[\sum_{i=1}^{6} A_i(s,t)M_i]u(\vec{p}_1), \tag{6.11}$$

$$\langle\gamma N|T_1|\pi N\rangle = \bar{u}(\vec{p}_1)[\sum_{i=1}^{6} D_i(s,t)M_i]u(\vec{p}_2). \tag{6.12}$$

Application of eqs.(6.9) and (6.10) connects A_i and D_i by

$$A_i = D_i, \quad i = 1,3,4,6,$$
$$A_i = -D_i, \quad i = 2,5. \tag{6.13}$$

Eqs.(6.11) and (6.12), with the identification of eq.(6.13) can be inserted in the unitarity equation (6.5). A similar procedure can be followed for T_2. The result is that relations of the imaginary parts of all A_i to the sum of matrix elements and of the real parts of all A_i to the sum of matrix elements are obtained as in eq.(6.7) and eq.(6.8). Unfortunately the usual conventions lead to a relation for the real parts of A_i for the T_1 amplitudes and to a relation for the imaginary parts of the T_2 amplitudes. Inclusion of a factor i in the amplitudes can remove this unelegant convention. As an overall factor of i is immaterial for the physical results, one continues to use the A_i and M_i as defined in section 4, but uses nevertheless the discontinuity of the T_1 matrix element as given by the imaginary parts of the amplitudes $A_1, \ldots A_6$ and by the real parts of $A_7, \ldots A_{12}$ for the T_2 matrix element.

Assuming that the only singularities which occur in the amplitudes are given by the unitarity equation (6.4), or eqs. (6.7) and (6.8), one is led via the Cauchy theorem to expressions like

$$f(z) = \frac{1}{2\pi i} \int_a^\infty \frac{f(x'+i\varepsilon)-f(x'-i\varepsilon)}{x'-z}\, dx, \tag{6.14}$$

when $f(z)\to 0$, for $|z|\to\infty$. Here the branch cut extends from a to ∞.

For the $\mathcal{T}$-even amplitudes, it follows from the reality of the amplitudes below the cut that

$$f(z) = f^*(z^*), \tag{6.15}$$

and for the $\mathcal{T}$-odd amplitudes, it follows from the imaginary character of the amplitudes below the cut that

$$f(z) = -f^*(z^*). \tag{6.16}$$

In the former case eq.(6.14) reduces to

$$f(x+i\varepsilon) = \frac{1}{\pi}\int_a^\infty \frac{\mathrm{Im}\, f\,(x')}{x'-x-i\varepsilon}dx', \tag{6.17}$$

and in the latter case to

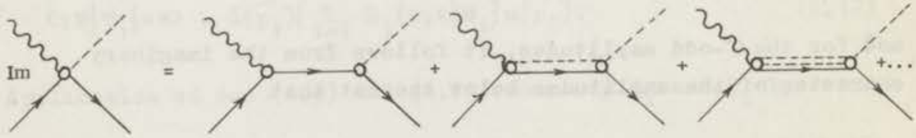
$$f(x+i\varepsilon) = \frac{1}{\pi i}\int_a^\infty \frac{\mathrm{Re}\, f\,(x')}{x'-x-i\varepsilon}dx'. \tag{6.18}$$

The last equation gives for the function $g(z) = i\, f(z)$ eq.(6.17) back. So it is useful to consider the amplitudes $A_1, \ldots A_6$ and $iA_7, \ldots iA_{12}$, which obey eq.(6.17). Moreover from eqs.(6.7) and (6.8) it is seen that the connection of the discontinuities of both sets of amplitudes to the strong interaction matrix elements $\langle \pi N | T^+ | n \rangle$ is the same.

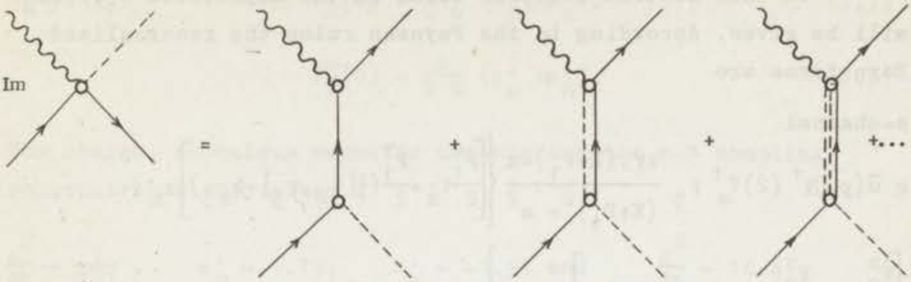
To find the discontinuity, one has to insert all possible physical states into eq.(6.7). The lowest possible invariant mass of the intermediate state is $(m + \mu)^2$, as this is the minimal physical value for s. For every new possible many-particle-state a new branchpoint is created. Although eq.(6.7) should be applied only for physical s-values, one can also use it as a recipe to find the pole terms (extended unitarity). For a one-particle intermediate state the discontinuity becomes a δ-function in s, giving rise to a pole.

Use of eq.(6.7) in the above sense for all three channels gives the singularities of the amplitudes $A_i(s,t,u)$ in the variables s,t and u, or as only two are independent variables, in s and t. This is presented diagrammatically in the following figures, where only the lowest possible intermediate states are indicated.

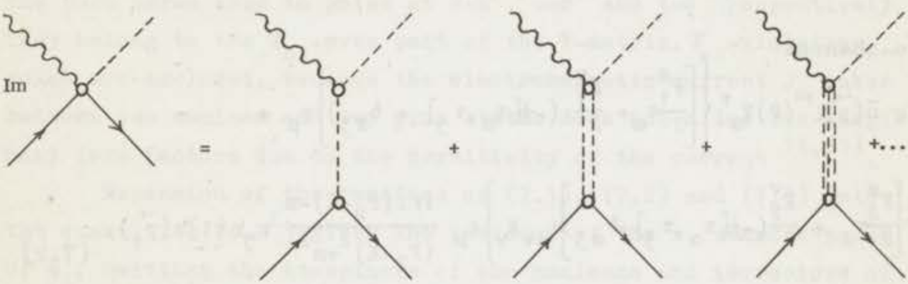
s-channel



u-channel



t-channel



One sees that poles arise in the s- and u-channel through one-nucleon states and cuts by π-N states. In the t-channel there is a pole due to the one-pion state and there are cuts due to the many-pion states. G-parity gives in the t-channel a restriction. For the isoscalar amplitudes the G-parity of the intermediate state must be +1, for the isovector ones -1, thus restricting the possible intermediate states to an even or odd number of pions. The two-pion state is often approximated by a ρ-meson state, the three-pion state by an ω-meson state.

Although the above discussion is given for photoproduction, everything can be repeated for electroproduction, starting from

$$\langle \pi N e^- | T | e^- N \rangle - \langle \pi N e^- | T^\dagger | e^- N \rangle = i \sum_n \langle \pi N e^- | T^\dagger | e^- n \rangle \langle n e^- | T | e^- N \rangle ,$$

where $\langle \pi N e^- | T^\dagger | e^- n \rangle$ is considered as a strong interaction matrix element i.e. the electron remains in the same state.

7 POLE TERMS

In this section the pole terms in the amplitudes $B_1, \ldots B_8$ will be given. According to the Feynman rules the renormalized Born terms are

s-channel

$$g\,\bar{u}(\vec{p}_2)\chi^{\dagger}(2)V_{\alpha}^{\dagger}\gamma_5\,\frac{i\gamma.(K+P_1)-m}{(K+P_1)^2+m^2}\left\{\left[\frac{F_1^s}{2}\tau_\alpha+\frac{F_1^v}{2}\left(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}\right)\right]\gamma_\mu+\right.$$

$$\left.\left[\frac{F_2^s}{2}\tau_\alpha+\frac{F_2^v}{2}\left(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}\right)\right]\sigma_{\mu\nu}K_\nu\right\}\varepsilon_\mu\chi(1)u(\vec{p}_1), \qquad (7.1)$$

u-channel

$$g\,\bar{u}(\vec{p})\chi^{\dagger}(2)V_{\alpha}^{\dagger}\left\{\left[\frac{F_1^s}{2}\tau_\alpha+\frac{F_1^v}{2}\left(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}\right)\right]\gamma_\mu+\right.$$

$$\left.\left[\frac{F_2^s}{2}\tau_\alpha+\frac{F_2^v}{2}\left(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}\right)\right]\sigma_{\mu\nu}K_\nu\right\}\varepsilon_\mu\,\frac{i\gamma.(P_2-K)-m}{(P_2-K)^2+m^2}\,\gamma_5\chi(1)u(\vec{p}_1), \qquad (7.2)$$

t-channel

$$i\,g\,\bar{u}(\vec{p}_2)\chi^{\dagger}(2)V_{\alpha}^{\dagger}\gamma_5\,\tfrac{1}{2}[\tau_\alpha,\tau_3]\frac{1}{(Q-K)^2+\mu^2}\left[2(\varepsilon.Q)-\varepsilon.K\right]F_\pi\chi(1)u(\vec{p}_1). \qquad (7.3)$$

Here $F_1^s(K^2)$, $F_1^v(K^2)$ are the isoscalar and isovector electric form factors, $F_2^s(K^2)$, $F_2^v(K^2)$ the isoscalar and isovector magnetic form factors, which are related to the electric and magnetic form factors of the neutron and proton by

$$F_{1,2}^v(K^2) = F_{1,2}^p(K^2) - F_{1,2}^n(K^2), \qquad (7.4)$$

$$F_{1,2}^s(K^2) = F_{1,2}^p(K^2) + F_{1,2}^n(K^2). \qquad (7.5)$$

The normalization of these form factors and of the pion electromagnetic form factor is given by

$$F_1^{v,s}(0) = F_\pi(0) = e, \qquad (7.6)$$

$$F_2^V(0) = \frac{e}{2\,m}\,(\mu'_p - \mu'_n), \tag{7.7}$$

$$F_2^S(0) = \frac{e}{2\,m}\,(\mu'_p + \mu'_n). \tag{7.8}$$

The charge, anomalous magnetic moments and the π-N coupling constant have the values

$$\frac{e^2}{4\pi} = \frac{1}{137}, \quad \mu'_p = 1.79, \quad \mu'_n = -1.91 \text{ and } \quad \frac{g^2}{4\pi} = 14.4.$$

The Born terms lead to poles at $s=m^2$, $u=m^2$ and $t=\mu^2$ respectively. They belong to the $\mathcal{T}_{st}$-even part of the T-matrix. $\mathcal{T}_{st}$-violating poles are excluded, because the electromagnetic current J_μ taken between two nucleon and two pion states does not allow for imaginary form factors due to the hermiticity of the current [14,11].

Expansion of the residues of (7.1), (7.2) and (7.3) into the quantities $(\varepsilon.N^i)$ gives the residues of the s, u and t poles of B_i. Omitting the isospinors of the nucleons and isovectors of the pion, one obtains the pole terms

$$(7.1):\ \frac{1}{s-m^2}\,\frac{g}{2}\,\{F_1^S\,\tau_\alpha + F_1^V(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3})\}\ \bar{u}(\vec{p}_2)\{\varepsilon.N^1-\varepsilon.N^2$$
$$-\tfrac{1}{2}(\varepsilon.N^3)-\tfrac{1}{2}(\varepsilon.N^4)\}\ u(\vec{p}_1) + \frac{1}{s-m^2}\,\frac{g}{2}\{F_2^S\tau_\alpha + F_2^V(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3})\}\ x$$
$$\bar{u}(\vec{p}_2)\,[\,2m(\varepsilon.N^1)-m(\varepsilon.N^4)+2(\varepsilon.N^6)+(\varepsilon.N^8)]\,u(\vec{p}_1), \tag{7.9}$$

$$(7.2):\ \frac{1}{u-m^2}\ \frac{g}{2}\{\,F_1^S\tau_\alpha + F_1^V(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3})\}\ \bar{u}(\vec{p}_2)[\,\varepsilon.N^1-\varepsilon.N^2+\tfrac{1}{2}(\varepsilon.N^3)$$
$$-\tfrac{1}{2}(\varepsilon.N^4)]\,u(\vec{p}_1) + \frac{1}{u-m^2}\,\frac{g}{2}\{\,F_2^S\tau_\alpha + F_2^V(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3})\}\ x$$
$$\bar{u}(\vec{p}_2)[\,2m(\varepsilon.N^1)-m(\varepsilon.N^4)+2(\varepsilon.N^6)-(\varepsilon.N^8)]\,u(\vec{p}_1), \tag{7.10}$$

$$(7.3):\ g\,\tfrac{1}{2}[\tau_\alpha,\tau_3]\ \frac{1}{t-\mu^2}[\,-(\varepsilon.N^3)+\tfrac{1}{2}(\varepsilon.N^4)]\,F_\pi. \tag{7.11}$$

From the poles in the amplitudes B_i the poles in the Dennery or F.N.W. set of amplitudes are obtained. Of course, one can expand the residues of (7.1), (7.2) and (7.3), at $s = m^2$, $u = m^2$ or

$t = \mu^2$ immediately into the appropriate M_i, thus finding the residues of the amplitudes A_i.

8 SPECTRAL REPRESENTATIONS

In this section the Mandelstam representation is postulated for the amplitudes B_i, in first instance without overall subtraction constants. Then compatibility with current conservation requires for some amplitudes an overall subtraction constant. After obtaining a consistent set of double spectral representations for the amplitudes B_i, one-dimensional dispersion relations can be derived for the amplitudes A_i. The discussion is restricted to C-even amplitudes and can easily be extended to the C-odd ones.

The amplitudes $B_i(s,t,u,K^2)$ are free from kinematical singularities as Ball[15] has shown for $K^2 = 0$. It seems that his argument can be repeated for $K^2 \neq 0$. So it makes sense to postulate for these amplitudes the Mandelstam representation, which takes into account all singularities, which are given by unitarity in the three channels

$$B_i(s,t)=\frac{R_s^i}{s-m^2}+\frac{R_t^i}{t-\mu^2}+\frac{R_u^i}{u-m^2}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\frac{\rho_s^i(s')}{s'-s}+$$
$$\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\frac{\rho_t^i(t')}{t'-t}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du'\frac{\rho_u^i(u')}{u'-u}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds'\int_{4\mu^2}^{\infty} dt'\frac{b_{st}^i(s',t')}{(s'-s)(t'-t)}$$
$$+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds'\int_{(m+\mu)^2}^{\infty} du'\frac{b_{su}^i(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} du'\int_{4\mu^2}^{\infty} dt'\frac{b_{ut}^i(u',t')}{(u'-u)(t'-t)}. \tag{8.1}$$

The dependence of K^2 and isospin is not shown explicitly and u is a function of s and t, as given by eq.(2.4). As suggested by perturbation theory, the amplitudes which have a pole in one of the variables also, in general, will have a one-dimensional spectral representation in that variable. The regions, where b_{st}, b_{su} and b_{ut} are non zero, have been given by Ball [15]. The t-integration is for isoscalar amplitudes from $4\mu^2$ as indicated in eq.(8.1), but from $9\mu^2$ for isovector amplitudes $B_i^{\pm}$.

From eqs.(7.9), (7.10) and (7.11) the residues are found to be

$$R_s^i = R_{sc}^i + R_{sa}^i , \tag{8.2}$$

with

$$R_{sc}^1 = -R_{sc}^2 = -2R_{sc}^3 = -2R_{sc}^4 = \frac{g}{2} F_1^s,\ \frac{g}{2} F_1^v,\ \frac{g}{2} F_1^v,$$

$$R_{sa}^1 = -2R_{sa}^4 = mR_{sa}^6 = 2mR_{sa}^8 = mgF_2^s,\ mgF_2^v,\ mgF_2^v,$$

for (0), (+) and (-) amplitudes respectively.

In an analogous way

$$R_u^i = R_{uc}^i + R_{ua}^i , \tag{8.3}$$

with

$$R_{uc}^1 = -R_{uc}^2 = 2R_{uc}^3 = -2R_{uc}^4 = \frac{g}{2} F_1^s,\ \frac{g}{2} F_1^v,\ -\frac{g}{2} F_1^v,$$

$$R_{ua}^1 = -2R_{ua}^4 = mR_{ua}^6 = -2mR_{ua}^8 = mg\ F_2^s,\ mg\ F_2^v,\ -mg\ F_2^v,$$

for (0), (+) and (-) amplitudes respectively and

$$R_t^3 = -2R_t^4 = -g\ F_\pi , \tag{8.4}$$

for the (-) amplitudes. All other residues are zero. Therefore one also knows from eqs.(8.2), (8.3) and (8.4) which one-dimensional functions can be present.

From eq.(8.1) the subtraction constants in fixed-t and fixed-s dispersion relations (i.e. the value of $B_i(s,t)$ for $s\to\infty$ or $t\to\infty$) are given by

$$C_i(t) = \frac{R_t^i}{t-\mu^2} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\rho t^i(t')}{t'-t} , \tag{8.5}$$

and

$$C_i(s) = \frac{R_s^i}{s-m^2} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{\rho_s^i(s')}{s'-s} . \tag{8.6}$$

The question arises, whether eq.(8.1) is compatible with the current conservation equations

$$\frac{K^2}{2} B_1 + P.K\, B_2 + Q.K\, B_3 + K^2 B_4 = 0, \qquad (8.7)$$

$$B_5 + P.K\, B_6 + K^2 B_7 + Q.K\, B_8 = 0. \qquad (8.8)$$

The l.h.s. of these two equations consists of a combination of pole terms, one-dimensional spectral representations and two-dimensional spectral representations. It turns out that the combination of pole terms for (-) amplitudes gives rise to a constant term.

