ON THE RELATIVISTIC DYNAMICS OF SYSTEMS OF DIPOLES AND QUADRUPOLES

S. EMID

ON THE RELATIVISTIC DYNAMICS OF SYSTEMS OF DIPOLES AND QUADRUPOLES

ON THE RELATIVISTIC DYNAMICS OF SYSTEMS OF DIPOLES AND QUADRUPOLES

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE
RIJKSUNIVERSITEIT TE LEIDEN,
OP GEZAG VAN DE RECTOR MAGNIFICUS DR. C. SOETEMAN,
HOOGLERAAR IN DE FACULTEIT DER LETTEREN,
TEN OVERSTAAN VAN EEN COMMISSIE UIT DE SENAAT
TE VERDEDIGEN OP WOENSDAG 24 FEBRUARI 1971
TE KLOKKE 16.15 UUR

DOOR

SOEMAR EMID

GEBOREN TE COMMEWIJNE (SURINAME) IN 1941

1971

KONINKLIJKE DRUKKERIJ VAN DE GARDE N.V.
ZALTBOMMEL

Promotor: PROF. DR. P. MAZUR

Dit proefschrift is bewerkt onder leiding van

DR. J. VLIEGER

STELLINGEN

I

In de klassieke theorie is het mogelijk om de bijdrage van de intra-atomaire velden tot de massa en het inwendige impulsmoment van het atoom op manifest covariante wijze af te leiden.

Hoofdstuk IV van dit proefschrift.

II

In aanwezigheid van een dempingskoppel ten gevolge van straling is de atomaire energie-impulstensor niet symmetrisch.

Hoofdstuk IV van dit proefschrift.

III

De bewering van Rohrlich, dat het door Synge gedefinieerde absolute tweedimensionele oppervlak onafhankelijk is van de afstand tot de oorsprong, is onjuist.

F. Rohrlich, Classical charged particles, Addison-Wesley Publ. Comp. (Reading, 1965).

J. L. Synge, Relativity: The special theory, second edition, North-Holland Publ. Comp. (Amsterdam, 1965).

IV

Bij de afleiding van de bewegingsvergelijkingen voor een samengesteld systeem bestaande uit een aantal puntladingen is door Suttorp en de Groot gebruik gemaakt van een energie-impulstensor die niet voldoet aan de gestelde eisen voor een eindig systeem met een positief definiete energiedichtheid.

L. G. Suttorp en S. R. de Groot, Nuovo Cimento **65**A (1970) vgl. (31) p. 254.

V

Ten onrechte wordt door Wenckebach *et al.* opgemerkt, dat voor de vergelijkingen van Provotorov de frequentie ω_1, corresponderende met de amplitude van het hoogfrequente wisselveld, veel kleiner moet zijn dan het verschil Δ tussen de frequentie van dit wisselveld en de resonantiefrequentie waarop is ingestraald.

W. Th. Wenckebach, T. J. B. Swanenburg en N. J. Poulis, Physica **46** (1970) 303.
B. N. Provotorov, Zh. éksp. Fiz. **41** (1961) 1582, Sov. Phys. JETP **14** (1962) 1126.

VI

De relaxatiematrixvergelijkingen van Redfield kunnen op eenvoudige wijze enigszins algemener afgeleid worden.

A. G. Redfield, IBM J. Res. Develop. **1** (1957) 19, Advances in Magnetic Resonances **1** (1965) 1.
A. Abragam, The principles of nuclear magnetism, Oxford University Press (London, 1961).
C. P. Slichter, Principles of magnetic resonance, Harper (New York, 1964).

VII

Bij selectieve metingen van de transversale relaxatietijden aan kernspinresonantie-spectra van vloeistoffen, met behulp van een spin-echoprogramma in combinatie met een laagdoorlatend filter, is het scheidend vermogen sterk afhankelijk van de vorm van de inhomogeniteit van het statische magneetveld.

M. F. Augusteijn, S. Emid, A. F. Mehlkopf en J. Smidt, wordt gepubliceerd.

VIII

Bij de methode van Freeman en Hill voor het bepalen van de kernspinroosterrelaxatietijden wordt de gemiddelde waarde van het signaal buiten resonantie verwaarloosd ten opzichte van die van het signaal in resonantie. Dit is in het algemeen niet geoorloofd.

R. Freeman en H. D. W. Hill, J. Chem. Phys. **51** (1969) 3140.

IX

Ook indien stralingseffecten in rekening worden gebracht, kan de middelingsprocedure van de Groot en Suttorp worden toegepast op de atomaire energie-impulstensor.

S. R. de Groot en L. G. Suttorp, Physica **39** (1968) 28.

X

Het verdient aanbeveling om een verdampingskaart van Suriname te maken met behulp van de verdampingsformule van Penman.

H. L. Penman, Proc. Roy. Soc. A **193** (1948) 120.

[illegible] plitude van het hoogfrequente wisselveld, veel kleiner [illegible] teld, is tegen de frequentie van dit wisselveld en de resonantiefrequentie waarop is ingesteld.

W. Th. Wenckebach, T. J. B. Swanenburg en N. J. Poulis, Physica 46 (1970) 303.
B. N. Provotorov, Zh. eksp. Fiz. 41 (1961) 1582; Sov. Phys. JETP 14 (1962) 1126.

VI

De relaxatiematrixvergelijkingen van Redfield kunnen op eenvoudige wijze empirisch algemeen afgeleid worden.

A. G. Redfield, IBM J. Res. Develop. 1 (1957) 19; Advances in Magnetic Resonance 1 (1965) 1.
A. Abragam, The principles of nuclear magnetism, Oxford University Press (London, 1961).
C. P. Slichter, Principles of magnetic resonance, Harper (New York, 1964).

VII

Bij selectieve metingen van de transversale relaxatietijden aan [illegible] resonantie-spectra van vloeistoffen, met behulp van een spin-echo-programma in combinatie met een laagdoorlatend filter, is het schijnbaar verloop sterk afhankelijk van de vorm van de inhomogeniteit van het statische magneetveld.

M. E. [illegible], E. Bosch, A. E. Mehlkopf en J. Smidt, wordt gepubliceerd.

VIII

Bij de methode van Freeman en Hill voor het bepalen van de transversale relaxatietijd wordt de gemiddelde waarde van het signaal tijdens precessie [illegible] ten opzichte van die van het signaal in resonantie. Dit is in het algemeen niet gerechtvaardigd.

R. Freeman en H. D. W. Hill, J. Chem. Phys. 54 (1971) [illegible].

Aan mijn ouders

Aan mijn vrouw

CONTENTS

INTRODUCTION

Recently de Groot and Suttorp[1]) have derived the macroscopic energy-momentum tensor in polarized media. The derivation proceeds in two steps. First the energy-momentum laws on the atomic level are derived from the microscopic force law for charged particles (electrons and nuclei). Then by covariant statistical averaging they obtain the macroscopic energy-momentum conservation laws which contain the material and field parts of the macroscopic energy-momentum tensor. Explicit expressions are given for both parts in terms of statistical averages of atomic quantities.

The advantage of their approach is that the electromagnetic energy-momentum tensor is obtained in an unambiguous way. The correct form of this tensor has been a controversial issue in the past. In connection herewith they give an extensive account of the historical developments on this subject, in particular in their last paper. The treatment of de Groot and Suttorp may also be considered as the relativistic generalization of Mazur and de Groot's[2]) non-relativistic derivation of the ponderomotive force and pressure in a dielectric.

As a model for atoms and molecules de Groot and Suttorp[1]) consider stable groups of point particles, which are described classically. In this model the atoms and molecules are supposed to carry electric and magnetic dipole moments. The atomic electric quadrupole moment is not considered in their treatment and radiative effects are also neglected.

In order to describe the motion of the atom (molecule) as a whole de Groot and Suttorp define an approximate centre of gravity, which is in certain respects a generalization of the notion of the centre of gravity in classical non-relativistic mechanics. Since only the rest masses of the constituent particles of the atom are used in the definition of this reference point, the contributions of the kinetic energies and intra-atomic fields to the atomic mass have to be taken into account afterwards. Moreover, the intra-atomic electromagnetic fields are supposed to be of non-relativistic nature in the rest frame of the atom, so that the Darwin approximation can be used. Explicit expressions for the contributions of these fields to the energy-momentum and angular momentum laws are given in this approximation.

A complication of the centre of gravity definition of de Groot and Suttorp is the fact that the total atomic energy-momentum tensor is symmetric only if certain terms of intra-atomic origin are neglected. They demonstrate that a symmetric tensor can be obtained if the atomic energy is properly localized. This is realized by a change in the definition of the reference point.

From a formal point of view a definition of the centre of gravity in which only the rest masses are used is rather inconvenient in a relativistic theory of interacting particles, as is already clear from Einstein's fundamental energy-mass relation. An appropriate definition of the centre of gravity for a classical relativistic system having an internal angular momentum is not readily formulated. One can try to generalize the classical non-relativistic notion of the centre of gravity. A careful analysis by Møller[3]) indicates that for these systems it is not possible to define a unique centre of gravity (in the sense that it is the centre of energy in the Lorentz frame in which the system is momentarily at rest). Equations of motion are, however, given for so-called pseudo-centres of gravity. A different definition of the centre of gravity is used by *e.g.* Dixon[4]) and more recently also by Suttorp and de Groot[5]). Although from the point of view of uniqueness the latter definition is perhaps to be preferred, it also leads in the general case to a complicated type of equations of motion. However, for systems of the dimension of atoms and molecules it can be shown that both definitions lead to the same equations of motion.

Closely connected to the problem of defining an appropriate (pseudo) centre of gravity of a composite system, interacting through the electromagnetic fields of its constituents, is the question whether the renormalization effects of these fields can be taken into account in a general way without resorting to approximations or making assumptions of a non-relativistic nature. In the first chapters of this thesis this is realized in a formal way, whereas in the last chapter we give a general derivation of this possibility.

The atomic energy-momentum tensor, derived by de Groot and Suttorp, was also obtained by Vlieger[6]) in the first of a series of papers on the relativistic dynamics of polarized systems using the theory of Møller[3]) mentioned above. This tensor is found by first deriving the relativistic atomic equations of motion with the help of Møller's equations for the pseudo-centres of gravity. Vlieger also uses a classical model of atoms and molecules and supposes that these particles carry only electric and magnetic dipole moments. Radiative effects are neglected. In Vlieger's paper this is a consequence of the fact that Møller's theory is only valid for finite systems, *i.e.* systems with an energy-momentum tensor which vanishes outside a finite region is space at any time. Within this limitation the intra-atomic fields are taken into account covariantly, but explicit expressions are not given. In contrast to the results of de Groot and Suttorp the total atomic energy-momentum tensor obtained by Vlieger is directly symmetric as a result of a more adequate definition of the reference point of the atoms.

The atomic equations of motion obtained by Vlieger, however, still contain a kind of "Zitterbewegung" (trembling motion), since he uses Møller's equations of motion for the pseudo-centres of gravity, which contain this rather unphysical effect. In fact, such a motion is also present in the results of de Groot and Suttorp. The elimination of the trembling motion from the equations of motion will be the first subject of this thesis.

A second point will be the extension of the equations to the case where the atoms (or molecules) also possess, in addition to their dipole moments, an electric quadrupole moment which is of the same order of magnitude as the magnetic dipole moment. Finally, we treat the theory for radiating atoms and molecules.

We take the same classical model for atoms and molecules as in Vlieger's work[6]). For the cases mentioned above we derive relativistic atomic equations of motion from the sub-atomic energy-momentum and angular momentum laws. The equations are then used to derive the atomic energy-momentum tensor for a system consisting of a large number of dipole and quadrupole atoms (or molecules). Statistical averaging will not be treated in this thesis.

After summarizing in chapter I the results obtained by Vlieger[6]) we show in chapter II, by means of an iteration procedure, that in the case of atoms and molecules the terms which describe the trembling motion in the equations of motion are negligibly small. This enables us to write down equations of motion of the usual second order type and an internal angular momentum balance equation of the first order.

In chapter III we apply the theory to the case where the atoms and molecules also possess, in addition to their electric and magnetic dipole moments, electric quadrupole moments. It is found that the field part of the atomic energy-momentum tensor is not of the same form as in the case of pure dipoles, and is also no longer expressible *only* in terms of quantities appearing in the atomic field equations.

Radiative effects are neglected in chapters I–III, and therefore the atomic energy-momentum tensors found in these chapters are always symmetric.

In chapter IV we extend the theory, developed in chapters I and II, to the case of radiating dipole atoms and molecules. We use Dirac's[7]) covariant decomposition of retarded electromagnetic fields into a plus (or "self") part (half the sum of retarded and advanced fields) and a minus (or "radiative") part (half the difference of retarded and advanced fields). It then appears to be possible to define a sub-atomic energy-momentum tensor of the radiating atom (or molecule) in which the intra-atomic field contribution is derived explicitly with the help of the plus parts of the fields. This tensor has the important property of being finite, as required in the theory of Møller[3]). Furthermore it appears that the minus field may be treated as an *external* field in the derivation of the equations of motion and the internal angular momentum balance equation.

The atomic mass and internal angular momentum defined with the above sub-atomic energy-momentum tensor are found to be renormalized with respect to the effects of the intra-atomic field in a manifestly covariant way. From the general expressions of these quantities we calculate explicitly, in appendix II of chapter IV, the contributions of the intra-atomic fields to the order c^{-2} and find agreement with *e.g.* reference 1.

Next, the equations of motion and the internal angular momentum balance equation are derived by the method of the foregoing chapters. In appendix III a procedure is given for the calculation of the minus fields occurring in these equations. In order to simplify the rather complicated equations we assume that all terms originating from these fields which contain time derivatives of the four-velocity can be neglected. (Physically this means that we neglect radiative effects on the equations of motion due to the barycentric accelerations of the atoms ("Bremsstrahlung"), and consider only the damping due to the vibrations of their dipole moments.) It is found that the minus field contribution to the equations of motion can be devided into a part which may be interpreted as the radiation reaction force and another part which is a total time derivative. An analogous division into a radiation reaction torque and a total time derivative is performed in the internal angular momentum balance equation. In appendix IV a justification for the interpretation of the reaction force and reaction torque is given by relating these quantities to the radiation of energy-momentum and angular momentum respectively.

In the presence of radiation the atomic energy-momentum tensor is no longer symmetric, as need not be the case for non-closed systems. The treatment of radiating atoms which also have electric quadrupole moments is indicated at the end of chapter IV.

Parts of the contents of this thesis have been published in Physica (Physica **41** (1969) 368, **42** (1969) 12). The remainder will appear shortly.

REFERENCES

1) de Groot, S. R. and Suttorp, L. G., Physica **37** (1967) 284, 297, Physica **39** (1968) 28, 41, 77, 84.
2) Mazur, P. and de Groot, S. R., Physica **22** (1956) 657.
3) Møller, C., Ann. Inst. H. Poincaré **11** (1949–50) 251.
4) Dixon, W. G., Nuovo Cimento **38** (1965) 1616.
5) Suttorp, L. G. and de Groot, S. R., Nuovo Cimento **65**A (1970) 245.
6) Vlieger, J., Physica **37** (1967) 165.
7) Dirac, P. A. M., Proc. Roy. Soc. (London) A **167** (1938) 148.

CHAPTER I

MØLLER'S EQUATIONS OF MOTION FOR ELECTRIC AND MAGNETIC DIPOLE ATOMS AND THE ENERGY-MOMENTUM TENSOR

Synopsis

In this chapter we briefly summarize the results obtained by Vlieger in the first of a series of papers on the relativistic dynamics of polarized systems. Møller's relativistic equations of motion for finite systems with an internal angular momentum in an arbitrary external (non-gravitational) field of force are applied to the special model of (non-radiating) electric and magnetic dipole atoms (or molecules) in an electromagnetic field. The resulting equations of motion are used in order to derive the relativistic atomic energy-momentum tensor for a system consisting of these dipoles. This tensor has the same form as the atomic energy-momentum tensor obtained earlier by de Groot and Suttorp. But, since these authors use a definition for the reference point within the atoms which is an approximation of the one used by Vlieger, several quantities appearing in their final expression for the atomic energy-momentum tensor are approximations of the corresponding quantities in Vlieger's tensor.

§ 1. *Introduction.* In a paper on the relativistic dynamics of systems with an internal angular momentum Møller[1]) has derived the equations of motion for such systems in an arbitrary (non-gravitational) field of force. The systems are assumed to be finite, *i.e.* their energy-momentum tensors are zero outside a finite sphere at any time. For the description of the motion of these systems it was necessary to give a proper definition of the centre of gravity in the theory of special relativity. In the careful analysis of Møller's paper[1]) it appeared not possible to define a unique centre of gravity for such systems (in the sense that it is the centre of energy in the Lorentz frame in which the system is momentarily at rest). Equations of motion are, however, written down for so-called pseudo-centres of gravity and will be called Møller's equations of motion in this thesis. (For a brief survey, see § 2 of this chapter.)

Vlieger[2]) * has applied Møller's theory to the special model of electric and magnetic dipole point atoms (or molecules) in an external electromagnetic field and so derived the relativistic equations of motion for these dipoles (§ 3).

From these equations he derived the energy-momentum tensor for a system consisting of N interacting dipoles (§ 4). This tensor appears to be of exactly the same form as the atomic energy-momentum tensor which has been derived earlier by de Groot and Suttorp[3]). This is not self-evident, since these authors have not used pseudo-centres of gravity as reference points within the atoms, but some approximation, and have also made certain approximations in their calculations (*e.g.* of the intra-atomic fields). Yet the only difference between the tensor of de Groot and Suttorp and that of Vlieger appears to be that certain quantities in the expression of de Groot and Suttorp are approximations of the corresponding quantities in Vlieger's tensor.

§ 2. *Møller's equations of motion.* Consider, in the special theory of relativity, an arbitrary system subjected to a given external (non-gravitational) field of force, with a density described by the four-vector $f^\alpha(x)$, with $\alpha = 0, 1, 2, 3$ and $x = (ct, \boldsymbol{x})$ the time-space coordinates. The energy-momentum law of this system is in covariant form:

$$\partial_\beta T^{\alpha\beta} = f^\alpha, \tag{1}$$

where $T^{\alpha\beta}(x)$ is the energy-momentum tensor of the system which is supposed symmetric and $\partial_\alpha = (\partial/\partial ct, \partial/\partial\boldsymbol{x})$. Furthermore this tensor is assumed to be zero at any time outside a finite region in ordinary space, which is Møller's definition of a finite system. Taking into account the symmetry of $T^{\alpha\beta}$ we obtain from (1) the angular momentum balance:

$$\partial_\gamma(x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}) = x^\alpha f^\beta - x^\beta f^\alpha. \tag{2}$$

Let Ω be a four-dimensional cylindrical region, in which $T^{\alpha\beta} \neq 0$, around the world-line L of a representative point $x_{(\mathrm{r})}(\tau)$ of the system, where τ is the corresponding eigentime, and bounded by two three-dimensional hyperplanes $V(\tau)$ and $V(\tau + \mathrm{d}\tau)$, perpendicular to L, in $x_{(\mathrm{r})}(\tau)$ and $x_{(\mathrm{r})}(\tau + \mathrm{d}\tau)$. If one now integrates (1) and (2) over Ω, one obtains the following equations[1]):

$$\frac{\mathrm{d}P^\alpha}{\mathrm{d}\tau} = F^\alpha, \tag{3A}$$

$$\frac{\mathrm{d}M^{\alpha\beta}}{\mathrm{d}\tau} = D^{\alpha\beta}, \tag{3B}$$

* Since in the next chapters we frequently refer to Vlieger's work we have summarized, for the benefit of the reader, his results in this chapter.

where the four-vectors defined by the surface integrals:

$$P^{\alpha}(\tau) = -\frac{1}{c^2}\int\limits_{V(\tau)} T^{\alpha\beta}(x)\, u_{\beta}(\tau)\, \mathrm{d}V \tag{4}$$

and

$$F^{\alpha}(\tau) = \int\limits_{V(\tau)} f^{\alpha}(x)\left[1 + \frac{1}{c^2}\{x^{\gamma} - x^{\gamma}_{(\mathrm{r})}(\tau)\}\, \dot{u}_{\gamma}(\tau)\right] \mathrm{d}V \tag{5}$$

are respectively the energy-momentum of the system (as a whole) and the external force, acting upon it, whereas the tensors defined by the relations:

$$M^{\alpha\beta}(\tau) = -\frac{1}{c^2}\int\limits_{V(\tau)} \{x^{\alpha}T^{\beta\gamma}(x) - x^{\beta}T^{\alpha\gamma}(x)\}\, u_{\gamma}(\tau)\, \mathrm{d}V \tag{6}$$

and

$$D^{\alpha\beta}(\tau) = \int\limits_{V(\tau)} \{x^{\alpha}f^{\beta}(x) - x^{\beta}f^{\alpha}(x)\}\left[1 + \frac{1}{c^2}\{x^{\gamma} - x^{\gamma}_{(\mathrm{r})}(\tau)\}\, \dot{u}_{\gamma}(\tau)\right] \mathrm{d}V \tag{7}$$

represent respectively its angular momentum and the moment of the external force, both with respect to the origin of the Lorentz frame. In the above expressions $\mathrm{d}V$ is the magnitude of a surface element of $V(\tau)$. Furthermore $u(\tau) = \mathrm{d}x_{(\mathrm{r})}(\tau)/\mathrm{d}\tau$ is the four-velocity of the representative point, whereas $\dot{u}(\tau) = \mathrm{d}u(\tau)/\mathrm{d}\tau$ is its acceleration. Introducing the internal angular momentum:

$$\begin{aligned} \Omega^{\alpha\beta}(\tau) \equiv -\frac{1}{c^2}\int\limits_{V(\tau)} [\{x^{\alpha} - x^{\alpha}_{(\mathrm{r})}(\tau)\}\, T^{\beta\gamma}(x) - \\ - \{x^{\beta} - x^{\beta}_{(\mathrm{r})}(\tau)\}\, T^{\alpha\gamma}(x)]\, u_{\gamma}(\tau)\, \mathrm{d}V \end{aligned} \tag{8}$$

of the system with respect to $x_{(\mathrm{r})}(\tau)$, one may alternatively write equation (3B), with the help of (3A), as

$$\frac{\mathrm{d}\Omega^{\alpha\beta}}{\mathrm{d}\tau} = d^{\alpha\beta} - (u^{\alpha}P^{\beta} - u^{\beta}P^{\alpha}), \tag{9}$$

where

$$\begin{aligned} d^{\alpha\beta} \equiv \int\limits_{V(\tau)} [\{x^{\alpha} - x^{\alpha}_{(\mathrm{r})}(\tau)\}\, f^{\beta}(x) - \{x^{\beta} - x^{\beta}_{(\mathrm{r})}(\tau)\}\, f^{\alpha}(x)] \\ \cdot\left[1 + \frac{1}{c^2}\{x^{\gamma} - x^{\gamma}_{(\mathrm{r})}(\tau)\}\, \dot{u}_{\gamma}(\tau)\right] \mathrm{d}V \end{aligned} \tag{10}$$

is the moment of the external force with respect to $x_{(\mathrm{r})}(\tau)$.

As reference point Møller[1]) takes a so-called pseudo-centre of gravity $X^\alpha(\tau)$, defined by the condition:

$$\Omega^{\alpha\beta} U_\beta = 0, \tag{11}$$

which means that $X^\alpha(\tau)$ is the centre of mass (or energy) in the momentary rest frame of inertia of the system, *i.e.* in the Lorentz frame, for which $U^\alpha(\tau) \equiv \mathrm{d}X^\alpha(\tau)/\mathrm{d}\tau = (c, 0, 0, 0)$. Introducing the rest mass $M^*(\tau)$ of the system by means of:

$$P^\alpha U_\alpha = -M^* c^2, \tag{12}$$

Møller[1]) obtains with (3A), (9) and (11) the following equations of motion for the system as a whole:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(M^* U^\alpha) + \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\Omega^{\alpha\beta}\dot{U}_\beta) = F^\alpha - \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(d^{\alpha\beta}U_\beta), \tag{13A}$$

$$\dot{\Omega}^{\alpha\beta} + \frac{1}{c^2} U^\alpha \Omega^{\beta\gamma}\dot{U}_\gamma - \frac{1}{c^2} U^\beta \Omega^{\alpha\gamma}\dot{U}_\gamma = \Delta^\alpha{}_\varepsilon \Delta^\beta{}_\zeta d^{\varepsilon\zeta}, \tag{13B}$$

where we have introduced the tensor:

$$\Delta^\alpha{}_\beta \equiv \delta^\alpha{}_\beta + \frac{1}{c^2} U^\alpha U_\beta, \tag{14}$$

with $\delta^\alpha{}_\beta$ the elements of the unit four-tensor.

In § 3 and § 4 we shall give the results obtained by Vlieger[2]) for the equations of motion (13A) and (13B) in the case that the system is a (non-radiating) charged point atom, with electric and magnetic dipole moments $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, negligible higher order atomic moments, moving in an external electromagnetic field of force.

§ 3. *Equations of motion for electric and magnetic dipole atoms.* In the classical model of atoms we can characterize the electromagnetic properties of the atom by a charge-current density four-vector $j^\alpha(x)$ and a polarization tensor $m^{\alpha\beta}(x)$. The density of the external electromagnetic force, acting upon this atom can then be written as:

$$f^\alpha(x) = f^{\alpha\beta}(x)\{j_\beta(x)/c + \partial_\gamma m_\beta{}^\gamma(x)\}, \tag{15}$$

with $f^{\alpha\beta}(x)$ the tensor of the external electromagnetic field and c the velocity of light. For a point atom with charge e, electric and magnetic dipole moments $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, one has, if one neglects all higher order moments (*cf.* reference 4)):

$$j^\alpha(x)/c = e \int_{-\infty}^{+\infty} U^\alpha(\tau)\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau, \tag{16}$$

$$m_\alpha{}^\beta(x) = \int_{-\infty}^{+\infty} \mu_\alpha{}^\beta(\tau)\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau, \tag{17}$$

where $\mu_\alpha{}^\beta(\tau)$ is a tensor, depending on $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ and U^α. The explicit form of this tensor can be calculated with the help of the equations (83) and (84) of reference 4, but it is of no importance for the time being. As remarked by Vlieger[2]), the essential point of the expression (17) for the *approximate* polarization tensor is the fact that it does not contain any derivatives of the δ-function, in contrast with the *general* expression for this tensor, as given in reference 4.

Substituting (15), with (16) and (17), into the right-hand sides of (5) and (10) one obtains:

$$F^\alpha(\tau) = e \int\limits_{V(\tau)} \int\limits_{-\infty}^{+\infty} f^{\alpha\beta}(x) \left[1 + \frac{1}{c^2} \{x^\delta - X^\delta(\tau)\} \dot{U}_\delta(\tau) \right]$$
$$\cdot U_\beta(\tau')\, \delta^{(4)}\{X(\tau') - x\}\, \mathrm{d}V\, \mathrm{d}\tau' +$$
$$+ \int\limits_{V(\tau)} \int\limits_{-\infty}^{+\infty} f^{\alpha\beta}(x) \left[1 + \frac{1}{c^2} \{x^\delta - X^\delta(\tau)\} \dot{U}_\delta(\tau) \right]$$
$$\cdot \mu_\beta{}^\gamma(\tau')\, \partial_\gamma \delta^{(4)}\{X(\tau') - x\}\, \mathrm{d}V\, \mathrm{d}\tau', \tag{18}$$

$$d^{\alpha\beta}(\tau) = e \int\limits_{V(\tau)} \int\limits_{-\infty}^{+\infty} [\{x^\alpha - X^\alpha(\tau)\} f^{\beta\gamma}(x) - \{x^\beta - X^\beta(\tau)\} f^{\alpha\gamma}(x)]$$
$$\cdot \left[1 + \frac{1}{c^2} \{x^\delta - X^\delta(\tau)\} \dot{U}_\delta(\tau) \right] U_\gamma(\tau')\, \delta^{(4)}\{X(\tau') - x\}\, \mathrm{d}V\, \mathrm{d}\tau' +$$
$$+ \int\limits_{V(\tau)} \int\limits_{-\infty}^{+\infty} [\{x^\alpha - X^\alpha(\tau)\} f^{\beta\gamma}(x) - \{x^\beta - X^\beta(\tau)\} f^{\alpha\gamma}(x)]$$
$$\cdot \left[1 + \frac{1}{c^2} \{x^\delta - X^\delta(\tau)\} \dot{U}_\delta(\tau) \right] \mu_\gamma{}^\varepsilon(\tau')\, \partial_\varepsilon \delta^{(4)}\{X(\tau') - x\}\, \mathrm{d}V\, \mathrm{d}\tau'. \tag{19}$$

Since the hyperplane $V(\tau)$ is characterized by the orthogonality condition:

$$\{x^\alpha - X^\alpha(\tau)\}\, U_\alpha(\tau) = 0, \tag{20}$$

(18) and (19) can be rewritten as volume integrals over the total time-space, by introducing the function $\delta[-(1/c)\{x^\alpha - X^\alpha(\tau)\}\, U_\alpha(\tau)]$. We then have:

$$F^\alpha(\tau) = e \int^{\infty} \int\limits_{-\infty}^{+\infty} f^{\alpha\beta}(x) \left[1 + \frac{1}{c^2} \{x^\delta - X^\delta(\tau)\} \dot{U}_\delta(\tau) \right] U_\beta(\tau')\, \delta^{(4)}\{X(\tau') - x\}$$
$$\cdot \delta \left[-\frac{1}{c} \{x^\varepsilon - X^\varepsilon(\tau)\}\, U_\varepsilon(\tau) \right] \mathrm{d}^{(4)}x\, \mathrm{d}\tau' +$$

$$+\int_{-\infty}^{\infty}\int^{+\infty} f^{\alpha\beta}(x)\left[1+\frac{1}{c^2}\{x^\delta - X^\delta(\tau)\}\,\dot{U}_\delta(\tau)\right]\mu_\beta{}^\gamma(\tau')\,[\partial_\gamma\delta^{(4)}\{X(\tau')-x\}]$$

$$\cdot\,\delta\left[-\frac{1}{c}\{x^\varepsilon - X^\varepsilon(\tau)\}\,U_\varepsilon(\tau)\right]\mathrm{d}^{(4)}x\,\mathrm{d}\tau', \tag{21}$$

$$d^{\alpha\beta}(\tau) = e\int_{-\infty}^{\infty}\int^{+\infty}[\{x^\alpha - X^\alpha(\tau)\}\,f^{\beta\gamma}(x) - \{x^\beta - X^\beta(\tau)\}\,f^{\alpha\gamma}(x)]$$

$$\cdot\left[1+\frac{1}{c^2}\{x^\delta - X^\delta(\tau)\}\,\dot{U}_\delta(\tau)\right]U_\gamma(\tau')\,\delta^{(4)}\{X(\tau')-x\}\cdot$$

$$\cdot\,\delta\left[-\frac{1}{c}\{x^\varepsilon - X^\varepsilon(\tau)\}\,U_\varepsilon(\tau)\right]\mathrm{d}^{(4)}x\,\mathrm{d}\tau' + \int_{-\infty}^{\infty}\int^{+\infty}[\{x^\alpha - X^\alpha(\tau)\}\,f^{\beta\gamma}(x)$$

$$-\{x^\beta - X^\beta(\tau)\}\,f^{\alpha\gamma}(x)]\left[1+\frac{1}{c^2}\{x^\delta - X^\delta(\tau)\}\,\dot{U}_\delta(\tau)\right]$$

$$\cdot\,\mu_\gamma{}^\zeta(\tau')\,[\partial_\zeta\delta^{(4)}\{X(\tau')-x\}]\,\delta\left[-\frac{1}{c}\{x^\varepsilon - X^\varepsilon(\tau)\}\,U_\varepsilon(\tau)\right]\mathrm{d}^{(4)}x\,\mathrm{d}\tau'. \tag{22}$$

The right-hand sides of (21) and (22) can be integrated subsequently over x^α and τ'. (For details of these calculations see reference 2.) The results are:

$$F^\alpha(\tau) = \frac{e}{c}\,f^{\alpha\beta}\{X(\tau)\}\,U_\beta(\tau) - \frac{1}{c}\,[\partial_\gamma f^{\alpha\beta}\{X(\tau)\}]\,\mu_\beta{}^\gamma(\tau)$$

$$-\frac{1}{c^3}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,[f^{\alpha\beta}\{X(\tau)\}\,\mu_\beta{}^\gamma(\tau)\,U_\gamma(\tau)], \tag{23}$$

and:

$$d^{\alpha\beta}(\tau) = \frac{1}{c}\,[f^{\alpha\gamma}\{X(\tau)\}\,\mu_\gamma{}^\delta(\tau)\,\Delta_\delta{}^\beta(\tau) - f^{\beta\gamma}\{X(\tau)\}\,\mu_\gamma{}^\delta(\tau)\,\Delta_\delta{}^\alpha(\tau)], \tag{24}$$

which are respectively the external electromagnetic force and its moment acting on the dipole. With (23) and (24) the equations of motion (13A) and (13B) become for electric and magnetic dipoles in an external electromagnetic field:

$$c\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(M^*U^\alpha) + \frac{1}{c}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(\Omega^{\alpha\beta}\dot{U}_\beta) = e\,f^{\alpha\beta}U_\beta - (\partial_\gamma f^{\alpha\beta})\,\mu_\beta{}^\gamma$$

$$-\frac{1}{c^2}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\{(f^{\alpha\gamma}\mu_\gamma{}^\beta - f^{\beta\gamma}\mu_\gamma{}^\alpha)\,U_\beta\} + \frac{1}{c^4}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(U^\alpha U_\beta f^{\beta\gamma}\mu_\gamma{}^\delta U_\delta), \tag{25A}$$

$$c\,\Delta^\alpha{}_\varepsilon\Delta^\beta{}_\zeta\dot{\Omega}^{\varepsilon\zeta} = \Delta^\alpha{}_\varepsilon\Delta^\beta{}_\zeta(f^{\varepsilon\gamma}\mu_\gamma{}^\zeta - f^{\zeta\gamma}\mu_\gamma{}^\varepsilon). \tag{25B}$$

From these equations Vlieger[2]) derived the energy-momentum tensor for a system consisting of these dipoles. The main steps in this derivation are given in the next section. A comparison is made with the expression for the atomic energy-momentum tensor of de Groot and Suttorp[3]).

In the above treatment it is tacitly assumed that a tensor $T^{\alpha\beta}$ exists in the case of atoms and molecules, which satisfies the requirements in § 2. This means that radiative effects are neglected, but renormalization effects are taken into account.

§ 4. *The energy-momentum tensor for a system of dipoles.* We consider a system, consisting of N point atoms, numbered by the index k. The rest mass of the k^{th} atom will be denoted by $m^*_{(k)}$ (instead of M^*), its charge by $e_{(k)}$ (instead of e), whereas the tensor $\mu_\alpha{}^\beta$ becomes $\mu_{(k)\alpha}{}^\beta$, and $\Omega^{\alpha\beta}$: $\Omega^{\alpha\beta}_{(k)}$. For the centre of gravity of the k^{th} atom we write $R^\alpha_{(k)}$ (instead of X^α), for its velocity $U^\alpha_{(k)}$, for the tensor $\Delta^\alpha{}_\beta$: $\Delta^\alpha_{(k)\beta}$, and the eigentime of atom k becomes $\tau_{(k)}$. Supposing that there are no external fields acting from outside the system, the (external) field acting on atom k is the sum of the partial electromagnetic fields $f_{(l)}{}^{\alpha\beta}$ due to the other atoms l ($\neq k$):

$$f^{\alpha\beta}(R_{(k)}) = \sum_{l(\neq k)} f_{(l)}{}^{\alpha\beta}(R_{(k)}). \tag{26}$$

With these new notations, the equations of motion (41) become:

$$\begin{aligned} c\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{m^*_{(k)}U^\alpha_{(k)}\} + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{\Omega^{\alpha\beta}_{(k)}\dot{U}_{(k)\beta}\} = e_{(k)}\sum_{l(\neq k)} f^{\alpha\beta}_{(l)}(R_{(k)})\,U_{(k)\beta} - \\ - \sum_{l(\neq k)}\{\partial_{(k)\gamma} f^{\alpha\beta}_{(l)}(R_{(k)})\}\,\mu_{(k)\beta}{}^\gamma - \\ - \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}[\sum_{l(\neq k)}\{f^{\alpha\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^\beta - f^{\beta\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^\alpha\}\,U_{(k)\beta}] + \\ + \frac{1}{c^4}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{\sum_{l(\neq k)} U^\alpha_{(k)}U_{(k)\beta} f^{\beta\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^\delta U_{(k)\delta}\}, \end{aligned} \tag{27A}$$

$$c\Delta^\alpha_{(k)\varepsilon}\Delta^\beta_{(k)\zeta}\dot{\Omega}^{\varepsilon\zeta}_{(k)} = \Delta^\alpha_{(k)\varepsilon}\Delta^\beta_{(k)\zeta}[\sum_{l(\neq k)}\{f^{\varepsilon\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^\zeta - f^{\zeta\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^\varepsilon\}]. \tag{27B}$$

Multiplying both members of these equations by the four-dimensional δ-function $\delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}$, integrating partially over $\tau_{(k)}$, and summing the result over k, one obtains the following result (for detailed calculations see reference 2):

$$\partial_\beta\Big[\sum_k\Big\{c\int_{-\infty}^{+\infty} m^*_{(k)}\delta^{(4)}(R_{(k)} - R)\,\mathrm{d}\tau_{(k)}\Big\}u^\alpha_{(k)}\,u^\beta_{(k)} +$$

$$+\frac{1}{c^2}\sum_k\left\{c\int_{-\infty}^{+\infty}\Omega^{\alpha\gamma}_{(k)}\delta^{(4)}(R_{(k)}-R)\,\mathrm{d}\tau_{(k)}\right\}\{D_{(k)}\,u_{(k)\gamma}\}\,u^{\beta}_{(k)}\Big]=$$

$$=\frac{1}{c}\sum_{k,l,(k\neq l)}f^{\alpha\beta}_{(l)}(R)\,j_{(k)\beta}(R)-\sum_{k,l,(k\neq l)}\{\partial_\gamma f^{\alpha\beta}_{(l)}(R)\}\,m_{(k)\beta}{}^{\gamma}(R)-$$

$$-\partial_\beta\Bigg[\frac{1}{c^2}\sum_{k,l,(k\neq l)}\{f^{\alpha\gamma}_{(l)}(R)m_{(k)\gamma}{}^{\delta}(R)-f^{\delta\gamma}_{(l)}(R)m_{(k)\gamma}{}^{\alpha}(R)\}\,u_{(k)\delta}\,u^{\beta}_{(k)}-$$

$$-\frac{1}{c^4}\sum_{k,l,(k\neq l)}\{u_{(k)\gamma}\,f^{\gamma\delta}_{(l)}(R)m_{(k)\delta}{}^{\varepsilon}(R)\,u_{(k)\varepsilon}\}\,u^{\alpha}_{(k)}\,u^{\beta}_{(k)}\Bigg],\tag{28}$$

where $u^{\alpha}_{(k)}=U^{\alpha}_{(k)}\{\tau_{(k)}(t)\}$, and the operator $D_{(k)}\equiv u^{\alpha}_{(k)}\partial_\alpha$, whereas $j_{(k)\alpha}(R)$ is the charge-current density vector of the k^{th} atom (as a whole):

$$j_{(k)\alpha}(R)/c\equiv e_{(k)}\int_{-\infty}^{+\infty}U_{(k)\alpha}(\tau_{(k)})\,\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\}\,\mathrm{d}\tau_{(k)},\tag{29}$$

(*cf.* eq. (16)), and $m_{(k)\alpha}{}^{\beta}$ the polarization of this atom:

$$m_{(k)\alpha}{}^{\beta}(R)\equiv\int_{-\infty}^{+\infty}\mu_{(k)\alpha}{}^{\beta}(\tau_{(k)})\,\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\}\,\mathrm{d}\tau_{(k)},\tag{30}$$

(*cf.* eq. (17)).