For eq. (8.7):

$$\frac{K^2}{2} R_s^1\left\{\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right\} - R_c^1 \frac{u-s}{4}\left\{\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right\} - \tfrac{1}{2}Q.K\, R_{sc}^1\left\{\frac{1}{s-m^2} \mp \frac{1}{u-m^2}\right\} - \frac{K^2}{2} R_{sc}^1\left\{\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right\} - \frac{K^2}{2} R_{sa}^1\left\{\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right\} +$$

$$(-2K.Q + K^2)\frac{R_t^4}{t-\mu^2} = 0,\ 0,\ -\frac{g}{2}(F_\pi - F_1^v), \qquad (8.9)$$

for (+), (0) and (-) cases respectively.

For eq.(8.8):

$$\frac{u-s}{4} R_{sa}^6\left\{\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right\} + \frac{Q.K}{2} R_{sa}^6\left\{\frac{1}{s-m^2} \mp \frac{1}{u-m^2}\right\} = 0, 0, -gF_2^v, \qquad (8.10)$$

for (+), (0) and (-) cases respectively. Similar equations for the one-dimensional spectral representations arise. This is due to crossing symmetry and the fact that the relation between all $\rho_i(s)$, all $\rho_i(t)$ and all $\rho_i(u)$ is the same as the relation between the residues. Compatibility with eqs.(8.7) and (8.8) is obtained only, when all one-dimensional representations are taken to be zero. As for the double spectral representations, by appropriate choice of the density functions, agreement with current conservation can be obtained. Thus compatibility with eq. (8.7) and eq.(8.8) requires the introduction of an overall subtraction constant in two of the (-) amplitudes. In eq.(8.7) a constant must be added to one of the amplitudes in order to cancel $-\frac{g}{2}(F_\pi - F_V)$. As all B_i are free from kinematical singularities, this constant must be added to B_1^- or B_4^-. It can be added only to a function even under interchange of s and u. Thus B_4^- has to contain an overall subtraction constant $\frac{g}{2}\,\frac{F_\pi - F_1^V}{2}$. Similarly B_5^- or B_7^- must have an overall subtraction constant $g\,F_2^V$ or $g\,\frac{F_2^V}{K^2}$ respectively. As B_5^- is eliminated, B_5^- is chosen to contain this subtraction constant.

Thus the representation (8.1) is compatible with current conservation after inclusion of overall subtraction constants in B_4^- and B_5^- and leaving out all one-dimensional dispersion relations. One can then choose the double spectral functions of B_3 and B_5 in such a way that for given B_i $(i \neq 3,5)$ eqs.(8.7) and (8.8) hold. Only when Q.K = 0, there is a condition to be fulfilled between the other B_i

$$\frac{K^2}{2} B_1 + P.K\, B_2 + K^2 B_4 = 0. \tag{8.11}$$

This will be the case for the fixed-t dispersion relations, which will be used hereafter.

To summarize one has to use the Mandelstam representation

$$B_i(s,t) = C^i + \frac{R_s^i}{s-m^2} + \frac{R_t^i}{t-\mu^2} + \frac{R_u^i}{u-m^2} + \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \, \frac{b_{st}^i(s',t')}{(s'-s)(t'-t)} +$$

$$\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{(m+\mu)^2}^{\infty} du' \, \frac{b_{su}^i(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} du' \int_{4\mu^2}^{\infty} dt' \frac{b_{ut}^i(u',t')}{(u'-u)(t'-t)}, \tag{8.12}$$

where all quantities are defined in eqs.(8.2), (8.3) and (8.4), except C^i, which is zero for all B_i but B_4^- and B_5^-

$$C^{4-} = \frac{g}{2}\,\frac{F_\pi - F_1^V}{2}, \tag{8.13}$$

$$C^{5-} = g\,F_2^V. \tag{8.14}$$

From the Mandelstam representation (8.12), the Mandelstam representation for A_1, A_3, A_4 and A_6 can be obtained via eq.(4.19).

Without further assumptions, this is not possible for A_2 and A_5, as they contain kinematical singularities in the t-variable. Fixed-t dispersion relations can be obtained for all A_i, using eq.(8.12). Subtraction constants $C^i + \frac{R^i_t}{t-\mu^2}$ occur. The resulting one-dimensional representation is given in the next section.

9 FIXED-t DISPERSION RELATIONS

In this section the fixed-t dispersion relations, which will be used hereafter, will be written in a compact form. From the arguments in section 8, from the pole terms in section 7 and by using crossing symmetry one obtains

$$\mathrm{Re}\ \tilde{A}(s,t) = \{\frac{1}{s-m^2} + [\xi]\ \frac{1}{u-m^2}\}\ \tilde{\Gamma}(t) + \frac{(1-\xi)}{2}\ \frac{\tilde{\Gamma}_t}{t-\mu^2} +$$

$$\frac{P}{\pi} \int_{(m+\mu)^2}^{\infty} ds'\ \{\frac{1}{s'-s} + [\xi]\ \frac{1}{s'-u}\}\ \mathrm{Im}\ \tilde{A}(s',t), \tag{9.1}$$

where the vector notation for A_i and the matrix $[\xi]$ from section 5 are used. The residue vector $\tilde{\Gamma}(t)$ has elements

$$\begin{aligned}
\Gamma^{\pm,0}_1 &= \frac{g}{2}\ F^{v,s}_1(K^2)\ , \\
\Gamma^{\pm,0}_2 &= -g\ \frac{F^{v,s}_1(K^2)}{t-\mu^2}\ , \\
\Gamma^{\pm,0}_3 &= \Gamma^{\pm,0}_4 = -\frac{g}{2}\ F^{v,s}_2(K^2), \\
\Gamma^{\pm,0}_5 &= \tfrac{1}{2}\ \Gamma^{\pm,0}_2\ , \\
\Gamma^{\pm,0}_6 &= 0.
\end{aligned} \tag{9.2}$$

$\tilde{\Gamma}_t$ is zero except for the 5th element

$$\Gamma_t = \frac{2g}{K^2}\ [F_\pi(K^2) - F^v_1(K^2)]. \tag{9.3}$$

For the F.N.W. set $\frac{1}{t-\mu^2}$ in $\Gamma_2^{\pm,0}$ must be changed into $\frac{1}{t-\mu^2+K^2}$, while $\Gamma_5^{\pm,0}$ vanishes and Γ_t becomes $\frac{2g}{K^2}\{\frac{F_\pi}{t-\mu^2} - \frac{F_1^v}{t-\mu^2+K^2}\}$. This set of equations satisfies eq.(4.20) as can be seen by multiplication of A_2' and A_5' by Q.K. The pole terms satisfy eq.(4.20), so the whole equation satisfies eq.(4.20). This may also be noticed in the following way. As

$$M_2' = 2i\gamma_5\,[(P.\varepsilon)(Q.K)-(P.K)(Q.\varepsilon)] = -2i\gamma_5(P.K)(Q.\varepsilon),$$

$$M_5' = i\gamma_5\,[(K.\varepsilon)(Q.K)-K^2(Q.\varepsilon)] = -i\gamma_5\,K^2(Q.\varepsilon),$$

for Q.K = 0, A_2' and A_5' have an angular momentum decomposition such that eq.(4.20) is fulfilled. Therefore the set (9.1) satisfies eq.(4.20) for F.N.W. amplitudes and also for A_2 and A_5, as these can be obtained from the former ones.

In eq.(9.1) $\mathcal{C}$- (and $\mathcal{T}$-)invariance is assumed as will be done in the following sections. $\mathcal{C}$- (and $\mathcal{T}$-)odd amplitudes obey similar equations as eq.(9.1) without pole terms and with ξ replaced by $-\xi$.

10 COMMENTS

In this section some comments are made. In the first place on the standard procedure in the literature for the introduction of poles and subtraction constants. Secondly, the possibility of fixed-s dispersion relations is discussed.

The questions of pole terms and subtraction constants are in general two different questions. The poles are obtained from an expansion of the renormalized Born approximation, where the residue must be taken at the pole value. The question of subtraction is in most cases just an assumption. In the preceding section it was shown that current conservation entails the introduction of subtractions.

In the case of electro- and photoproduction the poles can be obtained in two ways. From the poles in B_i one obtains the poles in A_i, or one expands every Born approximation on its own in terms of the M_i at the pole value and one finds the same poles. This latter expansion is possible because at the pole value each Born approximation is conserved, as the exchanged particle is on the mass-shell.

The standard procedure in the literature is different. One takes the general Born approximation, consisting of three terms. This function is not expandable in the M_i because the general Born approximation is not conserved. Only in case of photoproduction it is conserved and the expansion of the total Born approximation in the M_i is possible, although not for each term separately. Thus one gets the impression that the t-pole is necessary for obtaining fixed-t dispersion relations in photoproduction [16]. This is not the case, although its presence in the t-channel plays a role via current conservation (eq.(8.7)). For electroproduction at $K^2 \neq 0$ one has to use a trick [10], when one wants to expand the total Born approximation in terms of the M_i. In fact, replacing ε by K in the total Born approximation one obtains

$$-i\gamma_5 g[F_\pi - F_1^V]\tfrac{1}{2}[\tau_\alpha, \tau_3]. \qquad (10.1)$$

One then adds a term [10]

$$i\gamma_5 \frac{g}{K^2}[F_\pi - F_1^V]\tfrac{1}{2}[\tau_\alpha, \tau_3](\varepsilon.K), \qquad (10.2)$$

to the total Born approximation, using the argument that it does not contribute as $\varepsilon.K = 0$. Expansion of the thus completed Born approximation gives the same pole terms and subtraction constants as in eq.(9.1). This is understandable, as eq.(8.9) is similar to a current conservation condition for the Born approximation. However, there is no reason that the general renormalized Born approximation must be conserved, as all orders of strong interactions are occurring. Moreover hadron current conservation has nothing to do with the condition $\varepsilon.K = 0$. This condition does not hold for the lepton vertex in neutrinoproduction of pions. Nevertheless, from the C.V.C. theory, one expects similar dispersion relations for electroproduction and the vector part of neutrinoproduction. In the method presented in section 8, this is automatically the case, but not for the trick of eq.(10.2).

The second comment is on the possibility of fixed-s dispersion relations. As the Mandelstam representation holds for A_1, A_3, A_4 and A_6, fixed-s dispersion relations can be derived for them. For A_2 and A_5, a difficulty arises from the kinematical singularity at $t=\mu^2$. Both amplitudes are written as a function which is a linear combination of amplitudes B_i divided by $t-\mu^2$. If one writes a dispersion relation for the numerator subtracted at $t=\mu^2$, one obtains

$$\mathrm{Re}\, A_2(s,t) = 2\,\frac{\mathrm{Re}\, B_2(s,t=\mu^2)}{t-\mu^2} + \frac{2R_u^2}{(u-m^2)(m^2-s-K^2)} +$$

$$\frac{P}{\pi}\int_{(m+\mu)^2}^{\infty} du'\,\frac{\mathrm{Im}\, A_2(s,u')}{u'-u} + \frac{P}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{\mathrm{Im}\, A_2(s,t')}{t'-t}\,, \tag{10.3}$$

$$\mathrm{Re}\, A_5(s,t) = \frac{[2(s-m^2)+K^2]\,\mathrm{Re}\, B_2(s,t=\mu^2)}{(t-\mu^2)K^2} + \frac{2gF_\pi}{K^2(t-\mu^2)} -$$

$$\frac{R_u^2}{(u-m^2)(m^2-s-K^2)} + \frac{P}{\pi}\int_{(m+\mu)^2}^{\infty} du'\,\frac{\mathrm{Im}\, A_5(s,u')}{u'-u} + \frac{P}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{\mathrm{Im}\, A_5(s,t')}{t'-t}\,. \tag{10.4}$$

Unless one knows $B_2(s,t=\mu^2)$, these relations are not of much interest. In the case of photoproduction, $B_2(s,t=\mu^2)$ is known from current conservation eq.(8.7)

$$\frac{m^2-s}{2}\, B_2(s,t=\mu^2) + (-2K.Q+K^2)\,\frac{R_t^4}{t-\mu^2} = 0, \tag{10.5}$$

$$B_2(s,t=\mu^2) = -\,\frac{ge}{s-m^2}\,. \tag{10.6}$$

So eq.(10.6) inserted in eq.(10.3) gives a fixed-s dispersion relation for A_2 in the case of photoproduction. The above procedure to subtract at the kinematical singularity leads to difficulties for A_2' and A_5', because $t=\mu^2-K^2$ may lie in the u' integration region.

CHAPTER III

MULTIPOLE AMPLITUDES

SUMMARY

In this chapter the fixed-t dispersion relations (eq. (9.1)) will be transformed into multipole dispersion relations. The latter are appropriate for the application of unitarity, which is discussed in section 14. In order to perform this projection of the fixed-t dispersion relations, an angular momentum analysis of the amplitudes A_i is needed. This is discussed in section 11. The angular momentum decomposition is formally applied in section 12 to give the multipole dispersion relations. The projected Born terms are given in section 13. In section 14 a condition from unitarity is derived, which in conjunction with multipole dispersion relations opens the possibility for a solution of the problem.

11 MULTIPOLE DECOMPOSITION

In order to find the angular momentum decomposition two-component Pauli spinors are introduced in the π-N centre-of-mass system. The T-matrix is then expressed with the help of a quantity F by

$$\bar{u}(\vec{p}_2)\sum_{i=1}^{6} A_i M_i \, u(\vec{p}_1) = \chi^{\dagger}(2)F\chi(1), \qquad (11.1)$$

for electroproduction and by

$$\bar{u}(\vec{p}_2)\sum_{i=1}^{4} A_i M_i \; u(\vec{p}_1) = \frac{4\pi W}{m}\chi^{\dagger}(2) F \chi(1), \qquad (11.2)$$

for photoproduction. The quantities χ are Pauli spinors for the case where the z-axis (along $\vec{p}_1$) is the axis of quantization. A different normalization for F is used in both cases in order to conform to the literature. In the following, the normalization of electroproduction is used, unless stated differently.

Just as for the invariant T-matrix a general form for F can be given, now using instead of γ-matrices the Pauli σ-matrices

$$F = i\vec{\sigma}.\vec{\varepsilon}\, F_1 + \vec{\sigma}.\hat{q}\;\vec{\sigma}.(\hat{k}\times\vec{\varepsilon})F_2 + i\,\vec{\sigma}.\hat{k}\;\hat{q}.\vec{\varepsilon}\, F_3 + i\,\vec{\sigma}.\hat{q}\;\hat{q}.\vec{\varepsilon}\, F_4 +$$

$$i\,\vec{\sigma}.\hat{k}\;\hat{k}.\vec{\varepsilon}\, F_5 + i\,\vec{\sigma}.\hat{q}\;\hat{k}.\vec{\varepsilon}\, F_6 - i\,\vec{\sigma}.\hat{q}\;\varepsilon_0\, F_7 - i\,\vec{\sigma}.\hat{k}\;\varepsilon_0\, F_8 . \qquad (11.3)$$

Again current conservation gives two equations

$$F_1 + \hat{k}.\hat{q}\, F_3 + F_5 - \frac{k_0}{k} F_8 = 0, \qquad (11.4)$$

$$\hat{k}.\hat{q}\, F_4 + F_6 - \frac{k_0}{k} F_7 = 0. \qquad (11.5)$$

One can eliminate F_7 and F_8 to get

$$F = i\,\vec{\sigma}.\vec{a}\, F_1 + \vec{\sigma}.\hat{q}\;\vec{\sigma}.(\hat{k}\times\vec{a})F_2 + i\,\vec{\sigma}.\hat{k}\;\hat{q}.\vec{a}\, F_3 + i\,\vec{\sigma}.\hat{q}\;\hat{q}.\vec{a}\, F_4 +$$

$$i\,\vec{\sigma}.\hat{k}\;\hat{k}.\vec{a}\, F_5 + i\,\vec{\sigma}.\hat{q}\;\hat{k}.\vec{a}\, F_6 , \qquad (11.6)$$

where

$$a_\mu = \varepsilon_\mu - \frac{\varepsilon_0}{k_0} K_\mu . \qquad (11.7)$$

This is the choice of Dennery, which amounts to virtual photons with transverse and longitudinal components. On the other hand, elimination of F_5 and F_6 gives

$$F = i\,\vec{\sigma}.\vec{b}\,F_1 + \vec{\sigma}.\hat{q}\,\vec{\sigma}.(\hat{k}\times\vec{b})F_2 + i\,\vec{\sigma}.\vec{k}\,\hat{q}.\vec{b}\,F_3 + i\,\vec{\sigma}.\hat{q}\,\hat{q}.\vec{b}\,F_4$$

$$- i\,\vec{\sigma}.\hat{q}\,b_0\,F_7 - i\,\vec{\sigma}.\hat{k}\,b_0\,F_8, \qquad (11.8)$$

where

$$b_\mu = \varepsilon_\mu - \frac{\vec{\varepsilon}.\vec{k}}{k}\,K_\mu\,. \qquad (11.9)$$

With this choice, one has transverse and scalar virtual photons. This choice is adopted in the rest of this chapter because of the simplicity of the angular momentum decomposition of F_7 and F_8 versus the one of F_5 and F_6.

Of course both choices can simply be obtained from eq. (11.3) by the appropriate addition of λk_μ to ε_μ, which is allowed by current conservation. Such an addition changes ε_μ into a_μ or b_μ, so that $a.k = b.k = 0$ is no longer valid for electroproduction. For photoproduction it still holds, expressing the transversality of the real photon.