In equation (28):

$$c\int_{-\infty}^{+\infty}m^{*}_{(k)}(\tau_{(k)})\,\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\}\,\mathrm{d}\tau_{(k)}=$$

$$=m^{*}_{(k)}(t)\sqrt{1-\boldsymbol{v}^2_{(k)}(t)/c^2}\,\delta^{(3)}\{\boldsymbol{R}_{(k)}(t)-\boldsymbol{R}\}\equiv\rho^{*\prime}_{(k)}(R)\tag{31}$$

is the rest mass density of atom k, *i.e.* the mass density in the momentary rest frame of inertia of this atom, $\boldsymbol{v}_{(k)}(t)$ is the velocity of the atom, whereas:

$$c\int_{-\infty}^{+\infty}\Omega^{\alpha\beta}_{(k)}(\tau_{(k)})\,\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\}\,\mathrm{d}\tau_{(k)}=$$

$$=\Omega^{\alpha\beta}_{(k)}(t)\sqrt{1-\boldsymbol{v}^2_{(k)}(t)/c^2}\,\delta^{(3)}\{\boldsymbol{R}_{(k)}(t)-\boldsymbol{R}\}\equiv\sigma^{*\alpha\beta}_{(k)}(R)\tag{32}$$

is a tensor, which in the atomic rest frame, represents the angular momentum density, (the "atomic angular momentum density tensor"). Introducing (31) and (32) in (28), and also using the Maxwell equations:

$$\partial_\beta f_{(k)\alpha}{}^{\beta}=j_{(k)\alpha}/c+\partial_\beta m_{(k)\alpha}{}^{\beta},\tag{33A}$$

$$\partial_\alpha f_{(k)\beta\gamma}+\partial_\beta f_{(k)\gamma\alpha}+\partial_\gamma f_{(k)\alpha\beta}=0,\tag{33B}$$

we can derive the following result:

$$\partial_\beta\left[\sum_k \rho^{*\prime}_{(k)} u^\alpha_{(k)} u^\beta_{(k)} + \frac{1}{c^2}\sum_k \sigma^{*\alpha\gamma}_{(k)}\{D_{(k)}u_{(k)\gamma}\}\, u^\beta_{(k)}\right] =$$

$$= -\partial_\beta\left[\sum_{k,l,(k\neq l)} f^{\alpha\gamma}_{(l)} h^\beta_{(k)\gamma} - \tfrac{1}{4}\sum_{k,l,(k\neq l)} f_{(l)\gamma\delta} f^{\gamma\delta}_{(k)} g^{\alpha\beta} + \right.$$

$$+\frac{1}{c^2}\sum_{k,l,(k\neq l)} \{f^{\alpha\gamma}_{(l)} m_{(k)\gamma\delta} - m^{\alpha\gamma}_{(k)} f_{(l)\gamma\delta}\}\, u^\delta_{(k)} u^\beta_{(k)} -$$

$$\left. -\frac{1}{c^4}\sum_{k,l,(k\neq l)} \{u^\gamma_{(k)} f_{(l)\gamma\delta} m^\delta_{(k)\varepsilon} u^\varepsilon_{(k)}\}\, u^\alpha_{(k)} u^\beta_{(k)}\right], \tag{34}$$

where

$$h^{\alpha\beta}_{(k)} \equiv f^{\alpha\beta}_{(k)} - m^{\alpha\beta}_{(k)}, \tag{35}$$

and where $g^{\alpha\beta}$ has the elements $g^{00} = -1$, $g^{ii} = 1$ (for $i = 1, 2, 3$) and $g^{\alpha\beta} = 0$ (for $\alpha \neq \beta$).

The atomic energy-momentum tensor of the field is defined by:

$$t^{\alpha\beta}_{(\mathrm{f})} \equiv \sum_{k,l,(k\neq l)}\left[f^{\alpha\gamma}_{(l)} h^\beta_{(k)\gamma} - \tfrac{1}{4}\{f_{(l)\gamma\delta} f^{\gamma\delta}_{(k)}\} g^{\alpha\beta} + \right.$$

$$+\frac{1}{c^2}\{f^{\alpha\gamma}_{(l)} m_{(k)\gamma\delta} - m^{\alpha\gamma}_{(k)} f_{(l)\gamma\delta}\}\, u^\delta_{(k)} u^\beta_{(k)} -$$

$$\left. -\frac{1}{c^4}\{u^\gamma_{(k)} f_{(l)\gamma\delta} m^\delta_{(k)\varepsilon} u^\varepsilon_{(k)}\}\, u^\alpha_{(k)} u^\beta_{(k)}\right], \tag{36}$$

This tensor is exactly the same as the one calculated by de Groot and Suttorp for these atoms. The atomic material energy-momentum tensor is defined by:

$$t^{*\alpha\beta}_{(\mathrm{m})} \equiv \sum_k\left[\rho^{*\prime}_{(k)} u^\alpha_{(k)} u^\beta_{(k)} + \frac{1}{c^2}\sigma^{*\alpha\gamma}_{(k)}\{D_{(k)}u_{(k)\gamma}\}\, u^\beta_{(k)}\right]. \tag{37}$$

Equation (34) can then be interpreted as the conservation law for energy-momentum at the atomic level for a system consisting of dipole atoms:

$$\partial_\beta\{t^{*\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}\} = 0. \tag{38}$$

The tensor $t^{*\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}$ is not symmetric, which has the disadvantage that the conservation law (38) does not imply the conservation of total angular momentum of the system (*cf.* reference 5). One can, however, add a divergence-free tensor to $t^{*\alpha\beta}_{(\mathrm{m})}$, and so obtain a new material energy-momentum tensor $t^{\alpha\beta}_{(\mathrm{m})}$, such that the total energy-momentum tensor $t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}$ is symmetric. Vlieger has used the same method as de Groot and Suttorp, but without making any approximations.

First one derives from (27B) the atomic angular momentum balance:

$$\sum_k \Delta^{\alpha}_{(k)\varepsilon} \Delta^{\beta}_{(k)\zeta} \partial_\gamma \{u^{\gamma}_{(k)} \sigma^{*\varepsilon\zeta}_{(k)}\} = \sum_{k,l,(k\neq l)} \Delta^{\alpha}_{(k)\varepsilon} \Delta^{\beta}_{(k)\zeta} \{m^{\varepsilon\gamma}_{(k)} f^{\zeta}_{(l)\gamma} - f^{\varepsilon\gamma}_{(l)} m^{\zeta}_{(k)\gamma}\}, \tag{39}$$

using (30) and (32). Now the right-hand side of this equation is equal to twice the antisymmetric part of the tensor $t^{\alpha\beta}_{(\mathrm{f})}$, eq. (36). For the left-hand side of (39) one finds:

$$\sum_k \Delta^{\alpha}_{(k)\varepsilon} \Delta^{\beta}_{(k)\zeta} \partial_\gamma \{u^{\gamma}_{(k)} \sigma^{*\varepsilon\zeta}_{(k)}\} = -\{t^{*\alpha\beta}_{(\mathrm{m})} - t^{*\beta\alpha}_{(\mathrm{m})}\} + \sum_k \partial_\gamma \{u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}, \tag{40}$$

using (37), the antisymmetry of $\sigma^{*\alpha\beta}_{(k)}$, and the property

$$\sigma^{*\alpha\beta}_{(k)} u_{(k)\beta} = 0, \tag{41}$$

which follows from (32), together with (11). The left-hand side of (39) is therefore equal to minus twice the antisymmetric part of the tensor $t^{*\alpha\beta}_{(\mathrm{m})} - \frac{1}{2} \sum_k \partial_\gamma \{u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}$. We still have the liberty of adding an arbitrary symmetric tensor to this, for which we take: $\frac{1}{2} \sum_k \partial_\gamma \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)}\}$. We then define:

$$t^{\alpha\beta}_{(\mathrm{m})} \equiv t^{*\alpha\beta}_{(\mathrm{m})} + \frac{1}{2} \sum_k \partial_\gamma \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)} - u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}, \tag{42}$$

as the new atomic material energy-momentum tensor of the system, and this is allowed, since the tensor added to $t^{*\alpha\beta}_{(\mathrm{m})}$ in (42) is divergence-free, so that both $t^{*\alpha\beta}_{(\mathrm{m})}$ and $t^{\alpha\beta}_{(\mathrm{m})}$ lead to the same physical results. The left-hand side of (39) is then equal to minus twice the antisymmetric part of the tensor $t^{\alpha\beta}_{(\mathrm{m})}$, and the atomic angular momentum balance therefore expresses the fact that the total atomic energy-momentum tensor $t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}$ is *symmetric*. The conservation law (38) for energy-momentum can then be written as:

$$\partial_\beta \{t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}\} = 0. \tag{43}$$

If we now substitute (37) into (42), we get:

$$t^{\alpha\beta}_{(\mathrm{m})} = \sum_k \Big[\rho^{*\prime}_{(k)} u^{\alpha}_{(k)} u^{\beta}_{(k)} - \tfrac{1}{2} \Delta^{\alpha}_{(k)\varepsilon} \Delta^{\beta}_{(k)\zeta} \partial_\gamma \{u^{\gamma}_{(k)} \sigma^{*\varepsilon\zeta}_{(k)}\} + \tfrac{1}{2} \partial_\gamma \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)}\} +$$
$$+ \frac{1}{2c^2} \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)}\} \{D_{(k)} u_{(k)\gamma}\} \Big], \tag{44}$$

and this has exactly the same *form* as the final atomic material energy-momentum tensor derived by de Groot and Suttorp[3]). However, as a consequence of the different definition of the centre of gravity of the atoms and the various approximations made by these authors, they obtain expressions for the rest mass-density (denoted by $\rho^{\mathrm{t}\prime}_{(k)}$) and the atomic angular momentum density ($\sigma^{+\alpha\beta}_{(k)}$) which are approximations of the expressions $\rho^{*\prime}_{(k)}$ and $\sigma^{*\alpha\beta}_{(k)}$ used in this chapter. (*E.g.* they obtain the contribution of the intra-atomic field to $\rho^{*\prime}_{(k)}$ only within the Coulomb-approximation.) But taking into account the limits of the approximations made by de Groot and Suttorp, there appears to be a complete agreement between their results and those given in the present chapter, and this is, of course, very satisfactory.

REFERENCES

1) Møller, C., Ann. Inst. H. Poincaré **11** (1949–50) 251.
2) Vlieger, J., Physica **37** (1967) 165.
3) de Groot, S. R. and Suttorp, L. G., Physica **37** (1967) 284, 297.
4) de Groot, S. R. and Suttorp, L. G., Physica **31** (1965) 1713.
5) Møller, C., The theory of relativity, Oxford University Press (London, 1952).

Chapter II

SIMPLIFICATION OF MØLLER'S EQUATIONS OF MOTION AND THE ENERGY-MOMENTUM TENSOR

Synopsis

The classical equations of motion for electric and magnetic dipole atoms (or molecules) in an external electromagnetic field of force, treated in the previous chapter, are simplified by showing that certain terms, which contain an unphysical trembling motion ("Zitterbewegung"), are completely negligible with respect to the other terms in these equations. The resulting equations are used in order to derive the relativistic atomic energy-momentum tensor for a system, consisting of these dipole atoms. The field part of this tensor has exactly the same form as obtained before, but the material part is slightly different as a consequence of the simplification in the equations of motion. The same symmetrization procedure, as used in the preceding chapter, can be applied to the total energy-momentum tensor. Radiative effects are neglected throughout the theory.

§ 1. *Introduction.* In the preceding chapter, in the following denoted by I, we have treated the equations of motion of electric and magnetic dipole atoms in an external electromagnetic field of force[1]). The treatment was based on Møller's theory[2]) of the dynamics of relativistic systems with an internal angular momentum in an arbitrary (non-gravitational) field of force. In this theory the equations of motion and of the change of the intrinsic angular momentum are found, using the definition * (I.11) for the centre of gravity or rather, using Møller's words, the pseudo-centres of gravity of the system. These equations were then applied to the special model of electric and magnetic dipole atoms (or molecules) and used in order to obtain the relativistic energy-momentum tensor for a system, consisting of N of these atoms. It was found that this four-tensor is exactly of the same *form* as the one derived by de Groot and Suttorp[3, 4]), although these authors have used a slightly

* Formulae of chapter I will be denoted here by (I.1), (I.2), etc.

different definition for the reference point within the atom from the one used here.

The motion of the pseudo-centres of gravity, following from Møller's equations[2]), is still rather unphysical as it contains, superposed on what may be interpreted as the real translational motion of the system, a very small (order of the Compton-wavelength) trembling motion (frequency higher than 10^{21}), the so-called "Zitterbewegung", which has no physical interpretation and is purely a consequence of the definition (I. 11) of these reference points. It will be shown in § 2 that from the above equations of motion, which are third order differential equations, one can derive, by using an iterative procedure proposed by Plahte[5]), equations of the usual second order type, which possess solutions which also satisfy the original equations, but which do not give rise to a trembling motion and can therefore be considered as the correct equations, describing the motion of the system *as a whole* in an external field of force. It appears that for atoms or molecules this iteration method is rapidly convergent, and gives already in lowest order negligible results. The equations of motion (and of course also the angular momentum equation) are therefore simplified, and this is also the case with the atomic energy-momentum tensor derived with the help of these equations. In §3 we find that the field part of this tensor is exactly the same as obtained in I, but that its material part is slightly different. It is shown that the total energy-momentum tensor can be obtained in a symmetrical form, using the same procedure as in I, § 4.

The theory developed in the present chapter is based on a classical model of the atoms or molecules. Radiative effects are neglected, as this has also been done in I. We shall treat this latter point in chapter IV.

§ 2. *Simplification of Møller's equations of motion in the case of electric and magnetic dipole atoms.* In I we have summarize the derivation[1]) of the equations of motion (I.25) of electric and magnetic dipole atoms in a given external electromagnetic field of force, starting from Møller's equations (I.13) for the motion of arbitrary relativistic systems with an internal angular momentum. The latter equations were derived by Møller[2]), using the condition (*cf.* (I.11))

$$\Omega^{\alpha\beta} U_\beta = 0 \tag{1}$$

for the reference point within the system. It follows from the definition (I.8) of $\Omega^{\alpha\beta}$ (internal angular momentum four-tensor with respect to the reference point) and the fact that U_β is the derivative of the time-space coordinate four-vector X_β of this point with respect to the eigentime τ, that this condition has the form of a first-order differential equation in X, and does therefore not define one single reference point, but an infinity of these points, which are called pseudo-centres of gravity by Møller. This is the reason, that the differential equations (I.13) for the motion of these pseudo-centres of gravity are not of the second order in X, as usual, but of the third order. Consequently the energy-momentum four-tensor derived in I with the help of these equations, contains a material part, which does possess not only velocity-dependent terms, but also terms depending on the accelerations of the atoms (*cf.* (I.44)). This result has also been obtained by de Groot and Suttorp[3,4]), using, however, a definition for the reference points within the atoms (molecules) different from ours.

The problem of solving the equations (I.13) and in particular the physical significance of the solutions obtained, have been a matter of long discussions in the past, (see refs. 6 and 7). One obtains a trembling motion of a very high frequency ($> 10^{21}$), the so-called "Zitterbewegung", superposed on a motion of a much smoother character, (in the free case a straight line). From Møller's point of view[2]), however, this trembling motion is *only* a consequence of the definition (1) of the reference points, and has therefore no physical meaning*. Moreover he finds a very small amplitude of the trembling motion, namely of the order of the Compton wave length, which is for an atom about 10^{-13} cm, much smaller than its dimensions. It is therefore evident, that one must try to find new differential equations, which describe the above mentioned smooth motion, but not the unphysical trembling motion.

A straightforward way to achieve this is the following iteration procedure, which is formally analogous to the method of Plahte[5]) for the elimination of the "Zitterbewegung" of the classical spinning electron. Consider the equation of motion (I. 13A) and the internal angular momentum equation (I.13B):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(M^* U^\alpha) + \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\Omega^{\alpha\beta}\dot{U}_\beta) = F^\alpha - \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(d^{\alpha\beta}U_\beta), \tag{2A}$$

$$\dot{\Omega}^{\alpha\beta} + \frac{1}{c^2}U^\alpha\Omega^{\beta\gamma}\dot{U}_\gamma - \frac{1}{c^2}U^\beta\Omega^{\alpha\gamma}\dot{U}_\gamma = \Delta^\alpha{}_\varepsilon\Delta^\beta{}_\zeta d^{\varepsilon\zeta}, \tag{2B}$$

where M^* is the rest mass of the system, defined by (I.12), F^α the four-vector of the external force acting upon it (eq. (I.5)), $d^{\alpha\beta}$ the four-tensor of the moment of this force with respect to the reference point (eq. I.10)), and

$$\Delta^\alpha{}_\beta \equiv \delta^\alpha{}_\beta + \frac{1}{c^2}U^\alpha U_\beta, \tag{3}$$

with $\delta^\alpha{}_\beta$ the elements of the unit four-tensor. (The dots mean differentiation with respect to the eigentime τ.) Since it is the second term at the left-hand

* One should also note that a motion of the extreme high frequency of the trembling motion, has never been observed in physics.

side of eq. (2A), which causes the microscopically small trembling motion, one can obviously write eq. (2A) in lowest order by neglecting this term, and then solve $\dot{U}_\alpha$ as a function of X_α and U_α from the approximate second order differential equation, which is left. If this result is now substituted for $\dot{U}_\beta$ and $\dot{U}_\gamma$ into eqs. (2A) and (2B), one obtains in first approximation the influence of the external field on the motion through the "Zitter-term" $c^{-2}(\mathrm{d}/\mathrm{d}\tau)(\Omega^{\alpha\beta}\dot{U}_\beta)$, and the terms $c^{-2}U^\alpha\Omega^{\beta\gamma}\dot{U}_\gamma$ and $c^{-2}U^\beta\Omega^{\alpha\gamma}\dot{U}_\gamma$, causing the so-called Thomas-precession of the internal angular momentum of the system. In this way one obtains for (2A) a second- and for (2B) a first-order differential equation, and one can easily see that this set of equations describes, within the approximation of their derivation, the motion of the system (with intrinsic angular momentum) in the external field of force *without*, however, the unphysical trembling motion. With the above mentioned iteration method of Plahte[5]), one could also obtain higher approximations of this set of differential equations, but we shall show below that for atoms or molecules in an electromagnetic field the first approximation is already negligibly small. The equations of Møller for the motion of electric and magnetic dipole atoms, treated in I, can therefore be simplified.

In order to prove the above statements, we recall that for these dipole atoms (*cf.* eqs. (I.23) and (I.24)):

$$F^\alpha = \frac{e}{c} f^{\alpha\beta}U_\beta - \frac{1}{c}(\partial_\gamma f^{\alpha\beta})\,\mu_\beta{}^\gamma - \frac{1}{c^3}\frac{\mathrm{d}}{\mathrm{d}\tau}(f^{\alpha\beta}\mu_\beta{}^\gamma U_\gamma), \tag{4}$$

and:

$$d^{\alpha\beta} = \frac{1}{c}(f^{\alpha\gamma}\mu_\gamma{}^\delta\Delta_\delta{}^\beta - f^{\beta\gamma}\mu_\gamma{}^\delta\Delta_\delta{}^\alpha), \tag{5}$$

where e is the total charge of the atom (or molecule), $f^{\alpha\beta}$ the tensor of the external electromagnetic field $((f^{01}, f^{02}, f^{03}) = \boldsymbol{e}$ and $(f^{23}, f^{31}, f^{12}) = \boldsymbol{b}$, with $\boldsymbol{e}$ the electric and $\boldsymbol{b}$ the magnetic field strength) and $\mu^{\alpha\beta}$ a tensor given by*:

$$\mu^{\alpha\beta} \equiv \textstyle\sum_i e_{(i)}(x^\alpha_{(i)}U^\beta - x^\beta_{(i)}U^\alpha) + \frac{1}{2}\sum_i e_{(i)}(x^\alpha_{(i)}\dot{x}^\beta_{(i)} - x^\beta_{(i)}\dot{x}^\alpha_{(i)}), \tag{6}$$

where $e_{(i)}$ are the charges of the constituent particles (electrons and nuclei) of the atom (or molecule), $x^\alpha_{(i)}$ the relative time-space coordinates of these particles with respect to the centre of gravity of the atom and $\dot{x}^\alpha_{(i)}$ the derivatives of these quantities with respect to the eigentime τ of this reference point. Substituting eqs. (4) and (5) into (2), we obtain, after

* This formula is directly found by comparing eq. (17) of ref. 8 with eq. (17) of chapter I. We have only changed the notations in (6).

multiplying by c:

$$c \frac{\mathrm{d}}{\mathrm{d}\tau} (M^* U^\alpha) + \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}\tau} (\Omega^{\alpha\beta} \dot{U}_\beta) = e f^{\alpha\beta} U_\beta - (\partial_\gamma f^{\alpha\beta}) \mu_\beta{}^\gamma -$$

$$- \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\alpha\gamma} \mu_\gamma{}^\beta - f^{\beta\gamma} \mu_\gamma{}^\alpha) U_\beta\} + \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} (U^\alpha U_\beta f^{\beta\gamma} \mu_\gamma{}^\delta U_\delta), \tag{7A}$$

$$c\dot{\Omega}^{\alpha\beta} + \frac{1}{c} U^\alpha \Omega^{\beta\gamma} \dot{U}_\gamma - \frac{1}{c} U^\beta \Omega^{\alpha\gamma} \dot{U}_\gamma = \Delta^\alpha{}_\varepsilon \Delta^\beta{}_\zeta (f^{\varepsilon\gamma} \mu_\gamma{}^\zeta - f^{\zeta\gamma} \mu_\gamma{}^\varepsilon), \tag{7B}$$

(*cf.* (I.25)). Omitting the second term on the left-hand side of eq. (7A), we find in lowest order approximation:

$$c \frac{\mathrm{d}}{\mathrm{d}\tau} (M^* U^\alpha) \simeq e f^{\alpha\beta} U_\beta - (\partial_\gamma f^{\alpha\beta}) \mu_\beta{}^\gamma -$$

$$- \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\alpha\gamma} \mu_\gamma{}^\beta - f^{\beta\gamma} \mu_\gamma{}^\alpha) U_\beta\} + \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} (U^\alpha U_\beta f^{\beta\gamma} \mu_\gamma{}^\delta U_\delta). \tag{8}$$

Let us now first disregard the terms with $\mu^{\alpha\beta}$ at the right-hand side of this equation. (We shall see below that they give rise in the iterated equations to terms of third and fourth order in the internal variables $x^\alpha_{(i)}$ and $\dot{x}^\alpha_{(i)}$, which are neglected in the present theory.) One can then easily solve $\dot{U}^\alpha$ from the remaining equation, obtaining:

$$\dot{U}^\alpha \simeq \frac{e}{M^* c} f^{\alpha\beta} U_\beta - \frac{\dot{M}^*}{M^*} U^\alpha. \tag{9}$$

Substituting this result into (7A) and (7B), we get

$$c \frac{\mathrm{d}}{\mathrm{d}\tau} (M^* U^\alpha) + \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{e}{M^*} \Omega^{\alpha\beta} f_\beta{}^\gamma U_\gamma \right) \simeq e f^{\alpha\beta} U_\beta -$$

$$- (\partial_\gamma f^{\alpha\beta}) \mu_\beta{}^\gamma - \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\alpha\gamma} \mu_\gamma{}^\beta - f^{\beta\gamma} \mu_\gamma{}^\alpha) U_\beta\} +$$

$$+ \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} (U^\alpha U_\beta f^{\beta\gamma} \mu_\gamma{}^\delta U_\delta), \tag{10A}$$

$$c\dot{\Omega}^{\alpha\beta} + \frac{e}{M^* c^2} U^\alpha \Omega^{\beta\gamma} f_\gamma{}^\delta U_\delta - \frac{e}{M^* c^2} U^\beta \Omega^{\alpha\gamma} f_\gamma{}^\delta U_\delta \simeq$$

$$\simeq \Delta^\alpha{}_\varepsilon \Delta^\beta{}_\zeta (f^{\varepsilon\gamma} \mu_\gamma{}^\zeta - f^{\zeta\gamma} \mu_\gamma{}^\varepsilon), \tag{10B}$$

where we have used the relation (1). If we introduce the tensor:

$$\tilde{\mu}^{\alpha\beta} \equiv \mu^{\alpha\beta} - \frac{e}{M^*} \Omega^{\alpha\beta}, \tag{11}$$

eqs. (10A) and (10B) can be written as

$$c \frac{\mathrm{d}}{\mathrm{d}\tau} (M^* U^\alpha) \simeq e f^{\alpha\beta} U_\beta - (\partial_\gamma f^{\alpha\beta})\, \mu_\beta{}^\gamma - \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} (f^{\alpha\gamma} \mu_\gamma{}^\beta U_\beta) +$$

$$+ \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} (U_\beta f^{\beta\gamma} \tilde{\mu}_\gamma{}^\delta \Delta_\delta{}^\alpha), \tag{12A}$$

$$c\dot{\Omega}^{\alpha\beta} \simeq (f^{\alpha\gamma} \mu_\gamma{}^\beta - f^{\beta\gamma} \mu_\gamma{}^\alpha) + \frac{1}{c^2} (f^{\alpha\gamma} U^\beta - f^{\beta\gamma} U^\alpha)\, \mu_\gamma{}^\delta U_\delta +$$

$$+ \frac{1}{c^2} U_\delta f^{\delta\gamma} (\tilde{\mu}_\gamma{}^\beta U^\alpha - \tilde{\mu}_\gamma{}^\alpha U^\beta), \tag{12B}$$

using also the relation (1).

Now the four-tensor $\Omega^{\alpha\beta}$ is given by (I.8), and taking for the energy-momentum tensor $T^{\alpha\beta}(x)$, appearing in this formula, in *first* approximation only its material part:

$$T^{\alpha\beta}_{(\mathrm{m})}(x) = c \sum_i \int_{-\infty}^{+\infty} m_{(i)} \frac{\mathrm{d}X^\alpha_{(i)}}{\mathrm{d}\tau_{(i)}} \frac{\mathrm{d}X^\beta_{(i)}}{\mathrm{d}\tau_{(i)}} \delta^{(4)}\{X_{(i)}(\tau_{(i)}) - x\}\, \mathrm{d}\tau_{(i)}, \tag{13}$$

where $m_{(i)}$ are the rest masses of the constituent particles of the atom (molecule), $X^\alpha_{(i)}$ their time-space coordinates and $\tau_{(i)}$ their eigentimes, one derives in a straightforward way, making also use of eq. (1) and the relation $x^\alpha_{(i)} U_\alpha = 0$ (*cf.* (I.20)), the expression

$$\Omega^{\alpha\beta} \simeq \sum_i m_{(i)} (x^\alpha_{(i)} \dot{x}^\gamma_{(i)} \Delta^\beta{}_\gamma - x^\beta_{(i)} \dot{x}^\gamma_{(i)} \Delta^\alpha{}_\gamma), \tag{14A}$$

or

$$\Omega^{\alpha\beta} \simeq \sum_i m_{(i)} (x^\varepsilon_{(i)} \dot{x}^\gamma_{(i)} - \dot{x}^\varepsilon_{(i)} x^\gamma_{(i)})\, \Delta^\alpha{}_\varepsilon \Delta^\beta{}_\gamma, \tag{14B}$$

(*cf.* also the results obtained by de Groot and Suttorp[3,4]).

One easily verifies, with the help of the antisymmetry of the field tensor $f^{\alpha\beta}$ and the property $\Delta_\alpha{}^\gamma \Delta_\gamma{}^\beta = \Delta_\alpha{}^\beta$, that the last terms at the right-hand sides of eqs. (12A) and (12B) respectively can be rewritten in the following forms:

$$\frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} (U_\beta f^{\beta\gamma} \tilde{\mu}_\gamma{}^\delta \Delta_\delta{}^\alpha) = \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \left\{ U_\beta f^{\beta\gamma} \left(\mu_\varepsilon{}^\zeta \Delta^\varepsilon{}_\gamma \Delta^\delta{}_\zeta - \frac{e}{M^*} \Omega_\gamma{}^\delta \right) \Delta_\delta{}^\alpha \right\}, \tag{15A}$$

and:

$$\frac{1}{c^2} U_\delta f^{\delta\gamma} (\tilde{\mu}_\gamma{}^\beta U^\alpha - \tilde{\mu}_\gamma{}^\alpha U^\beta) = \frac{1}{c^2} U_\delta f^{\delta\gamma} \left\{ \left(\mu_\varepsilon{}^\zeta \Delta^\varepsilon{}_\gamma \Delta^\beta{}_\zeta - \right. \right.$$

$$\left. \left. - \frac{e}{M^*} \Omega_\gamma{}^\beta \right) U^\alpha - \left(\mu_\varepsilon{}^\zeta \Delta^\varepsilon{}_\gamma \Delta^\alpha{}_\zeta - \frac{e}{M^*} \Omega_\gamma{}^\alpha \right) U^\beta \right\}. \tag{15B}$$

Futhermore it can be shown (see appendix) that, for *atoms* and *molecules*, the following inequality holds for each α and β:

$$\left| \frac{e}{M^*} \Omega^{\alpha\beta} \right| \ll |\mu^{\varepsilon\zeta} \Delta^{\alpha}{}_{\varepsilon} \Delta^{\beta}{}_{\zeta}|. \tag{16}$$

We may therefore neglect the terms containing $\Omega^{\alpha\beta}$ at the right-hand sides of eqs. (13A) and (13B) with respect to the terms with $\mu^{\alpha\beta}$, which means that we may replace the tensor $\tilde{\mu}^{\alpha\beta}$ to the expressions at the left-hand sides of these equations in very good approximation by $\mu^{\alpha\beta}$. Eqs. (12A) and(12B) then finally become:

$$c \frac{\mathrm{d}}{\mathrm{d}\tau} (M^* U^{\alpha}) \simeq e f^{\alpha\beta} U_{\beta} - (\partial_{\gamma} f^{\alpha\beta}) \mu_{\beta}{}^{\gamma} -$$

$$- \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\alpha\gamma} \mu_{\gamma}{}^{\beta} - f^{\beta\gamma} \mu_{\gamma}{}^{\alpha}) U_{\beta}\} + \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} (U^{\alpha} U_{\beta} f^{\beta\gamma} \mu_{\gamma}{}^{\delta} U_{\delta}), \tag{17A}$$

$$c \dot{\Omega}^{\alpha\beta} \simeq \Delta^{\alpha}{}_{\varepsilon} \Delta^{\beta}{}_{\zeta} (f^{\varepsilon\gamma} \mu_{\gamma}{}^{\zeta} - f^{\zeta\gamma} \mu_{\gamma}{}^{\varepsilon}). \tag{17B}$$

These are the simplified Møller-equations in the case of dipole atoms (or molecules), describing their relativistic motion with a very high degree of accuracy, however, without the physically irrelevant trembling motion.

We have two remarks to make about the above derivation. First of all we note that the contribution of the intra-atomic field to the tensor $\Omega^{\alpha\beta}$ has been neglected in our calculations. De Groot and Suttorp[4]) have discussed this contribution in the so-called "Darwin-approximation", but it appears to be much smaller than that of $T^{\alpha\beta}_{(\mathrm{m})}$, calculated above. The (strong) inequality (16) will therefore certainly also hold for the exact angular momentum tensor $\Omega^{\alpha\beta}$.

As a second point there is the fact that we have taken into account only the effect of the Lorentz-force term of eq. (8) in the iteration procedure and neglected the effect of the terms containing $\mu^{\alpha\beta}$. In particular when the total atomic charge $e = 0$, these are the only terms left at the right-hand side of (8). If we now consider *e.g.* the term $-(\partial_{\gamma} f^{\alpha\beta}) \mu_{\beta}{}^{\gamma}$ in (8), this will give rise to:

$$- \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \left\{ \frac{1}{M^*} \Omega^{\alpha\beta} (\partial_{\delta} f_{\beta}{}^{\gamma}) \mu_{\gamma}{}^{\delta} \right\} \tag{18A}$$

in the iterated equation of motion (*cf.* (10A)), and to:

$$- \frac{1}{M^* c^2} U^{\alpha} \Omega^{\beta\gamma} (\partial_{\varepsilon} f_{\gamma}{}^{\delta}) \mu_{\delta}{}^{\varepsilon} + \frac{1}{M^* c^2} U^{\beta} \Omega^{\alpha\gamma} (\partial_{\varepsilon} f_{\gamma}{}^{\delta}) \mu_{\delta}{}^{\varepsilon} \tag{18B}$$

in the angular momentum equation (*cf.* (10B)). The characteristic quantity appearing in (18) is

$$\frac{1}{M^*}\,\Omega^{\alpha\beta}\mu^{\gamma\delta}, \tag{19}$$

a third and fourth order quantity in the internal atomic variables (just as a magnetic quadrupole or an electric octupole). For consistency with the dipole approximation, made in the present chapter, the terms (18) have therefore to be neglected in the iterated equations of motion. Along the same lines one proves that also, the other $\mu^{\alpha\beta}$-terms in (8) give rise to negligible effects in these equations.

Since the left-hand side of eq. (17A) does not contain the intrinsic angular momentum tensor $\Omega^{\alpha\beta}$, as was the case in the original Møller-equations (2A) and (7A), or the iterated equation (10A), the equations (17A) and (17B) appear at first sight to be uncoupled. However, as $\Omega^{\alpha\beta}$ is related to (*cf.* (14)):

$$\mu^{\alpha\beta}_{(2)} \equiv \tfrac{1}{2}\sum_i e_{(i)}(x^{\alpha}_{(i)}\dot{x}^{\beta}_{(i)} - x^{\beta}_{(i)}\dot{x}^{\alpha}_{(i)}), \tag{20}$$

which is a part of the tensor $\mu^{\alpha\beta}$, see eq. (6), appearing at the right-hand side of the equation of motion (17A), we come to the conclusion that, in reality, there is no question of uncoupling eqs. (17A) and (17B). On the contrary, one should also write down a differential equation for the first term at the right-hand side of eq. (6), *i.e.* for:

$$\mu^{\alpha\beta}_{(1)} \equiv \sum_i e_{(i)}(x^{\alpha}_{(i)}U^{\beta} - x^{\beta}_{(i)}U^{\alpha}), \tag{21}$$

in order to solve (17A). The reason that we do not derive such equation here is that we shall always consider the right-hand side of (17A) (and (17B)) as *given*, as was also the case in the original Møller-equations (2A) and (2B).

Since the trembling motion originated from the special choice (1) of the reference points, and had to be eliminated afterwards as an unphysical motion, the question arises whether other conditions than (1) could be used, which lead directly to "physical" second order equations of motion without "Zitter-terms"? This has been investigated by *e.g.* Dixon[9]) using the following condition for the reference point:

$$\Omega^{\alpha\beta}P_{\beta} = 0, \tag{22}$$

where P_{β} is the energy-momentum four-vector of the system. In contrast with (1), eq. (22) defines one single reference point: the centre of energy of the system in the Lorentz frame, for which the space-part of P_{β} (*i.e.* the momentum) is zero. In the special case of a (charged) system with only a magnetic dipole moment proportional to the internal angular momentum the condition (22) leads immediately to second order equations of motion. In the more general case, however, (22) leads to a complicated set of equations of motion, as follows from the results obtained recently by Suttorp and de Groot[10]) (*cf.* eqs. (46)–(48) of that reference). It can be shown that within the approximations made in the present chapter their equations reduce to our equations (17). Although in some cases eq. (22) would have been a slightly better starting point for the derivation of the equations of motion than (1), we have taken the latter condition, since we could then prove in chapter I, that one obtains formally the same results as de Groot and Suttorp[3,4]) for the atomic energy-momentum tensor of a system of dipoles.

§ 3. *The atomic energy-momentum tensor.* In a completely analogous way as in I, § 4, one can derive the energy-momentum four-tensor for a system of N dipole atoms or molecules, starting now from the equations (17). It is evident, that the field part $t^{\alpha\beta}_{(\mathrm{f})}$, of this tensor will be the same as in I, since the Møller-equations (7) and the simplified type (17) do not differ in their field parts. We therefore obtain again the expression (I.36) for $t^{\alpha\beta}_{(\mathrm{f})}$. However, since the "material parts" of eqs. (7A) and (17A) (*i.e.* the left-hand sides) are different, this will also be the case with the material parts of the energy-momentum tensors, derived with the help of these equations. It is easily seen that, in the material part, found with the simplified Møller-equations (17A), the "acceleration term" will be lacking, so that instead of (I.37), we simply have:

$$t^{*\alpha\beta}_{(\mathrm{m})} = \textstyle\sum_k \rho^{*\prime}_{(k)} u^{\alpha}_{(k)} u^{\beta}_{(k)}, \tag{23}$$

where $\rho^{*\prime}_{(k)}$ is the rest mass density of atom k ($k = 1, 2, \ldots, N$), defined by (I.31), and $u^{\alpha}_{(k)}$ the four-velocity of this atom.

In a completely similar way as in chapter I, one can again symmetrize the total energy-momentum tensor, by adding the divergence-free tensor:

$$\tfrac{1}{2} \textstyle\sum_k \partial_\gamma \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)} - u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\} \tag{24}$$

to its material part (see (I.42)), so that this now becomes

$$t^{\alpha\beta}_{(\mathrm{m})} = \textstyle\sum_k [\rho^{*\prime}_{(k)} u^{\alpha}_{(k)} u^{\beta}_{(k)} + \tfrac{1}{2}\partial_\gamma \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)} - u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}], \tag{25}$$

with $\sigma^{*\alpha\beta}_{(k)}$ the atomic angular momentum density tensor, defined by (I.32). This symmetry can be proved again by applying the atomic intrinsic angular momentum balance, which follows from eq. (17B). (Note that this balance equation will have a different form from (I.39)). We know that the symmetric total energy-momentum tensor $t^{\alpha\beta} \equiv t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}$ has the advantage of implying *both* the conservation law of total energy-momentum (*cf.* (I.43)) *and* of total angular momentum of the system of dipole atoms (*cf.* refs. 3 and 4). The material energy-momentum tensor (25) is slightly different from the expression in chapter I (eq. (I.44)), which had been derived earlier by de Groot and Suttorp[3,4]). As we have seen, this is due to the elimination of the unphysical trembling motion of the (pseudo)-centres of gravity of the atoms.

§ 4. *Concluding remark.* The theory has been developed in the previous and present chapters only for the case of electric and magnetic *dipole* atoms or molecules. The electric quadrupole moments of the atoms, which are of the same (second) order in the internal atomic variables as the magnetic dipole moments, have always been neglected. As Rosenfeld[11]) has pointed out, this is in general certainly not allowed. We shall therefore investigate in the next chapter (III), what are the changes in the theory, if atomic or molecular electric quadrupole moments are also taken into account.

APPENDIX

Proof of the inequality (16). We first write formula (14B) in the following form:

$$\Omega^{\alpha\beta} \simeq \Sigma_i' \, m_{\text{el}}(x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)}) \, \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta} + \\ + \Sigma_i'' \, M_{(i)}(x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)}) \, \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}, \tag{A.1}$$

where m_{el} and $M_{(i)}$ are the rest masses of an electron and the i^{th} nucleus respectively, whereas the symbol Σ' denotes summation over the electrons and Σ'' over the nuclei. Multiplying this expression with e/M^* and using the fact that* $|(e/M^*) \, m_{\text{el}}| \leqslant (m_{\text{el}}/m_{\text{pr}}) \, |e_{\text{el}}|$ and $|eM_{(i)}/M^*| \lesssim e_{(i)}$, with e_{el} the (negative) elementary charge, m_{pr} the rest mass of the proton and $e_{(i)}$ the (positive) charge of the i^{th} nucleus, we obtain the following inequality for each α and β:

$$\left| \frac{e}{M^*} \, \Omega^{\alpha\beta} \right| \lesssim \frac{m_{\text{el}}}{m_{\text{pr}}} \, |e_{\text{el}} \, \Sigma_i' \, (x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)}) \, \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}| + \\ + \Sigma_i'' \, |e_{(i)}(x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)}) \, \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}|. \tag{A.2}$$

We now compare the right-hand side of this inequality with the absolute value of the corresponding element of the tensor $\mu^{\varepsilon\zeta}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}$, which can be written as

$$\mu^{\varepsilon\zeta}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta} = \tfrac{1}{2}e_{\text{el}} \, \Sigma_i' \, (x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)}) \, \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta} + \\ + \tfrac{1}{2} \, \Sigma_i'' \, e_{(i)}(x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)}) \, \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}, \tag{A.3}$$

where we have used the fact that $\mu^{\varepsilon\zeta}_{(1)}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta} = 0$, which follows from eqs. (21) and $\Delta^{\alpha}{}_{\beta}U^{\beta} = 0$. In the momentary atomic rest frame the tensor $\mu^{\varepsilon\zeta}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}$ possesses only nonzero space-space components (just as the tensor $\Omega^{\alpha\beta}$), and one can prove, using formula (56) of ref. 8, that these are the components of the magnetic dipole moment of the atom, as long as one neglects moments of higher than the second order in the internal variables. Since this atomic

* The validity of these inequalities is easily checked by means of elementary calculations.

(molecular) magnetic dipole moment is the sum of an electronic and a nuclear moment, and since the nuclear contribution is according to Van Vleck[12]) completely negligible with respect to the electronic contribution, we can neglect the second sum of (A.3) against the first, so that we get:

$$|\mu^{\varepsilon\zeta}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}| \approx \tfrac{1}{2}\,|\,e_{\mathrm{el}}\,\textstyle\sum_i' \,(x^{\varepsilon}_{(i)}\dot{x}^{\zeta}_{(i)} - \dot{x}^{\varepsilon}_{(i)}x^{\zeta}_{(i)})\,\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}|. \tag{A.4}$$

We then see, that the first sum at the right-hand side of (A.2) is much smaller than the second member of (A.4), because the factor $(m_{\mathrm{el}}/m_{\mathrm{pr}})$ is of the order 10^{-3}. But the second sum of (A.2), which is again a nuclear contribution, is also negligibly small, so that we may finally conclude, that:

$$\left|\frac{e}{M^*}\,\Omega^{\alpha\beta}\right| \ll |\mu^{\varepsilon\zeta}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}|. \tag{A.5}$$

This proves the validity of the inequality (16).

Remark. The reason, that we have rewritten the last terms of eqs. (12A) and (12B) in the forms as given by the right-hand sides of eqs. (15A) and (15B) is, that we cannot apply the above arguments of Van Vleck to the tensor $\mu^{\alpha\beta}$. Though $\mu^{\alpha\beta}_{(1)}$ gives no contribution to those terms, the remaining part $\mu^{\alpha\beta}_{(2)}$ has nonzero space-time components in the momentary atomic rest-frame which depends on the atomic (molecular) electric quadrupole moment. For this moment, however, it is generally no longer true that the nuclear contributions are negligible with respect to the electronic contributions. This is of course irrelevant in the present chapter, where we have always neglected electric quadrupole moments in the dipole approximation but the proof given above of the possibility of replacing $\tilde{\mu}^{\alpha\beta}$ by $\mu^{\alpha\beta}$ in the iterated Møller-equations is also valid if these moments are no longer neglected, as will be done chapter III.

REFERENCES

1) Vlieger, J., Physica **37** (1967) 165.
2) Møller, C., Ann. Inst. H. Poincaré **11** (1949–50) 251.
3) de Groot, S. R. and Suttorp, L. G., Physica **37** (1967) 284.
4) de Groot, S. R. and Suttorp, L. G., Physica **37** (1967) 297.
5) Plahte, E., Nuovo Cimento Suppl. **4** (1966) 291.
6) Mathisson, M., Acta phys. Polon. **6** (1937) 163, 218.
7) Weyssenhoff, J. and Raabe, A., Acta phys. Polon. **9** (1947) 7; Weyssenhoff, J., Acta phys. Polon. **9** (1947) 26, 34.
8) de Groot, S. R. and Suttorp, L. G., Physica **31** (1965) 1713.
9) Dixon, W. G., Nuovo Cimento **38** (1965) 1616.
10) Suttorp, L. G. and de Groot, S. R., Nuovo Cimento **65**A (1970) 245.
11) Rosenfeld, L., Theory of electrons, North-Holland Publ. Co. (Amsterdam, 1951).
12) Van Vleck, J. H., The theory of electric and magnetic susceptibilities, Clarendon Press (Oxford, 1932) 259, 277.

Chapter III

THE CASE OF ATOMS AND MOLECULES WITH ELECTRIC AND MAGNETIC DIPOLE MOMENTS AND ELECTRIC QUADRUPOLE MOMENTS

Synopsis

The classical relativistic equations of motion for electric and magnetic dipole atoms or molecules in an external electromagnetic field of force, given in two previous chapters (I and II), are extended in the present chapter to the case that these atoms possess also electric quadrupole moments. Again Møller's equations of motion for relativistic systems with an internal angular momentum are taken as the starting point. We only consider the form of these equations in which the term describing the unphysical trembling motion of the atoms is eliminated (see chapter II). The resulting equations of motion are used in order to derive the relativistic atomic energy-momentum tensor for a system, consisting of a (large) number of these atoms. It is found that the field part of this tensor is not of the same form as in the pure dipole case and also no longer expressible *only* in terms of quantities appearing in the atomic field equations.

§ 1. *Introduction*. In the two previous chapters (denoted hereafter by I and II), we have treated the classical relativistic equations of motion for electric and magnetic dipole atoms (or molecules) in an external electromagnetic field of force[1,2]). As a basis we have chosen Møller's theory[3]) on the relativistic dynamics of systems with an internal angular momentum. Since the equations of motion, obtained in I, contained an unphysical trembling motion, this had to be eliminated. This was done in chapter II, where we have derived simplified Møller-equations, describing the "real" motion of the dipole atoms (molecules).

In the present chapter we shall generalize the latter equations to the case that the atoms (molecules) possess, in addition to their electric (magnetic) dipole moments, also an electric quadrupole moment. According to Rosenfeld[4]) it is necessary to take into account the electric quadrupole moments of the atoms, as soon as one considers their magnetic dipole moments, since both kinds of moments are of the same (second) order in the internal atomic variables (relative coordinates and velocities of the constituent particles of the atoms) and may give rise to effects, which are of the same order of magnitude. The model of dipole atoms, considered in I and II, was therefore somewhat artificial.

In § 2 we shall calculate, using the same technique as in reference 1, the additional terms in the equation of motion (and the internal angular momentum equation) due to the atomic (molecular) electric quadrupole moment.

In § 3 we then obtain again the relativistic energy-momentum tensor for a system, consisting of N atoms (molecules) of the kind considered in the present chapter. In an analogous way to that followed in I (and II) this four-tensor can be symmetrized, with the difference that we have to add a divergence-free tensor not only to the material part of the energy-momentum tensor, but also to its field part. In contrast with what has been found in the pure dipole case, this field part is no longer expressible *only* in terms of quantities appearing in the atomic field equations (*i.e.* the atomic field and polarization tensors), but also contains terms depending on the electric quadrupole moment density of the atoms or molecules.

Just as in the two preceding chapters, the present theory is one in which radiative effects are neglected throughout. We treat these effects in chapter IV.

§ 2. *Equations of motion.* From I, § 3 we know that for the calculations of the external electromagnetic force F^α and its moment $d^{\alpha\beta}$, acting upon the atom (or molecule), it is necessary to use the expressions (I.16) and (I.17)* for the charge-current density vector $j^\alpha(x)$ and polarization tensor $m^{\alpha\beta}(x)$ of this atom (molecule). As we have stressed in I, it is essential in these calculations that this polarization tensor is of the form (I.17) in the dipole case, and does not contain any derivatives of the δ-function, appearing in this formula. As soon as we also take into account the atomic (molecular) electric quadrupole moment, however, (I.17) is no longer the correct expression for the polarization tensor, but this must be replaced by:

$$m^{\alpha\beta}(x) = \int_{-\infty}^{+\infty} \mu^{\alpha\beta}(\tau)\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau + \partial_\gamma \int_{-\infty}^{+\infty} \chi^{\alpha\beta\gamma}(\tau)\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau, \quad (1)$$

where $\mu^{\alpha\beta}(\tau)$ is a four-tensor, depending on the atomic (molecular) electric and magnetic dipole moments $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, electric quadrupole moment q, and furthermore on the atomic four-velocity $U^\alpha(\tau) = \mathrm{d}X^\alpha(\tau)/\mathrm{d}\tau$ and acceleration $\mathrm{d}U^\alpha(\tau)/\mathrm{d}\tau$, whereas the third order tensor $\chi^{\alpha\beta\gamma}(\tau)$ depends only on q and U^α. The symbol $X(\tau)$ is an abbreviation for the time-space coordinates $X^\alpha(\tau)$ ($\alpha = 0, 1, 2, 3$) of the reference point, describing the position of the atom

* Formulae and equations of chapters I and II will be denoted in the following by (I.1), (II.1), *etc.*

as a function of its eigentime τ (see I and II), x stands for the time-space coordinates $x^{\alpha} = (ct\ \boldsymbol{x})$ of an arbitrary point and ∂_{α} for differentiation with respect to these coordinates: $\partial_{\alpha} = (\partial/\partial ct, \partial/\partial \boldsymbol{x})$; ($c$ is velocity of light).

Formula (1) is immediately obtained from the general expression for the atomic polarization tensor, derived by de Groot and Suttorp[5]) (see formula (17) of that reference). If one considers only terms up to the second order in internal atomic parameters and expresses these quantities by means of a Lorentz-transformation in the corresponding quantities in the atomic frame (see ref. 5) one easily proves the validity of eq. (1) and all statements made about it. It is of no use to take into account terms of higher than the second order in the general expression (17) of de Groot and Suttorp[5]), since they give rise only to terms depending on higher order atomic moments than $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ and q, which will not be considered in the present chapter. From eqs. (17) and (27) of ref. 5 one still obtains the following relation:

$$\chi^{\alpha\beta\gamma}(\tau)\, U_{\gamma}(\tau) = 0, \tag{2}$$

which will be important to simplify the results below.

Let:

$$j^{\alpha}(x) \equiv ce \int_{-\infty}^{+\infty} U^{\alpha}(\tau)\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau \tag{3}$$

(*cf.* (I.16)) be the charge-current density vector of the atom (molecule) as a whole (e is total atomic (molecular) charge), then the density of the external electromagnetic force acting upon it can be written as (*cf.* (I.15)):

$$f^{\alpha}(x) = f^{\alpha\beta}(x)\{j_{\beta}(x)/c + \partial_{\gamma} m_{\beta}{}^{\gamma}(x)\}, \tag{4}$$

with $f^{\alpha\beta}(x)$ the tensor of the external electromagnetic field:

$$(f^{01}, f^{02}, f^{03}) = \boldsymbol{e} \qquad \text{and} \qquad (f^{23}, f^{31}, f^{12}) = \boldsymbol{b},$$

with $\boldsymbol{e}$ the electric and $\boldsymbol{b}$ the magnetic field strength. If we now substitute eqs. (1) and (3) into the right-hand side of (4), and calculate the external force $F^{\alpha}(\tau)$, (I.5), and its moment $d^{\alpha\beta}(\tau)$, (I.10), with respect to the centre of gravity of the atom (molecule), we obtain for the terms, depending on e and $\mu^{\alpha\beta}(\tau)$, of course, the same results as in chapter I, *i.e.* (I.23) and (I.24):

$$F^{\alpha}(\tau) = \frac{e}{c} f^{\alpha\beta}\{X(\tau)\}\, U_{\beta}(\tau) - \frac{1}{c}\, [\partial_{\gamma} f^{\alpha\beta}\{X(\tau)\}]\, \mu_{\beta}{}^{\gamma}(\tau) -$$

$$- \frac{1}{c^3} \frac{\mathrm{d}}{\mathrm{d}\tau} [f^{\alpha\beta}\{X(\tau)\}\, \mu_{\beta}{}^{\gamma}(\tau)\, U_{\gamma}(\tau)] + \ldots, \tag{5}$$

and:

$$d^{\alpha\beta}(\tau) = \frac{1}{c} [f^{\alpha\gamma}\{X(\tau)\}\, \mu_{\gamma}{}^{\delta}(\tau)\, \Delta_{\delta}{}^{\beta}(\tau) - f^{\beta\gamma}\{X(\tau)\}\, \mu_{\gamma}{}^{\delta}(\tau)\, \Delta_{\delta}{}^{\alpha}(\tau)] + \ldots, \tag{6}$$

where:

$$\Delta_\alpha{}^\beta(\tau) \equiv \delta_\alpha{}^\beta + \frac{1}{c^2}\, U_\alpha(\tau)\, U^\beta(\tau). \tag{7}$$

We are therefore only left with the evaluation of terms in $F^\alpha(\tau)$ and $d^{\alpha\beta}(\tau)$, containing $\chi^{\alpha\beta\gamma}(\tau)$, *i.e.* with the calculation of the integrals:

$$\begin{aligned} I_1 \equiv & \int\limits_{V(\tau)} \int\limits_{-\infty}^{+\infty} f^{\alpha\beta}(x) \left[1 + \frac{1}{c^2} \{x^\varepsilon - X^\varepsilon(\tau)\} \dot{U}_\varepsilon(\tau) \right] \cdot \\ & \cdot \chi_\beta{}^{\gamma\delta}(\tau')\, \partial_\gamma \partial_\delta \delta^{(4)}\{X(\tau') - x\}\, \mathrm{d}V\, \mathrm{d}\tau' = \\ = & \int\limits^{\infty} \int\limits_{-\infty}^{+\infty} f^{\alpha\beta}(x) \left[1 + \frac{1}{c^2} \{x^\varepsilon - X^\varepsilon(\tau)\} \dot{U}_\varepsilon(\tau) \right] \cdot \\ & \cdot \chi_\beta{}^{\gamma\delta}(\tau') [\partial_\gamma \partial_\delta \delta^{(4)}\{X(\tau') - x\}] \cdot \\ & \cdot \delta\left[-\frac{1}{c} \{x^\eta - X^\eta(\tau)\}\, U_\eta(\tau) \right] \mathrm{d}^{(4)}x\, \mathrm{d}\tau', \end{aligned} \tag{8}$$

and:

$$\begin{aligned} I_2 \equiv & \int\limits_{V(\tau)} \int\limits_{-\infty}^{+\infty} [\{x^\alpha - X^\alpha(\tau)\}\, f^{\beta\gamma}(x) - \{x^\beta - X^\beta(\tau)\}\, f^{\alpha\gamma}(x)] \cdot \\ & \cdot \left[1 + \frac{1}{c^2} \{x^\eta - X^\eta(\tau)\}\, \dot{U}_\eta(\tau) \right] \chi_\gamma{}^{\delta\varepsilon}(\tau')\, \partial_\delta \partial_\varepsilon \delta^{(4)}\{X(\tau') - x\}\, \mathrm{d}V\, \mathrm{d}\tau' = \\ = & \int\limits^{\infty} \int\limits_{-\infty}^{+\infty} [\{x^\alpha - X^\alpha(\tau)\}\, f^{\beta\gamma}(x) - \{x^\beta - X^\beta(\tau)\}\, f^{\alpha\gamma}(x)] \cdot \\ & \cdot \left[1 + \frac{1}{c^2} \{x^\eta - X^\eta(\tau)\}\, \dot{U}_\eta(\tau) \right] \chi_\gamma{}^{\delta\varepsilon}(\tau') \cdot \\ & \cdot [\partial_\delta \partial_\varepsilon \delta^{(4)}\{X(\tau') - x\}]\, \delta\left[-\frac{1}{c} \{x^\zeta - X^\zeta(\tau)\}\, U_\zeta(\tau) \right] \mathrm{d}^{(4)}x\, \mathrm{d}\tau', \end{aligned} \tag{9}$$

where we have used the same notation as in chapter I (*cf.* eqs. (I.18), (I.19), (I.20) and (I.21)).

Integrating the right-hand sides of eqs. (8) and (9) twice partially with respect to x^α, then integrating over these variables, and finally applying the formulae:

$$\int\limits_{-\infty}^{+\infty} g(\tau')\, \delta\left[-\frac{1}{c} \{X^\alpha(\tau') - X^\alpha(\tau)\}\, U_\alpha(\tau) \right] \mathrm{d}\tau' = \frac{1}{c} [g(\tau')]_{\tau'=\tau}, \tag{10}$$

$$\int_{-\infty}^{+\infty} g(\tau')\,\delta'\left[-\frac{1}{c}\{X^{\alpha}(\tau')-X^{\alpha}(\tau)\}\,U_{\alpha}(\tau)\right]\mathrm{d}\tau' =$$

$$= -\frac{1}{c^2}\left[\frac{\mathrm{d}g(\tau')}{\mathrm{d}\tau'}\right]_{\tau'=\tau}, \tag{11}$$

where $\delta'(y) \equiv \mathrm{d}\delta(y)/\mathrm{d}y$, (for derivation see reference 1), and:

$$\int_{-\infty}^{+\infty} g(\tau')\,\delta''\left[-\frac{1}{c}\{X^{\alpha}(\tau')-X^{\alpha}(\tau)\}\,U_{\alpha}(\tau)\right]\mathrm{d}\tau' =$$

$$=\left[\frac{1}{c^3}\,\frac{\mathrm{d}^2 g(\tau')}{\mathrm{d}\tau'^2}-\frac{1}{c^5}\,g(\tau')\,\dot{U}^{\alpha}(\tau)\,\dot{U}_{\alpha}(\tau)\right]_{\tau'=\tau}, \tag{12}$$

with $\delta''(y) \equiv \mathrm{d}^2\delta(y)/\mathrm{d}y^2$, (see appendix), we obtain the following results:

$$I_1 = \frac{1}{c}\,[\partial_\gamma\partial_\delta f^{\alpha\beta}\{X(\tau)\}]\,\chi_\beta{}^{\gamma\delta}(\tau) +$$

$$+\frac{1}{c^3}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,[[U_\delta(\tau)\,\partial_\gamma f^{\alpha\beta}\{X(\tau)\}+U_\gamma(\tau)\,\partial_\delta f^{\alpha\beta}\{X(\tau)\}]\,\chi_\beta{}^{\gamma\delta}(\tau)] +$$

$$+\frac{1}{c^5}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\,[f^{\alpha\beta}\{X(\tau)\}\,\chi_\beta{}^{\gamma\delta}(\tau)]\,U_\gamma(\tau)\,U_\delta(\tau)\right], \tag{13}$$

and:

$$I_2 = -\frac{1}{c}\,[[\partial_\delta f^{\alpha\gamma}\{X(\tau)\}]\{\chi_\gamma{}^{\delta\varepsilon}(\tau)+\chi_\gamma{}^{\varepsilon\delta}(\tau)\}\,\varDelta_\varepsilon{}^{\beta}(\tau)-[\partial_\delta f^{\beta\gamma}\{X(\tau)\}]\cdot$$

$$\cdot\{\chi_\gamma{}^{\delta\varepsilon}(\tau)+\chi_\gamma{}^{\varepsilon\delta}(\tau)\}\,\varDelta_\varepsilon{}^{\alpha}(\tau)] -$$

$$-\frac{1}{c^3}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,[[f^{\alpha\gamma}\{X(\tau)\}\{\chi_\gamma{}^{\beta\delta}(\tau)+\chi_\gamma{}^{\delta\beta}(\tau)\} -$$

$$-f^{\beta\gamma}\{X(\tau)\}\{\chi_\gamma{}^{\alpha\delta}(\tau)+\chi_\gamma{}^{\delta\alpha}(\tau)\}]\,U_\delta(\tau)] -$$

$$-\frac{1}{c^5}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,[[f^{\alpha\gamma}\{X(\tau)\}\,U^{\beta}(\tau)-f^{\beta\gamma}\{X(\tau)\}\,U^{\alpha}(\tau)]\,\chi_\gamma{}^{\delta\varepsilon}(\tau)\,U_\delta(\tau)\,U_\varepsilon(\tau)] -$$

$$-\frac{1}{c^5}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\,[f^{\alpha\gamma}\{X(\tau)\}\,\chi_\gamma{}^{\delta\varepsilon}(\tau)]\,U^{\beta}(\tau) -\right.$$

$$\left.-\frac{\mathrm{d}}{\mathrm{d}\tau}\,[f^{\beta\gamma}\{X(\tau)\}\,\chi_\gamma{}^{\delta\varepsilon}(\tau)]\,U^{\alpha}(\tau)\right]U_\delta(\tau)\,U_\varepsilon(\tau). \tag{14}$$

The rather lengthy, but straightforward calculations have been omitted here.

The expressions (13) and (14) can now be simplified by using the relation (2). We then find:

$$I_1 = \frac{1}{c}\left[\partial_\gamma\partial_\delta f^{\alpha\beta}\{X(\tau)\}\right]\chi_\beta{}^{\gamma\delta}(\tau) + \frac{1}{c^3}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left[\partial_\delta f^{\alpha\beta}\{X(\tau)\}\right]\chi_\beta{}^{\gamma\delta}(\tau)\, U_\gamma\right] +$$

$$+ \frac{1}{c^5}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[f^{\alpha\beta}\{X(\tau)\}\,\dot{\chi}_\beta{}^{\gamma\delta}(\tau)\, U_\gamma(\tau)\, U_\delta(\tau)\right], \tag{15}$$

and:

$$I_2 = -\frac{1}{c}\left[\left[\partial_\delta f^{\alpha\gamma}\{X(\tau)\}\right]\chi_\gamma{}^{\varepsilon\delta}(\tau)\,\Delta_\varepsilon{}^\beta(\tau) - \left[\partial_\delta f^{\beta\gamma}\{X(\tau)\}\right]\chi_\gamma{}^{\varepsilon\delta}(\tau)\,\Delta_\varepsilon{}^\alpha(\tau)\right] -$$

$$-\frac{1}{c}\left[\left[\partial_\delta f^{\alpha\gamma}\{X(\tau)\}\right]\chi_\gamma{}^{\delta\beta}(\tau) - \left[\partial_\delta f^{\beta\gamma}\{X(\tau)\}\right]\chi_\gamma{}^{\delta\alpha}(\tau)\right] -$$

$$-\frac{1}{c^3}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left[f^{\alpha\gamma}\{X(\tau)\}\,\chi_\gamma{}^{\delta\beta}(\tau) - f^{\beta\gamma}\{X(\tau)\}\,\chi_\gamma{}^{\delta\alpha}(\tau)\right] U_\delta(\tau)\right] -$$

$$-\frac{1}{c^5}\left[f^{\alpha\gamma}\{X(\tau)\}\, U^\beta(\tau) - f^{\beta\gamma}\{X(\tau)\}\, U^\alpha(\tau)\right]\dot{\chi}_\gamma{}^{\delta\varepsilon}(\tau)\, U_\delta(\tau)\, U_\varepsilon(\tau), \tag{16}$$

where the dots denote differentiation with respect to τ.

From eqs. (5), (6), (15) and (16) we obtain, in the approximation, that we have an atom (or molecule) with only electric and magnetic dipole moments and electric quadrupole moment, the following results:

$$F^\alpha(\tau) = \frac{e}{c} f^{\alpha\beta}\{X(\tau)\}\, U_\beta(\tau) - \frac{1}{c}\left[\partial_\gamma f^{\alpha\beta}\{X(\tau)\}\right]\mu_\beta{}^\gamma(\tau) +$$

$$+ \frac{1}{c}\left[\partial_\gamma\partial_\delta f^{\alpha\beta}\{X(\tau)\}\right]\chi_\beta{}^{\gamma\delta}(\tau) - \frac{1}{c^3}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[f^{\alpha\beta}\{X(\tau)\}\,\mu_\beta{}^\gamma(\tau)\, U_\gamma(\tau)\right] +$$

$$+ \frac{1}{c^3}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left[\partial_\delta f^{\alpha\beta}\{X(\tau)\}\right]\chi_\beta{}^{\gamma\delta}(\tau)\, U_\gamma(\tau)\right] +$$

$$+ \frac{1}{c^5}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[f^{\alpha\beta}\{X(\tau)\}\,\dot{\chi}_\beta{}^{\gamma\delta}(\tau)\, U_\gamma(\tau)\, U_\delta(\tau)\right], \tag{17}$$

and:

$$d^{\alpha\beta}(\tau) = \frac{1}{c}\left[f^{\alpha\gamma}\{X(\tau)\}\,\mu_\gamma{}^\delta(\tau)\,\Delta_\delta{}^\beta(\tau) - f^{\beta\gamma}\{X(\tau)\}\,\mu_\gamma{}^\delta(\tau)\,\Delta_\delta{}^\alpha(\tau)\right] -$$

$$-\frac{1}{c}\left[\left[\partial_\delta f^{\alpha\gamma}\{X(\tau)\}\right]\chi_\gamma{}^{\varepsilon\delta}(\tau)\,\Delta_\varepsilon{}^\beta(\tau) - \left[\partial_\delta f^{\beta\gamma}\{X(\tau)\}\right]\chi_\gamma{}^{\varepsilon\delta}(\tau)\,\Delta_\varepsilon{}^\alpha(\tau)\right] -$$

$$- \frac{1}{c} \left[[\partial_\delta f^{\alpha\gamma}\{X(\tau)\}] \, \chi_\gamma{}^{\delta\beta}(\tau) - [\partial_\delta f^{\beta\gamma}\{X(\tau)\}] \, \chi_\gamma{}^{\delta\alpha}(\tau) \right] -$$

$$- \frac{1}{c^3} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[[f^{\alpha\gamma}\{X(\tau)\} \, \chi_\gamma{}^{\delta\beta}(\tau) - f^{\beta\gamma}\{X(\tau)\} \, \chi_\gamma{}^{\delta\alpha}(\tau)] \, U_\delta(\tau) \right] -$$

$$- \frac{1}{c^5} [f^{\alpha\gamma}\{X(\tau)\} \, U^\beta(\tau) - f^{\beta\gamma}\{X(\tau)\} \, U^\alpha(\tau)] \, \dot{\chi}_\gamma{}^{\delta\varepsilon}(\tau) \, U_\delta(\tau) \, U_\varepsilon(\tau), \quad (18)$$

for the external electromagnetic force and its moment, acting upon this atom (molecule).

Substituting the last two formulae into the right-hand sides of the Møller-equations (I.13A) and (I.13B) (or (II.2A) and (II.2B)), and taking into account that we may neglect the second term at the left-hand side of (I.13A) (or (II.2A)) and the second and third at the same side of (I.13B) (or (II.2B)) for an atom (molecule) which, in addition to its charge and electric (magnetic) dipole moment, may also possess an electric quadrupole moment (see chapter II for the proof and in particular the remark at the end of the appendix in that chapter), we obtain the following (simplified) forms for the equation of motion and the internal angular momentum equation for the atom (molecule) in an external electromagnetic field:

$$c \frac{\mathrm{d}}{\mathrm{d}\tau} (M^* U^\alpha) = e f^{\alpha\beta} U_\beta - (\partial_\gamma f^{\alpha\beta}) \, \mu_\beta{}^\gamma +$$

$$+ (\partial_\gamma \partial_\delta f^{\alpha\beta}) \, \chi_\beta{}^{\gamma\delta} - \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\alpha\gamma} \mu_\gamma{}^\beta - f^{\beta\gamma} \mu_\gamma{}^\alpha) \, U_\beta\} +$$

$$+ \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} [\{(\partial_\delta f^{\alpha\gamma}) \, \chi_\gamma{}^{\beta\delta} - (\partial_\delta f^{\beta\gamma})(\chi_\gamma{}^{\alpha\delta} + \chi_\gamma{}^{\delta\alpha})\} \, U_\beta] +$$

$$+ \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} (U^\alpha U_\beta f^{\beta\gamma} \mu_\gamma{}^\delta U_\delta) - \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} \{U^\alpha U_\beta (\partial_\delta f^{\beta\gamma}) \, \chi_\gamma{}^{\varepsilon\delta} U_\varepsilon\} +$$

$$+ \frac{1}{c^4} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[U_\beta \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\alpha\gamma} \chi_\gamma{}^{\delta\beta} - f^{\beta\gamma} \chi_\gamma{}^{\delta\alpha}) \, U_\delta\} \right] -$$

$$- \frac{1}{c^6} \frac{\mathrm{d}}{\mathrm{d}\tau} (U^\alpha U_\beta f^{\beta\gamma} \dot{\chi}_\gamma{}^{\delta\varepsilon} U_\delta U_\varepsilon), \quad (19\mathrm{A})$$

$$c \, \dot{\Omega}^{\alpha\beta} = \Delta^\alpha{}_\eta \Delta^\beta{}_\zeta (f^{\eta\gamma} \mu_\gamma{}^\zeta - f^{\zeta\gamma} \mu_\gamma{}^\eta) -$$

$$- \Delta^\alpha{}_\eta \Delta^\beta{}_\zeta \{(\partial_\delta f^{\eta\gamma}) \, \chi_\gamma{}^{\zeta\delta} - (\partial_\delta f^{\zeta\gamma}) \, \chi_\gamma{}^{\eta\delta}\} -$$

$$- \{\Delta^\alpha{}_\eta (\partial_\delta f^{\eta\gamma}) \chi_\gamma{}^{\delta\beta} - \Delta^\beta{}_\zeta (\partial_\delta f^{\zeta\gamma}) \, \chi_\gamma{}^{\delta\alpha}\} -$$

$$- \frac{1}{c^2} \Delta^\alpha{}_\eta \Delta^\beta{}_\zeta \frac{\mathrm{d}}{\mathrm{d}\tau} \{(f^{\eta\gamma} \chi_\gamma{}^{\delta\zeta} - f^{\zeta\gamma} \chi_\gamma{}^{\delta\eta}) \, U_\delta\}, \quad (19\mathrm{B})$$

using the properties:

$$\Delta^{\alpha}{}_{\beta} U^{\beta} = 0, \qquad \Delta^{\alpha}{}_{\gamma} \Delta^{\gamma}{}_{\beta} = \Delta^{\alpha}{}_{\beta}, \tag{20}$$

and eq. (2).

In the next section we shall derive, from eqs. (19A) and (19B), the atomic (molecular) energy-momentum four-tensor for a system of atoms (or molecules), taking into account the atomic (molecular) electric and magnetic dipole moments, as well as the electric quadrupole moments.