The connection between Dirac and Pauli spinors (appendix A) relates the invariant amplitudes A_i and the centre-of-mass quantities F_i. In matrix notation

$$\widetilde{F}(s,t) = [C^{-1}(s)][B(s,t)]\,\widetilde{A}(s,t), \qquad (11.10)$$

$$\widetilde{A}(s,t) = [B^{-1}(s,t)][C(s)]\,\widetilde{F}(s,t), \qquad (11.11)$$

where F_1, F_2, F_3, F_4, F_7, F_8 are the components of $\widetilde{F}$ and the matrices $[B]$ and $[B^{-1}]$ are

$$
[B(s,t)] = \left[\begin{array}{c|c|c|c|c|c}
1 & 0 & -\frac{K.Q}{W-m} & W-m+\frac{K.Q}{W-m} & 0 & \frac{K^2}{W-m} \\
\hline
-1 & 0 & -\frac{K.Q}{W+m} & W+m+\frac{K.Q}{W+m} & 0 & \frac{K^2}{W+m} \\
\hline
0 & \frac{1}{W+m}\left[W^2-m^2+\frac{K^2}{2}\right] & 1 & -1 & -\frac{K^2}{W+m} & 0 \\
\hline
0 & -\frac{1}{W-m}\left[W^2-m^2+\frac{K^2}{2}\right] & 1 & -1 & \frac{K^2}{W-m} & 0 \\
\hline
-(E_1-m) & \begin{array}{l}\frac{1}{2k_o}\left[k^2(3K.Q+2k_oW)\right. \\ \left.-\vec{k}.\vec{q}(2(W^2-m^2)+K^2)\right]\end{array} & \begin{array}{c}K.Q \\ +q_o(W-m)\end{array} & \begin{array}{c}-K.Q-q_o(W-m) \\ +(E_1-m)(W+m)\end{array} & \begin{array}{c}q_oK^2 \\ -k_oK.Q\end{array} & -(E_1-m)(W+m) \\
\hline
E_1+m & \begin{array}{l}-\frac{1}{2k_o}\left[k^2(3K.Q+2k_oW)\right. \\ \left.-\vec{k}.\vec{q}(2(W^2-m^2)+K^2)\right]\end{array} & \begin{array}{c}K.Q \\ +q_o(W+m)\end{array} & \begin{array}{c}-K.Q-q_o(W+m) \\ +(E_1+m)(W-m)\end{array} & \begin{array}{c}-q_oK^2 \\ +k_oK.Q\end{array} & -(E_1+m)(W-m)
\end{array}\right] \qquad (11.13)
$$

$$
[B^{-1}(s,t)] =
\begin{array}{r|c|c|c|c|c|c|}
\hline
\frac{1}{4W^2k_o} & (W-m)\left[(W+m)^2 - K^2 - 2m\frac{K^2}{k^2}(E_1+m)\right] & -(W+m)\frac{k_o}{E_1+m} \times \left[W^2-m^2+K^2\right] & 2m(W+m) \times \left[K.Q - \frac{K^2}{k^2}\vec{q}.\vec{k}\right] & 2m(W-m) \times \left[K.Q - \frac{K^2}{k^2}\vec{q}.\vec{k}\right] & 2mK^2\frac{k_o}{k^2} & 2mK^2\frac{k_o}{k^2} \\
\hline
\frac{K^2}{2k_oW^2(t-\mu^2)} & (W-m)\left[1 - \frac{(E_1+m)(W-m)}{k^2}\right] & k_o\frac{(W+m)}{E_1+m} & (W^2-m^2) \times \left[\frac{K.Q}{K^2} - \frac{\vec{q}.\vec{K}}{k^2}\right] & (W^2-m^2) \times \left[-\frac{K.Q}{K^2} + \frac{\vec{q}.\vec{K}}{k^2}\right] & -(W+m)\frac{k_o}{k^2} & (W-m)\frac{k_o}{k^2} \\
\hline
\frac{1}{4W^2k_o} & (W-m)\left[W+m - (E_1+m)\frac{K^2}{k^2}\right] & k_o\frac{(W+m)^2}{E_1+m} & (W+m) \times \left[K.Q + 2k_oW - \frac{K^2}{k^2}\vec{q}.\vec{K}\right] & (W-m) \times \left[K.Q + 2k_oW - \frac{K^2}{k^2}\vec{q}.\vec{K}\right] & K^2\frac{k_o}{k^2} & K^2\frac{k_o}{k^2} \\
\hline
\frac{1}{4W^2k_o} & (W-m)\left[W+m - (E_1+m)\frac{K^2}{k^2}\right] & k_o\frac{(W+m)^2}{E_1+m} & (W+m) \times \left[K.Q - \frac{K^2}{k^2}\vec{q}.\vec{K}\right] & (W-m) \times \left[K.Q - \frac{K^2}{k^2}\vec{q}.\vec{K}\right] & K^2\frac{k_o}{k^2} & K^2\frac{k_o}{k^2} \\
\hline
\frac{\beta}{2W^2k_o(t-\mu^2)} & (W-m)\left[1 - \frac{(E_1+m)(W-m)}{k^2}\right] & k_o\frac{(W+m)}{E_1+m} & (W^2-m^2) \times \left[\delta/\beta - \frac{\vec{q}.\vec{K}}{k^2}\right] & (W^2-m^2) \times \left[-\delta/\beta + \frac{\vec{q}.\vec{K}}{k^2}\right] & -(W+m)\frac{k_o}{k^2} & (W-m)\frac{k_o}{k^2} \\
\hline
\frac{W^2-m^2}{4W^2k_o} & -1 + \frac{(E_1+m)(W-m)}{k^2} & \frac{k_o}{E_1+m} & -\frac{K.Q}{W-m} + (W+m)\frac{\vec{q}.\vec{K}}{k^2} & -\frac{K.Q}{W+m} + (W-m)\frac{\vec{q}.\vec{K}}{k^2} & -\frac{k_o}{k^2} & -\frac{k_o}{k^2} \\
\hline
\end{array}
\qquad (11.14)
$$

where $\delta = \frac{3}{2}K.Q+k_oW$ and $\beta = W^2-m^2+\frac{K^2}{2}$. The diagonal matrix $[C]$ is given by

$$[C(s)] = \frac{2m}{(W-m)[(E_1+m)(E_2+m)]^{\frac{1}{2}}} \begin{bmatrix} 1 & & & & & \\ & \frac{(E_1+m)(E_2+m)(W-m)}{qk(W+m)} & & & & \\ & & \frac{(E_1+m)(W-m)}{qk(W+m)} & & & \\ & & & \frac{(E_2+m)}{q^2} & & \\ & & & & \frac{(E_2+m)(W-m)}{q} & \\ & & & & & \frac{(E_1+m)(W-m)}{k} \end{bmatrix}. \tag{11.12}$$

Inspection of eqs.(11.12) and (11.13) reveals the properties

$$\begin{aligned} C_1(W) &= -C_2(-W), \quad B_{1j}(W) = -B_{2j}(-W), \quad j = 1,\ldots 6, \\ C_3(W) &= C_4(-W), \quad B_{3j}(W) = B_{4j}(-W), \quad j = 1,\ldots 6, \\ C_5(W) &= C_6(-W), \quad B_{5j}(W) = B_{6j}(-W), \quad j = 1,\ldots 6. \end{aligned} \tag{11.15}$$

By a lengthy but straightforward calculation [19] one can connect $\widetilde{F}$ with the eigenamplitudes of parity and angular momentum.

For electroproduction there are six types of transitions possible with orbital angular momentum ℓ and definite parity. They are classified according to the character of the photon (transverse or scalar) and the total angular momentum $J=\ell\pm\frac{1}{2}$ of the final π-N state. The transverse photon states can be either electric or magnetic with parity $(-1)^L$ and $(-1)^{L+1}$ respectively, where L is the total angular momentum of the photon. In this way, one obtains the following table (in photoproduction only the transverse states contribute):

Table I

Multipole states for electro- and photoproduction

J	L	Parity $-(-1)^{\ell}$	Multipole transition	Notation	Lowest value of ℓ permitted
$\ell+\frac{1}{2}$	$L=J+\frac{1}{2}=\ell+1$	$(-1)^L$	electric $2^{\ell+1}$	$E_{\ell+}$	0
$\ell-\frac{1}{2}$	$L=J-\frac{1}{2}=\ell-1$	$(-1)^L$	electric $2^{\ell-1}$	$E_{\ell-}$	2
$\ell+\frac{1}{2}$	$L=J-\frac{1}{2}=\ell$	$-(-1)^L$	magnetic 2^{ℓ}	$M_{\ell+}$	1
$\ell-\frac{1}{2}$	$L=J+\frac{1}{2}=\ell$	$-(-1)^L$	magnetic 2^{ℓ}	$M_{\ell-}$	1
$\ell+\frac{1}{2}$	$L=J+\frac{1}{2}=\ell+1$	$(-1)^L$	scalar $2^{\ell+1}$	$S_{\ell+}$	0
$\ell-\frac{1}{2}$	$L=J-\frac{1}{2}=\ell-1$	$(-1)^L$	scalar $2^{\ell-1}$	$S_{\ell-}$	1

Sometimes longitudinal multipoles $L_{\ell\pm}$ are used in the literature, which are related to the scalar ones by

$$L_{\ell\pm} = \frac{k_o}{k} S_{\ell\pm} \; . \qquad (11.16)$$

These longitudinal multipoles must have zeros for $k_o=0$. This, however, will not be the case in non-analytical calculations. Therefore, singularities in terms like $\frac{L_{\ell\pm}}{k_o}$ could occur in the numerically computed cross-sections. It is because of this that scalar multipoles are used here.

The result of the angular momentum decomposition in matrix notation is

$$\widetilde{F} = \sum_{\ell=0}^{\infty} \begin{bmatrix} G_\ell(x) & 0 \\ 0 & H_\ell(x) \end{bmatrix} \widetilde{M}_\ell(s), \qquad (11.17)$$

where the vector $\widetilde{M}_\ell$ has as its components $E_{\ell+}$, $E_{\ell-}$, $M_{\ell+}$, $M_{\ell-}$, $S_{\ell+}$, $S_{\ell-}$ and where G_ℓ and H_ℓ are the matrices

$$G_\ell = \begin{bmatrix} P'_{\ell+1} & P'_{\ell-1} & \ell P'_{\ell+1} & (\ell+1)P'_{\ell-1} \\ 0 & 0 & (\ell+1)P'_\ell & \ell P'_\ell \\ P''_{\ell+1} & P''_{\ell-1} & -P''_{\ell+1} & P''_{\ell-1} \\ -P''_\ell & -P''_\ell & P''_\ell & -P''_\ell \end{bmatrix}, \qquad (11.18)$$

$$H_\ell = \begin{bmatrix} -(\ell+1)P'_\ell & \ell P'_\ell \\ (\ell+1)P'_{\ell+1} & -\ell P'_{\ell-1} \end{bmatrix}. \qquad (11.19)$$

Here derivatives $P'_\ell(x)$ of Legendre polynomials occur. The argument x is $\cos\theta$, which is related to t and s (eq.(2.7)). Now eq.(11.17) may be inverted using the relations

$$P'_{2\ell}(x) = \sum_{k=1,3,\ldots}^{2\ell-1} (2k+1)P_k(x) , \qquad (11.20a)$$

$$P'_{2\ell+1}(x) = \sum_{k=0,2,\ldots}^{2\ell} (2k+1)P_k(x) , \qquad (11.20b)$$

for $\ell=0,1,2,\ldots$ The result is

$$\widetilde{M}_\ell(s) = \int_{-1}^{+1} dx \begin{bmatrix} D_\ell(x) & 0 \\ 0 & E_\ell(x) \end{bmatrix} \widetilde{F}(s,t), \qquad (11.21)$$

where

$$D_\ell = \begin{bmatrix} \frac{1}{2(\ell+1)}\{P_\ell, -P_{\ell+1}, \frac{\ell}{2\ell+1}(P_{\ell-1}-P_{\ell+1}), \frac{\ell+1}{2\ell+3}(P_\ell-P_{\ell+2})\} \\ \frac{1}{2\ell}\{P_\ell, -P_{\ell-1}, \frac{\ell+1}{2\ell+1}(P_{\ell+1}-P_{\ell-1}), \frac{\ell}{2\ell-1}(P_\ell-P_{\ell-2})\} \\ \frac{1}{2(\ell+1)}\{P_\ell, -P_{\ell+1}, \frac{1}{2\ell+1}(P_{\ell+1}-P_{\ell-1}), 0\} \\ \frac{1}{2\ell}\{-P_\ell, P_{\ell-1}, \frac{1}{2\ell+1}(P_{\ell-1}-P_{\ell+1}), 0\} \end{bmatrix}, \qquad (11.22)$$

and

$$E_\ell = \begin{bmatrix} \frac{1}{2(\ell+1)}\{P_{\ell+1}, P_\ell\} \\ \frac{1}{2\ell}\{P_{\ell-1}, P_\ell\} \end{bmatrix} \qquad (11.23)$$

The multipole formalism has been chosen here, because this is the usual description in the literature [6,10]. However, the helicity formalism of Jacob and Wick [20] is useful in some particular applications. So therefore the connection between the two formalisms is given in appendix C.

12 MULTIPOLE DISPERSION RELATIONS

In this section the dispersion relations satisfied by the multipole amplitudes $\widetilde{M}_\ell$ are derived from the considerations of sections 9 and 11.

From the fixed-t dispersion relation eq.(9.1) and eqs. (11.10), (11.11), (11.17) and (11.21) one finds for $\widetilde{M}_\ell$

$$\mathrm{Re}\widetilde{M}_\ell(s) = \int_{-1}^{1} dx \begin{bmatrix} D_\ell(x) & 0 \\ 0 & E_\ell(x) \end{bmatrix} [C^{-1}(s)][B(s,t)]\{\frac{\widetilde{\Gamma}(t)}{s-m^2} + [\xi]\frac{\widetilde{\Gamma}(t)}{u-m^2} + \frac{(1-\xi)}{2}\frac{\widetilde{\Gamma}_t}{t-\mu^2}\} + \frac{P}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \int_{-1}^{1} dx \begin{bmatrix} D_\ell(x) & 0 \\ 0 & E_\ell(x) \end{bmatrix} [C^{-1}(s)][B(s,t)] \times$$

$$\{\frac{1}{s'-s} + [\xi]\frac{1}{s'-u}\}[B^{-1}(s',t)][C(s')]\sum_{\ell'=0}^{\infty} \begin{bmatrix} G_{\ell'}(x') & 0 \\ 0 & H_{\ell'}(x') \end{bmatrix} \mathrm{Im}\widetilde{M}_{\ell'}(s'). \qquad (12.1)$$

Here

$$x' = \frac{kq}{k'q'}x + \frac{k_o'q_o'-k_oq_o}{k'q'} \qquad (12.2)$$

In this dispersion relation, the expansion in terms of $\mathrm{Im}\widetilde{M}_{\ell'}(s')$ is assumed to converge even outside the physical region ($|x'|>1$). Due to the kinematical singularities in A_2 and A_5 and the influence of the t-cut, this is certainly not true everywhere. However, it is known from the pion-nucleon phase shift analysis [21] carried out using partial-wave dispersion relations that the s-channel partial-wave dispersion relations appear to be well satisfied up to pion lab. energies of ~1400 MeV. The analysis of Donnachie and Shaw [24,25] also indicates the validity of this approach up to 450 MeV in photoproduction. The t-cut comes mainly from ω and ρ exchange and this effect is known to be small[25].

For more detailed arguments, see ref. 26.

In practice, it is more convenient to use as integration variable W', the c.m.energy, in the multipole dispersion relations. For further discussion only the case of photoproduction will be considered. The development of electroproduction dispersion relations is in no way different, only very much more lengthy. Equation (12.1) is rewritten as

$$\mathrm{Re}\,\tilde{\mathcal{M}}_\ell(W) = \tilde{B}_\ell(W) + \frac{P}{\pi}\int_{m+\mu}^{\infty} dW' \frac{\mathrm{Im}\,\tilde{\mathcal{M}}_\ell(W')}{W'-W} + \tilde{C}_\ell(W), \tag{12.3}$$

where

$$\tilde{\mathcal{M}}_\ell(W) = \frac{W}{k\zeta q^\ell}\begin{bmatrix} E_{\ell+} \\ (E_2+m)E_\ell \\ M_{\ell+} \\ (E_2+m)M_\ell \end{bmatrix}, \tag{12.4}$$

with

$$\zeta = [(W+m)^2-\mu^2]^{\frac{1}{2}},$$

and

$$\tilde{B}_\ell(W) = \frac{W}{k\zeta q^\ell}\begin{bmatrix} E^B_{\ell+} \\ (E_2+m)E^B_\ell \\ M^B_{\ell+} \\ (E_2+m)M^B_{\ell-} \end{bmatrix}, \tag{12.5}$$

where the $\left(E_{\ell\pm}{}^B, M_{\ell\pm}{}^B\right)$ are the projected Born terms of eq.(12.1). The matrix $\tilde{C}_\ell$ is given by

$$\tilde{C}_\ell(W) = \frac{1}{\pi}\int_{m+\mu}^{\infty} dW' \sum_{\ell'} \tilde{K}_{\ell\ell'}(W,W')\,\mathrm{Im}\,\tilde{\mathcal{M}}_{\ell'}(W'). \tag{12.6}$$

It contains the crossed channel contributions to $\mathrm{Re}\,\tilde{\mathcal{M}}_\ell$ as well as the contribution of s-channel multipoles, other than the direct term. The kinematical factors $1/q^\ell$ give $\tilde{\mathcal{M}}_\ell$ the proper threshold behaviour. They appear quite naturally as factors in the dispersion relations (see next section).

In practice, only states with $\ell' = 0$ or 1 need be retained in eq.(12.6), and of these the electric dipole E_{0+} transitions and the magnetic dipole M^3_{1+} transition are by far the most important. The multipoles with higher ℓ' values can be neglected firstly because their maximum values are very much less than, for example, the maximum value of the M^3_{1+} transition, and secondly because their maxima occur at fairly high energies and in general the kernels $\widetilde{K}_{\ell\ell'}(W,W')$ are rapidly decreasing functions of W'. This dominance in the crossed channel of multipoles leading to s- and p-wave states only, is paralleled in pion-nucleon scattering, where the only important contributions to the crossed channel come from the resonant P_{33} and the low energy s-waves[27]. Explicit formulae for the kernels, which are of use below 1 GeV, can be found in reference 28 and are left out here for reasons of space.