§ 3. *The atomic energy-momentum tensor.* We now consider a system, consisting of N atoms (molecules), numbered by the index k with electric dipole moments $\boldsymbol{\mu}_{(k)}$, magnetic dipole moments $\boldsymbol{\nu}_{(k)}$, electric quadrupole moments $q_{(k)}$ and negligible higher order moments. Just as in chapter I, § 4, we change the notation, denoting the rest mass of the k^{th} atom (molecule) by $m^*_{(k)}$ and its charge by $e_{(k)}$, while the tensors $\mu^{\alpha\beta}$, $\chi^{\alpha\beta\gamma}$, *etc.* become $\mu^{\alpha\beta}_{(k)}$, $\chi^{\alpha\beta\gamma}_{(k)}$, *etc.* For the time-space coordinates of the reference point describing the position of the k^{th} atom (molecule) we write $R^{\alpha}_{(k)}$ (instead of X^{α}), for its four-velocity $U^{\alpha}_{(k)}$, *etc.*, while the eigentime of atom k becomes $\tau_{(k)}$. Supposing that there are no external fields acting from outside on the system, the (external) field acting upon atom k is the sum of the partial fields $f^{\alpha\beta}_{(l)}$, due to the other atoms l $(\neq k)$:

$$f^{\alpha\beta}(R_{(k)}) = \sum_{l(\neq k)} f^{\alpha\beta}_{(l)}(R_{(k)}). \tag{21}$$

With these new notations, the equation of motion (19A) becomes:

$$\begin{aligned}
c \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} (m^*_{(k)} U^{\alpha}_{(k)}) &= e_{(k)} \sum_{l(\neq k)} f^{\alpha\beta}_{(l)}(R_{(k)})\, U_{(k)\beta} - \\
&- \sum_{l(\neq k)} \{\partial_{(k)\gamma} f^{\alpha\beta}_{(l)}(R_{(k)})\}\, \mu_{(k)\beta}{}^{\gamma} + \sum_{l(\neq k)} \{\partial_{(k)\gamma}\partial_{(k)\delta} f^{\alpha\beta}_{(l)}(R_{(k)})\}\, \chi_{(k)\beta}{}^{\gamma\delta} - \\
&- \frac{1}{c^2} \sum_{l(\neq k)} \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} [\{f^{\alpha\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\beta} - f^{\beta\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\alpha}\}\, U_{(k)\beta}] + \\
&+ \frac{1}{c^2} \sum_{l(\neq k)} \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} [[\{\partial_{(k)\delta} f^{\alpha\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\beta\delta} - \{\partial_{(k)\delta} f^{\beta\gamma}_{(l)}(R_{(k)})\} \cdot \\
&\cdot \{\chi_{(k)\gamma}{}^{\alpha\delta} + \chi_{(k)\gamma}{}^{\delta\alpha}\}]\, U_{(k)\beta}] + \\
&+ \frac{1}{c^4} \sum_{l(\neq k)} \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} \{U^{\alpha}_{(k)} U_{(k)\beta} f^{\beta\gamma}_{(l)}(R_{(k)}) \mu_{(k)\gamma}{}^{\delta} U_{(k)\delta}\} - \\
&- \frac{1}{c^4} \sum_{l(\neq k)} \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} [U^{\alpha}_{(k)} U_{(k)\beta} \{\partial_{(k)\delta} f^{\beta\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\varepsilon\delta} U_{(k)\varepsilon}] +
\end{aligned}$$

$$+\frac{1}{c^4}\sum_{l(\neq k)}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\cdot$$

$$\cdot\left[U_{(k)\beta}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\left[\{f_{(l)}^{\alpha\gamma}(R_{(k)})\,\chi_{(k)\gamma}{}^{\delta\beta}-f_{(l)}^{\beta\gamma}(R_{(k)})\,\chi_{(k)\gamma}{}^{\delta\alpha}\}\,U_{(k)\delta}\right]\right]-$$

$$-\frac{1}{c^6}\sum_{l(\neq k)}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{U_{(k)}^{\alpha}U_{(k)\beta}f_{(l)}^{\beta\gamma}(R_{(k)})\,\dot{\chi}_{(k)\gamma}{}^{\delta\varepsilon}U_{(k)\delta}U_{(k)\varepsilon}\},\tag{22A}$$

and the internal angular momentum balance equation (19B):

$$c\,\dot{\Omega}_{(k)}^{\alpha\beta}=\Delta_{(k)\eta}^{\alpha}\Delta_{(k)\zeta}^{\beta}\sum_{l(\neq k)}\{f_{(l)}^{\eta\gamma}(R_{(k)})\,\mu_{(k)\gamma}{}^{\zeta}-f_{(l)}^{\zeta\gamma}(R_{(k)})\,\mu_{(k)\gamma}{}^{\eta}\}-$$

$$-\Delta_{(k)\eta}^{\alpha}\Delta_{(k)\zeta}^{\beta}\sum_{l(\neq k)}[\{\partial_{(k)\delta}f_{(l)}^{\eta\gamma}(R_{(k)})\}\,\chi_{(k)\gamma}{}^{\zeta\delta}-\{\partial_{(k)\delta}f_{(l)}^{\zeta\gamma}(R_{(k)})\}\,\chi_{(k)\gamma}{}^{\eta\delta}]-$$

$$-\Delta_{(k)\eta}^{\alpha}\sum_{l(\neq k)}\{\partial_{(k)\delta}f_{(l)}{}^{\eta\gamma}(R_{(k)})\}\,\chi_{(k)\gamma}{}^{\delta\beta}+$$

$$+\Delta_{(k)\zeta}^{\beta}\sum_{l(\neq k)}\{\partial_{(k)\delta}f_{(l)}^{\zeta\gamma}(R_{(k)})\}\,\chi_{(k)\gamma}{}^{\delta\alpha}-\frac{1}{c^2}\Delta_{(k)\eta}^{\alpha}\,\Delta_{(k)\zeta}^{\beta}\sum_{l(\neq k)}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\cdot$$

$$\cdot\,[\{f_{(l)}^{\eta\gamma}(R_{(k)})\,\chi_{(k)\gamma}{}^{\delta\zeta}-f_{(l)}^{\zeta\gamma}(R_{(k)})\,\chi_{(k)\gamma}{}^{\delta\eta}\}\,U_{(k)\delta}].\tag{22B}$$

We shall now follow the same procedure as indicated in § 4 of chapter I (*cf.* ref. 1): we multiply both members of eq. (22A) and of eq. (22B) by the four-dimensional δ-function $\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\}$, integrate over $\tau_{(k)}$, and sum the result over k. We then obtain, after partial integration with respect to $\tau_{(k)}$, and with:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\,\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\}=-U_{(k)}^{\alpha}(\tau_{(k)})\,\partial_\alpha\delta^{(4)}\{R_{(k)}(\tau_{(k)})-R\},\tag{23}$$

the following results:

$$\partial_\beta\left\{c\sum_k\int_{-\infty}^{+\infty}m_{(k)}^{*}U_{(k)}^{\alpha}U_{(k)}^{\beta}\delta^{(4)}(R_{(k)}-R)\,\mathrm{d}\tau_{(k)}\right\}=$$

$$=\sum_{k,l,(k\neq l)}e_{(k)}\int_{-\infty}^{+\infty}f_{(l)}^{\alpha\beta}(R_{(k)})\,U_{(k)\beta}\delta^{(4)}(R_{(k)}-R)\,\mathrm{d}\tau_{(k)}-$$

$$-\sum_{k,l,(k\neq l)}\int_{-\infty}^{+\infty}\{\partial_{(k)\gamma}f_{(l)}^{\alpha\beta}(R_{(k)})\}\,\mu_{(k)\beta}{}^{\gamma}\delta^{(4)}(R_{(k)}-R)\,\mathrm{d}\tau_{(k)}+$$

$$+\sum_{k,l,(k\neq l)}\int_{-\infty}^{+\infty}\{\partial_{(k)\gamma}\partial_{(k)\delta}f_{(l)}^{\alpha\beta}(R_{(k)})\}\,\chi_{(k)\beta}{}^{\gamma\delta}\delta^{(4)}(R_{(k)}-R)\,\mathrm{d}\tau_{(k)}-$$

$$- \partial_\beta \Bigg[\frac{1}{c^2} \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} \{ f^{\alpha\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\delta} - f^{\delta\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\alpha} \}\, U_{(k)\delta} U^{\beta}_{(k)} \cdot$$

$$\cdot \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} - \frac{1}{c^2} \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} [\{\partial_{(k)\varepsilon} f^{\alpha\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\delta\varepsilon} -$$

$$- \{\partial_{(k)\varepsilon} f^{\delta\gamma}_{(l)}(R_{(k)})\} \{\chi_{(k)\gamma}{}^{\alpha\varepsilon} + \chi_{(k)\gamma}{}^{\varepsilon\alpha}\}]\, U_{(k)\delta} U^{\beta}_{(k)} \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} -$$

$$- \frac{1}{c^4} \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} U^{\alpha}_{(k)} U_{(k)\delta} f^{\delta\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\varepsilon} U_{(k)\varepsilon} U^{\beta}_{(k)} \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} +$$

$$+ \frac{1}{c^4} \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} U^{\alpha}_{(k)} U_{(k)\delta} \{\partial_{(k)\varepsilon} f^{\delta\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\eta\varepsilon} U_{(k)\eta} U_{(k)}{}^{\beta} \cdot$$

$$\cdot \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} - \frac{1}{c^4} \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} [\{ f^{\alpha\gamma}_{(l)}(R_{(k)})\ \chi_{(k)\gamma}{}^{\varepsilon\delta} -$$

$$- f^{\delta\gamma}_{(l)}(R_{(k)})\, \chi_{(k)\gamma}{}^{\varepsilon\alpha} \}\, U_{(k)\varepsilon}]\, U_{(k)\delta} U^{\beta}_{(k)} \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} +$$

$$+ \frac{1}{c^6} \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} U^{\alpha}_{(k)} U_{(k)\delta} f^{\delta\gamma}_{(l)}(R_{(k)})\ \dot{\chi}_{(k)\gamma}{}^{\varepsilon\eta} U_{(k)\varepsilon} U_{(k)\eta} U_{(k)}{}^{\beta} \cdot$$

$$\cdot \delta^{(4)}\ (R_{(k)} - R)\, \mathrm{d}\tau_{(k)} \Bigg], \tag{24A}$$

$$\partial_\gamma \Big\{ c \sum_k \int\limits_{-\infty}^{+\infty} \Omega^{\alpha\beta}_{(k)} U^{\gamma}_{(k)} \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} \Big\} =$$

$$= \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} \Delta^{\alpha}_{(k)\eta} \Delta^{\beta}_{(k)\zeta} \{ f^{\eta\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\zeta} - f^{\zeta\gamma}_{(l)}(R_{(k)})\, \mu_{(k)\gamma}{}^{\eta} \} \cdot$$

$$\cdot \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} - \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} \Delta^{\alpha}_{(k)\eta} \Delta^{\beta}_{(k)\zeta} [\{\partial_{(k)\delta} f^{\eta\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\zeta\delta} -$$

$$- \{\partial_{(k)\delta} f^{\zeta\gamma}_{(l)}(R_{(k)})\}\ \chi_{(k)\gamma}{}^{\eta\delta}]\ \delta^{(4)}(R_{(k)} - R)\ \mathrm{d}\tau_{(k)} -$$

$$- \sum_{k,l,(k\neq l)} \int\limits_{-\infty}^{+\infty} \Delta^{\alpha}_{(k)\eta} \{\partial_{(k)\delta} f^{\eta\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\delta\beta} \delta^{(4)}(R_{(k)} - R\, \mathrm{d}\tau_{(k)} +$$

$$+ \sum_{k,l,(k\neq l)} \int_{-\infty}^{+\infty} \Delta^{\beta}_{(k)\zeta}\{\partial_{(k)\delta} f^{\zeta\gamma}_{(l)}(R_{(k)})\}\, \chi_{(k)\gamma}{}^{\delta\alpha} \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)} -$$

$$- \frac{1}{c^2} \sum_{k,l,(k\neq l)} \int_{-\infty}^{+\infty} \Delta^{\alpha}_{(k)\eta} \Delta^{\beta}_{(k)\zeta} \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} [\{f^{\eta\gamma}_{(l)}(R_{(k)})\, \chi_{(k)\gamma}{}^{\delta\zeta} -$$

$$- f^{\zeta\gamma}_{(l)}(R_{(k)})\, \chi_{(k)\gamma}{}^{\delta\eta}\}\, U_{(k)\delta}]\, \delta^{(4)}(R_{(k)} - R)\, \mathrm{d}\tau_{(k)}. \tag{24B}$$

The left-hand side of eq. (24A) can now be written (see chapter I, § 4) as the divergence $\partial_\beta t^{*\alpha\beta}_{(\mathrm{m})}$ of the tensor:

$$t^{*\alpha\beta}_{(\mathrm{m})} \equiv \sum_k \rho^{*\prime}_{(k)} u^{\alpha}_{(k)} u^{\beta}_{(k)}, \tag{25}$$

where

$$\rho^{*\prime}_{(k)}(R) \equiv c \int_{-\infty}^{+\infty} m^{*}_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)} \tag{26}$$

(*cf.* (I.31)) is the rest mass density of atom k (*i.e.* the mass density in the momentary rest frame of inertia of this atom), and $u^{\alpha}_{(k)}(t)$ is equal to $U^{\alpha}_{(k)}(\tau_{(k)})$ considered as a function of the time t, instead of the eigentime $\tau_{(k)}$, (*cf.* (I.28)). Similarly we obtain for the left-hand side of eq. (24B) the expression:

$$\sum_k \partial_\gamma \{u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}, \tag{27}$$

where:

$$\sigma^{*\alpha\beta}_{(k)}(R) \equiv c \int_{-\infty}^{+\infty} \Omega^{\alpha\beta}_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)} \tag{28}$$

(*cf.* (I.32)) is the tensor which in the atomic rest frame represents the angular momentum density (the "atomic angular momentum density tensor"). With these results, and using the same method by which we have derived (I.28) from (I.27A), eqs. (24A) and (24B) become:

$$\partial_\beta t^{*\alpha\beta}_{(\mathrm{m})}(R) = \frac{1}{c} \sum_{k,l,(k\neq l)} f^{\alpha\beta}_{(l)}(R)\, j_{(k)\beta}(R) - \sum_{k,l,(k\neq l)} \{\partial_\gamma f^{\alpha\beta}_{(l)}(R)\}\, M_{(k)\beta}{}^{\gamma}(R) +$$

$$+ \sum_{k,l,(k\neq l)} \{\partial_\gamma \partial_\delta f^{\alpha\beta}_{(l)}(R)\}\, X_{(k)\beta}{}^{\gamma\delta}(R) - \partial_\beta \left[\frac{1}{c^2} \sum_{k,l,(k\neq l)} \{f^{\alpha\gamma}_{(l)}(R)\, M_{(k)\gamma}{}^{\delta}(R) - \right.$$

$$- f^{\delta\gamma}_{(l)}(R)\, M_{(k)\gamma}{}^{\alpha}(R)\}\, u_{(k)\delta}\, u_{(k)}{}^{\beta} - \frac{1}{c^2} \sum_{k,l,(k\neq l)} \{\partial_\varepsilon f^{\alpha\gamma}_{(l)}(R)\, X_{(k)\gamma}{}^{\delta\varepsilon}(R) -$$

$$- \{\partial_\varepsilon f^{\delta\gamma}_{(l)}(R)\}\{X_{(k)\gamma}{}^{\alpha\varepsilon}(R) + X_{(k)\gamma}{}^{\varepsilon\alpha}(R)\}\, u_{(k)\delta}\, u_{(k)}{}^{\beta} -$$

$$- \frac{1}{c^4} \sum_{k,l,(k\neq l)} u^{\alpha}_{(k)} u_{(k)\delta} f^{\delta\gamma}_{(l)}(R)\, M_{(k)\gamma}{}^{\varepsilon}(R)\, u_{(k)\varepsilon} u_{(k)}{}^{\beta} +$$

$$+ \frac{1}{c^4} \sum_{k,l,(k\neq l)} u^{\alpha}_{(k)} u_{(k)\delta} \{\partial_\varepsilon f^{\delta\gamma}_{(l)}(R)\}\, X_{(k)\gamma}{}^{\eta\varepsilon}(R)\, u_{(k)\eta} u^{\beta}_{(k)} -$$

$$- \frac{1}{c^4} \sum_{k,l,(k\neq l)} \partial_\eta [\{f^{\alpha\gamma}_{(l)}(R)\, X_{(k)\gamma}{}^{\varepsilon\delta}(R) - f^{\delta\gamma}_{(l)}(R)\, X_{(k)\gamma}{}^{\varepsilon\alpha}(R)\}\, u_{(k)\varepsilon} u^{\eta}_{(k)}] \cdot$$

$$\cdot u_{(k)\delta} u^{\beta}_{(k)} + \frac{1}{c^6} \sum_{k,l,(k\neq l)} u^{\alpha}_{(k)} u_{(k)\delta} f^{\delta\gamma}_{(l)}(R) \cdot$$

$$\left. \cdot \partial_\zeta \{X_{(k)\gamma}{}^{\varepsilon\eta}(R)\, u^{\zeta}_{(k)}\}\, u_{(k)\varepsilon} u_{(k)\eta} u_{(k)}{}^{\beta} \right], \tag{29A}$$

$$\sum_k \partial_\gamma \{\sigma^{*\alpha\beta}_{(k)}(R)\, u_{(k)}{}^{\gamma}\} =$$

$$= \sum_{k,l,(k\neq l)} \Delta^{\alpha}_{(k)\eta} \Delta^{\beta}_{(k)\zeta} \{f^{\eta\gamma}_{(l)}(R)\, M_{(k)\gamma}{}^{\zeta}(R) - f^{\zeta\gamma}_{(l)}(R)\, M_{(k)\gamma}{}^{\eta}(R)\} -$$

$$- \sum_{k,l,(k\neq l)} \Delta^{\alpha}_{(k)\eta} \Delta^{\beta}_{(k)\zeta} [\{\partial_\delta f^{\eta\gamma}_{(l)}(R)\}\, X_{(k)\gamma}{}^{\zeta\delta}(R) - \{\partial_\delta f^{\zeta\gamma}_{(l)}(R)\} \cdot$$

$$\cdot X_{(k)\gamma}{}^{\eta\delta}(R)] - \sum_{k,l,(k\neq l)} \Delta^{\alpha}_{(k)\eta} \{\partial_\delta f^{\eta\gamma}_{(l)}(R)\}\, X_{(k)\gamma}{}^{\delta\beta}(R) +$$

$$+ \sum_{k,l,(k\neq l)} \Delta^{\beta}_{(k)\zeta} \{\partial_\delta f^{\zeta\gamma}_{(l)}(R)\}\, X_{(k)\gamma}{}^{\delta\alpha}(R) -$$

$$- \frac{1}{c^2} \sum_{k,l,(k\neq l)} \Delta^{\alpha}_{(k)\eta} \Delta^{\beta}_{(k)\zeta}\, \partial_\varepsilon [\{f^{\eta\gamma}_{(l)}(R)\, X_{(k)\gamma}{}^{\delta\zeta}(R) -$$

$$- f^{\zeta\gamma}_{(l)}(R)\, X_{(k)\gamma}{}^{\delta\eta}(R)\}\, u_{(k)\delta} u^{\varepsilon}_{(k)}], \tag{29B}$$

where $j^{\alpha}_{(k)}(R)$ is the charge-current density four-vector of the k^{th} atom (molecule) as a whole:

$$j^{\alpha}_{(k)}(R) \equiv c e_{(k)} \int_{-\infty}^{+\infty} U^{\alpha}_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, d\tau_{(k)}, \tag{30}$$

and where we have introduced, in addition, the tensors:

$$M^{\alpha\beta}_{(k)}(R) \equiv \int_{-\infty}^{+\infty} \mu^{\alpha\beta}_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, d\tau_{(k)}, \tag{31}$$

and:

$$X^{\alpha\beta\gamma}_{(k)}(R) \equiv \int_{-\infty}^{+\infty} \chi^{\alpha\beta\gamma}_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, d\tau_{(k)}. \tag{32}$$

For the polarization tensor (*cf.* eq. (1)) of the k^{th} atom (molecule) we have, with eqs. (31) and (32):

$$m_{(k)}^{\alpha\beta}(R) = M_{(k)}^{\alpha\beta}(R) + \partial_\gamma X_{(k)}^{\alpha\beta\gamma}(R). \tag{33}$$

Differentiating the third, fifth and seventh term at the right-hand side of eq. (29A) partially with respect to R^α, using eq. (33) and finally applying the Maxwell-equations (*cf.* (I.33A) and (I.33B)):

$$\partial_\beta f_{(k)\alpha}{}^{\beta} = j_{(k)\alpha}/c + \partial_\beta m_{(k)\alpha}{}^{\beta}, \tag{34A}$$

$$\partial_\alpha f_{(k)\beta\gamma} + \partial_\beta f_{(k)\gamma\alpha} + \partial_\gamma f_{(k)\alpha\beta} = 0, \tag{34B}$$

we can derive the following result:

$$\begin{aligned}
\partial_\beta t_{(\mathrm{m})}^{*\alpha\beta} = -\partial_\beta \Bigg[& \sum_{k,l,(k\neq l)} f_{(l)}^{\alpha\gamma} h_{(k)\gamma}^{\beta} - \tfrac{1}{4} \sum_{k,l,(k\neq l)} \{f_{(l)\gamma\delta} f_{(k)}^{\gamma\delta}\} g^{\alpha\beta} - \\
& - \sum_{k,l,(k\neq l)} \{\partial_\delta f_{(l)}^{\alpha\gamma}\} X_{(k)\gamma}{}^{\delta\beta} + \frac{1}{c^2} \sum_{k,l,(k\neq l)} \{f_{(l)}^{\alpha\gamma} m_{(k)\gamma}{}^{\delta} - \\
& - f_{(l)}^{\delta\gamma} m_{(k)\gamma}{}^{\alpha}\} u_{(k)\delta} u_{(k)}{}^{\beta} - \frac{1}{c^2} \sum_{k,l,(k\neq l)} [\partial_\varepsilon \{f_{(l)}^{\alpha\gamma} X_{(k)\gamma}{}^{\delta\varepsilon} - \\
& - f_{(l)}^{\delta\gamma} X_{(k)\gamma}{}^{\alpha\varepsilon}\} - \{\partial_\varepsilon f_{(l)}^{\delta\gamma}\} X_{(k)\gamma}{}^{\varepsilon\alpha}] u_{(k)\delta} u_{(k)}^{\beta} - \\
& - \frac{1}{c^4} \sum_{k,l,(k\neq l)} u_{(k)}^{\alpha} u_{(k)\delta} f_{(l)}^{\delta\gamma} m_{(k)\gamma}{}^{\varepsilon} u_{(k)\varepsilon} u_{(k)}^{\beta} + \\
& + \frac{1}{c^4} \sum_{k,l,(k\neq l)} u_{(k)}^{\alpha} u_{(k)\delta} \partial_\varepsilon \{f_{(l)}^{\delta\gamma} X_{(k)\gamma}{}^{\eta\varepsilon}\} u_{(k)\eta} u_{(k)}^{\beta} - \\
& - \frac{1}{c^4} \sum_{k,l,(k\neq l)} \partial_\eta [\{f_{(l)}^{\alpha\gamma} X_{(k)\gamma}{}^{\varepsilon\delta} - f_{(l)}^{\delta\gamma} X_{(k)\gamma}{}^{\varepsilon\alpha}\} u_{(k)\varepsilon} u_{(k)}{}^{\eta}] u_{(k)\delta} u_{(k)}^{\beta} + \\
& + \frac{1}{c^6} \sum_{k,l,(k\neq l)} u_{(k)}^{\alpha} u_{(k)\delta} f_{(l)}^{\delta\gamma} \partial_\zeta \{X_{(k)\gamma}{}^{\varepsilon\eta} u_{(k)}^{\zeta}\} u_{(k)\varepsilon} u_{(k)\eta} u_{(k)}{}^{\beta} \Bigg] \equiv -\partial_\beta t_{(\mathrm{f})}^{*\alpha\beta},
\end{aligned} \tag{35}$$

where:

$$h_{(k)}^{\alpha\beta} \equiv f_{(k)}^{\alpha\beta} - m_{(k)}^{\alpha\beta} \tag{36}$$

(*cf.* (I.35)) and where $t_{(\mathrm{f})}^{*\alpha\beta}$ is an abbreviation for the expression between square brackets in the second member of (35); (the metric tensor $g^{\alpha\beta}$ has the elements $g^{00} = -1$, $g^{ii} = 1$ $(i = 1, 2, 3)$ and $g^{\alpha\beta} = 0$ $(\alpha \neq \beta)$). The tensor $t_{(\mathrm{m})}^{*\alpha\beta}$ may now be interpreted as the material part of the atomic (molecular) energy-momentum tensor of the system, and $t_{(\mathrm{f})}^{*\alpha\beta}$ as its field part. Eq. (35) then represents the conservation law of energy-momentum of the system at the atomic (molecular) level.

Eq. (29B) can be transformed, with the help of eq. (33), into:

$$\sum_k \partial_\gamma\{\sigma^{\star\alpha\beta}_{(k)} u_{(k)}{}^\gamma\} = \sum_{k,l,(k\neq l)} \Delta^\alpha_{(k)\eta}\Delta^\beta_{(k)\zeta}\{f^{\eta\gamma}_{(l)} m_{(k)\gamma}{}^\zeta - f^{\zeta\gamma}_{(l)} m_{(k)\gamma}{}^\eta\} -$$

$$- \sum_{k,l,(k\neq l)} \Delta^\alpha_{(k)\eta}\Delta^\beta_{(k)\zeta}\partial_\delta\{f^{\eta\gamma}_{(l)} X_{(k)\gamma}{}^{\zeta\delta} - f^{\zeta\gamma}_{(l)} X_{(k)\gamma}{}^{\eta\delta}\} -$$

$$- \sum_{k,l,(k\neq l)} \Delta^\alpha_{(k)\eta}(\partial_\delta f^{\eta\gamma}_{(l)})\, X_{(k)\gamma}{}^{\delta\beta} + \sum_{k,l,(k\neq l)} \Delta^\beta_{(k)\zeta}(\partial_\delta f^{\zeta\gamma}_{(l)})\, X_{(k)\gamma}{}^{\delta\alpha} -$$

$$- \frac{1}{c^2}\sum_{k,l,(k\neq l)} \Delta^\alpha_{(k)\eta}\Delta^\beta_{(k)\zeta}\partial_\varepsilon\{(f^{\eta\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\zeta} - f^{\zeta\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\eta})\, u_{(k)\delta} u^\varepsilon_{(k)}\}. \tag{37}$$

Now the left-hand side of this equation is equal to minus twice the antisymmetric part of the tensor (*cf.* (II.25)):

$$t^{\alpha\beta}_{(\mathrm{m})} \equiv t^{\star\alpha\beta}_{(\mathrm{m})} + \tfrac{1}{2}\sum_k \partial_\gamma\{u^\alpha_{(k)}\sigma^{\star\beta\gamma}_{(k)} + u^\beta_{(k)}\sigma^{\star\alpha\gamma}_{(k)} - u^\gamma_{(k)}\sigma^{\star\alpha\beta}_{(k)}\}, \tag{38}$$

which may again be interpreted as the atomic (molecular) energy-momentum tensor of the system, since its difference with the tensor $t^{\star\alpha\beta}_{(\mathrm{m})}$ is divergence-free, so that both $t^{\star\alpha\beta}_{(\mathrm{m})}$ and $t^{\alpha\beta}_{(\mathrm{m})}$ lead to the same physical results (*e.g.* the difference between the two tensors is irrelevant in the conservation law (35)).

The right-hand side of eq. (37) turns out to be equal to:

$$\{t^{\star\alpha\beta}_{(\mathrm{f})} - t^{\star\beta\alpha}_{(\mathrm{f})}\} - \sum_{k,l,(k\neq l)} \partial_\delta\{f^{\alpha\gamma}_{(l)} X_{(k)\gamma}{}^{\beta\delta} - f^{\beta\gamma}_{(l)} X_{(k)\gamma}{}^{\alpha\delta}\} -$$

$$- \frac{1}{c^2}\sum_{k,l,(k\neq l)} \partial_\varepsilon\{(f^{\alpha\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\beta} - f^{\beta\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\alpha})\, u_{(k)\delta} u_{(k)}{}^\varepsilon\},$$

or, in other words, equal to twice the antisymmetric part of the tensor:

$$\bar{t}^{\star\alpha\beta}_{(\mathrm{f})} \equiv t^{\star\alpha\beta}_{(\mathrm{f})} - \tfrac{1}{2}\sum_{k,l,(k\neq l)} \partial_\delta\{f^{\alpha\gamma}_{(l)} X_{(k)\gamma}{}^{\beta\delta} - f^{\beta\gamma}_{(l)} X_{(k)\gamma}{}^{\alpha\delta}\} -$$

$$- \frac{1}{2c^2}\sum_{k,l,(k\neq l)} \partial_\varepsilon\{(f^{\alpha\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\beta} - f^{\beta\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\alpha})\, u_{(k)\delta}\, u_{(k)}{}^\varepsilon\}. \tag{39}$$

Further, a procedure analogous to that of chapter I, § 4, can be applied in order to symmetrize the total energy-momentum tensor. We add the *symmetric* tensor:

$$s^{\alpha\beta}_{(\mathrm{f})} \equiv \tfrac{1}{2}\sum_{k,l,(k\neq l)} \partial_\delta\{f^{\alpha\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\beta} - f^{\delta\gamma}_{(l)} X_{(k)\gamma}{}^{\alpha\beta} + f^{\beta\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\alpha} - f^{\delta\gamma}_{(l)} X_{(k)\gamma}{}^{\beta\alpha}\} +$$

$$+ \frac{1}{2c^2}\sum_{k,l,(k\neq l)} \partial_\varepsilon\{(f^{\alpha\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\varepsilon} - f^{\varepsilon\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\alpha})\, u_{(k)\delta} u^\beta_{(k)} +$$

$$+ (f^{\beta\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\varepsilon} - f^{\varepsilon\gamma}_{(l)} X_{(k)\gamma}{}^{\delta\beta})\, u_{(k)\delta} u^\alpha_{(k)}\} \tag{40}$$

to $\bar{t}^{*\alpha\beta}_{(\mathrm{f})}$, eq. (39), and define the new tensor:

$$t^{\alpha\beta}_{(\mathrm{f})} \equiv \bar{t}^{*\alpha\beta}_{(\mathrm{f})} + s^{\alpha\beta}_{(\mathrm{f})} = t^{*\alpha\beta}_{(\mathrm{f})} - \tfrac{1}{2} \sum_{k,l,(k\neq l)} \partial_\delta \{ f^{\alpha\gamma}_{(l)} (X_{(k)\gamma}{}^{\beta\delta} - X_{(k)\gamma}{}^{\delta\beta}) -$$

$$- f^{\beta\gamma}_{(l)} (X_{(k)\gamma}{}^{\alpha\delta} + X_{(k)\gamma}{}^{\delta\alpha}) + f^{\delta\gamma}_{(l)} (X_{(k)\gamma}{}^{\alpha\beta} + X_{(k)\gamma}{}^{\beta\alpha}) \} -$$

$$- \frac{1}{2c^2} \sum_{k,l,(k\neq l)} \partial_\varepsilon \{ f^{\alpha\gamma}_{(l)} (X_{(k)\gamma}{}^{\delta\beta} u^{\varepsilon}_{(k)} - X_{(k)\gamma}{}^{\delta\varepsilon} u^{\beta}_{(k)})\, u_{(k)\delta} -$$

$$- f^{\beta\gamma}_{(l)} (X_{(k)\gamma}{}^{\delta\alpha} u^{\varepsilon}_{(k)} + X_{(k)\gamma}{}^{\delta\varepsilon} u^{\alpha}_{(k)})\, u_{(k)\delta} +$$

$$+ f^{\varepsilon\gamma}_{(l)} (X_{(k)\gamma}{}^{\delta\alpha} u^{\beta}_{(k)} + X_{(k)\gamma}{}^{\delta\beta} u^{\alpha}_{(k)})\, u_{(k)\delta} \}. \tag{41}$$

Since the difference between the tensors $t^{\alpha\beta}_{(\mathrm{f})}$ and $t^{*\alpha\beta}_{(\mathrm{f})}$ is divergence-free, we may also interpret $t^{\alpha\beta}_{(\mathrm{f})}$ as the atomic energy-momentum tensor of the field. Furthermore, it follows from the symmetry of $s^{\alpha\beta}_{(\mathrm{f})}$, that the antisymmetric parts of $t^{\alpha\beta}_{(\mathrm{f})}$ and $\bar{t}^{*\alpha\beta}_{(\mathrm{f})}$ are equal. The right-hand side of eq. (37) is, therefore, also equal to twice the antisymmetric part of the tensor $t^{\alpha\beta}_{(\mathrm{f})}$, and since the left-hand side of this equation is minus twice the antisymmetric part of $t^{\alpha\beta}_{(\mathrm{m})}$, we conclude, that it follows from the balance equation (37) for the intrinsic angular momentum of the atoms (molecules), that the total energy-momentum tensor $t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})}$ is *symmetric*. This symmetric tensor has the advantage that the conservation law of total energy-momentum:

$$\partial_\beta \{ t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})} \} = 0 \tag{42}$$

now implies also the conservation law of total angular momentum of the system plus the field (see *e.g.* ref. 6).

The explicit expression for the atomic field tensor $t^{\alpha\beta}_{(\mathrm{f})}$ is rather complicated. It follows from eqs. (41) and (35), that this tensor is the sum of two tensors:

$$t^{\alpha\beta}_{(\mathrm{f})} = t^{\alpha\beta}_{(\mathrm{f})\mathrm{I}} + t^{\alpha\beta}_{(\mathrm{f})\mathrm{II}}, \tag{43}$$

where:

$$t^{\alpha\beta}_{(\mathrm{f})\mathrm{I}} \equiv \sum_{k,l,(k\neq l)} [f^{\alpha\gamma}_{(l)} h^{\beta}_{(k)\gamma} - \tfrac{1}{4} \{ f_{(l)\gamma\delta} f^{\gamma\delta}_{(k)} \} g^{\alpha\beta} + \frac{1}{c^2} \{ f^{\alpha\gamma}_{(l)} m_{(k)\gamma\delta} -$$

$$- m^{\alpha\gamma}_{(k)} f_{(l)\gamma\delta} \} u^{\delta}_{(k)} u^{\beta}_{(k)} - \frac{1}{c^4} \{ u^{\gamma}_{(k)} f_{(l)\gamma\delta} m^{\delta}_{(k)\varepsilon} u^{\varepsilon}_{(k)} \} u^{\alpha}_{(k)} u^{\beta}_{(k)}] \tag{44A}$$

is of the same form as the field tensor in chapter I for pure dipole atoms (molecules) and depends *only* on quantities appearing in the atomic field equations, whereas the tensor:

$$
\begin{aligned}
t^{\alpha\beta}_{(\mathrm{f})\mathrm{II}} \equiv \sum_{k,l,(k\neq l)} & [-\{\partial_\delta f^{\alpha\gamma}_{(l)}\}\, X_{(k)\gamma^{\delta\beta}} - \tfrac{1}{2}\partial_\delta\{f^{\alpha\gamma}_{(l)}(X_{(k)\gamma^{\beta\delta}} - X_{(k)\gamma^{\delta\beta}}) - \\
& - f^{\beta\gamma}_{(l)}(X_{(k)\gamma^{\alpha\delta}} + X_{(k)\gamma^{\delta\alpha}}) + f^{\delta\gamma}_{(l)}(X_{(k)\gamma^{\alpha\beta}} + X_{(k)\gamma^{\beta\alpha}})\} + \\
& + \frac{1}{c^2}\{\partial_\varepsilon f^{\delta\gamma}_{(l)}\}\, X_{(k)\gamma^{\varepsilon\alpha}} u_{(k)\delta} u^{\beta}_{(k)} - \\
& - \frac{1}{c^2}\,\partial_\varepsilon\{f^{\alpha\gamma}_{(l)} X_{(k)\gamma^{\delta\varepsilon}} - f^{\delta\gamma}_{(l)} X_{(k)\gamma^{\alpha\varepsilon}}\} u_{(k)\delta} u^{\beta}_{(k)} - \\
& - \frac{1}{2c^2}\,\partial_\varepsilon\{f^{\alpha\gamma}_{(l)}(X_{(k)\gamma^{\delta\beta}} u^{\varepsilon}_{(k)} - X_{(k)\gamma^{\delta\varepsilon}} u^{\beta}_{(k)})\, u_{(k)\delta} - \\
& - f^{\beta\gamma}_{(l)}(X_{(k)\gamma^{\delta\alpha}} u^{\varepsilon}_{(k)} + X_{(k)\gamma^{\delta\varepsilon}} u^{\alpha}_{(k)})\, u_{(k)\delta} + f^{\varepsilon\gamma}_{(l)}(X_{(k)\gamma^{\delta\alpha}} u^{\beta}_{(k)} + \\
& + X_{(k)\gamma^{\delta\beta}} u^{\alpha}_{(k)}) u_{(k)\delta}\} + \frac{1}{c^4}\{\partial_\varepsilon(f^{\delta\gamma}_{(l)} X_{(k)\gamma^{\eta\varepsilon}})\, u_{(k)\delta} u_{(k)\eta}\}\, u^{\alpha}_{(k)} u^{\beta}_{(k)} - \\
& - \frac{1}{c^4}\,\partial_\eta\{(f^{\alpha\gamma}_{(l)} X_{(k)\gamma^{\varepsilon\delta}} - f^{\delta\gamma}_{(l)} X_{(k)\gamma^{\varepsilon\alpha}})\, u_{(k)\varepsilon} u_{(k)}{}^{\eta}\}\, u_{(k)\delta} u^{\beta}_{(k)} + \\
& + \frac{1}{c^6}\{u_{(k)\delta} f^{\delta\gamma}_{(l)} \partial_\zeta(X_{(k)\gamma^{\varepsilon\eta}} u^{\zeta}_{(k)})\, u_{(k)\varepsilon} u_{(k)\eta}\}\, u^{\alpha}_{(k)} u^{\beta}_{(k)}]
\end{aligned}
\tag{44B}
$$

is not expressible in those quantities. In contrast with $t^{\alpha\beta}_{(\mathrm{f})\mathrm{I}}$, the tensor $t^{\alpha\beta}_{(\mathrm{f})\mathrm{II}}$ contains only derivatives of field quantities, *etc.*, with respect to time and space coordinates.

If we had considered higher order electric and magnetic moments of the atoms (molecules) than electric (magnetic) dipole moments and electric quadrupole moments, we should have obtained further contributions to the field tensor $t^{\alpha\beta}_{(\mathrm{f})}$. The explicit evaluation of these rather complicated contributions will not be given here, since we will not need them for future purposes.