13 THE PROJECTED POLE TERMS

From eq.(12.1), it can be seen that the projected pole terms are obtained by a series of matrix multiplications, followed by integration. The integrals give rise to Legendre functions of the second kind

$$Q_\ell(y) = \tfrac{1}{2}\int_{-1}^{1} dx\, \frac{P_\ell(x)}{y-x} \quad . \tag{13.1}$$

These functions occur with arguments $\bar{q}_0$ and $\bar{E}_2$, defined by

$$t-\mu^2 = -2kq\left(\frac{2k_0q_0+K^2}{2kq} - x\right) = -2kq(\bar{q}_0-x), \tag{13.2}$$

and

$$u-m^2 = -2kq\left(\frac{2k_0E_2+K^2}{2kq} + x\right) = -2kq(\bar{E}_2+x). \tag{13.3}$$

Performing the matrix multiplications and integrating, yields the following expressions

$$E^B_{\ell+} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{2m}\frac{1}{2(\ell+1)}\Big(\frac{2\delta_{\ell 0}}{W^2-m^2}[\Gamma_1+(W-m)\Gamma_3-\xi(W+m)\Gamma_3]$$

$$-\xi[\Gamma_1-2m\Gamma_3]T_{\ell+} - \frac{(1-\xi)}{2}[4\Gamma_1+K^2\Gamma_t][\frac{\ell R^\pi_\ell}{(E_1+m)(W-m)} - \frac{q(\ell+1)R^\pi_{\ell+1}}{k(E_2+m)(W-m)}]$$

$$-2\xi[\Gamma_1-(W+m)\Gamma_3]\frac{\ell R^N_\ell}{(E_1+m)(W-m)} + 2\xi[\Gamma_1+(W-m)\Gamma_3]\frac{q(\ell+1)R^N_{\ell+1}}{k(E_2+m)(W-m)}\Big), \tag{13.4}$$

$$E^B_{\ell-} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{2m}\frac{1}{2\ell}\Big(-\xi[\Gamma_1-2m\Gamma_3]T_{\ell-} + \frac{(1-\xi)}{2}[4\Gamma_1+K^2\Gamma_t]\,x$$

$$[\frac{(\ell+1)R^\pi_\ell}{(E_1+m)(W-m)} - \frac{q\ell R^\pi_{\ell-1}}{k(E_2+m)(W-m)}] + 2\xi[\Gamma_1-(W+m)\Gamma_3]\frac{(\ell+1)R^N_\ell}{(E_1+m)(W-m)}$$

$$-2\xi[\Gamma_1+(W-m)\Gamma_3]\frac{q\ell R^N_{\ell-1}}{k(E_2+m)(W-m)}\Big), \tag{13.5}$$

$$M^B_{\ell+} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{2m}\frac{1}{2(\ell+1)}\Big(-\xi[\Gamma_1-2m\Gamma_3]T_{\ell+} +$$

$$\frac{(1-\xi)}{2}[4\Gamma_1+K^2\Gamma_t]\frac{R^\pi_\ell}{(E_1+m)(W-m)} + 2\xi[\Gamma_1-(W+m)\Gamma_3]\frac{R^N_\ell}{(E_1+m)(W-m)}\Big), \tag{13.6}$$

$$M^B_{\ell-} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{2m}\frac{1}{2\ell}\Big(\frac{2qk\,\delta_{\ell 1}}{(E_1+m)(E_2+m)(W-m)^2}[-\Gamma_1+(W+m)\Gamma_3$$

$$-\xi(W-m)\Gamma_3] + \xi[\Gamma_1-2m\Gamma_3]T_{\ell-} - \frac{(1-\xi)}{2}[4\Gamma_1+K^2\Gamma_t]\frac{R^\pi_\ell}{(E_1+m)(W-m)}$$

$$-2\xi[\Gamma_1-(W+m)\Gamma_3]\frac{R^N_\ell}{(E_1+m)(W-m)}\Big), \tag{13.7}$$

$$S^B_{\ell+} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{2m} \frac{1}{2(\ell+1)}\Bigg(\frac{2k\,\delta_{\ell 0}}{(E_1+m)(W^2-m^2)} [-\Gamma_1+(E_1+m)\Gamma_3$$

$$+\xi(W+m)\Gamma_3+\frac{(1-\xi)}{4}k_0(W+m)\Gamma_t] + \frac{(1-\xi)}{2}\frac{(2q_0-k_0)}{(W-m)}[\frac{K^2}{2}\Gamma_t+2\Gamma_1][\frac{Q_\ell(\bar{q}_0)}{q(E_1+m)}$$

$$-\frac{Q_{\ell+1}(\bar{q}_0)}{k(E_2+m)}] + \frac{(-1)^{\ell+1}\xi}{W-m}[-m(2q_0-k_0)\Gamma_3+(2q_0-W)\Gamma_1][\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)}$$

$$+\frac{Q_{\ell+1}(\bar{E}_2)}{k(E_2+m)}] + \frac{(-1)^{\ell+1}\xi}{W-m}[m\Gamma_1+(k_0W+K^2-\mu^2)\Gamma_3][\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)}-\frac{Q_{\ell+1}(\bar{E}_2)}{k(E_2+m)}]\Bigg), \tag{13.8}$$

$$S^B_{\ell-} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{2m} \frac{1}{2\ell}\Bigg(\frac{2q\,\delta_{\ell 1}}{(E_2+m)(W-m)^2} [\Gamma_1+(E_1-m)\Gamma_3$$

$$+\xi(W-m)\Gamma_3-\frac{(1-\xi)}{4}k_0(W-m)\Gamma_t] + \frac{(1-\xi)(2q_0-k_0)}{2(W-m)}[\frac{K^2}{2}\Gamma_t+2\Gamma_1][\frac{Q_\ell(\bar{q}_0)}{q(E_1+m)}$$

$$-\frac{Q_{\ell-1}(\bar{q}_0)}{k(E_2+m)}] + \frac{(-1)^{\ell+1}\xi}{W-m}[-m(2q_0-k_0)\Gamma_3+(2q_0-W)\Gamma_1][\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)}$$

$$+\frac{Q_{\ell-1}(\bar{E}_2)}{k(E_2+m)}] + \frac{(-1)^{\ell+1}\xi}{W-m}[m\Gamma_1+(k_0W+K^2-\mu^2)\Gamma_3][\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)}-\frac{Q_{\ell-1}(\bar{E}_2)}{k(E_2+m)}]\Bigg). \tag{13.9}$$

In these equations, the following notation has been used

$$R^N_\ell = \frac{(-1)^\ell}{(2\ell+1)}[Q_{\ell+1}(\bar{E}_2) - Q_{\ell-1}(\bar{E}_2)] \;, \tag{13.10}$$

$$R^\pi_\ell = \frac{1}{(2\ell+1)}[Q_{\ell+1}(\bar{q}_0) - Q_{\ell-1}(\bar{q}_0)] \;, \tag{13.11}$$

$$T_{\ell\pm} = (-1)^\ell[\frac{Q_\ell(\bar{E}_2)}{kq} - \frac{W+m}{(E_1+m)(E_2+m)(W-m)}Q_{\ell\pm1}(\bar{E}_2)] \;. \tag{13.12}$$

It should be kept in mind that Γ_1 and Γ_3 depend on $\pm$, 0 isospin indices, which have been suppressed here for convenience.

The pole terms for photoproduction are obtained by putting $K^2 = 0$, which simplifies the kinematics and makes the functions $\Gamma_1(K^2)$ and $\Gamma_3(K^2)$ constant. Extracting a factor of $eg/(2k\ m)$ from eqs.(13.4) through (13.7) and multiplying by $m/4\pi W$ due to the different normalization (see eq.(11.2)) the expressions of Schmidt and Guigay, as quoted in ref. 25, are reproduced namely

$$E^B_{\ell+} = \frac{1}{\ell+1}\frac{Z}{\mu}\Big(\delta_{\ell 0}\frac{(\xi-1)}{2}(\mu'-\frac{m}{W})+\delta_{\ell 0}\frac{(\xi+1)}{2}\frac{m}{W}(\mu'+1)-\xi(\mu'+1)\mathcal{T}_{\ell+} +$$

$$(\xi-1)[\frac{2m}{W+m}\ell R^\pi_\ell - \frac{2m}{W-m}\frac{q(\ell+1)}{(E_2+m)}R^\pi_{\ell+1}] - \xi[\mu'+\frac{2m}{W+m}]\ell R^N_\ell -$$

$$\xi[\mu'-\frac{2m}{W-m}]\frac{q(\ell+1)}{(E_2+m)}R^N_{\ell+1}\Big) , \qquad (13.13)$$

$$E^B_{\ell-} = \frac{1}{\ell}\frac{Z}{\mu}\Big(-\xi(\mu'+1)\mathcal{T}_{\ell-} + (1-\xi)[\frac{2m(\ell+1)}{W+m}R^\pi_\ell - \frac{2mq\ell R^\pi_{\ell-1}}{(W-m)(E_2+m)}]$$

$$+\xi[\mu'+\frac{2m}{W+m}](\ell+1)R^N_\ell + \xi[\mu'-\frac{2m}{W-m}]\frac{q\ell}{(E_2+m)}R^N_{\ell-1}\Big) , \qquad (13.14)$$

$$M^B_{\ell+} = \frac{1}{\ell+1}\frac{Z}{\mu}\Big(-\xi(1+\mu')\mathcal{T}_{\ell+} + \frac{(1-\xi)2m}{W+m}R^\pi_\ell + \xi[\mu'+\frac{2m}{W+m}]R^N_\ell\Big) , \qquad (13.15)$$

$$M^B_{\ell-} = \frac{1}{\ell}\frac{Z}{\mu}\Big(\delta_{\ell 1}\frac{q}{(E_2+m)}[\frac{(\xi-1)}{2}(\mu'+\frac{m}{W}) - \frac{(\xi+1)}{2}\frac{m}{W}(\mu'+1)]$$

$$+\xi(\mu'+1)\mathcal{T}_{\ell-} + \frac{(\xi-1)2m}{W+m}R^\pi_\ell - \xi(\mu'+\frac{2m}{W+m})R^N_\ell\Big) . \qquad (13.16)$$

Here we have defined

$$\mathcal{T}_{\ell\pm} = m\ k\ T_{\ell\pm} , \qquad (13.17)$$

$$\frac{Z}{\mu} = \frac{[W-m][(E_1+m)(E_2+m)]^{\frac{1}{2}}}{16\pi mkW}\,\frac{eg}{2} = \frac{\zeta}{16\pi mW}\,\frac{eg}{2} = \left(\frac{E_2+m}{2W}\right)^{\frac{1}{2}}\frac{1}{2}\,\frac{e}{(4\pi)^{\frac{1}{2}}}\,\frac{g}{(4\pi)^{\frac{1}{2}}}\,\frac{1}{2m}, \tag{13.18}$$

ζ being defined by eq.(12.4), and $\overline{E}_2$, $\bar{q}_0$ reducing to E_2/q and q_0/q respectively. Again the $\pm$, 0 indices have been omitted. The appropriate anomalous magnetic moment combinations are

$$\mu^{\pm} = \mu'_P - \mu'_N\ , \qquad \mu^0 = \mu'_P + \mu'_N\ . \tag{13.19}$$

From eqs.(13.13) - (13.16), it can be seen that at threshold factors can be taken out, which cancel the factors in eqs.(12.4) and (12.5).

14 CONDITIONS FROM UNITARITY

If we define the reaction matrix R in terms of the S-matrix by

$$S = 1 + iR, \tag{14.1}$$

then the unitarity of the S-matrix

$$S^{\dagger}S = 1, \tag{14.2}$$

together with time-reversal* and rotational invariance, leads to the condition for the R-matrix

$$\mathrm{Im}\langle\beta|R|\alpha\rangle = \tfrac{1}{2}\sum_{\nu}\langle\nu|R|\beta\rangle^{*}\langle\nu|R|\alpha\rangle\ , \tag{14.3}$$

where $|\nu\rangle$ is a complete set of states. To first order in e, when intermediate states containing photons may be neglected, eq. (14.3) imposes a phase condition on the individual multipole transition amplitudes in the region of partial-wave elasticity, namely that the phase of a multipole transition to a final pion-nucleon state is equal to the scattering phase shift of that

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* Using time-reversal invariance here requires no additional assumptions in our case, since it is already implicit in writing down the dispersion relations in section 9.

pion-nucleon state. This result was first derived by Watson[29] and is generally known as Watson's theorem. The result can be derived very elegantly[30], using the helicity formalism of appendix C, and we outline that method here.

Firstly we introduce the helicity amplitudes for pion-nucleon scattering from an initial state of helicity μ, angles (θ,φ) to a final state of helicity μ', angles (θ',φ') by

$$\langle\mu'|R|\mu\rangle = g_{\mu'\mu}(\theta',\varphi';\theta,\varphi) = \frac{1}{2\pi}\sum_J\sum_M(J+\tfrac{1}{2})\langle\mu'|T^J|\mu\rangle D^{J*}_{M,\mu'}(\varphi',\theta',-\varphi')$$

$$\times\ D^J_{M,\mu}(\varphi,\theta,-\varphi). \tag{14.4}$$

Retaining only intermediate states of one pion and one nucleon, then in terms of $f_{\mu,\lambda}(\theta,\varphi)$ of eq.(C.2) and of $g_{\mu',\mu}(\theta',\varphi';\theta,\varphi)$, eq.(14.3) becomes

$$\mathrm{Im}f_{\mu,\lambda}(\theta,\varphi) = \tfrac{1}{2}\sum_{\mu'}\int d\Omega'\, g^*_{\mu',\mu}(\theta',\varphi';\theta,\varphi)f_{\mu',\lambda}(\theta',\varphi'). \tag{14.5}$$

Substituting eqs.(C.2) and (14.4) in eq.(14.5), and using the orthonormality relation

$$\int d\Omega'\, D^{J*}_{M,N}(\varphi',\theta',-\varphi')D^{J'}_{M',N'}(\varphi',\theta',-\varphi') = \frac{4\pi}{(2J+1)}\delta_{JJ'}\delta_{MM'}\delta_{NN'}\,, \tag{14.6}$$

the unitarity relation is readily obtained for the helicity amplitudes

$$\mathrm{Im}\langle\mu|T^J|\lambda\rangle = \tfrac{1}{2}\sum_{\mu'}\langle\mu'|T^J|\mu\rangle^*\langle\mu'|T^J|\lambda\rangle\ . \tag{14.7}$$

Using eq.(C.10)-(C.13) and the parity relation of the scattering amplitudes

$$\langle\tfrac{1}{2}|T^J|\pm\tfrac{1}{2}\rangle = -\langle-\tfrac{1}{2}|T^J|\mp\tfrac{1}{2}\rangle\ , \tag{14.8}$$

gives immediately (an extra factor i must be included, because in section 6 this is done for A_i and therefore for M_ℓ as well)

$$\mathrm{Im}\begin{pmatrix} M_{\ell\pm} \\ E_{\ell\pm} \end{pmatrix} = \begin{pmatrix} M_{\ell\pm} \\ E_{\ell\pm} \end{pmatrix} \tfrac{1}{2}\{\langle\tfrac{1}{2}|T^J|\tfrac{1}{2}\rangle \pm \langle\tfrac{1}{2}|T^J|-\tfrac{1}{2}\rangle\}^* \quad . \tag{14.9}$$

Since the partial wave scattering amplitude is given by

$$f_{\ell\pm} = (\text{kinematical factor})\{\langle\tfrac{1}{2}|T^J|\tfrac{1}{2}\rangle \pm \langle\tfrac{1}{2}|T^J|-\tfrac{1}{2}\rangle\} , \tag{14.10}$$

and recalling that

$$f_{\ell\pm} = \frac{1}{q}\, e^{i\delta_{\ell\pm}} \sin\delta_{\ell\pm} , \tag{14.11}$$

Eq.(14.9) gives immediately that

$$\begin{pmatrix} M_{\ell\pm} \\ E_{\ell\pm} \end{pmatrix} = \begin{pmatrix} |M_{\ell\pm}| \\ |E_{\ell\pm}| \end{pmatrix} e^{i\delta_{\ell\pm} + in\pi} , \tag{14.12}$$

where n is an integer.

CHAPTER IV

SOLUTION METHOD AND RESULTS

SUMMARY

In this chapter a brief account is given of the methods used in the literature (section 15). Then in section 16 some information on π-N scattering is given. The conformal mapping approach, as applied for photoproduction in ref. 31, is discussed in section 17. In section 18 some of the results of numerical calculations with the method of section 17 are mentioned. For the full details of all calculations one is referred to ref. 31.

15 METHODS IN THE LITERATURE

Chew, Goldberger, Low and Nambu[6] were the first to write down fixed-t dispersion relations for photoproduction. They projected them for S- and P-wave multipoles, taking the static limit ($m \rightarrow \infty$). Solutions for the static limit and some corrections thereupon were obtained. Due to the important effect of the first π-N resonance (P_{33}) and the structure of the dispersion relation one can neglect other multipoles in the dispersion relation for M^{3}_{1+}. The dispersion equation for M^{3}_{1+} then has great similarity to the dispersion relation for f^{3}_{1+} (the P_{33} scattering amplitude). From this an approximative solution was obtained

$$M^3_{1+} = \frac{e}{2m}\left(\frac{\mu_p-\mu_n}{2f}\right)\frac{k}{q}\, f^3_{1+} = \frac{e}{2m}\left(\frac{\mu_p-\mu_n}{2f}\right)\frac{k}{q}\,\frac{e^{i\delta}\sin\delta}{q} , \qquad (15.1)$$

where δ is the P_{33} phase shift and f is related to the πNN coupling constant g through

$$f = g\,\frac{\mu}{2m} . \qquad (15.2)$$

The constant of proportionality in the first equation of eqs. (15.1) is just the ratio of the pole terms for M^3_{1+} and f^3_{1+}. The M^3_{1+} multipole is the dominant one in the region around the P_{33} resonance. Using eq. (15.1) approximate solutions for the other multipoles were obtained.

After this basic paper, many others followed. The main trend is only sketched here, referring only to published papers.

Numerical calculations of the C.G.L.N. theory were done by Höhler and Müllensiefen[32] and by Dietz, Höhler and Müllensiefen,[33] showing the importance of relativistic corrections to the static approximation of C.G.L.N. Calculations in the approximation of relativistic Born terms plus M^3_{1+} as given by eq. (15.1) were done by Höhler and Schmidt[34].

An other approach was started by Ball[15], who considered fixed-t dispersion relations for the amplitudes A_i. The amplitudes A^0 he considered as given by the poles and the t-channel contribution of the ρ-meson (which nowadays is considered as very small). For the $A^{\pm}$ amplitudes he evaluated the real part by fixed-t dispersion relations, using for Im A just Im M^3_{1+} as given by eq. (15.1).

The same procedure has been used in great detail by Schmidt[35] and Müllensiefen[36], leaving out the ρ-contribution which is after all an unknown parameter. The latter author includes also the phases for E_{0+}. The real part of E_{0+} is obtained by numerical projection from ReA, and Im E_{0+} can then be obtained from Watson's theorem (section 14).

The relativistic approaches mentioned above, using the static M^3_{1+} solution (15.1) give a reasonable description of photopion production below 400 MeV. Nevertheless there are discrepancies, and it is worthwhile to try to improve the calculations.

This was done by Donnachie and Shaw[25], who used multipole dispersion relations. They obtained a solution for M^3_{1+} different from eq. (15.1) by an iterative procedure, using eq. (15.1) as starting solution. Also E^1_{0+} and E^3_{0+} were solved for by an iterative procedure using the Born term plus M^3_{1+} contribution to these multipoles (via the kernels of eq. (12.6)) as starting solution. The other P-wave isovector multipoles were treated in Born approximation and with sometimes the influence of M^3_{1+} included. D-wave multipoles were taken in the Born approximation. For the isoscalar multipoles, E^0_{0+} was solved by the iterative procedure. The others were taken in the Born approximation. The solutions gave good agreement with experiment, though still some discrepancies remain.

It is the purpose of the following calculations to see how far the results can be improved by extending to F waves and by taking into account also the effects of $E^{0,1,3}_{0+}$ and M^3_{1+} on the other multipoles. Moreover on the basis of these solutions it is worthwhile to compare with all experimental information available and to give predictions in the not measured regions.

16 PHASE SHIFTS

In this section some information on the π-N phase shifts is given. From the tables of Donnachie[22], compiling the results of the phase shift analysis of Donnachie, Kirsopp, Lea and Lovelace[21], one can find the phases and inelasticities. It is useful to summarize in a table (table II) the important items i.e. whether the phase shift is important in the region under consideration, whether it leads to a resonance inside or outside this region and how large the inelasticity is. The inelasticity starts at the energy E_1 and becomes important at the energy E_2, where the inelasticity parameter η is 0.9. Between these two energies the Watson theorem still is a good approximation. Above E_2 it becomes worse.

In the calculations, the values of the phase shifts are used as given in ref. 22. The table II gives only qualitative and rough information. This can be used to estimate the importance of the various imaginary parts of the multipoles and therefore gives the basis for the approximations used in section 17.

T A B L E II

Phase shift	Characteristics	E_1	E_2	Related multipoles
S_{11}	resonance (970?)	460	560	E^0_{0+}, E^1_{0+}
P_{11}	resonance (750)	400	460	M^0_{1-}, M^1_{1-}
P_{13}	small	680	850	E^0_{1+}, M^0_{1+}, E^1_{1+}, M^1_{1+}
D_{13}	resonance (750)	520	650	E^0_{2-}, M^0_{2-}, E^1_{2-}, M^1_{2-}
D_{15}	small, but res.(1000)	730	850	E^0_{2+}, M^0_{2+}, E^1_{2+}, M^1_{2+}
F_{15}	small, but res.(1050)	730	850	E^0_{3-}, M^0_{3-}, E^1_{3-}, M^1_{3-}
F_{17}	small	950	1380	E^0_{3+}, M^0_{3+}, E^1_{3+}, M^1_{3+}
S_{31}	resonance (950)	560	730	E^3_{0+}
P_{31}	appreciable	850	1020	M^3_{1-}
P_{33}	resonance (345)	700	1020	E^3_{1+}, M^3_{1+}
D_{33}	small	520	750	E^3_{2-}, M^3_{2-}
D_{35}	small	850	1020	E^3_{2+}, M^3_{2+}
F_{35}	small	900	1100	E^3_{3-}, M^3_{3-}
F_{37}	small	900	1380	E^3_{3+}, M^3_{3+}

P_{33}, D_{13} and F_{15} are also called 1st 2nd and 3rd resonance.

17 SOLUTION METHOD WITH CONFORMAL MAPPING

After the preliminary discussions in section 15 about methods used in the literature, a new method applied in this thesis is presented in this section.

From eq.(12.3) and eq.(12.6) it is seen that every multipole M_ℓ (i.e. and element of 12.4; the script $\mathcal{M}_\ell$ notation is not used in this section) satisfies an integral equation of the form

$$\mathrm{Re}\, M_\ell(W) = B_\ell(W) + \frac{P}{\pi}\int_{(m+\mu)}^{\infty} dW' \frac{\mathrm{Im}\, M_\ell(W')}{W' - W}$$

$$+ \frac{1}{\pi}\sum_{\ell'}\int_{(m+\mu)}^{\infty} dW'\, K_{\ell\ell'}(W,W')\, \mathrm{Im}\, M_\ell(W'). \qquad (17.1)$$

Here the summation over ℓ' implies also a summation over all four multipoles belonging to a specific ℓ' (the matrix notation is not used in eq.(17.1)).

Defining

$$F_\ell(W) = B_\ell(W) + \frac{1}{\pi}\sum_{\ell\neq\ell'}\int_{(m+\mu)}^{\infty} dW'\, K_{\ell\ell'}(W,W')\, \mathrm{Im}\, M_\ell(W'), \qquad (17.2)$$

one obtains from eq.(17.1)

$$\mathrm{Re}\, M_\ell(W) = F_\ell(W) + \frac{P}{\pi}\int_{(m+\mu)}^{\infty} dW' \frac{\mathrm{Im}\, M_\ell(W')}{W' - W}$$

$$+ \frac{1}{\pi}\int_{(m+\mu)}^{\infty} K_{\ell\ell}(W,W')\, \mathrm{Im}\, M_\ell(W'). \qquad (17.3)$$

In eq.(17.2) the summation goes over all multipoles except the one multipole, which has B_ℓ as pole term. Supposing for the moment that $F_\ell(W)$ is known, one is left with an equation for $M_\ell(W)$. This equation can be written in a different form by a conformal mapping technique. For a more detailed account see the paper of

Lovelace[37] and his unpublished notes as quoted by Donnachie[22]. The main points are:

There exist two sets of functions $g_j^\ell(W)$ and $h_j^\ell(W)$ ($j = 1,2,\ldots \infty$) with the properties

$$g_j^\ell(W) = \frac{P}{\pi} \int_{(m+\mu)}^{\infty} dW' \frac{h_j^\ell(W')}{W' - W} , \tag{17.4}$$

and

$$\mathrm{Im}\, M_\ell(W) = \sum_{j=1}^{\infty} a_j^\ell \, h_j^\ell(W) , \tag{17.5}$$

where this series converges under very general conditions. Moreover $g_j^\ell(W)$ and $h_j^\ell(W)$ have the correct threshold behaviour and vanish asymptotically sufficiently rapidly. The functions h_j^ℓ and g_j^ℓ are related to Gegenbauer polynomials and hypergeometric functions which can be easily computed numerically. The threshold behaviour of Im M_ℓ can be obtained from Watson's theorem

$$\mathrm{Im}\, M_\ell(W) = \mathrm{Re}\, M_\ell(W) \tan \delta_\ell(W) . \tag{17.6}$$

The threshold behaviour is determined by $\tan \delta_\ell(W)$, which behaves like $q^{2\ell+1}$. So from the normalization (12.4) follows that Im $M_\ell(W)$ behaves like $q^{2\ell+1}$ at threshold. For $W \to \infty$ a bound can be derived from general assumptions[28]. The convention (12.4) means then that $M_\ell(W)$ vanishes rapidly at infinity. The functions $h_j^\ell(W)$ are chosen to have also such a behaviour for every j.

Truncating the series (17.5) at some value N, one writes eq.(17.3) in the form

$$\mathrm{Re}\, M_\ell(W) = F_\ell(W) + \sum_{j=1}^{N} a_j^\ell \, \{ g_j^\ell(W) + \tilde{g}_j^\ell(W) \} , \tag{17.7}$$

where

$$\tilde{g}_j^\ell(W) = \frac{1}{\pi} \int_{(m+\mu)}^{\infty} dW' \, K_{\ell\ell}(W,W') \, h_j^\ell(W') . \tag{17.8}$$

In these equations $g_j^\ell(W)$ and $\tilde{g}_j^\ell(W)$ are known and can therefore be computed at every W-value. Eq.(17.6) can also be translated in terms of a_j^ℓ

$$\sum_{j=1}^{N} a_j^{\ell} h_j^{\ell}(W) = \text{Re } M_\ell(W) \tan \delta_\ell(W) \ . \qquad (17.9)$$

Eqs.(17.7) and (17.9) give an equation for a_j^ℓ

$$\sum_{j=1}^{N} a_j^{\ell}\{h_j^{\ell}(W)-[g_j^{\ell}(W)+\tilde{g}_j^{\ell}(w)]\tan\delta_\ell(W)\}=F_\ell(W)\tan\delta_\ell(W). \qquad (17.10)$$

Once $F_\ell(W)$ is known eq.(17.10) can be solved for a_j^ℓ, when $\delta_\ell(W)$ is given for a set of W-values. On the computer this can be done by a fitting procedure. One then automatically obtains errors on a_j^ℓ from the errors on $\delta_\ell(W)$. For this fitting the knowledge of $\delta_\ell(W)$ is needed only in a small region (up to 1 GeV say). In the region up to 500 MeV the multipoles as obtained with these a_j^ℓ satisfy eqs.(17.3) and (17.6). The number N (N<< the number of W-values where δ_ℓ is known) is chosen to be as small as possible, but such that a good fit is obtained. Good means that it satisfies statistical criteria and that the obtained multipoles do not suffer from large oscillations at the higher energies.

The above discussion assumes that $F_\ell(W)$ is known. In practice one starts with a set of coupled equations (17.1). In the following, it is shown how this set is brought into equations like eq.(17.3) by some reasonable approximations. The first step is to truncate the infinite sum over ℓ' in eq.(17.1). In the region below 1 GeV, it is a reasonable approximation to keep only $\ell' = 0$ and 1 in the summation. From the phase shifts in section 16 it is seen that for higher ℓ-values the imaginary part becomes only appreciable for higher W'-values. The kernels are such that they decrease sufficiently with W', such that for W-values below 1 GeV the summation in eq.(17.1) can be truncated. Later on one can always try to correct this approximation by simulating distant effects by inclusion of a pole at very high energy (say at -5 GeV). Then one can fix the strength of this pole by demanding that Re $M_\ell(W)$ at high energies cancels this pole term.

The inclusion of the pole, with the obtained strength, leads to a reasonable high-energy behaviour. One is now left with a set of equations for the 12 isoscalar multipoles and a similar set for the 24 isovector multipoles. These equations are in part coupled.

In the case of isoscalar multipoles the important contribution to the third part of eq.(17.1)(called crossed-cut contributions) comes from the s-wave E^{o}_{0+} multipole. So one has a set of uncoupled equations for the multipoles. One solves for E^{o}_{0+}, making use of eq.(17.10) with $F_{\ell}(W) = E^{oB}_{0+}(W)$*). Then one knows for the other multipoles the forces, being given in the approximation of Born term and crossed-cut contribution from E^{o}_{0+} (i.e. in eq.(17.2) one takes only Im E^{o}_{0+}). How the multipoles are obtained with these forces is dealt with below, together with the isovector multipoles.

For the isovector multipoles the P_{33} resonance is of greatest importance. The dominance of this phase shift, the structure of the kernel, and Born-term for the M^{3}_{1+} equation make it possible to treat M^{3}_{1+} in isolation. That is, eq.(17.3) can be solved for M^{3}_{1+}, using as force the Born-term. From eq.(17.10) one then obtains M^{3}_{1+}. The only important other coupling of the multipoles is of E^{1}_{0+} and E^{3}_{0+} with each other and of the M^{3}_{1+}, E^{1}_{0+}, E^{3}_{0+} to all other multipoles. First one solves the coupled set of E^{1}_{0+} and E^{3}_{0+}, taking into account the known contribution of M^{3}_{1+}. This is done by starting with eq.(17.10) using as force the Born + M^{3}_{1+} combination. Then it is redone with inclusion in F^{3}_{0+} of E^{1}_{0+} and in F^{1}_{0+} of E^{3}_{0+}. The process is repeated until a consistent solution is found. For all other isovector multipoles the forces are now given by Born + M^{3}_{1+} + E^{1}_{0+} + E^{3}_{0+} contributions.

In order to see how these other multipoles can be dealt with, it is useful to classify the phase shifts in three groups.

1 The phase shift is small, both in the region under consideration and at higher energies.

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*) Note that this is the first element of the vector $\tilde{B}_{o}$ - eq. (12.5) - , as the script $\mathcal{M}_{\ell}$ notation is dropped in this section.

In this case, it follows from eq.(17.6) that the second term in eq.(17.1) (the rescattering term) can be neglected. The real part is given by the force (17.2) alone, and the imaginary part is obtained from the real part by eq.(17.6). For the scalar multipoles the force is given by (Born + E^0_{0+}), for the isovector multipoles by the (Born + M^3_{1+} + E^1_{0+} + E^3_{0+}) approximation.

This is done for all multipoles mentioned in table II in section 16, where as characteristic of the phase shift is given the title small.

2 The phase shift is small in the region under consideration, but may be large elsewhere.

In this case the rescattering may be non-negligible, although the contribution comes from a region outside one's control and can be only estimated crudely. This phenomenon one has for all phase shifts which are resonating near to the region of 0 - 500 MeV. In practice, this means the $M^{0,1}_{1-}$ transitions to a final P_{11} pion-nucleon state and the $E^{0,1}_{2-}$, $M^{0,1}_{2-}$ transitions to a final D_{13} pion-nucleon state.

To estimate the rescattering the following ansatz is made

$$M_{\ell\pm} = \lambda F_{\ell\pm} f_{\ell\pm} \tag{17.11}$$

where $F_{\ell\pm}$ is the force (eq.(17.2)) and $f_{\ell\pm}$ is the π-N scattering amplitude belonging to the final π-N state of the multipole transition. λ is a parameter which will be determined by a consistency condition. The form (17.11) is suggested by the C.G.L.N. solution for M^3_{1+} (see eq.(15.1)). It is used here to get a form for the imaginary part only i.e. to estimate the rescattering. To determine λ the imaginary part is inserted in eq.(17.3)(without a crossed-cut contribution) and the condition is imposed that Re $M_\ell(W)$ vanishes at the resonance position. Of course, this is already the case for eq.(17.11), but it is not necessarily the case for Re $M_\ell(W)$ calculated by eq.(17.3).

The ansatz (17.11) gives a rescattering contribution in eq.(17.3) at 500 MeV, which is of the order of 100% of the inhomogeneous term for the $M_{1-}^{0,1}$ multipoles and of the order of 40% in the case of $E_{2-}^{(0,1)}$ and $M_{2-}^{(0,1)}$ multipoles. Of course, the error, which one has to impose at this assumption, is difficult to estimate. It is taken to be 25% which was checked for E_{2-}^{1} in the following way. As this is the dominant multipole at the second resonance, the cross-section due to this multipole was calculated from eq.(17.11) at the resonance, giving 42 μb for π^+ and 21 μb for π^0 production. After taking off the background as well as possible, the corresponding experimental values are, 50 - 60 μb and 20 - 25 μb respectively.

3 The scattering phase shift is large in the region under consideration.

In this case, the rescattering term is important and the multipoles must be solved from eq.(17.10). The multipole transitions involved are the $E_{0+}^{(0,1)}$ transitions to the S_{11} pion-nucleon final state, the M_{1+}^{3}, E_{1+}^{3} transitions to the P_{33} state, the E_{0+}^{3} to the S_{31} state, and the M_{1-}^{3} transition to the P_{31} state.

To summarize, the order of events is as follows:

a Eq.(17.10) is used to solve for E_{0+}^{0}, with $F_{0+}^{0} = B_{0+}^{0}$

b Eq.(17.10) is used to solve for M_{1+}^{3}, with $F_{1+}^{3} = B_{1+}^{3}$

c Eq.(17.10) is used to solve for E_{0+}^{1} and E_{0+}^{3} with F_{0+}^{1} and F_{0+}^{3} containing the M_{1+}^{3} contributions as well as the Born terms. The equations are then resolved after inclusion of E_{0+}^{1} in F_{0+}^{3} and E_{0+}^{3} in F_{0+}^{1}. This process is repeated until a consistent solution is obtained.

d Eq.(17.10) is used to solve for E_{1+}^{3}, M_{1-}^{3} where the forces contain Born term and the E_{0+}^{1}, E_{0+}^{3}, M_{1+}^{3} contributions.

e Eq.(17.1) is used to evaluate the remaining multipoles with the rescattering term neglected except in the case of the M_{1-}^{0}, $M_{1-}^{1}, E_{2-}^{0}, E_{2-}^{1}, M_{2-}^{0}, M_{2-}^{1}$, where the ansatz (17.11) was used to estimate the rescattering term.