APPENDIX

Proof of formula (12). We shall first calculate the integral:

$$
\int_{-\infty}^{+\infty} g(\tau')\, \delta''\{h(\tau')\}\, \mathrm{d}\tau', \qquad \text{with} \qquad \delta''(y) = \mathrm{d}^2\delta(y)/\mathrm{d}y^2, \tag{A1}
$$

where $h(\tau')$ is monotonically increasing. Introducing:

$$
y = h(\tau'), \qquad \tau' = \tau'(y), \tag{A2}
$$

we get:

$$\int_{-\infty}^{+\infty} g(\tau')\,\delta''\{h(\tau')\}\,\mathrm{d}\tau' = \int_{h(-\infty)}^{h(+\infty)} g\{\tau'(y)\}\,\frac{\mathrm{d}^2\delta(y)}{\mathrm{d}y^2}\,\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}\,\mathrm{d}y =$$

$$= \int_{h(-\infty)}^{h(+\infty)} \left[\frac{\mathrm{d}^2 g\{\tau'(y)\}}{\mathrm{d}y^2}\,\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y} + 2\,\frac{\mathrm{d}g\{\tau'(y)\}}{\mathrm{d}y}\,\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2} + \right.$$

$$\left. + g\{\tau'(y)\}\,\frac{\mathrm{d}^3\tau'(y)}{\mathrm{d}y^3}\right]\delta(y)\,\mathrm{d}y =$$

$$= \left[\frac{\mathrm{d}^2 g\{\tau'(y)\}}{\mathrm{d}y^2}\,\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y} + 2\,\frac{\mathrm{d}g\{\tau'(y)\}}{\mathrm{d}y}\,\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2} + \right.$$

$$\left. + g\{\tau'(y)\}\,\frac{\mathrm{d}^3\tau'(y)}{\mathrm{d}y^3}\right]_{y=0}. \tag{A3}$$

We now have:

$$\frac{\mathrm{d}g\{\tau'(y)\}}{\mathrm{d}y} = g'\{\tau'(y)\}\,\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}, \tag{A4}$$

and:

$$\frac{\mathrm{d}^2 g\{\tau'(y)\}}{\mathrm{d}y^2} = g''\{\tau'(y)\}\left\{\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}\right\}^2 + g'\{\tau'(y)\}\,\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2}, \tag{A5}$$

so that eq. (A3) becomes:

$$\int_{-\infty}^{+\infty} g(\tau')\,\delta''\{h(\tau')\}\,\mathrm{d}\tau' =$$

$$= \left[g''\{\tau'(y)\}\left\{\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}\right\}^3 + 3g'\{\tau'(y)\}\,\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2}\cdot\right.$$

$$\left. \cdot\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y} + g\{\tau'(y)\}\,\frac{\mathrm{d}^3\tau'(y)}{\mathrm{d}y^3}\right]_{y=0}. \tag{A6}$$

If we now differentiate the identity:

$$h\{\tau'(y)\} \equiv y \tag{A7}$$

(which follows from eq. (A2)) three times with with respect to y, we obtain the following relations:

$$h'\{\tau'(y)\}\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}=1, \tag{A8}$$

$$h''\{\tau'(y)\}\left\{\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}\right\}^2+h'\{\tau'(y)\}\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2}=0, \tag{A9}$$

$$h'''\{\tau'(y)\}\left\{\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}\right\}^3+$$

$$+\,3h''\{\tau'(y)\}\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2}\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}+h'\{\tau'(y)\}\frac{\mathrm{d}^3\tau'(y)}{\mathrm{d}y^3}=0, \tag{A10}$$

from which it follows that:

$$\frac{\mathrm{d}\tau'(y)}{\mathrm{d}y}=\frac{1}{h'\{\tau'(y)\}}, \tag{A11}$$

$$\frac{\mathrm{d}^2\tau'(y)}{\mathrm{d}y^2}=-\frac{h''\{\tau'(y)\}}{[h'\{\tau'(y)\}]^3}, \tag{A12}$$

$$\frac{\mathrm{d}^3\tau'(y)}{\mathrm{d}y^3}=-\frac{h'''\{\tau'(y)\}\,h'\{\tau'(y)\}-3[h''\{\tau'(y)\}]^2}{[h'\{\tau'(y)\}]^5}. \tag{A13}$$

Substituting eqs. (A11)–(A13) into the right-hand side of eq. (A6), we get:

$$\int_{-\infty}^{+\infty} g(\tau')\,\delta''\{h(\tau')\}\,\mathrm{d}\tau'=\left[\frac{g''\{\tau'(y)\}}{[h'\{\tau'(y)\}]^3}-3\,\frac{g'\{\tau'(y)\}h''\{\tau'(y)\}}{[h'\{\tau'(y)\}]^4}-\right.$$

$$\left.-\,\frac{g\{\tau'(y)\}\,h'''\{\tau'(y)\}}{[h'\{\tau'(y)\}]^4}+3\,\frac{g\{\tau'(y)\}[h''\{\tau'(y)\}]^2}{[h'\{\tau'(y)\}]^5}\right]_{y=0}. \tag{A14}$$

If we now suppose that τ is the (only) value of τ' for which $h(\tau')=0$, it follows from eq. (A2) that $\tau'(0)=\tau$, so that eq. (A14) becomes:

$$\int_{-\infty}^{+\infty} g(\tau')\,\delta''\{h(\tau')\}\,\mathrm{d}\tau'=\left[\frac{g''(\tau')}{\{h'(\tau')\}^3}-3\,\frac{g'(\tau')\,h''(\tau')}{\{h'(\tau')\}^4}-\right.$$

$$\left.-\,\frac{g(\tau')\,h'''(\tau')}{\{h'(\tau')\}^4}+3\,\frac{g(\tau')\{h''(\tau')\}^2}{\{h'(\tau')\}^5}\right]_{\tau'=\tau}. \tag{A15}$$

Taking the special case that:

$$h(\tau')=-\frac{1}{c}\,\{X^\alpha(\tau')-X^\alpha(\tau)\}\,U_\alpha(\tau), \tag{A16}$$

we have:

$$h'(\tau') = -\frac{1}{c} U^\alpha(\tau') U_\alpha(\tau), \tag{A17}$$

$$h''(\tau') = -\frac{1}{c} \dot{U}^\alpha(\tau') U_\alpha(\tau), \tag{A18}$$

$$h'''(\tau') = -\frac{1}{c} \ddot{U}^\alpha(\tau') U_\alpha(\tau). \tag{A19}$$

The function $h(\tau')$, (A16), has indeed its (only) zero just for the value τ of the variable τ', and since furthermore:

$$U^\alpha(\tau) U_\alpha(\tau) = -c^2, \tag{A20}$$

and consequently:

$$\dot{U}^\alpha(\tau) U_\alpha(\tau) = 0, \tag{A21}$$

$$\ddot{U}^\alpha(\tau) U_\alpha(\tau) = -\dot{U}^\alpha(\tau) \dot{U}_\alpha(\tau), \tag{A22}$$

we find for the function (A16) the following properties:

$$[h'(\tau')]_{\tau'=\tau} = c, \tag{A23}$$

$$[h''(\tau')]_{\tau'=\tau} = 0, \tag{A24}$$

$$[h'''(\tau')]_{\tau'=\tau} = \frac{1}{c} \dot{U}^\alpha(\tau) \dot{U}_\alpha(\tau). \tag{A25}$$

Substituting eqs. (A23)–(A25), together with (A16), into (A15), we obtain:

$$\int_{-\infty}^{+\infty} g(\tau')\, \delta''\left[-\frac{1}{c}\{X^\alpha(\tau') - X^\alpha(\tau)\} U_\alpha(\tau)\right] d\tau' =$$

$$= \left[\frac{1}{c^3} g''(\tau') - \frac{1}{c^5} g(\tau') \dot{U}^\alpha(\tau) \dot{U}_\alpha(\tau)\right]_{\tau'=\tau}, \tag{A26}$$

which is just formula (12) of the present chapter.

REFERENCES

1) Vlieger, J., Physica **37** (1967) 165.
2) Vlieger, J. and Emid, S., Physica **41** (1969) 368.
3) Møller, C., Ann. Inst. H. Poincaré **11** (1949–50) 251.
4) Rosenfeld, L., Theory of electrons, North-Holland Publishing Company (Amsterdam, 1951).
5) de Groot, S. R. and Suttorp, L. G., Physica **31** (1965) 1713.
6) Møller, C., The theory of relativity, Oxford University Press (London, 1952).

Chapter IV

THE CLASSICAL EQUATIONS OF MOTION WITH RADIATION REACTION AND THE ENERGY-MOMENTUM TENSOR FOR ELECTRIC AND MAGNETIC DIPOLE ATOMS AND MOLECULES

Synopsis

The classical equations of motion for electric and magnetic dipole atoms (or molecules) in an external electromagnetic field of force, treated in chapters I and II on the basis of Møller's theory of the relativistic dynamics of systems with an internal angular momentum, are extended to the case that the reaction of radiation on the atoms is taken into account. To this end Møller's theory, which is valid only for *finite* systems (total energy-momentum tensor zero outside a finite region in space for arbitrary fixed time), is modified in order to be applicable to the case of *radiating* atoms (or molecules). Dirac's method of splitting retarded fields covariantly into self-parts (half sum of retarded and advanced fields) and "radiative" parts (half difference of retarded and advanced fields) is applied to the sub-atomic fields. It is proved that the sub-atomic self-force density can be written as minus the divergence of a symmetrical four-tensor, which is zero outside the atomic system and which, added to the sub-atomic material energy-momentum tensor, may be interpreted as the total energy-momentum tensor of the *finite* atomic system. With the help of the latter tensor the atomic mass, intrinsic angular momentum and centre of gravity are defined, using Møller's theory. The influence of the "radiative" part of the field on the centre of gravity motion of the atoms and the change of their intrinsic angular momentum is then analysed. The equation of motion and the intrinsic angular momentum balance equation, obtained for radiating charged dipole atoms are used in order to derive the relativistic atomic energy-momentum tensor for a system consisting of a large number of these atoms. In contrast with the tensors in the previous chapters, this tensor is no longer symmetrical.

The treatment of the present chapter could be extended to include the case in which the atoms also possess electric quadrupole moments.

§ 1. *Introduction.* In the first chapter, denoted hereafter by I, we have treated the classical equations of motion for electric and magnetic dipole atoms (or molecules) † in an external electromagnetic field of force[1]). The treatment was based on Møller's theory[2]) of the relativistic dynamics of *finite* systems †† with an internal angular momentum in an arbitrary (non-gravitational) field of force. In chapter II[3]) we have simplified Møller's equations of motion in the case of electric and magnetic dipole atoms, by showing that certain terms, which contain an unphysical trembling motion can be eliminated. In this way differential equations of the usual second order type were obtained for the description of the motion of the atom *as a whole* in a given external electromagnetic field of force. In chapter III[4]) we have extended the theory developed in I and II to the case, that the atoms carry, in addition to their electric and magnetic dipole moments, also electric quadrupole moments. In all previous chapters (I–III) radiative effects were neglected throughout, as was also the case in the work of de Groot and Suttorp[5]). In our treatment this was a consequence of the fact that Moller's theory[2]) is only valid for finite systems, *i.e.* for systems whose energy-momentum tensor is zero outside a finite region in space for arbitrary fixed times. It is well known that the energy-momentum tensor of the electromagnetic field of a (classically) radiating atom does not possess the latter property.

In the present chapter we shall take into account radiative effects in the theory. To this end we shall have to modify Møller's theory[2]). We shall apply Dirac's method[6]) of splitting covariantly (retarded) fields into self parts (half sum of retarded and advanced fields) and so-called "radiative" parts ‡ (half difference of retarded and advanced fields) to the sub-atomic fields. This leads to a corresponding splitting of the electromagnetic force density, acting upon the atom, into two parts. In section 2 we shall prove that the sub-atomic self-force density can be written as minus the divergence of a symmetric four-tensor, which is zero outside the atomic system. This tensor may be added to the sub-atomic material energy-momentum tensor, and the sum may be interpreted as the total energy-momentum tensor of the *finite* atomic system. With the help of the latter tensor, we shall define in section 3 the atomic mass, intrinsic angular momentum and centre of gravity, using Møller's theory[2]). Next, the influence of the remaining ("radiative") part of the field on the motion of this centre of gravity and on the change of this intrinsic angular momentum is

† In the present chapter we shall always write shortly atoms instead of atoms (or molecules), if this does not lead to ambiguity.

†† Systems with an energy-momentum tensor, which is zero outside a finite region in space for arbitrary fixed times.

‡ For reasons, which will become clear below, the word "radiative" is not properly chosen here, and we shall therefore avoid it as much as possible.

analysed. In this way the equation of motion and the intrinsic angular momentum balance equation for classical radiating charged dipole atoms are derived (in simplified form; see chapter II). These equations are an extension of the equations of motion, derived by Dirac[6]) for classical spinless point electrons, and analogous to those obtained by Bhabha and Harish-Chandra[7]) for point particles, possessing higher electric and magnetic multipole moments. In contrast with the results obtained by those authors, our results are, however, free from (self-energy) divergencies, since we have used Møller's theory[2]) as the starting point, which is a theory for *extended* particles. Furthermore we neglect radiative effects due to the accelerations of the centres of gravity of the atoms ("Bremsstrahlung").

The equations of motion and the intrinsic angular momentum balance equation, derived in section 3 are used in section 4 in order to derive the relativistic energy-momentum tensor for a system, consisting of a large number of radiating dipole atoms. In contrast with the atomic energy-momentum tensors in the previous chapters (I–III), the one obtained here is no longer symmetrical, as this need not be the case for non-closed systems[8]).

For simplicity's sake we have treated only the case for electric and magnetic dipole atoms, neglecting electric quadrupole moments. Using the results obtained in chapter III, we could, however, generalize the theory of the present chapter to include also atomic electric quadrupole moments. This will be indicated at the end of this chapter (section 5).

§ 2. *Modified theory of Møller.* We consider a classical radiating atomic system, moving in an external electromagnetic field. Since this system may not be considered as *finite* in the sense that its total (sub-atomic) energy-momentum tensor vanishes sufficiently rapidly† outside the system for arbitrary times, we cannot apply Møller's theory in the form as presented in ref. 2. We shall therefore have to modify that theory, in order to apply it to the case of radiating atoms.

We shall use Dirac's covariant definition[6]) of the self part and the "radiation" part of a retarded field. The self part is defined as half the sum of the retarded and the advanced electromagnetic field, produced by the (atomic) system ††:

$$f^{\alpha\beta}_{(+)}(x) = \tfrac{1}{2}\{f^{\alpha\beta}_{(\mathrm{ret})}(x) + f^{\alpha\beta}_{(\mathrm{adv})}(x)\}, \tag{1}$$

† The field part of this tensor is proportional to R^{-2} (R is distance from the centre of the atom) at large distances for the *radiating* atom, since it is a quadratic function of the electric and magnetic fields [see *e.g.* formula (27) of ref. 9], which are themselves proportional to R^{-1} at large distances. The surface integral over a large sphere in space is therefore non-vanishing in this case, which makes impossible the application of Møller's theory[2]).

†† where not explained, we use the same notation and symbols as in I–III.

whereas the "radiation" part is taken as half the difference of these two fields†:

$$f^{\alpha\beta}_{(-)}(x) = \tfrac{1}{2}\{f^{\alpha\beta}_{(\mathrm{ret})}(x) - f^{\alpha\beta}_{(\mathrm{adv})}(x)\}. \tag{2}$$

These fields may be written in terms of the four-potentials $A^{\alpha}_{(+)}(x)$ and $A^{\alpha}_{(-)}(x)$:

$$f^{\alpha\beta}_{(\pm)}(x) = \partial^{\alpha} A^{\beta}_{(\pm)}(x) - \partial^{\beta} A^{\alpha}_{(\pm)}(x), \tag{3}$$

with

$$A^{\alpha}_{(+)}(x) = \frac{1}{4\pi c} \int^{\infty} i^{\alpha}(x')\, \delta\{(x-x')^2\}\, \mathrm{d}^{(4)}x', \tag{4}$$

and

$$A^{\alpha}_{(-)}(x) = \frac{1}{4\pi c} \int^{\infty} i^{\alpha}(x')\, \Delta(x-x')\, \mathrm{d}^{(4)}x', \tag{5}$$

where

$$\Delta(x-x') \equiv \delta\{(x-x')^2\} \frac{x^0 - x'^0}{|x^0 - x'^0|} \tag{6}$$

and where $i^{\alpha}(x)$ is the (sub-atomic) charge-current density four-vector, (for derivation, see *e.g.* ref. 10). We shall, for the time being, make no assumptions about the form of $i^{\alpha}(x)$††, other than that it is zero outside a thin tube in space-time (of the order of the atomic dimensions for arbitrary fixed time).

The force density of the electromagnetic field produced by the atom may also be split into a "plus" part

$$f^{\alpha}_{(+)}(x) = \frac{1}{c} f^{\alpha\beta}_{(+)}(x)\, i_{\beta}(x) = \frac{1}{c} \{\partial^{\alpha} A^{\beta}_{(+)}(x) - \partial^{\beta} A^{\alpha}_{(+)}(x)\}\, i_{\beta}(x), \tag{7}$$

and a "minus" part

$$f^{\alpha}_{(-)}(x) = \frac{1}{c} f^{\alpha\beta}_{(-)}(x)\, i_{\beta}(x) = \frac{1}{c} \{\partial^{\alpha} A^{\beta}_{(-)}(x) - \partial^{\beta} A^{\alpha}_{(-)}(x)\}\, i_{\beta}(x). \tag{8}$$

We first consider the expression (7). It follows from the law of conservation of charge:

$$\partial^{\alpha} i_{\alpha}(x) = 0, \tag{9}$$

† Dirac's definition of the "radiation" part is in fact two times the right-hand side of eq. (2). As it will appear below, however, both definitions do not lead in the equations of motion merely to terms, describing the damping by *radiation*, *i.e.* transport of electromagnetic momentum and energy at very large distances from the atoms. We shall therefore avoid the use of the word "radiation" in connection with the fields $f^{\alpha\beta}_{(-)}(x)$ or $2f^{\alpha\beta}_{(-)}(x)$.

†† We may have a discrete as well as a continuous charge-current distribution in space.

that:

$$f^{\alpha}_{(+)}(x) = -\frac{1}{c}\,\partial_\beta\{A^{\alpha}_{(+)}(x)\,i^\beta(x)\} + \frac{1}{c}\,\{\partial^\alpha A^\beta_{(+)}(x)\}\,i_\beta(x). \tag{10}$$

The first term at the right-hand side of this equation has the form of the divergence of a tensor, which vanishes outside the tube in space–time within which $i^\alpha(x) \neq 0$, and we shall now prove, using the fact that the δ-function in eq. (4) is invariant for the interchange of the variables x and x', that also the last term of eq. (10) can be written in that form. Substituting eq. (4) into the last term of eq. (10), we get:

$$\frac{1}{c}\{\partial^\alpha A^\beta_{(+)}(x)\}\,i_\beta(x) = \frac{1}{2\pi c^2}\int^{\infty} i_\beta(x)\,i^\beta(x')(x^\alpha - x'^\alpha)\,\delta'\{(x-x')^2\}\,\mathrm{d}^{(4)}x'$$

$$= \frac{1}{2\pi c^2}\int^{\infty}\!\!\int^{\infty} i_\beta(x'')\,i^\beta(x')(x''^\alpha - x'^\alpha)\,\delta'\{(x''-x')^2\}\,\delta^{(4)}(x''-x)\,\mathrm{d}^{(4)}x'\,\mathrm{d}^{(4)}x'', \tag{11}$$

with: $\delta'(y) \equiv \mathrm{d}\delta(y)/\mathrm{d}y$. Interchanging the variables x' and x'' in the last member of (11), we find:

$$\frac{1}{c}\{\partial^\alpha A^\beta_{(+)}(x)\}\,i_\beta(x) = -\frac{1}{2\pi c^2}\int^{\infty}\!\!\int^{\infty} i_\beta(x'')\,i^\beta(x')(x''^\alpha - x'^\alpha)$$

$$\times\,\delta'\{(x''-x')^2\}\,\delta^{(4)}(x'-x)\,\mathrm{d}^{(4)}x'\,d^{(4)}x'', \tag{12}$$

where we have used the above-mentioned property of the δ-function appearing in (4). If we now take half the sum of the expressions (11) and (12), we obtain:

$$\frac{1}{c}\{\partial^\alpha A^\beta_{(+)}(x)\}\,i_\beta(x) = \frac{1}{4\pi c^2}\int^{\infty}\!\!\int^{\infty} i_\beta(x'')\,i^\beta(x')(x''^\alpha - x'^\alpha)$$

$$\times\,\delta'\{(x''-x')^2\}\{\delta^{(4)}(x''-x) - \delta^{(4)}(x'-x)\}\,\mathrm{d}^{(4)}x'\,\mathrm{d}^{(4)}x'', \tag{13}$$

and it is easily seen that the right-hand side of this equation is the divergence of a tensor, if we develop the function $\delta^{(4)}(x'-x)$ into a Taylor series around the point x'':

$$\frac{1}{c}\{\partial^\alpha A^\beta_{(+)}(x)\}\,i_\beta(x) = \frac{1}{4\pi c^2}\int^{\infty}\!\!\int^{\infty} i_\beta(x'')\,i^\beta(x')(x''^\alpha - x'^\alpha)$$

$$\times\,\delta'\{(x''-x')^2\}\Bigg[\delta^{(4)}(x''-x) - \sum_{n=0}^{\infty}\frac{1}{n!}(x''^{\lambda_1} - x'^{\lambda_1})$$

$$\times \ldots (x''^{\lambda_n} - x'^{\lambda_n})\,\partial_{\lambda_1}\ldots\partial_{\lambda_n}\delta^{(4)}(x''-x)\Bigg]\,\mathrm{d}^{(4)}x'\,\mathrm{d}^{(4)}x''$$

$$= -\frac{1}{4\pi c^2} \sum_{n=1}^{\infty} \frac{1}{n!}\, \partial_{\lambda_1} \dots \partial_{\lambda_n}$$

$$\times \int\!\!\int^{\infty\,\infty} (x''^{\lambda_1} - x'^{\lambda_1}) \dots (x''^{\lambda_n} - x'^{\lambda_n})\, i_\beta(x'')\, i^\beta(x')$$

$$\times (x''^\alpha - x'^\alpha)\, \delta'\{(x'' - x')^2\}\, \delta^{(4)}(x'' - x)\, \mathrm{d}^{(4)}x'\, \mathrm{d}^{(4)}x''$$

$$= -\frac{1}{4\pi c^2}\, \partial_{\lambda_1} \Bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\, \partial_{\lambda_2} \dots \partial_{\lambda_{n+1}} \int^{\infty} (x^\alpha - x'^\alpha)$$

$$\times (x^{\lambda_1} - x'^{\lambda_1}) \dots (x^{\lambda_{n+1}} - x'^{\lambda_{n+1}})\, i_\beta(x)\, i^\beta(x')\, \delta'\{(x - x')^2\}\, \mathrm{d}^{(4)}x' \Bigg], \quad (14)$$

where we have finally integrated over x''. Making the substitutions: $\lambda_1 \to \beta, \lambda_2 \to \lambda_1, \dots, \lambda_{n+1} \to \lambda_n$ and $\beta \to \gamma$ in the last member of (14), we get:

$$\frac{1}{c}\{\partial^\alpha A^\beta_{(+)}(x)\}\, i_\beta(x) = -\frac{1}{4\pi c^2}\, \partial_\beta \Bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\, \partial_{\lambda_1} \dots \partial_{\lambda_n}$$

$$\times \int^{\infty} (x^{\lambda_1} - x'^{\lambda_1}) \dots (x^{\lambda_n} - x'^{\lambda_n})\, i_\gamma(x)\, i^\gamma(x')$$

$$\times (x^\alpha - x'^\alpha)(x^\beta - x'^\beta)\, \delta'\{(x - x')^2\}\, \mathrm{d}^{(4)}x' \Bigg], \quad (15)$$

which is indeed the divergence of a four-tensor, vanishing outside the above-mentioned tube in space–time.

Substituting this result into the right-hand side of eq. (10) and using eq. (4), we find that the sub-atomic self-force density $f^\alpha_{(+)}(x)$ can be written as minus the diverence of a tensor $T'^{\alpha\beta}_{(+)}(x)$:

$$f^\alpha_{(+)}(x) = -\partial_\beta T'^{\alpha\beta}_{(+)}(x), \quad (16)$$

with

$$T'^{\alpha\beta}_{(+)}(x) = \frac{1}{4\pi c^2} \int^{\infty} i^\alpha(x')\, i^\beta(x)\, \delta\{(x - x')^2\}\, \mathrm{d}^{(4)}x'$$

$$+ \frac{1}{4\pi c^2} \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\, \partial_{\lambda_1} \dots \partial_{\lambda_n} \int^{\infty} (x^{\lambda_1} - x'^{\lambda_1}) \dots (x^{\lambda_n} - x'^{\lambda_n})$$

$$\times i_\gamma(x)\, i^\gamma(x')(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)\, \delta'\{(x - x')^2\}\, \mathrm{d}^{(4)}x'. \quad (17)$$

Using this result, one could now write the sub-atomic energy-momentum law:

$$\partial_\beta T^{\alpha\beta}_{(\mathrm{m})}(x) = f^\alpha_{(+)}(x) + f^\alpha_{(-)}(x) + f^\alpha(x), \quad (18)$$

where $T^{\alpha\beta}_{(\mathrm{m})}(x)$ is the sub-atomic material energy-momentum tensor and $f^\alpha(x)$ is the force density of the external electromagnetic field, in the following form:

$$\partial_\beta\{T^{\alpha\beta}_{(\mathrm{m})}(x) + T'^{\alpha\beta}_{(+)}(x)\} = f^{\alpha}_{(-)}(x) + f^{\alpha}(x), \tag{19}$$

and define

$$T'^{\alpha\beta}(x) \equiv T^{\alpha\beta}_{(\mathrm{m})}(x) + T'^{\alpha\beta}_{(+)}(x) \tag{20}$$

as the total sub-atomic energy-momentum tensor. This tensor would, however, not be symmetric, which is rather inconvenient, since then the energy-momentum law does not directly imply the angular momentum balance in the form of eq. (2) of chapter I. We therefore try to add a *divergence-free* tensor $B^{\alpha\beta}(x)$ to $T'^{\alpha\beta}(x)$, such that

$$T^{\alpha\beta}(x) \equiv T'^{\alpha\beta}(x) + B^{\alpha\beta}(x) \tag{21}$$

is *symmetric*. This tensor $T^{\alpha\beta}(x)$ can then be interpreted as the sub-atomic energy-momentum tensor, and used for the derivation of the equations of motion of the atoms.

Since the tensor $T^{\alpha\beta}_{(\mathrm{m})}(x)$ will be supposed symmetric, we have in fact to symmetrize only the tensor $T'^{\alpha\beta}_{(+)}(x)$. We shall use a method analogous to that of chapters I and III for the symmetrization of the atomic energy-momentum tensors. As:

$$T^{\alpha\beta}_{(+)}(x) \equiv T'^{\alpha\beta}_{(+)}(x) + B^{\alpha\beta}(x) = T'^{\beta\alpha}_{(+)}(x) + B^{\beta\alpha}(x), \tag{22}$$

we find for the antisymmetric part of $B^{\alpha\beta}(x)$, using (17):

$$\begin{aligned} B^{\alpha\beta}_{(\mathrm{a})}(x) &\equiv \tfrac{1}{2}\{B^{\alpha\beta}(x) - B^{\beta\alpha}(x)\} \\ &= -\frac{1}{8\pi c^2}\int^{\infty}\{i^{\alpha}(x')\, i^{\beta}(x) - i^{\beta}(x')\, i^{\alpha}(x)\}\, \delta\{(x-x')^2\}\, \mathrm{d}^{(4)}x'. \end{aligned} \tag{23}$$

The tensor $B^{\alpha\beta}(x)$ is therefore of the form:

$$\begin{aligned} B^{\alpha\beta}(x) = B^{\alpha\beta}_{(\mathrm{s})}(x) &- \frac{1}{8\pi c^2}\int^{\infty}\{i^{\alpha}(x')\, i^{\beta}(x) - i^{\beta}(x')\, i^{\alpha}(x)\} \\ &\times \delta\{(x-x')^2\}\, \mathrm{d}^{(4)}x', \end{aligned} \tag{24}$$

where $B^{\alpha\beta}_{(\mathrm{s})}(x)$ is the symmetric part of the tensor $B^{\alpha\beta}(x)$. Since:

$$\partial_\beta B^{\alpha\beta}(x) = 0, \tag{25}$$

our problem is to find a symmetric tensor $B^{\alpha\beta}_{(\mathrm{s})}(x)$, of which the divergence is given by:

$$\begin{aligned} \partial_\beta B^{\alpha\beta}_{(\mathrm{s})}(x) = \frac{1}{8\pi c^2}\,\partial_\beta &\int^{\infty}\{i^{\alpha}(x')\, i^{\beta}(x) - i^{\beta}(x')\, i^{\alpha}(x)\} \\ &\times \delta\{(x-x')^2\}\, \mathrm{d}^{(4)}x'. \end{aligned} \tag{26}$$

Introducing again the function $\delta^{(4)}(x'' - x)$ in the right-hand side of (26), we get:

$$\partial_\beta B^{\alpha\beta}_{(s)}(x) = \frac{1}{8\pi c^2}\,\partial_\beta \int\!\!\int\limits^{\infty\,\infty} \{i^\alpha(x')\, i^\beta(x'') - i^\beta(x')\, i^\alpha(x'')\}$$

$$\times\, \delta\{(x'' - x')^2\}\, \delta^{(4)}(x'' - x)\, \mathrm{d}^{(4)}\, x'\, \mathrm{d}^{(4)}\, x''$$

$$= \frac{1}{8\pi c^2}\,\partial_\beta \int\!\!\int\limits^{\infty\,\infty} \{i^\alpha(x')\, i^\beta(x'')\, \delta\{(x'' - x')^2\}$$

$$\times\, \{\delta^{(4)}\,(x'' - x) - \delta^{(4)}\,(x' - x)\}\, \mathrm{d}^{(4)}\, x'\, \mathrm{d}^{(4)}\, x''$$

$$= -\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma \Bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\,\partial_{\lambda_n}$$

$$\times \int\!\!\int\limits^{\infty\,\infty} (x''^{\lambda_1} - x'^{\lambda_1})\,\ldots\,(x''^{\lambda_n} - x'^{\lambda_n})\, i^\alpha(x')\, i^\beta(x'')$$

$$\times\, (x''^\gamma - x'^\gamma)\, \delta\{(x'' - x')^2\}\, \delta^{(4)}\,(x'' - x)\, \mathrm{d}^{(4)}\, x'\, \mathrm{d}^{(4)}\, x''\Bigg]$$

$$= -\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma \Bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\,\partial_{\lambda_n}$$

$$\times \int\limits^{\infty} (x^{\lambda_1} - x'^{\lambda_1})\,\ldots\,(x^{\lambda_n} - x'^{\lambda_n})\, i^\alpha(x')\, i^\beta(x)$$

$$\times\, (x^\gamma - x'^\gamma)\, \delta\{(x - x')^2\}\, \mathrm{d}^{(4)}\, x'\Bigg], \tag{27}$$

where we have used a procedure analogous to that in eqs. (11)–(15). If we now interchange the indices β and γ in the last member of (27), we find that:

$$\partial_\beta B^{\alpha\beta}_{(s)}(x) = -\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma \Bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\,\partial_{\lambda_n}$$

$$\times \int\limits^{\infty} (x^{\lambda_1} - x'^{\lambda_1})\,\ldots\,(x^{\lambda_n} - x'^{\lambda_n})\, i^\alpha(x')(x^\beta - x'^\beta)\, i^\gamma(x)$$

$$\times\, \delta\{(x - x')^2\}\, \mathrm{d}^{(4)}x'\Bigg]$$

$$= -\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma \Bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\,\partial_{\lambda_n} \int\limits^{\infty} (x^{\lambda_1} - x'^{\lambda_1})\,\ldots\,(x^{\lambda_n} - x'^{\lambda_n})$$

$$\times\, \{i^\alpha(x')(x^\beta - x'^\beta) + i^\beta(x')(x^\alpha - x'^\alpha)\}\, i^\gamma(x)\, \delta\{(x - x')^2\}\, \mathrm{d}^{(4)}x'\Bigg], \tag{28}$$

where in the last member we have added the integral:

$$-\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\partial_{\lambda_n}\int^{\infty}(x^{\lambda_1}-x'^{\lambda_1})\right.$$

$$\left.\ldots(x^{\lambda_n}-x'^{\lambda_n})(x^\alpha-x'^\alpha)\,i^\beta(x')\,i^\gamma(x)\,\delta\{(x-x')^2\}\,\mathrm{d}^{(4)}x'\right], \tag{29}$$

which is proved to be zero in appendix I.

Since the right-hand side of (28) is the divergence of a *symmetric* tensor, we may put:

$$B^{\alpha\beta}_{(\mathrm{s})}(x) = -\frac{1}{8\pi c^2}\,\partial_\gamma\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\partial_{\lambda_n}\int^{\infty}(x^{\lambda_1}-x'^{\lambda_1})\ldots(x^{\lambda_n}-x'^{\lambda_n})\right.$$

$$\left.\times\{i^\alpha(x')(x^\beta-x'^\beta)+i^\beta(x')(x^\alpha-x'^\alpha)\}\,i^\gamma(x)\,\delta\{(x-x')^2\}\,\mathrm{d}^{(4)}x'\right]. \tag{30}$$

We then find with (24) for $B^{\alpha\beta}(x)$:

$$B^{\alpha\beta}(x) = -\frac{1}{8\pi c^2}\int^{\infty}\{i^\alpha(x')\,i^\beta(x)-i^\beta(x')\,i^\alpha(x)\}\,\delta\{(x-x')^2\}\,\mathrm{d}^{(4)}x'$$

$$-\frac{1}{8\pi c^2}\,\partial_\gamma\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\partial_{\lambda_n}\int^{\infty}(x^{\lambda_1}-x'^{\lambda_1})\ldots(x^{\lambda_n}-x'^{\lambda_n})\right.$$

$$\left.\times\{i^\alpha(x')(x^\beta-x'^\beta)+i^\beta(x')(x^\alpha-x'^\alpha)\}\,i^\gamma(x)\,\delta\{(x-x')^2\}\,\mathrm{d}^{(4)}x'\right]. \tag{31}$$

If this divergence-free tensor is added to the tensor $T'^{\alpha\beta}_{(+)}(x)$, eq. (17), we finally obtain the tensor $T^{\alpha\beta}_{(+)}(x)$, as follows from eq. (22):

$$T^{\alpha\beta}_{(+)}(x) = \frac{1}{8\pi c^2}\int^{\infty}\{i^\alpha(x')\,i^\beta(x)+i^\beta(x')\,i^\alpha(x)\}\,\delta\{(x-x')^2\}\,\mathrm{d}^{(4)}\,x'$$

$$+\frac{1}{4\pi c^2}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\partial_{\lambda_n}\int^{\infty}(x^{\lambda_1}-x'^{\lambda_1})\ldots(x^{\lambda_n}-x'^{\lambda_n})$$

$$\times\, i_\gamma(x)\,i^\gamma(x')(x^\alpha-x'^\alpha)(x^\beta-x'^\beta)\,\delta'\{(x-x')^2\}\,\mathrm{d}^{(4)}x'$$

$$-\frac{1}{8\pi c^2}\,\partial_\gamma\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\partial_{\lambda_1}\ldots\partial_{\lambda_n}\int^{\infty}(x^{\lambda_1}-x'^{\lambda_1})\ldots(x^{\lambda_n}-x'^{\lambda_n})\right.$$

$$\left.\times\{i^\alpha(x')(x^\beta-x'^\beta)+i^\beta(x')(x^\alpha-x'^\alpha)\}\,i^\gamma(x)\,\delta\{(x-x')^2\}\,\mathrm{d}^{(4)}x'\right]. \tag{32}$$

The symmetric tensor:

$$T^{\alpha\beta}(x) = T^{\alpha\beta}_{(\mathrm{m})}(x) + T^{\alpha\beta}_{(+)}(x) \tag{33}$$

[*cf.* eqs. (29), (21) and (22)] will now be interpreted as the total sub-atomic energy-momentum tensor, and the energy-momentum law (19) takes the form:

$$\partial_\beta T^{\alpha\beta}(x) = f^\alpha_{(-)}(x) + f^\alpha(x). \tag{34}$$

This equation can then be used as the starting point for the derivation of the equation of motion and the intrinsic angular momentum balance equation of a radiating atom (*cf.* refs. 1 and 2). It should be stressed once more, that the results obtained in this section are *independent* of the special form of the sub-atomic charge-current density.

§3. *Equations of motion.* The atomic mass is defined by:

$$M^*(\tau) \equiv \frac{1}{c^4} \int^{\infty} T^{\alpha\beta}(x)\, U_\beta(\tau)\, U_\alpha(\tau)\, \delta\left[-\frac{1}{c}\{x^\gamma - X^\gamma(\tau)\}\, U_\gamma(\tau)\right] \mathrm{d}^{(4)}x. \tag{35}$$

We use the same definition and notation here as in chapter I. The intrinsic angular momentum tensor is given by [*cf.* (I.8)]:

$$\Omega^{\alpha\beta}(\tau) \equiv -\frac{1}{c^2} \int^{\infty} [\{x^\alpha - X^\alpha(\tau)\}\, T^{\beta\gamma}(x) - \{x^\beta - X^\beta(\tau)\}\, T^{\alpha\gamma}(x)]$$

$$\times\, U_\gamma(\tau)\, \delta\left[-\frac{1}{c}\{x^\delta - X^\delta(\tau)\}\, U_\delta(\tau)\right] \mathrm{d}^{(4)}x \tag{36}$$

and the (pseudo) centre of gravity $X^\alpha(\tau)$ of the atom is defined by the relation [*cf.* (I.11)]:

$$\Omega^{\alpha\beta} U_\beta = 0. \tag{37}$$

In appendix II we shall prove that the mass, the intrinsic angular momentum and the centre of gravity of the atom, found with the tensor (33), if we retain only terms up to order c^{-2} (c = velocity of light), are of the usual forms obtained in this approximation [see also the results of de Groot and Suttorp[5]]. Furthermore it appears that higher order corrections (due to the retardation of the self-force) are in general very small, so that the tensor (33) may indeed be considered as a correct expression for the energy-momentum tensor on the sub-atomic level.