In the evaluation of experimental quantities (cross-sections, polarizations), the higher multipoles ($\ell \geqq 4$) are taken in the Born approximation only. In practice, this is achieved by using the full (unprojected) Born term, with the multipole Born terms for $\ell \leqq 3$ subtracted.

By the errors on the coefficients a_j^{ℓ} one obtains automatically errors on the multipoles.

18 RESULTS

In this section, a selection of the results for experimental quantities is shown (figs. 2-10). A discussion of the general features of the results will be given. A detailed account of all results obtained is published elsewhere [31]. Tables of multipoles and figures of experimental quantities up to 500 MeV are given there. For the experimental quantities the formulae of appendix D are used.

The solution for M_{1+}^{3} differs from the C.G.L.N. solution. For the real part, the difference shows up most clearly away from the resonance position, for the imaginary part at the resonance position. Away from resonance the difference is 10%, at resonance about 5%. The E_{1+}^{3}/M_{1+}^{3} ratio is very similar to the one of Finkler[17] and to that obtained phenomenologically by Donnachie and Shaw[24]. This ratio becomes very small at resonance, but does not change sign. A feature which is not present in the Finkler solution is the increase of the ratio beyond the resonance.

At 500 MeV the $E_{2-}^{0,1}$ are much larger than the $M_{2-}^{0,1}$ multipoles. It is expected that the resonance D_{13} shows up most clearly in photoproduction on neutrons as the isospin combination enhances the E_{2-} effects, whereas on protons there is a partial cancellation. The same is true for the M_{1-} transitions to the P_{11} state. At 500 MeV this is seen from

$$M_{1-}^{0} = -3.18 \times 10^{-3} \ , \ M_{1-}^{1} = 1.118 \times 10^{-2} \ ,$$

$$M_{1-}^{(\pi^+,n)} = 7.5 \times 10^{-4},\ M_{1-}^{(\pi^o,p)} = 5.4 \times 10^{-4},\ M_{1-}^{(\pi^-,p)} = 9.82 \times 10^{-3}\ .$$

(The units are $\hbar = c = \mu = 1$)
Consequently the P_{11} resonance should not show up strongly in photoproduction from protons. The D_{15} resonance is not likely to show up strongly, as at 500 MeV the contributions are even smaller than the ones of F_{15}.

So from the results up to 500 MeV, one gets already the impression that the $E_{2-}^{0,1}$ transitions to the D_{13} state will show a resonance most clearly. This is experimentally also the case.

As far as experimental quantities are concerned the following conclusions can be drawn. The theoretical errors are of the order of 10%. They arise from the error in the coupling constant g and more importantly from the errors in the pion-nucleon shifts. The first error is a correlated one, as all Born terms contain this coupling constant. The results are obtained by using the maximum and minimum value of g. In each case also the effects of the errors in the phase shifts are taken into account. Then cross-sections and other quantities are calculated with the multipoles and their errors. In the graphs, the region of the theoretical prediction is indicated. The experimental errors are as far as statistical errors are concerned mostly smaller, but systematic errors cause a spread of ± 20%. Comparison is made with all experimental data below 500 MeV for π^+ and π^o production from protons.

The process $\gamma + p \to \pi^+ + n$

The differential cross-sections are described very well by the theory. A peak in the forward direction shows up which is connected with the pion pole in the t-channel (figs. 3,4,5,6).

The predictions for the ratio $(\sigma_\perp - \sigma_{||})/(\sigma_\perp + \sigma_{||})$ are reasonable (fig. 2), but there is a systematic discrepancy, the origin of which is not clear. This ratio contains the differential cross-section $\sigma_\perp$ at an angle $\theta = 90^o$ for linearly polarized photons, with the polarization direction perpendicular to the plane, and $\sigma_{||}$, where the direction is in the plane.

The recoil neutron polarization is not measured yet. The most notable feature is that it may reach the large value of 80% at 400 MeV and for θ around 90^o.

T h e p r o c e s s $\gamma + p \to \pi^o + n$

The experimental data are less numerous than in the case of π^+ production. At low energies only, there is a distinct descrepancy between the theory and experiment (in the region of 160 to 200 MeV (see fig. 3). This is due to the almost complete cancellation of the S-waves (it may be seen from the figures that the differential cross-sections are also 1/10 of the π^+ ones). This cancellation makes the calculations strongly dependent on the P-waves in this region, which are very small. The uncertain M_{1-} is as important as the M^3_{1+}, which causes the troubles, as is seen from the table below

T a b l e III

	E_{0+}	E_{1+}	E_{2-}	M_{1-}	M_{1+}
0	$-.129.10^{-2}$	0	$-.007.10^{-2}$	$-.034.10^{-2}$	$0.016.10^{-2}$
1	$4.413.10^{-2}$	$0.222.10^{-2}$	$0.119.10^{-2}$	$0.212.10^{-2}$	$-0.295.10^{-2}$
3	$-2.075.10^{-2}$	$-0.106.10^{-2}$	$-.067.10^{-2}$	$-0.290.10^{-2}$	$0.439.10^{-2}$
π^+	$2.877.10^{-2}$	$0.155.10^{-2}$	$0.078.10^{-2}$	$0.188.10^{-2}$	$-0.323.10^{-2}$
π^o	$-.041.10^{-2}$	$0.003.10^{-2}$	$-0.012.10^{-2}$	$-0.157.10^{-2}$	$0.210.10^{-2}$

The leading multipoles at 160 MeV photon lab. energy. The first column gives the isospin character of the multipole. The units are $\hbar = \mu = c = 1$.

It is known[24)] that small adjustments to the E_{0+} and M_{1-} multipoles (of the order of a few percent in the E_{0+} multipoles, which has a negligible effect on π^+ production) are sufficient to produce agreement with experiment. Since the theory works so nicely everywhere else, one may try to explain the discrepancy at threshold by the neglect of the ω-meson exchange in the t-channel. This would only have an influence on π^0 production and not on π^+ production. In the case of π^+ production the ρ-exchange can safely be ignored. It is consistent with zero[25], which is also supported by the experimental upper limit on the width $\Gamma_{\rho\pi\gamma}$[18], namely $\Gamma_{\rho\pi\gamma} < 0.6$ MeV. For the ω-meson the width $\Gamma_{\omega\pi\gamma} = 1.2$ MeV makes a contribution of ω-exchange more probable.

However, the above noted discrepancy vanishes rather quickly when M^3_{1+} becomes more important. Then agreement is good (figs. 7,8). Above 400 MeV, however, there is a tendency that the theory gives too high cross-sections.

The ratio $(\sigma_\perp - \sigma_{||})/(\sigma_\perp + \sigma_{||})$ agrees well with experiment (fig. 9) as well as the recoil proton polarization (fig.10).

All the calculations give information also on photoproduction from neutrons. Comparison could be made with data extracted from photoproduction on deuterium. This is not a good test of the theory, however, as the extraction procedure contains additional uncertainties. Experiments on radiative π^- capture by hydrogen, therefore, would be very useful.

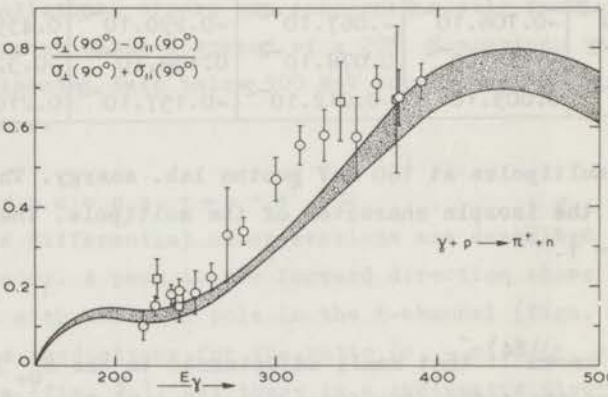

fig.2 The ratio $\frac{\sigma_\perp - \sigma_{||}}{\sigma_\perp + \sigma_{||}}$ for π^+ production.

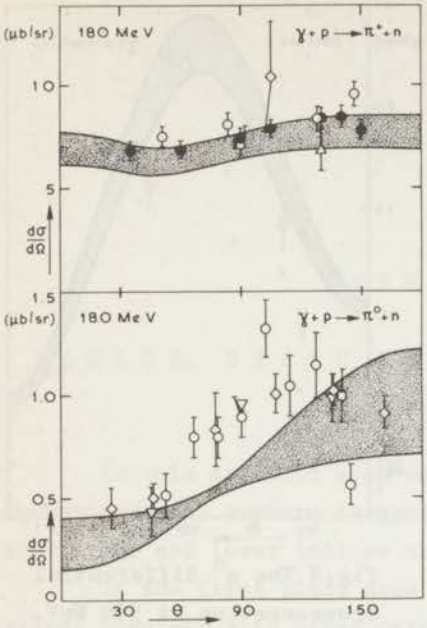

fig.3 The π^+ and π^0 differential cross-section at 180 MeV.

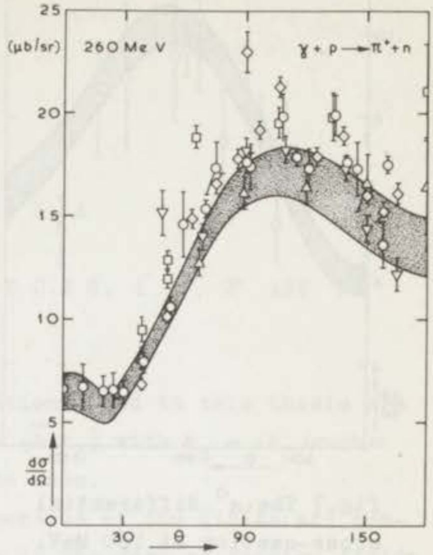

fig.4 The π^+ differential cross-section at 260 MeV.

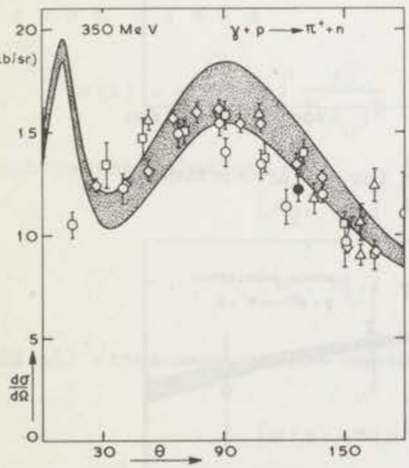

fig.5 The π^+ differential cross-section at 350 MeV.

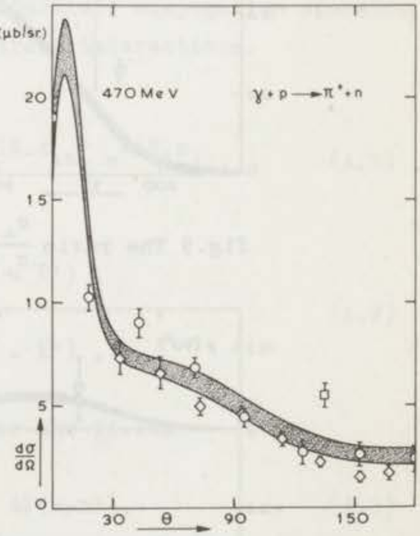

fig.6 The π^+ differential cross-section at 470 MeV.

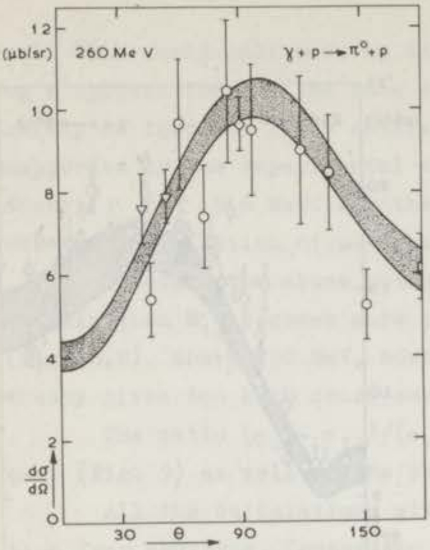

fig.7 The π^0 differential cross-section at 260 MeV.

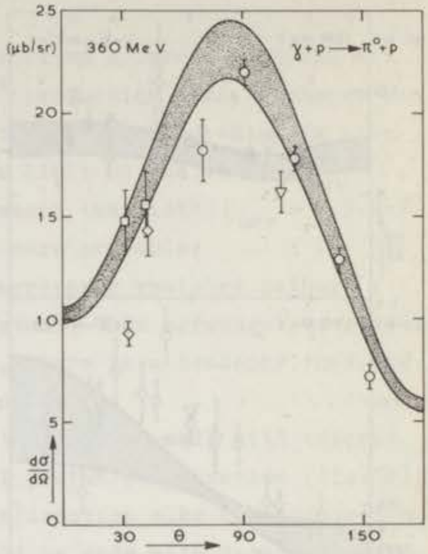

fig.8 The π^0 differential cross-section at 360 MeV.

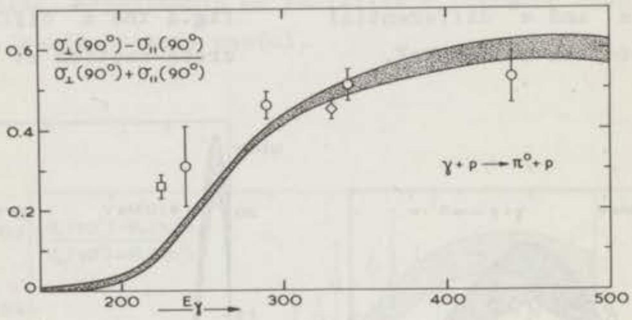

fig.9 The ratio $\frac{\sigma_\perp - \sigma_{\parallel}}{\sigma_\perp + \sigma_{\parallel}}$ for π^0 production.

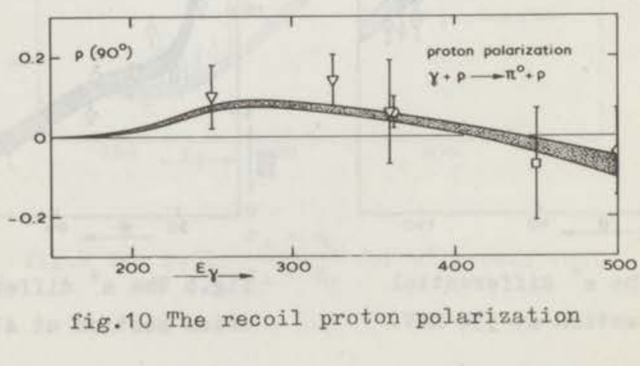

fig.10 The recoil proton polarization

A P P E N D I X A

F I E L D S, D I R A C M A T R I C E S, $\mathcal{C}$, $\mathcal{P}$ AND $\mathcal{T}$

In this appendix some conventions used in this thesis are summarized. For vectors is used $K = (\vec{k}, k_4)$ with $k_4 = ik_0$ such that upper and lower indices are the same.

In the first place some properties of the fields are compiled, together with Dirac matrices in the case of fermion fields. Then the definitions of the space reflection operator $\mathcal{P}$, the time-reversal operator $\mathcal{T}$ and the charge conjugation operator $\mathcal{C}$ are given. The latter is often denoted by $\mathcal{C}_{st}$ in the text, expressing that one considers the particle-antiparticle conjugation operator, belonging to and conserved by the strong interactions.

B o s o n f i e l d

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{(2k_0)^{\frac{1}{2}}} \left(a_{\vec{k}}\, e^{iK.x} + b^{*}_{\vec{k}}\, e^{-iK.x}\right) . \qquad (A.1)$$

Commutation relations:

$$[a_{\vec{k}}, a^{*}_{\vec{k}'}] = \delta(\vec{k} - \vec{k}') ,$$
$$[b_{\vec{k}}, b^{*}_{\vec{k}'}] = \delta(\vec{k} - \vec{k}') , \qquad (A.2)$$

and all other commutators vanish. For the fields

$$[\varphi(x), \varphi^{*}(x')] = i\Delta(x-x'), \qquad (A.3)$$

with $\Delta(x-x') = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2k_0} \left(e^{-iK.(x'-x)} - e^{iK.(x'-x)}\right)$. (A.4)

E l e c t r o m a g n e t i c   f i e l d

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\vec{k}}{(2k_0)^{\frac{1}{2}}} \left(A_\mu(\vec{k})e^{iK.x} + A^*_\mu(\vec{k})e^{-iK.x}\right). \qquad (A.5)$$

With use of four polarization vectors ε^λ_μ

$$\varepsilon^1_4 = \varepsilon^2_4 = \varepsilon^3_4 = 0,\ \vec{\varepsilon}^{\,1}.\vec{k} = \vec{\varepsilon}^{\,2}.\vec{k} = 0,\ \vec{\varepsilon}^{\,3} = \frac{\vec{k}}{k_0},\ \vec{\varepsilon}^{\,4} = 0,\ \varepsilon^4_4 = 1, \qquad (A.6)$$

one writes

$$A_\mu(x) = \sum_{\lambda=1}^{4} \frac{1}{(2\pi)^{3/2}}\int \frac{d\vec{k}}{(2k_0)^{\frac{1}{2}}} \varepsilon^\lambda_\mu \left(a^\lambda_{\vec{k}}\, e^{iK.x} + a^{\lambda *}_{\vec{k}} e^{-iK.x}\right). \qquad (A.7)$$

For the summation over polarization states, one can either sum over four states and use

$$\sum_{\lambda=1}^{4} \varepsilon^\lambda_\mu \varepsilon^\lambda_\nu = \delta_{\mu\nu}\ , \qquad (A.8)$$

or sum over the transverse states and use

$$\sum_{\lambda=1}^{2} \varepsilon^\lambda_i\ \varepsilon^\lambda_j = \delta_{ij} - \frac{k_i k_j}{k_0^2}\ . \qquad (A.9)$$

In calculations this gives the same answer, as can be seen in the Gupta-Bleuler formalism.