We now turn to the first term of the right-hand side of eq. (34), *i.e.* to the expression for the force density of the minus part of the field produced by the atom. This term is given by eq. (8), together with eqs. (5) and (6):

$$f^\alpha_{(-)}(x) = \frac{1}{c} \{\partial^\alpha A^\beta_{(-)}(x) - \partial^\beta A^\alpha_{(-)}(x)\}\, i_\beta(x). \tag{38}$$

From now on we shall only consider the case of charged electric and magnetic dipole point atoms. We then have for $i^\alpha(x)$ [see (I.16) and (I.17[†])]:

$$i^\alpha(x) = ce \int_{-\infty}^{+\infty} U^\alpha(\tau)\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau$$
$$+ c \int_{-\infty}^{+\infty} \mu^{\alpha\beta}(\tau)\, \partial_\beta\, \delta^{(4)}\{X(\tau) - x\}\, \mathrm{d}\tau. \tag{39}$$

If we substitute eq. (39) with x' instead of x into eq. (5) and perform (partial) integrations with respect to x', we get the following expression for the potential $A^\alpha_{(-)}(x)$ of a charged dipole point atom:

$$A^\alpha_{(-)}(x) = \frac{e}{4\pi} \int_{-\infty}^{+\infty} \delta[\{x - X(\tau)\}^2]\, s\{x^0 - X^0(\tau)\}\, U^\alpha(\tau)\, \mathrm{d}\tau$$
$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta'[\{x - X(\tau)\}^2]\, s\{x^0 - X^0(\tau)\}\, \mu^{\alpha\beta}(\tau)\, \{x_\beta - X_\beta(\tau)\}\, \mathrm{d}\tau, \tag{40}$$

where $s\{x^0 - X^0(\tau)\} = (x^0 - X^0(\tau))/|x^0 - X^0(\tau)|$. Terms with derivatives of the sign-function s are zero. This is best proved by calculating the potential $A^\alpha_{(-)}(x)$ first for an extended dipole atom, (instead of the second integral of (39) one then has $c\partial_\beta m^{\alpha\beta}(x)$, where $m^{\alpha\beta}$ is non-singular). In this way one easily finds, that the term with the derivative of the function s is equal to zero. One can then finally consider the limiting case of point dipole atoms. By differentiation of (40) we get for the minus field tensor:

$$f^{\alpha\beta}_{(-)}(x) = \frac{e}{2\pi} \int_{-\infty}^{+\infty} \delta'[\{x - X(\tau)\}^2]\, s\{x^0 - X^0(\tau)\}$$
$$\times [\{x^\alpha - X^\alpha(\tau)\}\, U^\beta(\tau) - \{x^\beta - X^\beta(\tau)\}\, U^\alpha(\tau)]\, \mathrm{d}\tau$$
$$- \frac{1}{\pi} \int_{-\infty}^{+\infty} \delta'[\{x - X(\tau)\}^2]\, s\{x^0 - X^0(\tau)\}\, \mu^{\alpha\beta}(\tau)\, \mathrm{d}\tau$$
$$+ \frac{1}{\pi} \int_{-\infty}^{+\infty} \delta''[\{x - X(\tau)\}^2]\, s\{x^0 - X^0(\tau)\}[\{x^\alpha - X^\alpha(\tau)\}\, \mu^{\beta\gamma}(\tau)$$
$$- \{x^\beta - X^\beta(\tau)\}\, \mu^{\alpha\gamma}(\tau)]\, \{x_\gamma - X_\gamma(\tau)\}\, \mathrm{d}\tau. \tag{41}$$

With eq. (41) for the field tensor $f^{\alpha\beta}_{(-)}(x)$ and eq. (39) for the sub-atomic charge-current density $i^\alpha(x)$ substituted into eq. (38), we obtain the force density $f^\alpha_{(-)}(x)$ of the minus part of the field, produced by the atom.

[†] Formulae of chapters I, II and III will be denoted by (I.1), (II.1), (III.1), *etc.*

We can now derive the equation of motion and the intrinsic angular momentum balance equation for radiating charged dipole atoms by the same method as in I, sections 2 and 3, starting with eq. (34). Since later on the minus part of the electromagnetic field, produced by an atom, will appear to be *non-singular* at the position of this atom (*cf.* also ref. 7), it may be treated in the same way as the external part of the electromagnetic field. We therefore obtain the following result for the equation of motion [*cf.* (I.25A)]:

$$\begin{aligned} c\,\frac{\mathrm{d}}{\mathrm{d}\tau}(M^{*}U^{\alpha}) + \frac{\mathrm{d}}{\mathrm{d}\tau}(\Omega^{\alpha\beta}\dot{U}_{\beta}) = \Big[& e f^{\alpha\beta}_{(-)}U_{\beta} - (\partial_{\gamma}f^{\alpha\beta}_{(-)})\,\mu_{\beta}{}^{\gamma} \\ & - \frac{1}{c^{2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\{(f^{\alpha\gamma}_{(-)}\mu_{\gamma}{}^{\beta} - f^{\beta\gamma}_{(-)}\mu_{\gamma}{}^{\alpha})\,U_{\beta}\} + \frac{1}{c^{4}}\frac{\mathrm{d}}{\mathrm{d}\tau}(U^{\alpha}U_{\beta}f^{\beta\gamma}_{(-)}\mu_{\gamma}{}^{\delta}U_{\delta})\Big] \\ & + \Big[e f^{\alpha\beta}U_{\beta} - (\partial_{\gamma}f^{\alpha\beta})\,\mu_{\beta}{}^{\gamma} - \frac{1}{c^{2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\{(f^{\alpha\gamma}\mu_{\gamma}{}^{\beta} - f^{\beta\gamma}\mu_{\gamma}{}^{\alpha})\,U_{\beta}\} \\ & + \frac{1}{c^{4}}\frac{\mathrm{d}}{\mathrm{d}\tau}(U^{\alpha}U_{\beta}f^{\beta\gamma}\mu_{\gamma}{}^{\delta}U_{\delta})\Big], \end{aligned} \tag{42A}$$

where $f^{\alpha\beta}_{(-)} = f^{\alpha\beta}_{(-)}\{X(\tau)\}$ and $f^{\alpha\beta} = f^{\alpha\beta}\{X(\tau)\}$, and for the intrinsic angular momentum balance equation we get [*cf.* (I.25B)]:

$$\begin{aligned} c\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}\dot{\Omega}^{\varepsilon\zeta} = {} & \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}(f^{\varepsilon\gamma}_{(-)}\mu_{\gamma}{}^{\zeta} - f^{\zeta\gamma}_{(-)}\mu_{\gamma}{}^{\varepsilon}) \\ & + \Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}\,(f^{\varepsilon\zeta}\mu_{\gamma}{}^{\zeta} - f^{\zeta\gamma}\mu_{\gamma}{}^{\varepsilon}), \end{aligned} \tag{42B}$$

where:

$$\Delta^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \frac{1}{c^{2}}\,U^{\alpha}U_{\beta}. \tag{43}$$

Except for the terms, containing the tensor $f^{\alpha\beta}_{(-)}$, the eqs. (42) are of exactly the same form as the eqs. (I.25). The terms containing $f^{\alpha\beta}_{(-)}$ describe the "radiative" effects.

Let us first investigate, whether the eqs. (42) can be simplified in an analogous way as was done in chapter II with the eqs. (II.7). We note that the field $f^{\alpha\beta}_{(-)}$ and its derivatives have finite values on the world line of the representative point of the atom, *i.e.* its centre of gravity (*cf.* also ref. 7). The main point in the simplification of the equations of motion in chapter II was the validity of the strong inequality [*cf.* (II.16)]:

$$\left|\frac{e}{M^{*}}\,\Omega^{\alpha\beta}\right| \ll |\mu^{\varepsilon\zeta}\Delta^{\alpha}{}_{\varepsilon}\Delta^{\beta}{}_{\zeta}|, \tag{44}$$

and one easily finds, going through the details of the derivation, that also in the present case this inequality simplifies the eqs. (42). But there was a second point in the iteration procedure of II, namely that we neglected terms of third and higher order in the internal variables. *E.g.* we neglected terms of the type (II.18). In the present case these terms become:

$$-\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{1}{M^*}\Omega^{\alpha\beta}(\partial_\delta f_{(-)\beta}{}^\gamma + \partial_\delta f_\beta{}^\gamma)\,\mu_\gamma{}^\delta\right\}, \tag{45A}$$

and:

$$-\frac{1}{M^*c^2}(U^\alpha\Omega^{\beta\gamma} - U^\beta\Omega^{\alpha\gamma})(\partial_\varepsilon f_{(-)\gamma}{}^\delta + \partial_\varepsilon f_\gamma{}^\delta)\,\mu_\delta{}^\varepsilon. \tag{45B}$$

As the derivatives of the field tensor $f^{\alpha\beta}_{(-)}$ remain finite on the world line of the atom, producing this field, terms of the forms (45A) and (45B) are again of the third and fourth order and can therefore also be dropped from the equations of motion and the angular momentum equation. We may, therefore, conclude that eqs. (42) can be written in the following simplified forms [*cf.* (II.17)]:

$$\begin{aligned}
&c\frac{\mathrm{d}}{\mathrm{d}\tau}(M^*U^\alpha)\\
&\quad=\Big[ef^{\alpha\beta}_{(-)}U_\beta - (\partial_\gamma f^{\alpha\beta}_{(-)})\,\mu_\beta{}^\gamma - \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\{(f^{\alpha\gamma}_{(-)}\mu_\gamma{}^\beta - f^{\beta\gamma}_{(-)}\mu_\gamma{}^\alpha)\,U_\beta\}\\
&\quad+\frac{1}{c^4}\frac{\mathrm{d}}{\mathrm{d}\tau}(U^\alpha U_\beta f^{\beta\gamma}_{(-)}\mu_\gamma{}^\delta U_\delta)\Big] + \Big[ef^{\alpha\beta}U_\beta - (\partial_\gamma f^{\alpha\beta})\,\mu_\beta{}^\gamma\\
&\quad-\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\{(f^{\alpha\gamma}\mu_\gamma{}^\beta - f^{\beta\gamma}\mu_\gamma{}^\alpha)\,U_\beta\} + \frac{1}{c^4}\frac{\mathrm{d}}{\mathrm{d}\tau}(U^\alpha U_\beta f^{\beta\gamma}\mu_\gamma{}^\delta U_\delta)\Big],
\end{aligned} \tag{46A}$$

and:

$$c\dot{\Omega}^{\alpha\beta} = \Delta^\alpha{}_\varepsilon\Delta^\beta{}_\zeta(f^{\varepsilon\gamma}_{(-)}\mu_\gamma{}^\zeta - f^{\zeta\gamma}_{(-)}\mu_\gamma{}^\varepsilon) + \Delta^\alpha{}_\varepsilon\Delta^\beta{}_\zeta(f^{\varepsilon\gamma}\mu_\gamma{}^\zeta - f^{\zeta\gamma}\mu_\gamma{}^\varepsilon). \tag{46B}$$

For further evaluation of the "radiative" contributions to the equations of motion we have to calculate formula (41) for the field tensor $f^{\alpha\beta}_{(-)}$ at $x = X(\tau)$:

$$\begin{aligned}
f^{\alpha\beta}_{(-)}\{X(\tau)\} &= \frac{e}{2\pi}\int_{-\infty}^{+\infty}\delta'[\{X(\tau) - X(\tau')\}^2]\,s\{X^0(\tau) - X^0(\tau')\}\\
&\quad\times[\{X^\alpha(\tau) - X^\alpha(\tau')\}\,U^\beta(\tau') - \{X^\beta(\tau) - X^\beta(\tau')\}\,U^\alpha(\tau')]\,\mathrm{d}\tau'\\
&\quad-\frac{1}{\pi}\int_{-\infty}^{+\infty}\delta'[\{X(\tau) - X(\tau')\}^2]\,s\{X^0(\tau) - X^0(\tau')\}\,\mu^{\alpha\beta}(\tau')\,\mathrm{d}\tau'\\
&\quad+\frac{1}{\pi}\int_{-\infty}^{+\infty}\delta''[\{X(\tau) - X(\tau')\}^2]\,s\{X^0(\tau) - X^0(\tau')\}\\
&\quad\times[\{X^\alpha(\tau) - X^\alpha(\tau')\}\,\mu^{\beta\gamma}(\tau') - \{X^\beta(\tau) - X^\beta(\tau')\}\,\mu^{\alpha\gamma}(\tau')]\\
&\quad\times\{X_\gamma(\tau) - X_\gamma(\tau')\}\,\mathrm{d}\tau',
\end{aligned} \tag{47A}$$

and also its derivative at that point:

$$\partial_\gamma f^{\alpha\beta}_{(-)}\{X(\tau)\} = \frac{e}{2\pi}\int_{-\infty}^{+\infty} \delta'[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}$$

$$\times [\delta_\gamma{}^\alpha U^\beta(\tau') - \delta_\gamma{}^\beta U^\alpha(\tau')]\, \mathrm{d}\tau'$$

$$+\frac{e}{\pi}\int_{-\infty}^{+\infty} \delta''[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}$$

$$\times [\{X^\alpha(\tau)-X^\alpha(\tau')\}U^\beta(\tau')-\{X^\beta(\tau)-X^\beta(\tau')\}U^\alpha(\tau')]\{X_\gamma(\tau)-X_\gamma(\tau')\}\mathrm{d}\tau'$$

$$-\frac{2}{\pi}\int_{-\infty}^{+\infty} \delta''[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}\, \mu^{\alpha\beta}(\tau')\{X_\gamma(\tau)-X_\gamma(\tau')\}\mathrm{d}\tau'$$

$$+\frac{1}{\pi}\int_{-\infty}^{+\infty} \delta''[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}$$

$$\times [\{X^\alpha(\tau)-X^\alpha(\tau')\}\mu^\beta{}_\gamma(\tau') - \{X^\beta(\tau)-X^\beta(\tau')\}\mu^\alpha{}_\gamma(\tau')]\, \mathrm{d}\tau'$$

$$+\frac{1}{\pi}\int_{-\infty}^{+\infty} \delta''[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}$$

$$\times [\delta_\gamma{}^\alpha \mu^{\beta\delta}(\tau') - \delta_\gamma{}^\beta \mu^{\alpha\delta}(\tau')]\{X_\delta(\tau)-X_\delta(\tau')\}\, \mathrm{d}\tau'$$

$$+\frac{2}{\pi}\int_{-\infty}^{+\infty} \delta'''[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}[\{X^\alpha(\tau)-X^\alpha(\tau')\}\mu^{\beta\delta}(\tau')$$

$$-\{X^\beta(\tau)-X^\beta(\tau')\}\mu^{\alpha\delta}(\tau')]\{X_\delta(\tau)-X_\delta(\tau')\}\{X_\gamma(\tau)-X_\gamma(\tau')\}\,\mathrm{d}\tau', \tag{47B}$$

where $\delta_\gamma{}^\alpha$ is the four-dimensional Kronecker symbol. (Note that terms with derivatives of s always disappear.)

A method for the rather lengthy and tedious calculations† of the expressions (47A) and (47B) is given in appendix III. We shall note here only some general features of the calculations. We have to evaluate terms [call them $G^{(n)}(\tau)$] of the form:

$$G^{(n)}(\tau) = \int_{-\infty}^{+\infty} g(\tau', \tau)\, \delta^{(n)}[\{X(\tau)-X(\tau')\}^2]\, s\{X^0(\tau)-X^0(\tau')\}\, \mathrm{d}\tau', \tag{48}$$

† The calculations of (47B) are shortened by the fact that $\partial_\gamma f^{\alpha\beta}_{(-)}$ only appears contracted twice with the antisymmetric tensor $\mu_\beta{}^\gamma$ in the equation of motion, so that we do not have to calculate symmetric terms in β and γ in the expressions (47B).

where $\delta^{(n)}(y) = \mathrm{d}^n\delta(y)/\mathrm{d}y^n$ $(n = 1, 2, \ldots)$ and $g(\tau', \tau)$ may be a function of $U^\alpha(\tau'), \mu^{\alpha\beta}(\tau')$ and $\{X^\alpha(\tau) - X^\alpha(\tau')\}$. First we calculate the expression:

$$G(x) = \int_{-\infty}^{+\infty} g(\tau', x)\, \delta[\{x - X(\tau')\}^2]\, s\{x^0 - X^0(\tau')\}\, \mathrm{d}\tau' \tag{49}$$

for a nearby point x [of $X(\tau)$], which may conveniently be chosen such that:

$$\{x^\alpha - X^\alpha(\tau)\}\, U_\alpha(\tau) = 0. \tag{50}$$

Denoting the small separation between x and $X(\tau)$ in the momentary rest frame of the atom by ε (> 0), formula (49) can be written as:

$$G(\tau; \varepsilon) = \int_{-\infty}^{+\infty} g(\tau', \tau)\, \delta[\{X(\tau) - X(\tau')\}^2 + \varepsilon^2]\, s\{X^0(\tau) - X^0(\tau')\}\, \mathrm{d}\tau', \tag{51}$$

where we have neglected terms of higher order in ε, which is immaterial, since we shall finally take the limit as ε tends to zero.

We now expand eq. (51) as a Taylor series with respect to the retardation and advancement times σ, obtained as the solutions of:

$$[\{X(\tau) - X(\tau')\}^2 + \varepsilon^2]_{\tau' = \tau \mp \sigma} = 0 \qquad (\sigma > 0), \tag{52}$$

where the minus and plus signs refer to the retarded and advanced solutions respectively†. If we now invert σ as a function of ε from eq. (52), and substitute the result into the above-mentioned expansion of (51), we obtain a power series in ε. To find the required functions (48) we only have to differentiate the latter power series with respect to ε a number of times, *e.g.*:

$$\int_{-\infty}^{+\infty} g(\tau', \tau)\, \delta^{(n)}[\{X(\tau) - X(\tau')\}^2 + \varepsilon^2]\, s\{X^0(\tau) - X^0(\tau')\}\, \mathrm{d}\tau'$$

$$= \left(\frac{1}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right)^n \int_{-\infty}^{+\infty} g(\tau', \tau)\, \delta[\{X(\tau) - X(\tau')\}^2 + \varepsilon^2]\, s\{X^0(\tau) - X^0(\tau')\}\, \mathrm{d}\tau', \tag{53}$$

and let ε tend to zero at the end of the calculations, so that all terms of order ε and higher vanish and only those independent of ε remain in the final results‡. Various calculations are simplified by the special forms to be used for the functions $g(\tau', \tau)$. For details we refer to appendix III.

† Note that σ may be different in value for both solutions.

‡ Note that the method of calculating the field quantities (47A) and (47B) by means of the limiting process, described above, is *only* justified, since the fields $f^{\alpha\beta}_{(-)}(x)$ and $\partial_\gamma f^{\alpha\beta}_{(-)}(x)$ appear to be continuous functions of x not only outside, but also inside the atoms, if they are considered as very small extended systems. We shall not prove these properties here.

Now if we make the *assumption* (*cf*. appendix III), that all terms containing derivatives of the atomic four-velocity with respect to τ, are negligibly small compared to the other terms in the equations, we get for the minus part of the field tensor [*cf*. (A.53)]:

$$f_{(-)}^{\alpha\beta}\{X(\tau)\} = \frac{1}{4\pi}\left\{\frac{2}{3c^4}\overset{(3)}{\mu}{}^{\alpha\beta} - \frac{4}{3c^6}\left(U^\alpha \overset{(3)}{\mu}{}^{\beta\gamma} - U^\beta \overset{(3)}{\mu}{}^{\alpha\gamma}\right)U_\gamma\right\}, \tag{54}$$

where the argument of the atomic variables is τ and $\overset{(n)}{\mu}{}^{\alpha\beta} = \mathrm{d}^n\mu^{\alpha\beta}(\tau)/\mathrm{d}\tau^n$. We therefore have, using the antisymmetry of $\mu^{\alpha\beta}$:

$$ef_{(-)}^{\alpha\beta}U_\beta = -\frac{1}{4\pi}\left[\frac{2e}{3c^4}\overset{(3)}{\mu}{}^{\alpha\beta}U_\beta\right]. \tag{55}$$

In the same approximation we obtain [*cf*. (A.54)]:

$$-\left(\partial_\gamma f_{(-)}^{\alpha\beta}\right)\mu_\beta{}^\gamma = \frac{1}{4\pi}\left[\frac{1}{3c^6}\overset{(4)}{\mu}{}^{\alpha\beta}\mu_{\beta\gamma}U^\gamma + \frac{1}{3c^6}\mu^{\alpha\beta}\overset{(4)}{\mu}_{\beta\gamma}U^\gamma\right.$$
$$\left. + \frac{1}{3c^6}U^\alpha \overset{(4)}{\mu}{}^{\beta\gamma}\mu_{\gamma\beta} + \frac{2}{c^8}U^\alpha U_\beta \overset{(4)}{\mu}{}^{\beta\gamma}\mu_{\gamma\delta}U^\delta\right]. \tag{56}$$

With the last two formulae, we can write, again in the above approximation, the first two terms on the right-hand side of eq. (46A) in the following form:

$$ef_{(-)}^{\alpha\beta}U_\beta - \left(\partial_\gamma f_{(-)}^{\alpha\beta}\right)\mu_\beta{}^\gamma = \frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-\frac{2e}{3c^4}\ddot{\mu}^{\alpha\gamma}U_\gamma\right.$$
$$+ \frac{1}{3c^6}\overset{(3)}{\mu}{}^{\alpha\beta}\mu^{\beta\gamma}U^\gamma + \frac{1}{3c^6}\mu^{\alpha\beta}\overset{(3)}{\mu}_{\beta\gamma}U^\gamma + \frac{1}{3c^6}U^\alpha\mu^{\beta\gamma}\overset{(3)}{\mu}_{\gamma\beta}$$
$$+ \frac{2}{c^8}U^\alpha U_\beta \overset{(3)}{\mu}{}^{\beta\gamma}\mu_{\gamma\delta}U^\delta - \frac{1}{3c^6}\ddot{\mu}^{\alpha\beta}\dot{\mu}_{\beta\gamma}U^\gamma$$
$$\left. - \frac{1}{3c^6}\dot{\mu}^{\alpha\beta}\ddot{\mu}_{\beta\gamma}U^\gamma - \frac{1}{3c^6}U^\alpha\dot{\mu}^{\beta\gamma}\ddot{\mu}_{\gamma\beta} - \frac{2}{c^8}U^\alpha U_\beta\dot{\mu}^{\beta\gamma}\ddot{\mu}_{\gamma\delta}U^\delta\right]$$
$$+ \frac{1}{4\pi}\left[\frac{2}{3c^6}\ddot{\mu}^{\alpha\beta}\ddot{\mu}_{\beta\gamma}U^\gamma + \frac{1}{3c^6}U^\alpha\ddot{\mu}^{\beta\gamma}\ddot{\mu}_{\gamma\beta} + \frac{2}{c^8}U^\alpha U_\beta\ddot{\mu}^{\beta\gamma}\ddot{\mu}_{\gamma\delta}U^\delta\right]. \tag{57}$$

For the other terms at the right-hand side of (46A), containing the field tensor $f_{(-)}^{\alpha\beta}$, we obtain with eq. (54):

$$-\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(f_{(-)}^{\alpha\beta}\mu_\gamma{}^\beta - f_{(-)}^{\beta\gamma}\mu_\gamma{}^\alpha\right)U_\beta\right] + \frac{1}{c^4}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U^\alpha U_\beta f_{(-)}^{\beta\gamma}\mu_\gamma{}^\delta U_\delta\right]$$

$$= \frac{1}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[-\frac{2}{3c^6} \overset{(3)}{\mu}{}^{\alpha\gamma} \mu_{\gamma\beta} U^\beta - \frac{2}{3c^6} \overset{(3)}{\mu}{}^{\alpha\gamma} \mu_{\gamma\beta} U^\beta \right.$$

$$\left. - \frac{2}{c^8} U^\alpha U_\beta \overset{(3)}{\mu}{}^{\beta\gamma} \mu_{\gamma\delta} U^\delta \right], \tag{58}$$

where we have used the antisymmetry of $\mu^{\alpha\beta}$.

Adding eqs. (57) and (58) we obtain the sum of all terms depending on $f^{\alpha\beta}_{(-)}$ at the right-hand side of eq. (46A), and denoting this sum by $cF^\alpha_{(-)}$ we get after partial differentiations of some terms the following result:

$$cF^\alpha_{(-)} = cF^\alpha_{(\mathrm{d})} + c \frac{\mathrm{d}}{\mathrm{d}\tau} G^\alpha_{(-)}, \tag{59}$$

where we have introduced $F^\alpha_{(\mathrm{d})}$ and $G^{\alpha\beta}_{(-)}$:

$$F^\alpha_{(\mathrm{d})} \equiv \frac{1}{4\pi} \left[\frac{2}{3c^7} \ddot{\mu}^{\alpha\beta} \ddot{\mu}^{\beta\gamma} U^\gamma + \frac{1}{3c^7} U^\alpha \ddot{\mu}^{\beta\gamma} \ddot{\mu}_{\gamma\beta} + \frac{2}{c^9} U^\alpha U_\beta \ddot{\mu}^{\beta\gamma} \ddot{\mu}_{\gamma\delta} U^\delta \right], \tag{60A}$$

and

$$G^\alpha_{(-)} \equiv \frac{1}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[-\frac{2e}{3c^5} \dot{\mu}^{\alpha\gamma} U_\gamma - \frac{1}{3c^7} \ddot{\mu}^{\alpha\beta} \mu_{\beta\gamma} U^\gamma - \frac{1}{3c^7} \mu^{\alpha\beta} \ddot{\mu}_{\beta\gamma} U^\gamma \right]$$

$$+ \frac{1}{4\pi} \left[\frac{1}{3c^7} U^\alpha \overset{(3)}{\mu}{}^{\beta\gamma} \mu_{\gamma\beta} - \frac{1}{3c^7} U^\alpha \dot{\mu}^{\beta\gamma} \ddot{\mu}_{\gamma\beta} - \frac{2}{c^9} U^\alpha U_\beta \dot{\mu}^{\beta\gamma} \ddot{\mu}_{\gamma\delta} U^\delta \right]. \tag{60B}$$

We shall interpret $F^\alpha_{(\mathrm{d})}$ as the radiation reaction four-force. As is shown in appendix IV, this force can be related to the energy-momentum radiation rate.

We finally turn to eq. (46B). The sum of the terms at the right-hand side of this equation, containing the field tensor $f^{\alpha\beta}_{(-)}$ will be denoted here by the symbol $c\omega^{\alpha\beta}_{(-)}$ and becomes with eq. (54):

$$c\omega^{\alpha\beta}_{(-)} \equiv (f^{\alpha\gamma}_{(-)} \mu_\gamma{}^\beta - f^{\beta\gamma}_{(-)} \mu_\gamma{}^\alpha) + \frac{1}{c^2} U^\alpha U_\varepsilon (f^{\varepsilon\gamma}_{(-)} \mu_\gamma{}^\beta - f^{\beta\gamma}_{(-)} \mu_\gamma{}^\varepsilon)$$

$$+ \frac{1}{c^2} U^\beta U_\zeta (f^{\alpha\gamma}_{(-)} \mu_\gamma{}^\zeta - f^{\zeta\gamma}_{(-)} \mu_\gamma{}^\alpha) = c d^{\alpha\beta}_{(-)} + c \frac{\mathrm{d}}{\mathrm{d}\tau} \Lambda^{\alpha\beta}, \tag{61}$$

where we have introduced $d^{\alpha\beta}_{(-)}$ and $\Lambda^{\alpha\beta}$:

$$d^{\alpha\beta}_{(-)} \equiv -\frac{1}{4\pi} \left[\left\{ \frac{2}{3c^5} \ddot{\mu}^{\alpha\gamma} - \frac{4}{3c^7} (U^\alpha \ddot{\mu}^{\gamma\delta} - U^\gamma \ddot{\mu}^{\alpha\delta}) U_\delta \right\} \dot{\mu}_\gamma{}^\beta \right.$$

$$\left. - \left\{ \frac{2}{3c^5} \ddot{\mu}^{\beta\gamma} - \frac{4}{3c^7} (U^\beta \ddot{\mu}^{\gamma\delta} - U^\gamma \ddot{\mu}^{\beta\delta}) U_\delta \right\} \dot{\mu}_\gamma{}^\alpha \right], \tag{62A}$$

and

$$\Lambda^{\alpha\beta} \equiv \frac{1}{4\pi}\Bigg[\left\{\frac{2}{3c^5}\ddot{\mu}^{\alpha\gamma} - \frac{4}{3c^7}(U^\alpha\ddot{\mu}^{\gamma\delta} - U^\gamma\ddot{\mu}^{\alpha\delta})\,U_\delta\right\}\mu_\gamma{}^\beta$$

$$-\left\{\frac{2}{3c^5}\ddot{\mu}^{\beta\gamma} - \frac{4}{3c^7}(U^\beta\ddot{\mu}^{\gamma\delta} - U^\gamma\ddot{\mu}^{\beta\delta})\,U_\delta\right\}\mu_\gamma{}^\alpha$$

$$+\frac{2}{3c^7}(U^\beta\mu^{\alpha\gamma} - U^\alpha\mu^{\beta\gamma})\,\ddot{\mu}_{\gamma\delta}U^\delta + \frac{2}{3c^7}(U^\beta\ddot{\mu}^{\alpha\gamma} - U^\alpha\ddot{\mu}^{\beta\gamma})\,\mu_{\gamma\delta}U^\delta$$

$$-\frac{2}{3c^7}(U^\beta\dot{\mu}^{\alpha\gamma} - U^\alpha\dot{\mu}^{\beta\gamma})\,\dot{\mu}_{\gamma\delta}U^\delta\Bigg]. \tag{62B}$$

In appendix IV we have proved that the radiation rate of angular momentum is given by:

$$j^{\alpha\beta}_{(\mathrm{rad})} = -\,d^{\alpha\beta}_{(-)} + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{\frac{1}{4\pi}\,\frac{2e}{3c^3}\,\bar{\mu}^{\alpha\beta}\right\}, \tag{63}$$

where the tensor $\bar{\mu}^{\alpha\beta}$ is defined by eqs. (A.77)–(A.79). If we now define the tensors:

$$d^{\alpha\beta}_{(\mathrm{d})} \equiv d^{\alpha\beta}_{(-)} - \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{\frac{1}{4\pi}\,\frac{2e}{3c^3}\,\bar{\mu}^{\alpha\beta}\right\}, \tag{64A}$$

and

$$\Lambda^{\alpha\beta}_{(-)} \equiv \Lambda^{\alpha\beta} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{1}{4\pi}\,\frac{2e}{3c^3}\,\bar{\mu}^{\alpha\beta}\right\}, \tag{64B}$$

we can rewrite eq. (61) as

$$c\omega^{\alpha\beta}_{(-)} = c d^{\alpha\beta}_{(\mathrm{d})} + c\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\Lambda^{\alpha\beta}_{(-)}. \tag{65}$$

We then see that we may interpret the tensor $d^{\alpha\beta}_{(\mathrm{d})}$ as the radiation reaction torque acting on the atom.

From the eqs. (46A) and (46B), together with the results eqs. (59)–(65), we shall derive in the next section the form of the atomic energy-momentum tensor for a system of dipole atoms, which may radiate, in an analogous way as this has been done in chapter I, section 4 for a non-radiating system.

§ 4. *The atomic energy-momentum tensor.* Analogously as in chapter I, section 4 we now consider a system consisting of N point atoms, numbered by the index k. All atomic quantities and variables belonging to the k^{th} atom will be labeled again by the index k. The rest mass of the k^{th} atom will be denoted by $m^*_{(k)}$ (instead of M^*) and its centre of gravity by $R^\alpha_{(k)}$ (instead of X^α). If there are external fields acting from outside the system, the (external) field $f^{\alpha\beta}$ acting upon atom k consists of the following parts:

$$f^{\alpha\beta}(R_{(k)}) = \sum_{l(\neq k)} f^{\alpha\beta}_{(l)}(R_{(k)}) + f^{\alpha\beta}_{(e)}(R_{(k)}). \tag{66}$$

The first term on the right-hand side of (66) is the sum of the partial (retarded) electromagnetic fields $f^{\alpha\beta}_{(l)}$ due to the other atoms l ($\neq k$) and the last term is the external field from outside the system. With these notations and the abbreviations (60), (62) and (64), the equations of motion (46A) becomes:

$$\begin{aligned}
c\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}(m^*_{(k)}U^\alpha_{(k)}) &= c\,\{F^\alpha_{(\mathrm{d})}\}_{(k)} + c\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{G^\alpha_{(-)}\}_{(k)} \\
&+ \Bigg[e_{(k)}\sum_{l(\neq k)} f^{\alpha\beta}_{(l)}(R_{(k)})U_{(k)\beta} - \sum_{l(\neq k)}\{\partial_{(k)\gamma}f^{\alpha\beta}_{(l)}(R_{(k)})\}\,\mu_{(k)\beta}{}^{\gamma} \\
&- \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}[\sum_{l(\neq k)} f^{\alpha\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\beta} - f^{\beta\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\alpha}\}\,U_{(k)\beta}] \\
&+ \frac{1}{c^4}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{\sum_{l(\neq k)} U^\alpha_{(k)}U_{(k)\beta}f^{\beta\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\delta}U_{(k)\delta}\}\Bigg] \\
&+ \Bigg[e_{(k)}f^{\alpha\beta}_{(e)}(R_{(k)})\,U_{(k)\beta} - \{\partial_{(k)\gamma}f^{\alpha\beta}_{(e)}(R_{(k)}\}\,\mu_{(k)\beta}{}^{\gamma} \\
&- \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}[\{f^{\alpha\gamma}_{(e)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\beta} - f^{\beta\gamma}_{(e)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\alpha}\}\,U_{(k)\beta}] \\
&+ \frac{1}{c^4}\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{U^\alpha_{(k)}U_{(k)\beta}f^{\beta\gamma}_{(e)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\delta}U_{(k)}{}^{\delta}\}\Bigg],
\end{aligned} \tag{67A}$$

and the intrinsic angular momentum balance equation (46B) becomes:

$$\begin{aligned}
c\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\Omega^{\alpha\beta}_{(k)} &= c\{d^{\alpha\beta}_{(\mathrm{d})}\}_{(k)} + c\frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}}\{\Lambda^{\alpha\beta}_{(-)}\}_{(k)} \\
&+ \Delta^\alpha_{(k)\varepsilon}\Delta^\beta_{(k)\zeta}[\sum_{l(\neq k)}\{f^{\varepsilon\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\zeta} - f^{\zeta\gamma}_{(l)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\varepsilon}\} \\
&+ \{f^{\varepsilon\gamma}_{(e)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\zeta} - f^{\zeta\gamma}_{(e)}(R_{(k)})\,\mu_{(k)\gamma}{}^{\varepsilon}\}].
\end{aligned} \tag{67B}$$

Using the same method and definitions as in chapter I, section 4 (*cf.* ref. 1) we obtain, after multiplication of eqs. (67A) and (67B) by the four-dimensional δ-function $\delta^{(4)}\{R_{(k)}(\tau_k) - R\}$, integration over $\tau_{(k)}$ and summation over k, the following results:

$$\partial_\beta\{t^{*\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})} + t^{*\alpha\beta}_{(-)}\} = c\sum_k\int_{-\infty}^{+\infty}\{F^\alpha_{(\mathrm{d})}\}_{(k)}\delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\,\mathrm{d}\tau_{(k)}, \tag{68A}$$

and:

$$\partial_\gamma \sum_k \{u^\gamma_{(k)}\sigma^{*\alpha\beta}_{(k)}\} = \partial_\gamma \sum_k \{u^\gamma_{(k)}\lambda^{\alpha\beta}_{(-)(k)}\}$$

$$+ c \sum_k [\int_{-\infty}^{+\infty} \{d^{\alpha\beta}_{(\mathrm{d})}\}_{(k)}\delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)}] + \{t^{\alpha\beta}_{(\mathrm{f})} - t^{\beta\alpha}_{(\mathrm{f})}\}, \tag{68B}$$

where by definition (see I and also II):

$$t^{*\alpha\beta}_{(\mathrm{m})} \equiv \sum_k \rho^{*\prime}_{(k)} u^\alpha_{(k)} u^\beta_{(k)}, \tag{69A}$$

$$t^{\alpha\beta}_{(\mathrm{f})} \equiv \sum_{k,l(k\neq l)} [f^{\alpha\gamma}_{(l)} h^\beta_{(k)\gamma} - \tfrac{1}{4}\{f_{(l)\gamma\delta} f^{\gamma\delta}_{(k)}\} g^{\alpha\beta}$$

$$+ \frac{1}{c^2} \{f^{\alpha\gamma}_{(l)} m_{(k)\gamma\delta} - m^{\alpha\gamma}_{(k)} f_{(l)\gamma\delta}\} u^\delta_{(k)} u^\beta_{(k)}$$

$$- \frac{1}{c^4} \{u^\gamma_{(k)} f_{(l)\gamma\delta} m^{\delta\varepsilon}_{(k)} u_{(k)\varepsilon}\} u^\alpha_{(k)} u^\beta_{(k)}]$$

$$+ \sum_k [f^{\alpha\gamma}_{(k)} f^\beta_{(\mathrm{e})\gamma} + f^{\alpha\gamma}_{(\mathrm{e})} h^\beta_{(k)\gamma} - \tfrac{1}{2} f_{(\mathrm{e})\gamma\delta} f^{\gamma\delta}_{(k)}\} g^{\alpha\beta}$$

$$+ \frac{1}{c^2} \{f^{\alpha\gamma}_{(\mathrm{e})} m_{(k)\gamma\delta} - m^{\alpha\gamma}_{(k)} f_{(\mathrm{e})\gamma\delta}\} u^\delta_{(k)} u^\beta_{(k)}$$

$$- \frac{1}{c^4} \{u^\gamma_{(k)} f_{(\mathrm{e})\gamma\delta} m^{\delta\varepsilon}_{(k)} u_{(k)\varepsilon}\} u^\alpha_{(k)} u^\beta_{(k)}]$$

$$+ f^{\alpha\gamma}_{(\mathrm{e})} f^\beta_{(\mathrm{e})\gamma} - \tfrac{1}{4}\{f_{(\mathrm{e})\gamma\delta} f^{\gamma\delta}_{(\mathrm{e})}\} g^{\alpha\beta}, \tag{69B}$$

$$t^{*\alpha\beta}_{(-)} \equiv -c \sum_k \int_{-\infty}^{+\infty} G^\alpha_{(-)(k)}(\tau_{(k)})\, U^\beta_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)}, \tag{69C}$$

$$\lambda^{\alpha\beta}_{(-)(k)} \equiv c \int_{-\infty}^{+\infty} \Lambda^{\alpha\beta}_{(-)(k)} \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)}, \tag{69D}$$

with $F^\alpha_{(\mathrm{d})(k)}$, $d^{\alpha\beta}_{(\mathrm{d})(k)}$, $G^\alpha_{(-)(k)}$ and $\Lambda^{\alpha\beta}_{(-)(k)}$ given by eqs. (60), (62) and (64).