F e r m i o n   f i e l d

$$\psi(x,t) = \sum_j \int \frac{d\vec{k}}{(2\pi)^{3/2}} \left(\frac{m}{k_0}\right)^{\frac{1}{2}} \left(a_{\vec{k},j}\ e^{iK.x}\ u_j(\vec{k}) + b^*_{\vec{k},j}\ e^{-iK.x}\ v_j(\vec{k})\right), \qquad (A.10)$$

$$\{a^*_{\vec{k},j},\ a_{\vec{k}',j'}\} = \delta_{jj'}\delta(\vec{k}-\vec{k}')\ ,$$
$$\{b^*_{\vec{k},j},\ b_{\vec{k}',j'}\} = \delta_{jj'}\delta(\vec{k}-\vec{k}')\ , \qquad (A.11)$$

and the other anti-commutators vanish. For the fields,

$$\{\bar{\psi}_\alpha(x),\psi_\beta(x')\} = -iS_{\beta\alpha}(x'-x) = -i(\gamma_\nu\partial'_\nu - m)_{\beta\alpha}\Delta(x'-x),$$

$$\{\bar{\psi}_\alpha(x),\psi_\beta(x')\}_{x_o=x'_o} = (\gamma_4)_{\beta\alpha}\delta(\vec{x}-\vec{x}'), \tag{A.12}$$

where the Pauli adjoint is defined by $\bar{u} = u^*\gamma_4$.

D i r a c   e q u a t i o n

$$(\gamma_\mu\partial_\mu + m)\psi(x) = 0, \tag{A.13}$$

or equivalently

$$\begin{aligned} (i\gamma.K+m)u(\vec{k}) &= 0, \\ \bar{u}(\vec{k})(i\gamma.K+m) &= 0, \\ (i\gamma.K-m)v(\vec{k}) &= 0, \\ \bar{v}(\vec{k})(i\gamma.K-m) &= 0. \end{aligned} \tag{A.14}$$

D i r a c   m a t r i c e s

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \tag{A.15}$$

$$\gamma_\mu^\dagger = \gamma_\mu, \tag{A.16}$$

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}, \tag{A.17}$$

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A.18}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_i\sigma_\gamma = \delta_{i\gamma} + i\varepsilon_{i\gamma k}\sigma_k. \tag{A.19}$$

Transposed matrices:

$$\tilde{\gamma}_1 = -\gamma_1, \tilde{\gamma}_2 = \gamma_2, \tilde{\gamma}_3 = -\gamma_3, \tilde{\gamma}_4 = \gamma_4, \tilde{\gamma}_5 = \gamma_5. \tag{A.20}$$

Solution Dirac equation:

$$u_{1,2}(\vec{k}) = \left(\frac{m+k_0}{2m}\right)^{\frac{1}{2}} \begin{pmatrix} \chi_{1,2} \\ \frac{\vec{\sigma}.\vec{k}}{m+k_0}\chi_{1,2} \end{pmatrix}, \tag{A.21}$$

$$v_{1,2}(\vec{k}) = \pm\left(\frac{m+k_0}{2m}\right)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{\sigma}.\vec{k}}{m+k_0}\chi_{1,2} \\ \chi_{1,2} \end{pmatrix}, \tag{A.22}$$

where the (-) sign corresponds to v_2. When $\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\chi_2 = \begin{pmatrix}0\\1\end{pmatrix}$ are taken, u_1 and v_2 correspond to a spin state with z-component $+\frac{1}{2}$, whereas u_2 and v_1 correspond to $-\frac{1}{2}$.

Normalization:

$$u_i^* u_j = v_i^* v_j = \frac{k_0}{m}\,\delta_{ij},$$
$$\bar{u}_i u_j = -\bar{v}_i v_j = \delta_{ij}. \qquad (A.23)$$

Sum over polarizations:

$$\sum_{i=1,2} u_{i\beta}(\vec{k})\bar{u}_{i\alpha}(\vec{k}) = \frac{1}{2m}(m-i\gamma.K)_{\beta\alpha},$$
$$\sum_{i=1,2} v_{i\beta}(\vec{k})\bar{v}_{i\alpha}(\vec{k}) = \frac{1}{2m}(-m-i\gamma.K)_{\beta\alpha}. \qquad (A.24)$$

Special matrices:

$$\gamma_4: \quad \gamma_4 u_i(\vec{k}) = u_i(-\vec{k}),$$
$$\gamma_4 v_i(\vec{k}) = -v_i(-\vec{k}). \qquad (A.25)$$

$$C = \gamma_2\gamma_4: \; u_i(\vec{k}) = C\,\tilde{\bar{v}}_j(\vec{k}),$$
$$v_i(\vec{k}) = C\,\tilde{\bar{u}}_j(\vec{k}), \qquad (A.26)$$

$$C^{-1} = \tilde{C} = C^{\dagger} = -C \qquad (A.27)$$

$$C\gamma_\mu C^{-1} = -\tilde{\gamma}_\mu, \; C\gamma_5 C^{-1} = \tilde{\gamma}_5, \qquad (A.28)$$

$$T = -\gamma^1\gamma^3\gamma^4: \; T\,\tilde{\bar{u}}_i(\vec{k}) = (-1)^i u_j(-\vec{k}),$$
$$T\,\tilde{\bar{v}}_i(\vec{k}) = (-1)^j v_j(-\vec{k}), \qquad (A.29)$$

$$T^{-1} = \tilde{T} = T^{\dagger} = -T. \qquad (A.30)$$

D e f i n i t i o n o f $\mathcal{P}$, $\mathcal{C}$ a n d $\mathcal{T}$

$\mathcal{P}$ transforms a state into one with opposite momenta, but same spins. $\mathcal{C}$ transforms a state into one with all particles replaced by anti-particles. $\mathcal{T}$ transforms an ingoing state into an outgoing one with opposite momenta and spins, while it reverses the order of the operators. There is invariance under these operations, when the absolute value of the S-matrix between certain states is the same as for the transformed states.

For fermions the following phases are used (for particles with positive parity):

$$\mathcal{P} a_{\vec{p},j} \mathcal{P}^{-1} = a_{-\vec{p},j} , \quad \mathcal{P} b_{\vec{p},j} \mathcal{P}^{-1} = -b_{-\vec{p},j} , \qquad (A.31)$$

$$\mathcal{C} a_{\vec{k},j} \mathcal{C}^{-1} = b_{\vec{k},i} , \qquad (A.32)$$

$$\mathcal{T} a_{\vec{k},i} \mathcal{T}^{-1} = (-1)^{j} a^{*}_{-\vec{k},j} , \quad \mathcal{T} b^{*}_{\vec{k},i} \mathcal{T}^{-1} = (-1)^{i} b_{-\vec{k},j}, \qquad (A.33)$$

where in general

$$\mathcal{T} | \vec{p},\vec{s} \text{ in}\rangle = \langle -\vec{p},-\vec{s} \text{ out}| , \mathcal{T} | 0\rangle = \langle 0| .$$

For spinless bosons (with positive parity) the same equations (A.31), (A.32) and (A.33) hold without the sign factors. For photons minus signs occur for $\mathcal{P}$ and $\mathcal{C}$ operations and time reversal means replacement of ε_{μ} by $-\zeta_{\mu}\varepsilon_{\mu}$, where $\zeta_{\mu} = +1$ for $\mu = 1,2,3$ and $\zeta_{4} = -1$.

APPENDIX B

ISOSPIN CONVENTIONS

The proton and neutron field are combined to form an isodoublet nucleon field

$$N = \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \phi_p \\ \phi_n \end{pmatrix} = \phi_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi. \tag{B.1}$$

In matrix elements the wave function $\langle 0|\phi|N\rangle$ occurs. For the isospin part this means the occurrence of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for proton or neutron respectively, which is, in general, denoted by

For the pion fields one has

$$\begin{aligned} \varphi &= (\pi^+)^* = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2), \\ \varphi^* &= (\pi^-)^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2), \\ \varphi_3 &= \pi 0 , \end{aligned} \tag{B.2}$$

where the scalar fields

$$\varphi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^*), \quad \varphi_2 = \frac{i}{\sqrt{2}}(\varphi^* - \varphi), \quad \varphi_3, \tag{B.3}$$

are introduced. They are considered as three vector $\vec{\varphi}$ in isospin space. The wave function $\langle 0|\vec{\varphi}|\pi\rangle$ gives rise to a vector V in isospin space

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix},$$

for $\pi^{\pm}$ states and (B.4)

$$V = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

for a π^0 state.

APPENDIX C

THE HELICITY FORMALISM

The connection between the traditional multipole description (section 11) and the helicity formalism[20] is given for photopion production.

There are eight helicity amplitudes $f_{\mu,\lambda}(\theta,\varphi)$ where μ is the helicity of the final nucleon and $\lambda = \lambda_N - \lambda_\gamma$ is the helicity of the initial state along the nucleon momentum. Parity reduces the number of independent amplitudes to four by the relation

$$f_{\mu,\lambda}(\theta,\varphi) = -f_{-\mu,-\lambda}(\theta,\pi-\varphi) \ . \qquad (C.1)$$

The angular momentum decomposition is given by [20]

$$f_{\mu,\lambda}(\theta,\varphi) = \sum_{J=\frac{1}{2}}^{\infty} (J+\tfrac{1}{2}) \langle \mu \, | T^J | \lambda \rangle e^{i(\lambda-\mu)\varphi} \, d^J_{\lambda\mu}(\theta), \qquad (C.2)$$

and the helicity amplitudes normalized by

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left| f_{\mu\lambda}(\theta,\varphi) \right|^2 \ . \qquad (C.3)$$

The connection between the helicity formalism and the multipole formalism is most easily obtained by specifying a co-ordinate frame with $\vec{p}_1$ as the z axis and the xz plane as the production plane.Then one makes a rotation within this frame to connect the final nucleon helicity state with the z-component nucleon spin state [38]. Introducing a vector

$$\tilde{f} = (f_{1/2,3/2}, f_{1/2,-3/2}, f_{1/2,1/2}, f_{1/2,-1/2}) =$$

$$= (H^-, \Phi^+, \Phi^-, H^+) \,, \qquad (C.4)$$

(the latter notation is used in the paper by G. Zweig[39]) one obtains

$$\tilde{f} = \tilde{R}\,\tilde{F} \,, \qquad (C.5)$$

where $\tilde{F}$ is given in section 11 and

$$\tilde{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \cos\frac{\theta}{2}\sin\theta & \cos\frac{\theta}{2}\sin\theta \\ 0 & 0 & \sin\frac{\theta}{2}\sin\theta & -\sin\frac{\theta}{2}\sin\theta \\ 2\cos\frac{\theta}{2} & -2\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\sin\theta & \sin\frac{\theta}{2}\sin\theta \\ -2\sin\frac{\theta}{2} & -2\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\sin\theta & -\cos\frac{\theta}{2}\sin\theta \end{bmatrix} . \qquad (C.6)$$

The connection with the multipoles follows from eq.(11.21), namely

$$\tilde{M}_{\ell'}(s) = \int_{-1}^{+1} dx\, D_{\ell'}(x)\, \tilde{F}(s,t), \qquad (C.7)$$

and

$$\tilde{F} = \tilde{R}^{-1}\,\tilde{f} = \tilde{R}^{-1} \sum_{\ell=0}^{\infty} \tilde{O}_\ell \tilde{T}_\ell \,, \qquad (C.8)$$

where eq.(C.2) has been used in matrix form, with the diagonal matrix

$$\tilde{O}_\ell = \begin{bmatrix} -\sin\frac{\theta}{2}\left[\left(\frac{\ell}{\ell+2}\right)^{1/2} P'_{\ell+1} + \left(\frac{\ell+2}{\ell}\right)^{1/2} P'_\ell\right] & & & \\ & \cos\frac{\theta}{2}\left[-\left(\frac{\ell}{\ell+2}\right)^{1/2} P'_{\ell+1} + \left(\frac{\ell+2}{\ell}\right)^{1/2} P'_\ell\right] & & \\ & & \cos\frac{\theta}{2}\left[P'_{\ell+1} - P'_\ell\right] & \\ & & & \sin\frac{\theta}{2}\left[P'_{\ell+1} + P'_\ell\right] \end{bmatrix} , \qquad (C.9)$$

and $\ell = j - 1/2$. Substituting eq.(C.8) in eq.(C.7) and performing the integration leads to

$$E_{\ell+} = \frac{\sqrt{2}}{i}\,\frac{1}{4(\ell+1)}\left\{\left(\frac{\ell}{\ell+2}\right)^{1/2}[-T_\ell(1)+T_\ell(2)] + [T_\ell(3)-T_\ell(4)]\right\}\ , \quad \text{(C.10)}$$

$$E_{(\ell+1)-} = \frac{\sqrt{2}}{i}\,\frac{1}{4(\ell+1)}\left\{\left(\frac{\ell+2}{\ell}\right)^{1/2}[T_\ell(1)+T_\ell(2)] + [T_\ell(3)+T_\ell(4)]\right\}\ , \quad \text{(C.11)}$$

$$M_{\ell+} = \frac{\sqrt{2}}{i}\,\frac{1}{4(\ell+1)}\left\{\left(\frac{\ell+2}{\ell}\right)^{1/2}[T_\ell(1)-T_\ell(2)] + [T_\ell(3)-T_\ell(4)]\right\}\ , \quad \text{(C.12)}$$

$$M_{(\ell+1)-} = \frac{\sqrt{2}}{i}\,\frac{1}{4(\ell+1)}\left\{\left(\frac{\ell}{\ell+2}\right)^{1/2}[T_\ell(1)+T_\ell(2)] - [T_\ell(3)+T_\ell(4)]\right\}, \quad \text{(C.13)}$$

and the inverse

$$T_\ell(1) = \langle\tfrac{1}{2}|T_\ell|\tfrac{3}{2}\rangle = \frac{[\ell(\ell+2)]^{1/2}}{i\sqrt{2}}\left[E_{\ell+} - M_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}\right] \quad \text{,(C.14)}$$

$$T_\ell(2) = \langle\tfrac{1}{2}|T_\ell|-\tfrac{3}{2}\rangle = \frac{[\ell(\ell+2)]^{1/2}}{i\sqrt{2}}\left[M_{\ell+} - E_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}\right] \quad \text{,(C.15)}$$

$$T_\ell(3) = \langle\tfrac{1}{2}|T_\ell|\tfrac{1}{2}\rangle = \frac{1}{i\sqrt{2}}\left[(\ell+2)\left(M_{(\ell+1)-} - E_{\ell+}\right) - \ell\left(M_{\ell+} + E_{(\ell+1)-}\right)\right] \quad \text{,(C.16)}$$

$$T_\ell(4) = \langle\tfrac{1}{2}|T_\ell|-\tfrac{1}{2}\rangle = \frac{1}{i\sqrt{2}}\left[\ell\left(M_{\ell+} - E_{(\ell+1)-}\right) + (\ell+2)\left(M_{(\ell+1)-} + E_{\ell+}\right)\right]. \quad \text{(C.17)}$$

APPENDIX D

CROSS-SECTION AND POLARIZATION FORMULAE

The differential cross-section for the transition from an initial γ-N state i to a final pion-nucleon state f is given by

$$\frac{d\sigma}{d\Omega} = \frac{q}{k}|\langle\chi_f|F|\chi_i\rangle|^2, \tag{D.1}$$

where

$$F = i\vec{\sigma}.\vec{\varepsilon}F_1 + \vec{\sigma}.\hat{q}\,\vec{\sigma}.(\hat{k}\times\vec{\varepsilon})F_2 + i\vec{\sigma}.\hat{k}\,\hat{q}.\vec{\varepsilon}F_3 + i\vec{\sigma}.\hat{q}\,\hat{q}.\vec{\varepsilon}F_4. \tag{D.2}$$

First of all those cross-sections in which the polarization of the final nucleon is unobserved, will be evaluated. Summing over the final spin states yields

$$\sum_f\langle\chi_f|F|\chi_i\rangle^*\langle\chi_f|F|\chi_i\rangle = \langle\chi_i|F^\dagger F|\chi_i\rangle ,$$

where

$$F^\dagger F = |F_1|^2\{\vec{\varepsilon}^*.\vec{\varepsilon} + i\vec{\sigma}.(\vec{\varepsilon}^*\times\vec{\varepsilon})\} + |F_2|^2\{\vec{\sigma}.(\hat{k}\times\vec{\varepsilon}^*)\vec{\sigma}.(\hat{k}\times\vec{\varepsilon})\}$$

$$+ |F_3|^2\{\hat{q}.\vec{\varepsilon}^*\,\hat{q}.\vec{\varepsilon}\} + |F_4|^2\{\hat{q}.\vec{\varepsilon}^*\,\hat{q}.\vec{\varepsilon}\}$$

$$+ F_1^*F_2\{-i\sigma.(\hat{k}\times\vec{\varepsilon})\hat{q}.\vec{\varepsilon}^* + i\vec{\sigma}.\hat{q}\,\hat{k}(\vec{\varepsilon}\times\vec{\varepsilon}^*) - i\vec{\sigma}.\vec{\varepsilon}^*\hat{q}.(\hat{k}\times\vec{\varepsilon}) + (\vec{\varepsilon}^*\times\hat{q}).(\hat{k}\times\vec{\varepsilon})\}$$

$$+ F_1^* F_3 \{ i\vec{\sigma}.(\vec{\varepsilon}^* \times \hat{k}) \hat{q}.\vec{\varepsilon} \} + F_1^* F_4 \{ \hat{q}.\vec{\varepsilon}\ \hat{q}.\vec{\varepsilon}^* + i\vec{\sigma}.(\vec{\varepsilon}^* \times \hat{q}) \hat{q}.\vec{\varepsilon} \}$$

$$+ F_2^* F_3 \{ \hat{q}.\vec{\varepsilon}\ \hat{q}.\vec{\varepsilon}^* + i\vec{\sigma}.\hat{k}\ \hat{q}.(\hat{k} \times \vec{\varepsilon}^*) \hat{q}.\vec{\varepsilon} + i\vec{\sigma}.(\hat{k} \times \vec{\varepsilon}^*) \hat{q}.\hat{k}\ \hat{q}.\vec{\varepsilon} \}$$

$$+ F_2^* F_4 \{ \hat{q}.\vec{\varepsilon}\, i\vec{\sigma}.(\hat{k} \times \vec{\varepsilon}^*) \} + F_3^* F_4 \{ \hat{q}.\vec{\varepsilon}^* \hat{q}.\vec{\varepsilon} [\hat{q}.\hat{k} + i\vec{\sigma}.(\hat{k} \times \hat{q})] \}$$

+ Hermitian Conjugate of the off-diagonal elements. (D.3)

Choosing a co-ordinate frame in which the production plane is the x,z plane and introducing the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ $(=\hat{k})$ in the x,y and z directions respectively, then for right and left circularly polarized photons (helicity $\pm$ 1),

$$\vec{\varepsilon} = \cos\varphi\ \hat{e}_1 + \sin\varphi\ \hat{e}_2 = \frac{1}{\sqrt{2}} \{ e^{i\varphi} \vec{\varepsilon}_- - e^{-i\varphi} \vec{\varepsilon}_+ \} . \tag{D.4}$$

and for linearly polarized photons

$$= \cos\ \ \hat{e}_1 + \sin\ \ \hat{e}_2 \tag{D.5}$$

A P o l a r i z e d n u c l e o n, c i r c u l a r l y p o l a r i z e d p h o t o n

Introducing the initial nucleon polarization by

$$\vec{P} = \langle \chi_i | \vec{\sigma} | \chi_i \rangle , \tag{D.6}$$

then

$$\langle \chi_i | F_\pm^\dagger F_\pm | \chi_i \rangle = (1 \mp \hat{k}.\vec{P})\alpha + \beta \pm \sin\theta\, \hat{e}_1.\vec{P}\gamma + \sin\theta\, \hat{e}_2.\vec{P}\ \delta\ , \tag{D.7}$$

where

$$\alpha = |F_1|^2 + |F_2|^2 - 2\cos\theta\, \mathrm{Re} F_1^* F_2 + \sin^2\theta\, \mathrm{Re}\{ F_1^* F_4 + F_2^* F_3 \}\ , \tag{D.8}$$

$$\beta = \tfrac{1}{2}\sin^2\theta \{ |F_3|^2 + |F_4|^2 + 2\cos\theta\, \mathrm{Re} F_3^* F_4 \}\ , \tag{D.9}$$

$$\gamma = \mathrm{Re}\{ F_1^* F_3 - F_2^* F_4 \} + \cos\theta\, \mathrm{Re}\{ F_1^* F_4 - F_2^* F_3 \}\ , \tag{D.10}$$

ERRATUM

Eq.(D.5) should be replaced by eq.(D.4), whereas eq.(D.4) reads

$$\vec{\varepsilon}_{\pm} = \mp \frac{1}{\sqrt{2}} (\hat{e}_1 \pm i\, \hat{e}_2).$$

$$\delta = \mathrm{Im}\{F_1^* F_3 - F_2^* F_4\} + \cos\theta\, \mathrm{Im}\{F_1^* F_4 - F_2^* F_3\} - \sin^2\theta\, \mathrm{Im}\{F_3^* F_4\} \; . \qquad (D.11)$$

For the special case of the initial nucleon being polarized along, or opposite, to $\hat{k}$, one obtains respectively

$$\langle \chi_i | F_\pm^\dagger F_\pm | \chi_i \rangle = (1 \mp 1)\alpha + \beta \; , \qquad (D.12)$$

and

$$\langle \chi_i | F_\pm^\dagger F_\pm | \chi_i \rangle = (1 \pm 1)\alpha + \beta \; . \qquad (D.13)$$

Insertion of eqs.(D.7), (D.12) or (D.13) in eq.(D.1) gives the appropriate cross-sections.

B P o l a r i z e d n u c l e o n, u n p o l a r i z e d p h o t o n

$$\frac{d\sigma(\vec{P})}{d\Omega} = \frac{1}{2}\left\{\frac{d\sigma_+(\vec{P})}{d\Omega} + \frac{d\sigma_-(\vec{P})}{d\Omega}\right\} = \frac{q}{k}\{\alpha + \beta + \sin\theta\, \hat{e}_2 \vec{P} \delta\} . \qquad (D.14)$$

C U n p o l a r i z e d n u c l e o n, c i r c u l a r l y p o l a r i z e d p h o t o n

$$\frac{d\sigma_+}{d\Omega} = \frac{d\sigma_-}{d\Omega} = \frac{d\sigma_0}{d\Omega} = \frac{q}{k}\{\alpha + \beta\} \; , \qquad (D.15)$$

where $\frac{d\sigma_0}{d\Omega}$ is the cross-section for an unpolarized initial state. [Note that in the helicity formalism,

$$\frac{d\sigma_+}{d\Omega} = \frac{q}{2k}\{|f_{1/2,-1/2}|^2 + |f_{1/2,-3/2}|^2 + |f_{-1/2,-1/2}|^2 + |f_{-1/2,-3/2}|^2\}$$

$$= \frac{q}{2k}\{|H^+|^2 + |\Phi^+|^2 + |\Phi^-|^2 + |H^-|^2\} = \frac{d\sigma_-}{d\Omega}$$

which is in disagreement with the result obtained in the paper by G. Zweig[39).]

D U n p o l a r i z e d n u c l e o n, l i n e a r l y p o l a r i z e d p h o t o n

From eq.(D.5) it follows that

$$F^{\dagger}(\vec{\varepsilon})F(\vec{\varepsilon}) = \tfrac{1}{2}\{F_+^{\dagger}F_+ + F_-^{\dagger}F_- - 2\mathrm{Re}(e^{2i\varphi}F_+^{\dagger}F_-)\} \ ,$$

and consequently

$$\frac{d\sigma(\varepsilon)}{d\Omega} = \frac{d\sigma_0}{d\Omega} + \frac{q}{k}\cos 2\varphi \sin^2\theta\{\tfrac{1}{2}|F_3|^2 + \tfrac{1}{2}|F_4|^2 + \mathrm{Re}\{F_2^*F_3 + F_1^*F_4\}$$

$$+ \cos\theta \mathrm{Re}\{F_3^*F_4\}\} \ . \qquad (D.16)$$

Using eqs.(D.8), (D.9) and (D.15), eq.(D.16) can be recast easily into the form usually quoted[25] viz.

$$\frac{k}{q}\frac{d\sigma}{d\Omega} = \{|F_1|^2 + |F_2|^2 + \tfrac{1}{2}|F_3|^2 + \tfrac{1}{2}|F_4|^2 + \mathrm{Re}F_1F_4^* + \mathrm{Re}F_2F_3^*\}$$

$$+\{\mathrm{Re}F_3F_4^* - 2\mathrm{Re}F_1F_2^*\}\cos\theta - \{\tfrac{1}{2}|F_3|^2 + \tfrac{1}{2}|F_4|^2 + \mathrm{Re}F_1F_4^* + \mathrm{Re}F_2F_3^*\}\cos^2\theta$$

$$-\mathrm{Re}F_3F_4^*\cos^3\theta + \sin^2\theta\cos 2\varphi\{\tfrac{1}{2}|F_3|^2 + \tfrac{1}{2}|F_4|^2 + \mathrm{Re}F_2F_3^*$$

$$+\mathrm{Re}F_1F_4^* + \mathrm{Re}F_3F_4^*\cos\theta\} \ . \qquad (D.17)$$

E Recoil nucleon polarization

Finally we obtain the recoil nucleon polarization, for a completely unpolarized initial state

$$\vec{P}\frac{d\sigma_0}{d\Omega} = \frac{q}{4k}\sum_{\substack{\text{Initial}\\ \text{spins}}}\langle\chi_i|F^{\dagger}\vec{\sigma}F|\chi_i\rangle = \frac{q}{4k}\sum_{\substack{\text{Photon}\\ \text{spin}}}\mathrm{Tr}[F^{\dagger}\vec{\sigma}F] \ . \qquad (D.18)$$

Using eq.(D.2) and

$$\sum_{\substack{\text{photon}\\ \text{spin}}}\varepsilon_i^*\varepsilon_j = \delta_{ij} - \hat{k}_i\hat{k}_j \qquad (D.19)$$

yields

$$\vec{P}(\theta)\frac{d\sigma_0}{d\Omega} = \frac{q}{k}(\hat{k}\times\hat{q})\mathrm{Im}\{2F_1F_2^* + F_1F_3^* - F_2F_4^* + \cos\theta\{F_1F_4^* - F_2F_3^*\} - \sin^2\theta F_3F_4^*\} \ . \qquad (D.20)$$

If the polarization perpendicular to the production plane is measured, eq.(D.20) gives the usual form[25)]

$$P(\theta)\frac{k}{q}\,\frac{d\sigma_0}{d\Omega} = \sin\theta\,\mathrm{Im}\{(2F_1F_2^*+F_1F_3^*-F_2F_4^*-F_3F_4^*)+(F_1F_4^*-F_2F_3^*)\cos\theta$$

$$+F_3F_4^*\cos^2\theta\}. \qquad \text{(D.21)}$$

S A M E N V A T T I N G

In dit proefschrift wordt de berekening gegeven van de botsingsdoorsnede voor een specifiek proces nl. de produktie van een π-meson uit de botsing van een foton en een nucleon ($\gamma + N \rightarrow \pi + N$). Door te vergelijken met experimentele gegevens wordt inzicht in de betrouwbaarheid van de theoretische veronderstellingen verkregen. Bovendien is kennis van dit proces van belang wegens de samenhang met andere kwesties, zoals de eigenschappen van nucleon isobaren, het quark model, somregels, elektroproduktie van π-mesonen, Compton verstrooiing aan nucleonen en vectormeson koppelingskonstantes. Hier wordt in de inleiding op ingegaan.

Voor de berekening van de botsingsdoorsnede voor fotopionproduktie wordt de elektromagnetische interactie in eerste orde behandeld. De sterke interactie moet volledig in rekening gebracht worden, omdat een storingsrekening in de koppelingskonstante niet mogelijk is. Als eerste stap scheidt men de elektromagnetische en sterke interacties. Het probleem wordt aldus teruggebracht tot de berekening van matrixelementen van de elektromagnetische stroom, waarbij alleen sterke interacties een rol spelen. Dit wordt in hoofdstuk I beschreven.

In hoofdstuk II wordt de analytische structuur van de matrixelementen van de stroom bestudeerd als functie van de energie en impulsoverdracht. Nadat de singulariteiten m.b.v. de unitariteit van de S-matrix gevonden zijn, is het mogelijk dispersierelaties te postuleren, welke de analytische structuur in rekening brengen. Hoewel de gedachtengang eenvoudig is, brengt de uitwerking nogal wat complicaties met zich mee. Dit hangt samen met de optredende spins van de nucleonen en het foton en met de eis van ijkinvariantie.

In hoofdstuk III wordt de theorie in een vorm gebracht, die geschikt is voor berekening. Dit wordt mogelijk gemaakt door multipoolovergangen te beschouwen, zodat het proces geanalyseerd kan worden in termen van eigentoestanden van het totale impulsmoment.

In hoofdstuk IV wordt getoond, dat tot een bepaalde energie (nl. 500 MeV foton laboratorium energie) de dispersierelaties, die nu integraalvergelijkingen zijn geworden, opgelost kunnen worden. De essentiële stap is het leggen van een verband tussen fotopionproduktie en pion-nucleon verstrooiing m.b.v. de unitariteit van de S-matrix. De faseverschuivingen van het laatste proces vormen dan de informatie nodig voor de oplossing van de vergelijkingen. De verkregen resultaten worden vergeleken met de experimenteel gemeten grootheden. De overeenkomst blijkt goed te zijn.

Enkele appendices geven de gebruikte conventies, het verband van de multipolen met heliciteitsamplitudes en de uitdrukkingen voor de belangrijke experimentele grootheden.

Aangezien de theorie, zoals hier toegepast op fotopionproduktie, ook toegepast kan worden op elektropionproduktie ($e + N \rightarrow e + N + \pi$), is ook het formalisme voor dit proces gegeven. Een expliciete berekening is nu niet mogelijk, omdat de elektromagnetische vormfactoren van het neutron en pion nog niet bekend zijn.

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S T E L L I N G E N

I

Het verval van een π^o-meson in drie fotonen is niet bijzonder geschikt als toets voor schending van ladingsconjugatie-symmetrie.

F.A. Berends, Phys.Lett. 16, 178 (1965)

II

Bij de door Bernstein, Feinberg en Lee geschatte vertakkings-verhouding voor een π^o-meson tussen het verval in drie fotonen en dat in twee fotonen is een onjuiste impulsmomentbarrière gebruikt.

J.Bernstein, G.Feinberg and T.D.Lee, Phys.Rev. 139, B1650 (1965)

III

De interactielagrangiaan door Schechter gebruikt voor het verval van positronium in drie fotonen geeft een overgangswaarschijn-lijkheid, die gelijk is aan nul.

J.Schechter, Phys.Rev. 132, 841 (1963)

IV

Berekeningen van de levensduur van het η-meson, enerzijds uit de breedte van het $\pi^o \to 2\gamma$ verval met behulp van unitaire symmetrie en anderzijds uit de fractie in het η-verval, waarbij de overgang is $\eta \to \pi^+\pi^-\gamma$, met behulp van het "ρ-meson-dominantiemodel", kunnen zeer verschillende uitkomsten leveren.

F.A.Berends and P.Singer, Phys.Lett. 19, 249, 616(E) (1965)

V

Als tweede-klas vectorstromen voor zwakke interacties bestaan, kunnen zij de hoofdbijdrage leveren tot het proces, waarbij een neutrino op een nucleon een η- of X^o-meson produceert.

F.A.Berends and P.Singer, Nuovo Cimento 46, 90 (1966)

VI

Door gebruik te maken van het hermitisch karakter van de octetstromen kan het theorema van Ademollo en Gatto bewezen worden voor de absolute waarde van de vectorkoppelingsconstante, waarbij de eis van ladingsconjugatie-symmetrie kan vervallen.

M.Ademollo and R.Gatto, Phys.Rev.Lett. 13, 264 (1964)

VII

Bij de bewering van Fubini en Furlan, dat de renormalisatie van de zwakke vectorstroom, die aanleiding geeft tot verandering van vreemdheid, een tweede-orde effect is in de semi-sterke interacties, wordt gebruik gemaakt van tijdomkeervariantie.

S.Fubini and G.Furlan, Physics 1, 229 (1965)

VIII

Als de massaverschillen binnen unitaire multipletten eerste-orde effecten zijn van een semi-sterke interactie, dan worden de massaverschillen alleen veroorzaakt door het gedeelte van de interactie, dat ladingsconjugatie-symmetrisch is. Het SU(3) transformatiekarakter en de sterkte van het niet-ladingsconjugatie-symmetrisch gedeelte is niet bekend.

J.Prentki and M.Veltman, Phys.Lett. 15, 88 (1965)

IX

De resultaten van de dispersierelatietheorie voor fotopionproductie, als verkregen in dit proefschrift, zijn een indicatie, dat de electromagnetische interacties van hadronen in grote mate ladingsconjugatie-invariant zijn.

X

Er is in de groep SU(3) een met de Weylgroep isomorfe ondergroep, welke de Weyltransformaties induceert.

XI

Niet elke kanonieke transformatie kan gegenereerd worden met behulp van de door Goldstein aangegeven vier soorten genererende functies.

H. Goldstein, Classical mechanics, Ch. 8.

XII

Een analyse van het meest recente experiment van het verval $K^+ \rightarrow e^+ + \nu + \pi^+ + \pi^-$ toont aan, dat de vectorstroom veel groter is dan op diverse theoretische gronden werd verwacht.

XIII

Interactie van zeer energetische protonen met de zwarte straling van 3^oK zal aanleiding geven tot een verdichting in de protonendichtheid van primaire kosmische straling bij ongeveer 10^{20}eV. Dit heeft echter een te klein effect om de gemeten knik in het integrale spectrum bij 10^{18}eV te verklaren.

XIV

Het verdient aanbeveling alle onderzoek in Nederland op het gebied van de hoge-energiefysica, zowel theoretisch als experimenteel, te bundelen in één instelling.

F.A. Berends Leiden, 29 juni 1967

Op verzoek van de faculteit der Wiskunde en Natuurwetenschappen volgt hier een kort overzicht van mijn academische studie.

Na het behalen van het einddiploma Gymnasium β aan het Stedelijk Gymnasium te Arnhem in 1956, begon ik mijn studie aan de Rijksuniversiteit te Leiden. Het kandidaatsexamen natuur- en wiskunde (A') werd in mei 1959 afgelegd. Van november 1959 tot september 1960 was ik als assistent verbonden aan het Mathematisch Instituut. Sinds februari 1961 ben ik werkzaam aan het Instituut-Lorentz. Na mijn doctorale examens in de wiskunde en in de natuurkunde in juli 1962, begon ik onder leiding van Prof. Dr J.A.M. Cox onderzoekingen op het gebied van de hoge-energie fysica, o.a. op het terrein van de dispersierelaties. In oktober 1964 begon een verblijf op het CERN in Genève. Naast de gunstige omstandigheden, die een dergelijk speurcentrum biedt, noem ik in het bijzonder het nut, dat ik ondervond van de discussies met Prof. Dr M. Veltman en van de samenwerking met Dr P. Singer en Dr A. Donnachie. Het onderzoek daar begonnen, zet ik sinds september 1966 voort in Leiden.

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