Eqs. (68) together with the definitions (69) are already the atomic energy-momentum law and the atomic angular momentum balance of the system, but we shall still transform these equations, in order to make them more transparent. First we notice that, since according to (60B), $G^\alpha_{(-)(k)}$ can be written as:

$$G^\alpha_{(-)(k)} = \frac{\mathrm{d}}{\mathrm{d}\tau_{(k)}} H^\alpha_{(-)(k)} + K^\alpha_{(-)(k)}, \tag{70}$$

we can write $t^{*\alpha\beta}_{(-)}$, eq. (69C), in the approximation $\dot{U}^\alpha \simeq 0$, as:

$$t^{*\alpha\beta}_{(-)} = -c\partial_\gamma \sum_k \int_{-\infty}^{+\infty} H^\alpha_{(-)(k)}(\tau_{(k)})\, U^\beta_{(k)}(\tau_{(k)})\, U^\gamma_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)}$$

$$- c \sum_k \int_{-\infty}^{+\infty} K^{\alpha}_{(-)(k)}(\tau_{(k)})\, U^{\beta}_{(k)}(\tau_{(k)})\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)}$$
$$\equiv - \partial_\gamma \sum_k \{u^{\gamma}_{(k)} s^{\alpha\beta}_{(-)(k)}\} + t^{*\alpha\beta}_{(-)2} \equiv t^{*\alpha\beta}_{(-)1} + t^{*\alpha\beta}_{(-)2}. \tag{71}$$

It follows from (60B) and (70), that the tensor $t^{*\alpha\beta}_{(-)2}$ is *symmetric*. For the tensor $t^{*\alpha\beta}_{(-)1}$ we write:

$$t^{*\alpha\beta}_{(-)1} = - \partial_\gamma \sum_k \{u^{\gamma}_{(k)} s^{(\mathrm{s})\alpha\beta}_{(-)(k)}\} - \partial_\gamma \sum_k \{u^{\gamma}_{(k)} s^{(\mathrm{a})\alpha\beta}_{(-)(k)}\}, \tag{72}$$

where $s^{(\mathrm{s})\alpha\beta}_{(-)(k)} = \frac{1}{2}\{s^{\alpha\beta}_{(-)(k)} + s^{\beta\alpha}_{(-)(k)}\}$ is the symmetric part of $s^{\alpha\beta}_{(-)(k)}$ and $s^{(\mathrm{a})\alpha\beta}_{(-)(k)} = \frac{1}{2}\{s^{\alpha\beta}_{(-)(k)} - s^{\beta\alpha}_{(-)(k)}\}$ its antisymmetric part. We now add a divergence-free tensor to $t^{*\alpha\beta}_{(-)1}$ such that the sum is a symmetric tensor $t^{\alpha\beta}_{(-)1}$:

$$t^{\alpha\beta}_{(-)1} = t^{*\alpha\beta}_{(-)1} + \partial_\gamma \sum_k \{-u^{\alpha}_{(k)} s^{(\mathrm{a})\beta\gamma}_{(-)(k)} - u^{\beta}_{(k)} s^{(\mathrm{a})\alpha\gamma}_{(-)(k)} + u^{\gamma}_{(k)} s^{(\mathrm{a})\alpha\beta}_{(-)(k)}\} = t^{\beta\alpha}_{(-)1}. \tag{73}$$

We shall also add the divergence-free tensor:

$$\tfrac{1}{2}\partial_\gamma \sum_k \{-u^{\alpha}_{(k)} \lambda^{\beta\gamma}_{(-)(k)} - u^{\beta}_{(k)} \lambda^{\alpha\gamma}_{(-)(k)} + u^{\gamma}_{(k)} \lambda^{\alpha\beta}_{(-)(k)}\}, \tag{74}$$

to (73), and define the final "minus" tensor $t^{\alpha\beta}_{(-)}$:

$$t^{\alpha\beta}_{(-)} \equiv t^{\alpha\beta}_{(-)1} + t^{*\alpha\beta}_{(-)2}$$
$$+ \tfrac{1}{2}\partial_\gamma \sum_k \{-u^{\alpha}_{(k)} \lambda^{\beta\gamma}_{(-)(k)} - u^{\beta}_{(k)} \lambda^{\alpha\gamma}_{(-)(k)} + u^{\gamma}_{(k)} \lambda^{\alpha\beta}_{(-)(k)}\}. \tag{75}$$

The explicit form of this tensor can be obtained from the expressions given above. From (71), (73) and (75) we have:

$$\partial_\beta t^{\alpha\beta}_{(-)} = \partial_\beta t^{*\alpha\beta}_{(-)}, \quad \text{and} \quad t^{\alpha\beta}_{(-)} - t^{\beta\alpha}_{(-)} = \partial_\gamma \sum_k \{u^{\gamma}_{(k)} \lambda^{\alpha\beta}_{(-)(k)}\}. \tag{76}$$

Just as in chapters I and II, we also define a new material tensor $t^{\alpha\beta}_{(\mathrm{m})}$:

$$t^{\alpha\beta}_{(\mathrm{m})} \equiv t^{*\alpha\beta}_{(\mathrm{m})} + \tfrac{1}{2}\partial_\gamma \sum_k \{u^{\alpha}_{(k)} \sigma^{*\beta\gamma}_{(k)} + u^{\beta}_{(k)} \sigma^{*\alpha\gamma}_{(k)} - u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}, \tag{77}$$

for which it follows that:

$$\partial_\beta t^{\alpha\beta}_{(\mathrm{m})} = \partial_\beta t^{*\alpha\beta}_{(\mathrm{m})}, \quad \text{and} \quad t^{\alpha\beta}_{(\mathrm{m})} - t^{\beta\alpha}_{(\mathrm{m})} = - \partial_\gamma \sum_k \{u^{\gamma}_{(k)} \sigma^{*\alpha\beta}_{(k)}\}. \tag{78}$$

If we now define the total atomic energy-momentum tensor as

$$t^{\alpha\beta} \equiv t^{\alpha\beta}_{(\mathrm{m})} + t^{\alpha\beta}_{(\mathrm{f})} + t^{\alpha\beta}_{(-)}, \tag{79}$$

we get with (76), instead of eqs. (68A) and (68B):

$$\partial_\beta t^{\alpha\beta} = c \sum_k \int_{-\infty}^{+\infty} \{F^{\alpha}_{(\mathrm{d})}\}_{(k)}\, \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)} \tag{80A}$$

and

$$\{t^{\beta\alpha} - t^{\alpha\beta}\} = c \sum_k \int_{-\infty}^{+\infty} \{d^{\alpha\beta}_{(\mathrm{d})}\}_{(k)} \delta^{(4)}\{R_{(k)}(\tau_{(k)}) - R\}\, \mathrm{d}\tau_{(k)}. \tag{80B}$$

Eq. (80A) is the energy-momentum law for a system consisting of N *radiating* atoms. The right-hand side of this equation describes the radiation reaction on this system. The atomic energy-momentum tensor $t^{\alpha\beta}$ for such a system is *no longer symmetric* as this was the case for non-radiating atoms (see I, II and III). This asymmetry follows immediately from eq. (80B), which is equivalent with the intrinsic atomic angular momentum balance in the case one does not neglect the reaction of the radiation. Equations (80A) and (80B) may be considered as the direct generalizations of equation (I.43) for the *symmetric* tensor $t^{\alpha\beta} \equiv t^{\alpha\beta}_{(m)} + t^{\alpha\beta}_{(f)}$, eqs. (I.36) and (II.25).

§ 5. *Concluding remark.* The present treatment could be extended to the case that the atoms possess, in addition to their electric and magnetic dipole moments, also an electric quadrupole moment. The starting point would then be the equations of motion as derived in chapter III, eqs. (19A) and (19B), supplemented with analogous terms in $f^{\alpha\beta}_{(-)}$. Then the same procedure as in appendix III could be applied. One would now have to calculate not only the tensor $f^{\alpha\beta}_{(-)}$ and its first derivative, but also the second, and one should also take into account the influence of the tensor $\chi^{\alpha\beta\gamma}$ (see chapter III) on the results. We do not want to go into the details of these very complicated calculations here.

APPENDIX I

Proof that expression (29) *is zero.* We write for the expression (29):

$$-\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma\Bigg[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\partial_{\lambda_1}\dots\partial_{\lambda_n}\int\limits^{\infty}\!\!\int\limits^{\infty}(x''^{\lambda_1}-x'^{\lambda_1})\dots(x''^{\lambda_n}-x'^{\lambda_n})$$
$$\times\,(x''^{\alpha}-x'^{\alpha})\,i^{\beta}(x')\,i^{\gamma}(x'')\,\delta\{(x''-x')^2\}\,\delta^{(4)}(x''-x)\,\mathrm{d}^{(4)}x'\,\mathrm{d}^{(4)}x''\Bigg] \tag{A.1}$$

and make a Taylor series expansion of the four-dimensional delta function $\delta^{(4)}(x''-x)$ around the point $(x'+x'')/2$. This gives for (A.1):

$$-\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma\Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n+1)!}\,\frac{(-1)^m}{2^m\,m!}\,\partial_{\lambda_1}\dots\partial_{\lambda_n}\partial_{\mu_1}\dots\partial_{\mu_m}$$
$$\times\int\limits^{\infty}\!\!\int\limits^{\infty}(x''^{\lambda_1}-x'^{\lambda_1})\dots(x''^{\lambda_n}-x'^{\lambda_n})(x''^{\mu_1}-x'^{\mu_1})\dots(x''^{\mu_m}-x'^{\mu_m})$$
$$\times\,(x''^{\alpha}-x'^{\alpha})\,i^{\beta}(x')\,i^{\gamma}(x'')\,\delta\{(x''-x')^2\}\,\delta^{(4)}\!\left(\frac{x'+x''}{2}-x\right)\mathrm{d}^{(4)}x'\,\mathrm{d}^{(4)}x''\Bigg]$$
$$=-\frac{1}{8\pi c^2}\,\partial_\beta\partial_\gamma\Bigg[\sum_{p=0}^{\infty}\left\{\sum_{m=0}^{p}\frac{1}{(p+1-m)!}\,\frac{(-1)^m}{2^m\,m!}\right\}\partial_{\lambda_1}\dots\partial_{\lambda_p}$$

$$\times \int\limits^{\infty}\int\limits^{\infty} (x''^{\lambda_1} - x'^{\lambda_1}) \ldots (x''^{\lambda_p} - x'^{\lambda_p})(x''^{\alpha} - x'^{\alpha})\, i^{\beta}(x')\, i^{\gamma}(x'')$$

$$\times\, \delta\{(x''-x')^2\}\, \delta^{(4)}\left(\frac{x'+x''}{2} - x\right) \mathrm{d}^{(4)} x'\, \mathrm{d}^{(4)} x'' \Bigg]. \tag{A.2}$$

Now we know that

$$\sum_{m=0}^{p} \frac{1}{(p+1-m)!}\frac{(-1)^m}{2^m\, m!} = \sum_{m=0}^{p+1} \frac{1}{(p+1-m)!}\frac{(-1)^m}{2^m\, m!} - \frac{(-1)^{p+1}}{(p+1)!\,2^{p+1}}$$

$$= \frac{1}{(p+1)!}(1-\tfrac{1}{2})^{p+1} - \frac{1}{(p+1)!}(-\tfrac{1}{2})^{p+1}$$

$$= \frac{1}{(p+1)!}\{(\tfrac{1}{2})^{p+1} - (-\tfrac{1}{2})^{p+1}\}, \tag{A.3}$$

and this is zero for odd p. In the series in the last member of eq. (A.2) we are therefore only left with terms with even powers in p, which contain an odd number of factors $(x''^{\lambda_i} - x'^{\lambda_i})$. If we now interchange the integration variables x' and x'' in the last member of (A.2), and also the dummy indices β and γ, the only effect will be a change of sign (because of the odd number of above mentioned factors) and this proves that (A.2), and therefore the expression (29) is zero.

APPENDIX II

The mass, centre of gravity and intrinsic angular momentum. We shall show that the mass, centre of gravity and intrinsic angular momentum of an atom, found with the sub-atomic energy-momentum tensor

$$T^{\alpha\beta} = T^{\alpha\beta}_{(\mathrm{m})} + T^{\alpha\beta}_{(+)},$$

eq. (33), and calculated in the approximation c^{-2} (c = velocity of light), are of the usual forms obtained in the literature in this approximation. We shall be concerned here only with the contributions of $T^{\alpha\beta}_{(+)}$ to these quantities, since those of the material tensor $T^{\alpha\beta}_{(\mathrm{m})}$ are well known and need no further discussion.

The mass-contribution, due to this tensor $T^{\alpha\beta}_{(+)}$, eq. (32), is given by

$$M_{(+)}(\tau) = \frac{1}{c^4}\int^{\infty} T^{\alpha\beta}_{(+)}(x)\, U_{\beta}(\tau)\, U_{\alpha}(\tau)\, \delta\left[-\frac{1}{c}\{x^{\gamma} - X^{\gamma}(\tau)\}\, U_{\gamma}(\tau)\right] \mathrm{d}^{(4)}\, x, \tag{A.4}$$

as follows from eq. (35). Substituting (32) into (A.4), we get:

$$M_{(+)}(\tau) = \frac{1}{8\pi c^6} \int\limits^{\infty}\int\limits^{\infty} \{i^\alpha(x')\, i^\beta(x) + i^\beta(x')\, i^\alpha(x)\}\, U_\beta(\tau)\, U_\alpha(\tau)$$

$$\times\, \delta\{(x-x')^2\}\, \delta\left[-\frac{1}{c}\{x^\gamma - X^\gamma(\tau)\}\, U_\gamma(\tau)\right] \mathrm{d}^{(4)}x\, \mathrm{d}^{(4)}x'$$

$$+ \frac{1}{4\pi c^6} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int\limits^{\infty}\int\limits^{\infty} \partial_{\lambda_1} \ldots \partial_{\lambda_n} [(x^{\lambda_1} - x'^{\lambda_1}) \ldots (x^{\lambda_n} - x'^{\lambda_n})$$

$$\times\, i^\gamma(x)\, i^\gamma(x')(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)\, U_\beta(\tau)\, U_\alpha(\tau)$$

$$\times\, \delta'\{(x-x')^2\}]\, \delta\left[-\frac{1}{c}\{x^\delta - X^\delta(\tau)\}\, U_\delta(\tau)\right] \mathrm{d}^{(4)}x\, \mathrm{d}^{(4)}x'$$

$$- \frac{1}{8\pi c^6} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int\limits^{\infty}\int\limits^{\infty} \partial_\gamma \partial_{\lambda_1} \ldots \partial_{\lambda_n} [(x^{\lambda_1} - x'^{\lambda_1}) \ldots (x^{\lambda_n} - x'^{\lambda_n})$$

$$\times\, \{i^\alpha(x')(x^\beta - x'^\beta) + i^\beta(x')(x^\alpha - x'^\alpha)\}\, U_\beta(\tau)\, U_\alpha(\tau)\, i^\gamma(x)\, \delta\{(x-x')^2\}]$$

$$\times\, \delta\left[-\frac{1}{c}\{x^\delta - X^\delta(\tau)\}\, U_\delta(\tau)\right] \mathrm{d}^{(4)}x\, \mathrm{d}^{(4)}x'. \tag{A.5}$$

In an analogous way as we have proved in appendix I, that the expression (29) was zero, one can now prove that the last sum of (A.5) is zero. (This proof will be omitted here.)

The other integrals in (A.5) will be calculated in the momentary rest system (m.r.s.) of the reference point. First we rewrite the first sum at the right-hand side of (A.5) in the following form:

$$\frac{1}{8\pi c^6} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int\limits^{\infty}\int\limits^{\infty} \partial_{\lambda_1} \ldots \partial_{\lambda_n} [(x^{\lambda_1} - x'^{\lambda_1}) \ldots (x^{\lambda_n} - x'^{\lambda_n})$$

$$\times\, i_\gamma(x)\, i^\gamma(x')(x^\alpha - x'^\alpha)\, U_\alpha(\tau)\, U_\beta(\tau) [\partial^\beta\, \delta\{(x-x')^2\}]]$$

$$\times\, \delta\left[-\frac{1}{c}\{x^\delta - X^\delta(\tau)\}\, U_\delta(\tau)\right] \mathrm{d}^{(4)}\, x\, \mathrm{d}^{(4)}\, x', \tag{A.6}$$

using:

$$(x^\beta - x'^\beta)\, \delta'\{(x-x')^2\} = \tfrac{1}{2}\partial^\beta\, \delta\{(x-x')^2\}. \tag{A.7}$$

From (A.5) and (A.6) we obtain in the m.r.s. ($U^0 = c$; $U^1 = U^2 = U^3 = 0$):

$$M_{(+)}(t) = \frac{1}{4\pi c^2} \int\limits^{\infty} \int\limits_{-\infty}^{+\infty} \int\limits^{\infty} \rho^{(0)}(\boldsymbol{x}, t)\, \rho^{(0)}(\boldsymbol{x}', t')$$

$$\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x}$$

$$-\frac{1}{8\pi c^4}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial t}\right)^n\int^{\infty}\int_{-\infty}^{+\infty}\int^{\infty}(t-t')^n\, i_{\gamma}^{(0)}(\boldsymbol{x},t)\, i\gamma^{(0)}(\boldsymbol{x}',t')$$

$$\times(ct-ct')\frac{\partial}{\partial ct}\,\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x},\quad \text{(A.8)}$$

where $\rho^{(0)}$ is the sub-atomic charge density in the m.r.s. Quantities in the m.r.s. are indicated by the symbol (0). It is easily seen that the terms with $\lambda_t = 1, 2, 3$ in (A.6) disappear in the m.r.s., so that one is left only with time derivatives in the second integral of (A.8).

If the first triple integral of (A.8) is integrated over ct', we get:

$$\frac{1}{4\pi c^2}\int^{\infty}\int^{\infty}\left\{\frac{\rho^{(0)}(\boldsymbol{x},t)\,\rho^{(0)}(\boldsymbol{x}',t-|\boldsymbol{x}-\boldsymbol{x}'|/c)}{2\,|\boldsymbol{x}-\boldsymbol{x}'|}\right.$$

$$\left.+\frac{\rho^{(0)}(\boldsymbol{x},t)\,\rho^{(0)}(\boldsymbol{x}',t+|\boldsymbol{x}-\boldsymbol{x}'|/c)}{2\,|\boldsymbol{x}-\boldsymbol{x}'|}\right\}\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x},\quad \text{(A.9)}$$

which can be written as a power series in $|\boldsymbol{x}-\boldsymbol{x}'|$ by expansion of

$$\rho^{(0)}(\boldsymbol{x}',t\mp|\boldsymbol{x}-\boldsymbol{x}'|/c)$$

around t. To order c^{-2} we get from (A.9):

$$\frac{1}{4\pi c^2}\int^{\infty}\int^{\infty}\frac{\rho^{(0)}(\boldsymbol{x},t)\,\rho^{(0)}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x}.\quad \text{(A.10)}$$

For the second integral of (A.8) we can write:

$$-\frac{1}{8\pi c^4}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial t}\right)^n\int^{\infty}\int_{-\infty}^{+\infty}\int^{\infty}(t-t')^{n+1}\, i_{\gamma}^{(0)}(\boldsymbol{x},t)$$

$$i\gamma^{(0)}(\boldsymbol{x}',t')\frac{\partial}{\partial t}\,\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x}$$

$$=-\frac{1}{8\pi c^4}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial t}\right)^{n+1}\int^{\infty}\int_{-\infty}^{+\infty}\int^{\infty}(t-t')^{n+1}\, i_{\gamma}^{(0)}(\boldsymbol{x},t)$$

$$i\gamma^{(0)}(\boldsymbol{x}',t')\,\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x}$$

$$+\frac{1}{8\pi c^4}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial}{\partial t}\right)^n\int^{\infty}\int_{-\infty}^{+\infty}\int^{\infty}(t-t')^n\, i_{\gamma}^{(0)}(\boldsymbol{x},t)$$

$$i\nu^{(0)}(\boldsymbol{x}', t')\,\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x}$$

$$+\frac{1}{8\pi c^4}\sum_{0=n}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial t}\right)^n\int^{\infty}\int_{-\infty}^{+\infty}\int^{\infty}(t-t')^{n+1}\left\{\frac{\partial}{\partial t}i_\nu^{(0)}(\boldsymbol{x},t)\right\}$$

$$i\nu^{(0)}(\boldsymbol{x}', t')\,\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x}. \tag{A.11}$$

The first two sums on the right-hand side of this equation give together:

$$\frac{1}{8\pi c^4}\int^{\infty}\int_{-\infty}^{+\infty}\int^{\infty} i_\nu^{(0)}(\boldsymbol{x},t)\,i\nu^{(0)}(\boldsymbol{x}',t')$$

$$\times\,\delta\{-(ct-ct')^2+(\boldsymbol{x}-\boldsymbol{x}')^2\}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}ct'\,\mathrm{d}^{(3)}\boldsymbol{x}, \tag{A.12}$$

which is equal to:

$$-\frac{1}{8\pi c^2}\int^{\infty}\int^{\infty}\left\{\frac{\rho^{(0)}(\boldsymbol{x},t)\,\rho^{(0)}(\boldsymbol{x}',t-|\boldsymbol{x}-\boldsymbol{x}'|/c)}{2\,|\boldsymbol{x}-\boldsymbol{x}'|}\right.$$

$$\left.+\frac{\rho^{(0)}(\boldsymbol{x},t)\,\rho^{(0)}(\boldsymbol{x}',t+|\boldsymbol{x}-\boldsymbol{x}'|/c)}{2\,|\boldsymbol{x}-\boldsymbol{x}'|}\right\}\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x}$$

$$+\frac{1}{8\pi c^4}\int^{\infty}\int^{\infty}\left\{\frac{\boldsymbol{i}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{i}^{(0)}(\boldsymbol{x}',t-|\boldsymbol{x}-\boldsymbol{x}'|/c)}{2\,|\boldsymbol{x}-\boldsymbol{x}'|}\right.$$

$$\left.+\frac{\boldsymbol{i}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{i}^{(0)}(\boldsymbol{x}',t+|\boldsymbol{x}-\boldsymbol{x}'|/c)}{2|\boldsymbol{x}-\boldsymbol{x}'|}\right\}\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x}, \tag{A.13}$$

where $\boldsymbol{i}^{(0)}$ is the sub-atomic current density in the m.r.s. Up to order c^{-2} we have from (A.13):

$$-\frac{1}{8\pi c^2}\int^{\infty}\int^{\infty}\frac{\rho^{(0)}(\boldsymbol{x},t)\,\rho^{(0)}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x}. \tag{A.14}$$

The last sum at the right hand side of eq. (A.11) can be integrated for t' and gives:

$$-\frac{1}{16\pi c^3}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial t}\right)^n\int^{\infty}\int^{\infty}|\boldsymbol{x}-\boldsymbol{x}'|^n\left\{\frac{\partial}{\partial t}\rho^{(0)}(\boldsymbol{x},t)\right\}$$

$$\times\left\{\rho^{(0)}\left(\boldsymbol{x}',t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)+(-1)^{n+1}\rho^{(0)}\left(\boldsymbol{x}',t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)\right\}$$

$$\times\,\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x}+\frac{1}{16\pi c^5}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial t}\right)^n\int^{\infty}\int^{\infty}|\boldsymbol{x}-\boldsymbol{x}'|^n\left\{\frac{\partial}{\partial t}\boldsymbol{i}^{(0)}(\boldsymbol{x}',t)\right\}$$

$$\cdot\left\{\boldsymbol{i}^{(0)}\left(\boldsymbol{x}',t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)+(-1)^{n+1}\boldsymbol{i}^{(0)}\left(\boldsymbol{x}',t+\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)\right\}\mathrm{d}^{(3)}\boldsymbol{x}'\,\mathrm{d}^{(3)}\boldsymbol{x}, \tag{A.15}$$

which is negligible, if we consider only contributions to the mass of order c^{-2}. From (A.8), (A.10), (A.14) and (A.15) we get to order c^{-2}:

$$M_{(+)}(t) = \frac{1}{8\pi c^2} \int^{\infty}\!\!\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)\, \rho^{(0)}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \mathrm{d}^{(3)}\boldsymbol{x}'\, \mathrm{d}^{(3)}\boldsymbol{x}. \tag{A.16}$$

This is the well known result for the mass-correction, due to the electromagnetic self-forces on extended charges, calculated to order c^{-2} (see *e.g.* refs. 5 and 11). We could, of course, also calculate higher corrections to $M_{(+)}$ than of order c^{-2}. However, since for atoms the electromagnetic mass-correction (A.16) is already very small, compared to the mechanical mass, it is of not much use to evaluate such higher order contributions.

The pseudo centres of gravity of the atom are defined by the relation:

$$\Omega^{\alpha\beta} U_\beta = 0, \tag{A.17}$$

where $\Omega^{\alpha\beta}$ is given by eq. (36), (see chapters I and II). In the m.r.s. we get from (A.17):

$$\Omega^{k0(0)}(t) = \Omega^{k0(0)}_{(\mathrm{m})}(t) + \Omega^{k0(0)}_{(+)}(t) = 0; \qquad (k = 1, 2, 3), \tag{A.18}$$

with

$$\Omega^{k0(0)}_{(\mathrm{m})}(t) = \frac{1}{c} \int^{\infty} [x^k - X^{k(0)}(t)]\, T^{00(0)}_{(\mathrm{m})}(\boldsymbol{x}, t)\, \mathrm{d}^{(3)}\boldsymbol{x}, \tag{A.19}$$

and:

$$\Omega^{k0(0)}_{(+)}(t) = \frac{1}{c} \int^{\infty} [x^k - X^{k(0)}(t)]\, T^{00(0)}_{(+)}(\boldsymbol{x}, t)\, \mathrm{d}^{(3)}\boldsymbol{x}. \tag{A.20}$$

From (A.18)–(A.20) we get for the equation of the (pseudo) centres of gravity in the m.r.s.

$$\boldsymbol{X}^{(0)}(t) = \frac{\int^{\infty} \boldsymbol{x}\{T^{00(0)}_{(\mathrm{m})}(\boldsymbol{x}, t) + T^{00(0)}_{(+)}(\boldsymbol{x}, t)\}\, \mathrm{d}^{(3)}\boldsymbol{x}}{\int^{\infty} \{T^{00(0)}_{(\mathrm{m})}(\boldsymbol{x}, t) + T^{00(0)}_{(+)}(\boldsymbol{x}, t)\}\, \mathrm{d}^{(3)}\boldsymbol{x}}. \tag{A.21}$$

The contribution of $T^{00(0)}_{(+)}$ in the denominator is of order c^{-2} compared to that of $T^{00(0)}_{(\mathrm{m})}$ (which is of order c^2) and is given by c^2 times the right-hand side of eq. (A.16). We cannot, however, apply the result for $M_{(+)}$ directly to $T^{00(0)}_{(+)}$ in the numerator, since we have integrated over $\boldsymbol{x}$ in the calculations for the mass. But we can calculate $T^{00(0)}_{(+)}$ directly from eq. (32) We have done this. To the required order the result is:

$$T^{00(0)}_{(+)}(\boldsymbol{x}, t) = \frac{1}{8\pi} \int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)\, \rho^{(0)}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \mathrm{d}^{(3)}\boldsymbol{x}'. \tag{A.22}$$

If we compare this result with the formula for the centre of gravity given *e.g.* in ref. 11 in the same approximation with the help of the Darwin Lagrangian, we find agreement, except for the fact that all quantities in our formulae have to be considered in the m.r.s., whereas the formulae given in ref. 11 are supposed to be valid in an arbitrary frame, which is, however, only allowed within the approximations made above.

We shall now calculate the contribution of $T^{\alpha\beta}_{(+)}$ to the space–space part of $\Omega^{\alpha\beta}$, eq. (36). In the m.r.s. this contribution is given by:

$$\Omega^{kl(0)}_{(+)}(t) = \frac{1}{c}\int^{\infty} \{[x^k - X^{k(0)}(t)]\, T^{l0(0)}_{(+)}(\boldsymbol{x}, t) - [x^l - X^{l(0)}(t)]\, T^{k0(0)}_{(+)}(\boldsymbol{x}, t)\}\, \mathrm{d}^{(3)}\boldsymbol{x}. \tag{A.23}$$

Because $T^{l0}_{(\mathrm{m})}$ is of order c, as can easily be seen from eq. (II.13), we need $T^{l0}_{(\mathrm{m})}$ only to order c^{-1}. Starting from eq. (32), we get, after a rather lengthy calculation, the following results in the m.r.s.:

$$\begin{aligned} T^{l0(0)}_{(+)}(\boldsymbol{x}, t) &= \frac{1}{8\pi c}\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)\, i^{l(0)}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \mathrm{d}^{(3)}\boldsymbol{x}' \\ &+ \frac{1}{8\pi c}\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}', t)(x^l - x'^l)\, \boldsymbol{i}^{(0)}(\boldsymbol{x}', t)\cdot(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, \mathrm{d}^{(3)}\boldsymbol{x}' \\ &- \frac{1}{16\pi}\frac{\partial}{\partial t}\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)\, \rho^{(0)}(\boldsymbol{x}', t)(x^l - x'^l)}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \mathrm{d}^{(3)}\boldsymbol{x}' + \ldots, \end{aligned} \tag{A.24}$$

where the points indicate terms of the form

$$\frac{1}{c}\frac{\partial}{\partial x_{l'}}\int^{\infty} (x^{l'} - x'^{l'})(x^l - x'^l)\{\ldots\ldots\}\, \mathrm{d}^{(3)}\boldsymbol{x}', \tag{A.25}$$

which do not contribute to eq. (A.23). Substituting (A.24) into (A.23) we get:

$$\begin{aligned} \Omega^{kl(0)}_{(+)}(t) &= \frac{1}{8\pi c^2}\int^{\infty}\!\!\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \\ &\times \{[x^k - X^{k(0)}(t)]\, i^{l(0)}(\boldsymbol{x}', t) - (lk)\}\, \mathrm{d}^{(3)}\boldsymbol{x}'\, \mathrm{d}^{(3)}\boldsymbol{x} \\ &+ \frac{1}{8\pi c^2}\int^{\infty}\!\!\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}', t)\, \boldsymbol{i}^{(0)}(\boldsymbol{x}, t)\cdot(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \\ &\times \{[x^k - X^{k(0)}(t)](x^l - x'^l) - (lk)\}\, \mathrm{d}^{(3)}\boldsymbol{x}'\, \mathrm{d}^{(3)}\boldsymbol{x} \\ &- \frac{1}{16\pi c^2}\frac{\partial}{\partial t}\int^{\infty}\!\!\int^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)\, \rho^{(0)}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \\ &\times \{[x^k - X^{k(0)}(t)](x^l - x'^l) - (lk)\}\, \mathrm{d}^{(3)}\boldsymbol{x}'\, \mathrm{d}^{(3)}\boldsymbol{x}. \end{aligned} \tag{A.26}$$

By interchanging the variables $\boldsymbol{x}$ and $\boldsymbol{x}'$ and using the property:

$$\begin{aligned}[x'^k - X^{k(0)}(t)](x^l - x'^l) - (lk) \\ = [x^k + (x'^k - x^k) - X^{k(0)}(t)](x^l - x'^l) - (lk) \\ = [x^k - X^{k(0)}(t)](x^l - x'^l) - (lk),\end{aligned} \tag{A.27}$$

we can rewrite the second double integral at the rigth-hand side of eq. (A.26) as:

$$\frac{1}{8\pi c^2}\int\limits^{\infty}\int\limits^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)\, \boldsymbol{i}^{(0)}(\boldsymbol{x}', t)\cdot(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \times \{[x^k - X^{k(0)}(t)](x^l - x'^l) - (lk)\}\, \mathrm{d}^{(3)}\boldsymbol{x}'\, \mathrm{d}^{(3)}\boldsymbol{x}. \tag{A.28}$$

The last term at the right-hand side of eq. (A.26) is zero, because the expression changes sign if we interchange $\boldsymbol{x}$ and $\boldsymbol{x}'$ and use also (A.27).

If we introduce the tensor:

$$T(\boldsymbol{x} - \boldsymbol{x}') \equiv U + \frac{(\boldsymbol{x} - \boldsymbol{x}')(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2}, \tag{A.29}$$

where U is the 3-dimensional unit tensor, then eq. (A.26) can be written in the form:

$$\Omega^{kl(0)}_{(+)}(t) = \frac{1}{8\pi c^2}\int\limits^{\infty}\int\limits^{\infty} \frac{\rho^{(0)}(\boldsymbol{x}, t)}{|\boldsymbol{x} - \boldsymbol{x}'|}\{[x^k - X^{k(0)}(t)] \times [\boldsymbol{i}^{(0)}(\boldsymbol{x}', t)\cdot T(\boldsymbol{x} - \boldsymbol{x}')]^l - (lk)\}\, \mathrm{d}^{(3)}\boldsymbol{x}'\, \mathrm{d}^{(3)}\boldsymbol{x}, \tag{A.30}$$

which is equivalent to the result obtained by de Groot and Suttorp [see ref. 5, eq. (II.13)], the only difference being that we have arbitrary charge and current distributions, which may be continuous or may consist of points particles, subject only to the requirement that the distribution has a finite extension in space for arbitrary fixed time.

APPENDIX III

A method for calculating the "minus" field terms (47A) *and* (47B). In general we have to calculate functions of the form (*cf.* pgs. 61–62) †:

$$G^{(n)}(\tau; \varepsilon) = \int\limits_{-\infty}^{+\infty} g(\tau')\, \delta^{(n)}\{h(\tau')\}\, s\{X^0(\tau) - X^0(\tau')\}\, \mathrm{d}\tau', \quad (n = 0, 1, 2, \ldots), \tag{A.31}$$

† We shall write shortly $g(\tau')$ instead of $g(\tau', \tau)$, [see eq. (48)]. We shall also omit the index (n) for $n = 0$ in order to avoid confusion with the symbol (0) for quantities in the m.r.s. (*cf.* apps. II and IV). As in the main text we also use here n-dashes instead of the number n in the expression $\delta^{(n)}\{h(\tau')\}$, *e.g.* $\delta^{(1)} = \delta'$, *etc.*

where we define:

$$h(\tau') \equiv [X(\tau) - X(\tau')]^2 + \varepsilon^2, \qquad (\varepsilon > 0), \tag{A.32}$$

and the sign-function $s = +1$ when the argument is positive and $s = -1$ when the argument is negative. We can expand the retarded or advanced solution of eq. (A.32) in powers of the retardation and advancement time σ respectively; σ is taken positive, but may have different values for the retarded and advanced solutions. We get

$$[h(\tau')]_{\tau'=\tau\mp\sigma} = 0 = \sum_{n=0}^{\infty} \frac{(\mp 1)^n}{n!} \sigma^n [h^{(n)}(\tau')]_{\tau'=\tau};$$

$$h^{(n)}(\tau') = \frac{\mathrm{d}^n h(\tau')}{\mathrm{d}\tau'^n}. \tag{A.33}$$

Eq. (A.33) constitutes, together with (A.32), a relation between σ and ε. The upper sign indicates the retarded solution and the lower sign the advanced.

We shall use the following notations for the scalar product of two four-vectors U^α and V^α:

$$U^\alpha(\tau)\, V_\alpha(\tau) = (UV) \qquad \text{and} \qquad U^\alpha(\tau)\, U_\alpha(\tau) = (U)^2. \tag{A.34}$$

For the four-velocity we have the relations

$$(U)^2 = -c^2;\ (U\dot{U}) = 0;\ (U\ddot{U}) + (\dot{U})^2 = 0, \text{ etc.} \tag{A.35}$$

By differentiating relation (A.35) a number of times we can obtain other relations between the derivatives of the four-velocity. From (A.32) and (A.35) we get for $h^{(n)}(\tau')$:

$$[h(\tau')]_{\tau'=\tau} = \varepsilon^2;\ [h^{(1)}(\tau')]_{\tau'=\tau} = 0;$$

$$[h^{(2)}(\tau')]_{\tau'=\tau} = -2c^2,\ [h^{(3)}(\tau'))]_{\tau'=\tau} = 0,$$

$$[h^{(4)}(\tau')]_{\tau'=\tau} = -2(\dot{U})^2,\ [h^{(5)}(\tau')]_{\tau'=\tau} = -10(\dot{U}\ddot{U}),$$

$$[h^{(6)}(\tau')]_{\tau'=\tau} = -16(\ddot{U})^2 - 18(\dot{U}\overset{(3)}{U}), \tag{A36}$$

(where instead of n-dots we put the number (n) above U for $n \geqslant 3$).

From (A.33) and (A.36) it follows that:

$$\sigma^2 + \frac{(\dot{U})^2}{12c^2}\sigma^4 \mp \frac{(\dot{U}\ddot{U})}{12c^2}\sigma^5 + \left[\frac{(\ddot{U})^2}{45c^2} + \frac{(\dot{U}\overset{(3}{U})}{40c^2}\right]\sigma^6 = \left(\frac{\varepsilon}{c}\right)^2. \tag{A.37}$$

It turns out that in all cases to be considered here, σ will be required to order ε^5, so that eq. (A.37) is sufficient. But the procedure can be applied to any order. Eq. (37) can now be solved for σ as a function of ε by successive iterations, yielding the following result:

$$\sigma = \frac{\varepsilon}{c} - \frac{(\dot{U})^2}{24c^2}\left(\frac{\varepsilon}{c}\right)^3 \mp \frac{(\dot{U}\ddot{U})}{24c^2}\left(\frac{\varepsilon}{c}\right)^4$$

$$+ \frac{1}{2}\left[7\left(\frac{(\dot{U})^2}{24c^2}\right)^2 - \frac{(\ddot{U})^2}{45c^2} - \frac{(\dot{U}\overset{(3)}{U})}{40c^2}\right]\left(\frac{\varepsilon}{c}\right)^5. \tag{A.38}$$

We take $n = 0$ in (A.31) and integrate the expression:

$$G(\tau;\varepsilon) \equiv \int_{-\infty}^{+\infty} g(\tau')\,\delta\{h(\tau')\}\,s\{X^0(\tau) - X^0(\tau')\}\,\mathrm{d}\tau'$$

$$= \left[\frac{g(\tau')}{|h^{(1)}(\tau')|}\right]_{\tau'=\tau-\sigma} - \left[\frac{g(\tau')}{|h^{(1)}(\tau')|}\right]_{\tau'=\tau+\sigma}. \tag{A.39}$$

Each member of the right-hand side of this equation can now be expanded in a power series of σ. In the numerator we shall do this to order σ^p and in the denominator to order σ^q (p and q are to be determined below):

$$\left[\frac{g(\tau')}{|h^{(1)}(\tau')|}\right]_{\tau'=\tau\mp\sigma} = \left(\sum_{k=0}^{p} \frac{(\mp 1)^k}{k!}\,\sigma^k\,[g^{(k)}(\tau')]_{\tau'=\tau}\right)$$

$$\times \left|\sum_{l=0}^{q} \frac{(\mp 1)^l}{l!}\,\sigma^l [h^{(l+1)}(\tau')]_{\tau'=\tau}\right|^{-1}, \tag{A.40}$$

with $g^{(k)}(\tau') = \mathrm{d}^k g(\tau')/\mathrm{d}\tau'^k$ and for $k = 0$ we omit the index (k).

Since σ is a power series of ε, the right-hand side of eq. (A.40) can also be considered as a function of ε. Let m be the highest power of this function, then it follows from (A.40), together with (A.36), that in general we must take $p = m + 1$ and $q = m + 2$, while σ must be calculated to order $m + 2$ in ε. Now the highest value of n in the function $G^{(n)}(\tau)$ to be calculated is $n = 3$. From the relation:

$$G^{(n)}(\tau) = \lim_{\varepsilon\to 0}\left(\frac{1}{2\varepsilon}\,\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right)^n G(\tau;\varepsilon), \tag{A.41}$$

where the limit must be taken after the differentiation, it follows that $G(\tau;\varepsilon)$, eq. (A.39), as a series in powers of ε must be calculated to order $m = 6$ (in general $m = 2n$). So we must take $p = 7$ and $q = 8$.

If eq. (A.40) is calculated for these values of p and q, the result is

$$\left[\frac{g(\tau')}{|h^{(1)}(\tau')|}\right]_{\tau'=\tau\mp\sigma} = \frac{g}{2c^2\sigma} \mp \frac{g^{(1)}}{2c^2} + \left[-\frac{(\dot{U})^2}{12c^4}g + \frac{g^{(2)}}{4c^2}\right]\sigma$$

$$+ \left[\mp\frac{1}{12c^2}g^{(3)} \pm \frac{(\dot{U})^2}{12c^4}g^{(1)} \pm \frac{5}{48c^4}(\dot{U}\ddot{U})\,g\right]\sigma^2$$

$$
\begin{aligned}
&+\left[\frac{g^{(4)}}{48c^2}-\frac{(\dot U)^2}{24c^4}g^{(2)}-\frac{5}{48c^4}(\dot U\ddot U)\,g^{(1)}\right.\\
&\left.+\frac{g}{2c^2}\left\{\frac{h^{(6)}}{2c^2 5!}+\left(\frac{(\dot U)^2}{6c^2}\right)^2\right\}\right]\sigma^3\\
&+\left[\mp\frac{g^{(5)}}{2c^2 5!}\pm\frac{(\dot U)^2}{72c^4}g^{(3)}\pm\frac{5(\dot U\ddot U)}{96c^4}g^{(2)}\right.\\
&\mp\frac{g^{(1)}}{2c^2}\left\{\frac{h^{(6)}}{2c^2 5!}+\left(\frac{(\dot U)^2}{6c^2}\right)^2\right\}\\
&\left.\mp\frac{g}{2c^2}\left\{\frac{h^{(7)}}{2c^2 6!}+\frac{5(\dot U)^2(\dot U\ddot U)}{72c^4}\right\}\right]\sigma^4\\
&+\left[\frac{g^{(6)}}{2c^2 6!}-\frac{(\dot U)^2 g^{(4)}}{12c^4 4!}-\frac{5(\dot U\ddot U)}{12c^2 4!}g^{(3)}\right.\\
&+\frac{g^{(2)}}{4c^2}\left\{\frac{h^{(6)}}{2c^2 5!}+\left(\frac{(\dot U)^2}{6c^2}\right)^2\right\}+\left[\frac{g^{(1)}}{2c^2}\left\{\frac{h^{(7)}}{2c^2 6!}+\frac{5(\dot U)^2(\dot U\ddot U)}{72c^4}\right\}\right.\\
&\left.\left.+\frac{g}{2c^2}\left\{\frac{h^{(8)}}{2c^2 7!}-\frac{(\dot U)^2 h^{(6)}}{c^4 6!}-\left(\frac{(\dot U)^2}{6c^2}\right)^3+\left(\frac{5}{24c^2}\right)^2(\dot U\ddot U)^2\right\}\right]\right]\sigma^5\\
&+\left[\mp\frac{g^{(7)}}{2c^2 7!}\pm\frac{(\dot U)^2 g^{(5)}}{2c^4 6!}\pm\frac{5(\dot U\ddot U)}{48c^4 4!}g^{(4)}\right.\\
&\mp\frac{g^{(3)}}{2c^2 3!}\left\{\frac{h^{(6)}}{2c^2 5!}+\left(\frac{(\dot U)^2}{6c^2}\right)^2\right\}\mp\frac{g^{(2)}}{4c^2}\left\{\frac{h^{(7)}}{2c^2 6!}+\frac{5(\dot U)^2(\dot U\ddot U)}{3c^4 4!}\right\}\\
&\mp\frac{g^{(1)}}{2c^2}\left\{\frac{h^{(8)}}{2c^2 7!}-\frac{(\dot U)^2 h^{(6)}}{c^4 6!}-\left(\frac{(\dot U)^2}{6c^2}\right)^3+\left(\frac{5}{24c^2}\right)^2(\dot U\ddot U)^2\right\}\\
&\left.\pm\frac{g}{2c^2}\left\{-\frac{h^{(9)}}{2c^2 8!}+\frac{(\dot U)^2 h^{(7)}}{6c^4 6!}+\frac{5(\dot U\ddot U)\,h^{(6)}}{4c^4 6!}-\frac{5(\dot U\ddot U)}{8c^2}\left(\frac{(\dot U)^2}{6c^2}\right)^2\right\}\right]\sigma^6,
\end{aligned}
\tag{A.42}
$$

with $g^{(n)}=[g^{(n)}(\tau')]_{\tau'=\tau}$ and $h^{(n)}=[h^{(n)}(\tau')]_{\tau'=\tau}$; we omit the index for $n=0$.

For the function $G^{(n)}(\tau)$ we need only the following cases†:

$$G(\tau)=\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} g(\tau')\,\delta\{h(\tau')\}\,s\{X^0(\tau)-X^0(\tau')\}\,\mathrm{d}\tau', \tag{A.43}$$

$$G^{(1)}(\tau)=\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} g(\tau')\,\delta'\{h(\tau')\}\,s\{X^0(\tau)-X^0(\tau')\}\,\mathrm{d}\tau', \tag{A.44}$$

$$G^{(2)}(\tau)=\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} g(\tau')\,\delta''\{h(\tau')\}\,s\{X^0(\tau)-X^0(\tau')\}\,\mathrm{d}\tau',\ ([g(\tau')]_{\tau'=\tau}=0) \tag{A.45}$$

† See footnote on pg. 76.

$$G^{(3)}(\tau) = \lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} g(\tau')\, \delta'''\{h(\tau')\}\, s\{X^0(\tau) - X^0(\tau')\}\, \mathrm{d}\tau',$$

$$([g(\tau')]_{\tau'=\tau} = [g^{(1)}(\tau')]_{\tau'=\tau} = [g^{(2)}(\tau')]_{\tau'=\tau} = 0). \tag{A.46}$$

To obtain $G(\tau)$ we need $m = 0$, so that σ is required to second order in ε. From (A.38), (A.39) and (A.42) we then get

$$G(\tau) = -\frac{1}{c^2}[g^{(1)}(\tau')]_{\tau'=\tau}. \tag{A.47}$$

For $G^{(1)}(\tau)$ we have $m = 2$, so that we need eq. (A.42) to order σ^2 and eq. (A.38) to order ε^4. The result is

$$G^{(1)}(\tau) = \frac{(\dot{U}\dot{U})}{6c^6}[g(\tau')]_{\tau'=\tau} + \frac{(\dot{U})^2}{6c^6}[g^{(1)}(\tau')]_{\tau'=\tau} - \frac{1}{6c^4}[g^{(3)}(\tau')]_{\tau'=\tau}. \tag{A.48}$$

In order to calculate $G^{(2)}(\tau)$, subject to $[g(\tau')]_{\tau'=\tau} = 0$, we need eq. (A.42) to order σ^4 and eq. (A.38) to order ε^4, as can easily be seen.
The result is:

$$G^{(2)}(\tau) = \left[-\frac{(\dot{U})^4}{12c^{10}} + \frac{2(\dot{U})^2}{15c^8} + \frac{3(\dot{U}\overset{(3)}{U})}{15c^8}\right][g^{(1)}(\tau')]_{\tau'=\tau} + \frac{(\dot{U}\dot{U})}{4c^8}[g^{(2)}(\tau')]_{\tau'=\tau} + \frac{(\dot{U})^2}{12c^8}[g^{(3)}(\tau')]_{\tau'=\tau} - \frac{1}{60c^6}[g^{(5)}(\tau')]_{\tau'=\tau},\ ([g(\tau')]_{\tau'=\tau} = 0). \tag{A.49}$$

From eq. (A.42) and the conditions for $g(\tau')$ in (A.46), it follows that σ must be given to the fifth order in ε and eq. (A.42) to the sixth order in σ. The result for $G^{(3)}(\tau)$ is:

$$G^{(3)}(\tau) = \left[-\frac{5(\dot{U})^4}{72c^{12}} + \frac{4(\dot{U})^2}{45c^{10}} + \frac{(\dot{U}\overset{(3)}{U})}{10c^{10}}\right][g^{(3)}(\tau')]_{\tau'=\tau} + \frac{(\dot{U}\dot{U})}{12c^{10}}[g^{(4)}(\tau')]_{\tau'=\tau} + \frac{(\dot{U})^2}{60c^{10}}[g^{(5)}(\tau')]_{\tau'=\tau} - \frac{6}{7!c^8}[g^{(7)}(\tau')]_{\tau'=\tau},$$

$$([g(\tau')]_{\tau'=\tau} = [g^{(1)}(\tau')]_{\tau'=\tau} = [g^{(2)}(\tau')]_{\tau'=\tau} = 0). \tag{A.50}$$

With the help of eqs. (A.47)–(A.50) we can calculate the expressions (47A) and (47B). The result for $f^{\alpha\beta}_{(-)}\{X(\tau)\}$ is‡:

‡ Instead of n dots we put the number (n) above $\mu^{\alpha\beta}$ for $n \geqslant 3$.

$$f^{\alpha\beta}_{(-)}\{X(\tau)\} = \frac{1}{4\pi}\left\{\frac{2e}{3c^4}(-\dot{U}^\alpha U^\beta + U^\alpha \dot{U}^\beta)\right.$$

$$-\frac{2}{3}\left(\frac{1}{c^6}(\dot{U}\dot{U})\,\mu^{\alpha\beta} + \frac{1}{c^6}(\dot{U})^2\dot{\mu}^{\alpha\beta} - \frac{1}{c^4}\overset{(3)}{\mu^{\alpha\beta}}\right)$$

$$+\left[\frac{2}{c^8}(\dot{U}\dot{U})\,U^\alpha\mu^{\beta\gamma}U_\gamma + \frac{1}{c^8}(\dot{U})^2\,\dot{U}^\alpha\mu^{\beta\gamma}U_\gamma + \frac{1}{c^8}(\dot{U})^2\,U^\alpha\mu^{\beta\gamma}\dot{U}_\gamma\right.$$

$$+\frac{2}{c^8}(\dot{U})^2\,U^\alpha\dot{\mu}^{\beta\gamma}U_\gamma - \frac{1}{3c^6}U^\alpha\mu^{\beta\gamma}\overset{(3)}{U}_\gamma - \frac{1}{3c^6}\overset{(3)}{U}{}^\alpha\mu^{\beta\gamma}U_\gamma$$

$$-\frac{2}{3c^6}\ddot{U}^\alpha\mu^{\beta\gamma}\dot{U}_\gamma - \frac{2}{3c^6}\dot{U}^\alpha\mu^{\beta\gamma}\ddot{U}_\gamma - \frac{4}{3c^6}\ddot{U}^\alpha\dot{\mu}^{\beta\gamma}U_\gamma$$

$$-\frac{4}{3c^6}U^\alpha\dot{\mu}^{\beta\gamma}\ddot{U}_\gamma - \frac{4}{3c^6}U^\alpha\,\overset{(3)}{\mu^{\beta\gamma}}U_\gamma - \frac{2}{c^6}\dot{U}^\alpha\dot{\mu}^{\beta\gamma}\dot{U}_\gamma$$

$$\left.\left.-\frac{2}{c^6}\dot{U}^\alpha\ddot{\mu}^{\beta\gamma}U_\gamma - \frac{2}{c^6}U^\alpha\ddot{\mu}^{\beta\gamma}\dot{U}_\gamma\right]_{-(\beta\alpha)}\right\}, \tag{A.51}$$

where the subscripts $-(\beta\alpha)$ indicate that we subtract the expression between square brackets with α and β interchanged. This is the same result as obtained by Harish-Chandra[7]) in a somewhat different way. If $\partial_\gamma f^{\alpha\beta}_{(-)}\{X(\tau)\}$, eq. (47B), is calculated, or rather $[\partial_\gamma f^{\alpha\beta}_{(-)}\{X(\tau)\}]\,\mu_\beta{}^\gamma(\tau)$, which is a slightly simpler expression, because of the antisymmetry of $\mu_\beta{}^\gamma(\tau)$, we obtain:

$$-[\partial_\gamma f^{\alpha\beta}_{(-)}\{X(\tau)\}]\,\mu_\beta{}^\gamma(\tau)$$

$$= \frac{e}{4\pi c^4}\left\{\frac{1}{3c^2}(\dot{U}\ddot{U})\,\mu^{\alpha\gamma}U_\gamma + \frac{1}{3c^2}(\dot{U})^2\mu^{\alpha\gamma}\dot{U}_\gamma - \tfrac{1}{3}\mu^{\alpha\gamma}\overset{(3)}{U}_\gamma\right.$$

$$+U^\alpha\left[\frac{1}{c^4}(\dot{U})^2\,U_\beta\mu^{\beta\gamma}\dot{U}_\gamma - \frac{1}{c^2}U_\beta\mu^{\beta\gamma}\overset{(3)}{U}_\gamma - \frac{2}{3c^2}\dot{U}_\beta\mu^{\beta\gamma}\ddot{U}_\gamma\right]$$

$$\left.-\frac{4}{3c^2}\ddot{U}^\alpha U_\beta\mu^{\beta\gamma}\dot{U}_\gamma - \frac{2}{3c^2}\dot{U}^\alpha U_\beta\mu^{\beta\gamma}\ddot{U}_\gamma\right\}$$

$$+\frac{1}{4\pi c^6}\left\{\left[\frac{2}{3c^4}(\dot{U})^4 - \frac{16}{15c^2}(\ddot{U})^2 - \frac{6}{5c^2}(\dot{U}\overset{(3)}{U})\right]\mu^{\alpha\beta}\mu_{\beta\gamma}U^\gamma\right.$$

$$-\frac{2}{c^2}(\dot{U}\ddot{U})\,\mu^{\alpha\beta}\mu_{\beta\gamma}\dot{U}^\gamma - \frac{2}{c^2}(\dot{U}\ddot{U})\,\dot{\mu}^{\alpha\beta}\mu_{\beta\gamma}U^\gamma - \tfrac{2}{3}(\dot{U})^2\mu^{\alpha\beta}\mu_{\beta\gamma}\ddot{U}^\gamma$$

$$-\frac{1}{c^2}(\dot{U})^2\dot{\mu}^{\alpha\beta}\mu_{\beta\gamma}\dot{U}^\gamma - \frac{(\dot{U})^2}{c^2}\ddot{\mu}^{\alpha\beta}\mu_{\beta\gamma}U^\gamma + \tfrac{2}{15}\mu^{\alpha\beta}\mu_{\beta\gamma}\overset{(4)}{U}{}^\gamma$$

$$+\tfrac{1}{3}\dot{\mu}^{\alpha\beta}\mu_{\beta\gamma}\overset{(3)}{U}{}^\gamma + \tfrac{2}{3}\ddot{\mu}^{\alpha\beta}\mu_{\beta\gamma}\ddot{U}^\gamma + \tfrac{2}{3}\overset{(3)}{\mu^{\alpha\beta}}\mu_{\beta\gamma}\dot{U}^\gamma + \tfrac{1}{3}\overset{(4)}{\mu^{\alpha\beta}}\mu_{\beta\gamma}U^\gamma$$

$$
\begin{aligned}
&-\frac{2}{c^2}(\dot U\ddot U)\,\mu^{\alpha\beta}\dot\mu_{\beta\gamma}U^\gamma-\frac{1}{c^2}(\dot U)^2\,\mu^{\alpha\beta}\,\dot\mu_{\beta\gamma}\dot U^\gamma-\frac{1}{c^2}(\dot U)^2\,\mu^{\alpha\beta}\ddot\mu_{\beta\gamma}U^\gamma\\
&+\tfrac13\mu^{\alpha\beta}\dot\mu_{\beta\gamma}\overset{(3)}{U}{}^\gamma+\tfrac23\mu^{\alpha\beta}\ddot\mu_{\beta\gamma}\ddot U^\gamma+\tfrac23\mu^{\alpha\beta}\overset{(3)}{\mu}_{\beta\gamma}\dot U^\gamma+\tfrac13\mu^{\alpha\beta}\overset{(4)}{\mu}_{\beta\gamma}U^\gamma\\
&+\left[\frac{1}{3c^4}(\dot U)^4-\frac{8}{15c^2}(\ddot U)^2-\frac{3}{5c^2}(\dot U\overset{(3)}{U})\right]U^\alpha\mu^{\beta\gamma}\mu_{\gamma\beta}\\
&-\frac{2}{c^2}(\dot U\ddot U)\,U^\alpha\mu^{\beta\gamma}\dot\mu_{\gamma\beta}-\frac{1}{c^2}(\dot U)^2\,U^\alpha\mu^{\beta\gamma}\ddot\mu_{\gamma\beta}+\tfrac13U^\alpha\mu^{\beta\gamma}\overset{(4)}{\mu}_{\gamma\beta}\\
&-\frac{1}{c^2}(\dot U\ddot U)\,\dot U^\alpha\,\mu^{\beta\gamma}\mu_{\gamma\beta}-\frac{1}{c^2}(\dot U)^2\,\dot U^\alpha\mu^{\beta\gamma}\dot\mu_{\gamma\beta}+\tfrac23\dot U^\alpha\mu^{\beta\gamma}\overset{(3)}{\mu}_{\gamma\beta}\\
&-\frac{1}{3c^2}(\dot U)^2\,\ddot U^\alpha\mu^{\beta\gamma}\mu_{\gamma\beta}+\tfrac23\ddot U^\alpha\mu^{\beta\gamma}\ddot\mu_{\gamma\beta}+\tfrac13\,\overset{(3)}{U}{}^\alpha\mu^{\beta\gamma}\dot\mu_{\gamma\beta}+\tfrac1{15}\,\overset{(4)}{U}{}^\alpha\mu^{\beta\gamma}\mu_{\gamma\beta}\\
&+\left[\frac{10}{3c^4}(\dot U)^4-\frac{64}{15c^2}(\ddot U)^2-\frac{24}{5c^2}(\dot U\overset{(3)}{U})\right]U^\alpha U_\gamma U_\delta\mu^\gamma{}_\beta\mu^{\beta\delta}\\
&-\frac{1}{c^4}(\dot U\ddot U)\,\mu^\gamma{}_\beta[8(\dot U^\alpha U_\gamma U_\delta+U^\alpha\dot U_\gamma U_\delta\\
&+U^\alpha U_\gamma\dot U_\delta)\,\mu^{\beta\delta}+16U^\alpha U_\gamma U_\delta\dot\mu^{\beta\delta}]\\
&-\frac{1}{c^4}(\dot U)^2\,\mu^\gamma{}_\beta\,[\tfrac83(\ddot U^\alpha U_\gamma U_\delta+U^\alpha\ddot U_\gamma U_\delta+U^\alpha U_\gamma\ddot U_\delta)\,\mu^{\beta\delta}\\
&+4(\dot U^\alpha\dot U_\gamma U_\delta+\dot U^\alpha U_\gamma\dot U_\delta+U^\alpha\dot U_\gamma\dot U_\delta)\,\mu^{\beta\gamma}\\
&+8(\dot U^\alpha U_\gamma U_\delta+U^\alpha\dot U_\gamma U_\delta+U^\alpha U^\gamma\dot U_\delta)\,\dot\mu^{\beta\delta}+8U^\alpha U_\gamma U_\delta\ddot\mu^{\beta\delta}]\\
&+\frac{1}{c^2}\mu^\gamma{}_\beta[\tfrac23\,(\overset{(4)}{U}{}^\alpha U_\gamma U_\delta+U^\alpha\,\overset{(4)}{U}_\gamma U_\delta+U^\alpha U_\gamma\,\overset{(4)}{U}_\delta)\,\mu^{\beta\delta}\\
&+(\,\overset{(3)}{U}{}^\alpha\dot U_\gamma U_\delta+\overset{(3)}{U}{}^\alpha U_\gamma\dot U_\delta+\dot U^\alpha\,\overset{(3)}{U}_\gamma U_\delta+\dot U^\alpha\overset{(3)}{U}_\gamma\,U_\delta\\
&+U^\alpha\,\overset{(3)}{U}_\gamma\dot U_\delta+U^\alpha\dot U_\gamma\,\overset{(3)}{U}_\delta)\,\mu^{\beta\delta}\\
&+\tfrac43(\ddot U^\alpha\ddot U_\gamma U_\delta+\ddot U^\alpha U_\gamma\ddot U_\delta+U^\alpha\ddot U_\gamma\ddot U_\delta)\,\mu^{\beta\delta}+2(\ddot U^\alpha\dot U_\gamma\dot U^\delta\\
&+\dot U^\alpha\ddot U_\gamma\dot U_\delta+\dot U^\alpha\dot U_\gamma\ddot U_\delta)\,\mu^{\beta\delta}\\
&+2(U^\alpha\,\overset{(3)}{U}_\gamma U_\delta+U^\alpha U_\gamma\,\overset{(3)}{U}_\delta+\overset{(3)}{U}{}^\alpha U_\gamma U_\delta)\,\dot\mu^{\beta\delta}+2U^\alpha U^\gamma U_\delta\,\overset{(4)}{\mu}{}^{\beta\delta}\\
&+4(\ddot U^\alpha\dot U_\gamma U_\delta+\ddot U^\alpha U_\gamma\dot U_\delta+\dot U^\alpha\ddot U_\gamma U_\delta+\dot U^\alpha U_\gamma\ddot U_\delta\\
&+U^\alpha\ddot U_\gamma\dot U_\delta+U^\alpha\dot U_\gamma\ddot U_\delta)\,\dot\mu^{\beta\delta}\\
&+4(\ddot U^\alpha U_\gamma U_\delta+U^\alpha\ddot U_\gamma U_\delta+U^\alpha U_\gamma\ddot U_\delta)\,\ddot\mu^{\beta\delta}+4(\dot U^\alpha U_\gamma U_\delta\\
&+U^\alpha\dot U_\gamma U_\delta+U^\alpha U_\gamma\dot U_\delta)\,\overset{(3)}{\mu}{}^{\beta\delta}\\
&+6\dot U^\alpha\dot U_\gamma\dot U_\delta\dot\mu^{\beta\delta}+6(\dot U^\alpha\dot U_\gamma U_\delta+\dot U^\alpha U_\gamma\dot U_\delta+U^\alpha\dot U_\gamma\dot U_\delta)\,\ddot\mu^{\beta\delta}]\Big\}.
\end{aligned}
\qquad (A.52)
$$

So far we have carried out all calculations quite generally. Now we shall make, however, the *assumption* that we may neglect *all* terms in the above formulae, containing any derivative of the four-velocity U^α with respect to the eigentime τ. It can be shown, that in the case of atoms or molecules, with which we deal, this assumption is certainly justified. Physically it means that one neglects all radiative effects on the motion of the atoms (molecules) due to their *barycentric* accelerations, and only considers the damping due to the vibrations of their dipole moments.

Instead of (A.51) and (A.52), we then obtain the much simpler expressions:

$$f^{\alpha\beta}_{(-)}\{X(\tau)\} = \frac{1}{4\pi c^4}\left\{\tfrac{2}{3}\overset{(3)}{\mu}{}^{\alpha\beta} - \frac{4}{3c^2}\left(U^\alpha \overset{(3)}{\mu}{}^{\beta\gamma}U_\gamma - U^\beta \overset{(3)}{\mu}{}^{\alpha\gamma}U_\gamma\right)\right\}, \tag{A.53}$$

and

$$-\left[\partial_\gamma f^{\alpha\beta}_{(-)}\{X(\tau)\}\right]\mu_\beta{}^\gamma(\tau) = \frac{1}{4\pi c^6}\left\{\tfrac{1}{3}\overset{(4)}{\mu}{}^{\alpha\beta}\mu_{\beta\gamma}U^\gamma + \tfrac{1}{3}\mu^{\alpha\beta}\overset{(4)}{\mu}{}_{\beta\gamma}U^\gamma\right.$$

$$\left. + \tfrac{1}{3}U^\alpha\mu^{\beta\gamma}\overset{(4)}{\mu}{}_{\gamma\beta} + \frac{2}{c^2}U^\alpha U_\beta\mu^{\beta\gamma}\overset{(4)}{\mu}{}_{\gamma\delta}U^\delta\right\}, \tag{A.54}$$

results, which we could also have obtained, of course, by assuming from the beginning, that $\overset{(n)}{U}{}^\alpha = 0$ $(n = 1, 2, \ldots)$. We have preferred, however, the more elaborate way of calculation, made above, since we then also obtained formulae, which were valid in cases, where radiation damping effects, due to the barycentric accelerations of particles are of importance. In connection herewith we may remark that the calculations of Harish-Chandra [7]) were not sufficient to give the explicit expression for the derivative of the minus field tensor $\partial_\gamma f^{\alpha\beta}_{(-)}$. Therefore we have done the calculations once more, giving first a series expansion to an order that is in general sufficient, and then specializing in each case to the necessary order, so that all contributions of the same order are taken into account.

APPENDIX IV

Radiated energy-momentum and angular momentum. We shall follow an analogous procedure as Rohrlich [12]) and Cohn [13]) have applied calculating the radiation rate of classical point charges (see also ref. 14 for the extension to dipoles). We shall still have to carry these calculations a little further, since we also want to calculate the "radiation" of angular momentum.

For the charged dipole point atom the retarded four-potential at a point x is:

$$A^\alpha_{(\mathrm{ret})}(x) = \frac{e}{2\pi}\int_{-\infty}^{+\infty}\delta[\{x - X(\tau)\}^2]\,\theta\{x^0 - X^0(\tau)\}\,U^\alpha(\tau)\,\mathrm{d}\tau$$

$$+\frac{1}{\pi}\int_{-\infty}^{+\infty}\delta'[\{x-X(\tau)\}^2]\,\theta\{x^0-X^0(\tau)\}\,\mu^{\alpha\gamma}(\tau)(x_\gamma-X_\gamma(\tau))\,\mathrm{d}\tau, \tag{A.55}$$

where $\theta = 1$, when the argument is positive, and $\theta = 0$, when the argument is negative. Introducing the following notations and definitions[12]):

$$U^\alpha \equiv U^\alpha(\tau_{(\mathrm{ret})});\, n^\alpha\colon n^\alpha n_\alpha = 1,\, n^\alpha U_\alpha = 0;\, R^\alpha \equiv x^\alpha - X^\alpha(\tau_{(\mathrm{ret})});$$

$$R^\alpha R_\alpha = 0;\, R^\alpha = r\left(n^\alpha + \frac{1}{c}U^\alpha\right);\, r = n^\alpha R_\alpha = -\frac{1}{c}R^\alpha U_\alpha > 0;$$

$$\hat{R}^\alpha \equiv n^\alpha + \frac{1}{c}U^\alpha;\, \overset{(n)}{\mu}{}^{\alpha\beta} \equiv \overset{(n)}{\mu}{}^{\alpha\beta}(\tau_{(\mathrm{ret})}) = \left(\frac{\mathrm{d}^n\mu^{\alpha\beta}(\tau)}{\mathrm{d}\tau^n}\right)_{\tau=\tau_{(\mathrm{ret})}}, \tag{A.56}$$

where $\tau_{(\mathrm{ret})}$ is the retarded eigentime, we get, in the approximation that we may neglect all $\overset{(n)}{U}{}^\alpha$ $(n = 1, 2, \ldots)$, from (A.55) for the field tensor at x:

$$f^{\alpha\beta}_{(\mathrm{ret})}(x) = \frac{1}{4\pi}\left[\frac{e}{cr^2}U^\alpha n^\beta + \frac{1}{c^2r^2}\dot{\mu}^{\alpha\beta} + \frac{2}{c^2r^2}n^\alpha\dot{\mu}^{\beta\gamma}\hat{R}_\gamma\right.$$
$$\left.+\frac{2}{c^2r^2}\hat{R}^\alpha\dot{\mu}^{\beta\gamma}n_\gamma + \frac{1}{c^2r^2}\dot{\mu}^{\alpha\gamma}\hat{R}_\gamma\hat{R}^\beta + \frac{1}{c^3r}\hat{R}^\alpha\ddot{\mu}^{\beta\gamma}\hat{R}_\gamma\right]_{-(\beta\alpha)}, \tag{A.57}$$

where the subscripts $-(\beta\alpha)$ indicate that we subtract the expression between square brackets with α and β interchanged, and only terms of order r^{-1} and r^{-2} have been written down. (We shall not need terms of order r^{-3} and higher negative powers of r in the calculations below.)

The energy-momentum tensor of the electromagnetic field generated by the atom, at x is defined by‡:

$$T^{\alpha\beta}_{(\mathrm{em})}(x) \equiv -f^{\alpha\gamma}_{(\mathrm{ret})}(x)\,f_{(\mathrm{ret})\gamma}{}^{\beta}(x) - \tfrac{1}{4}g^{\alpha\beta}f^{\gamma\delta}_{(\mathrm{ret})}(x)\,f_{(\mathrm{ret})\gamma\delta}(x). \tag{A.58}$$

$T^{\alpha\beta}_{(\mathrm{em})}$ consists of terms proportional to r^{-2}, denoted by $T^{\alpha\beta}_{(\mathrm{em})}(r^{-2})$, and terms proportional to r^{-3}, denoted by $T^{\alpha\beta}_{(\mathrm{em})}(r^{-3})$, but also terms proportional to higher negative powers of r than the third, wich will not be written explicitly, since they will appear to be of no importance in the calculations below. We find from (A.57) and (A.58), using also the relations (A.56) and the antisymmetry of $\mu^{\alpha\beta}$, that:

$$T^{\alpha\beta}_{(\mathrm{em})}(r^{-2}) = -\frac{1}{(4\pi c^3r)^2}\hat{R}^\alpha\hat{R}^\beta\hat{R}_\gamma\ddot{\mu}^{\gamma\delta}\ddot{\mu}_{\delta\varepsilon}\hat{R}^\varepsilon. \tag{A.59}$$

‡ This definition is in accordance with our previous definition of the energy-momentum tensor of the electromagnetic field, but differs by a minus sign from definition (4.114) of ref. 12.

$$T^{\alpha\beta}_{(\mathrm{em})}(r^{-3}) = \frac{1}{(4\pi)^2\, r^3}\Big\{-\frac{e}{c^3}\,(\hat{R}^\alpha\dddot{\mu}^{\beta\gamma} + \hat{R}^\beta\dddot{\mu}^{\alpha\gamma})\,\hat{R}_\gamma$$

$$+\frac{2}{c^5}\,(\hat{R}^\alpha\dot{\mu}^{\beta\gamma} + \hat{R}^\beta\dot{\mu}^{\alpha\gamma})\,\ddot{\mu}_{\gamma\delta}\hat{R}^\delta + \frac{2}{c^6}\,(\hat{R}^\alpha\dot{\mu}^{\beta\gamma} + \hat{R}^\beta\dot{\mu}^{\alpha\gamma})\,\hat{R}_\gamma U_\delta\ddot{\mu}^{\delta\varepsilon}n_\varepsilon$$

$$+\frac{2}{c^6}\,(\hat{R}^\alpha\ddot{\mu}^{\beta\gamma} + \hat{R}^\beta\ddot{\mu}^{\alpha\gamma})\,\hat{R}_\gamma n_\delta\dot{\mu}^{\delta\varepsilon}U_\varepsilon + \frac{2}{c^6}(U^\alpha\hat{R}^\beta + U^\beta\hat{R}^\alpha)\hat{R}_\gamma\dot{\mu}^{\gamma\delta}\ddot{\mu}_{\delta\varepsilon}\hat{R}^\varepsilon$$

$$-\hat{R}^\alpha\hat{R}^\beta\Big(\frac{2e}{c^4}\,U_\gamma\ddot{\mu}^{\gamma\delta}n_\delta + \frac{2}{c^5}\,\hat{R}_\gamma\dot{\mu}^{\gamma\delta}\ddot{\mu}_{\delta\varepsilon}\hat{R}^\varepsilon + \frac{4}{c^5}\,n_\gamma\dot{\mu}^{\gamma\delta}\ddot{\mu}_{\delta\varepsilon}\hat{R}^\varepsilon\Big)\Big\}. \tag{A.60}$$

One can verify from (A.59) and (A.60), by putting $\alpha = \beta$, that the $g^{\alpha\beta}$-terms are zero.

We shall now give the expressions for the energy-momentum and angular momentum radiated in a proper time $d\tau$ by an arbitrary moving charged particle possessing electric and magnetic dipole moments [in the approximation that $\overset{(n)}{U}{}^\alpha$ are very small ($n = 1, 2, \ldots$)]. The basic result for the radiation of energy-momentum has already been described by Rohrlich[12]) for a charged point particle (see also refs. 15 and 16). Cohn[13]) extends the method to the radiation of angular momentum for a charged spinning particle.

The energy-momentum radiated in a proper time $d\tau$ by an arbitrary moving particle is defined by †:

$$dP^\alpha_{(\mathrm{rad})} = \lim_{r\to\infty}\frac{1}{c}\int_{dW\,(\mathrm{timelike})} T^{\alpha\beta}_{(\mathrm{em})}\,dW_\beta, \tag{A.61}$$

where dW is the intersection of a 3-dimensional timelike surface with two light cones from $X(\tau_{(\mathrm{ret})}) \equiv X$ and $X(\tau_{(\mathrm{ret})} + d\tau) \equiv X'$(see fig. 1); dW depends on $d\tau$ and is very far away from the world line of the particle

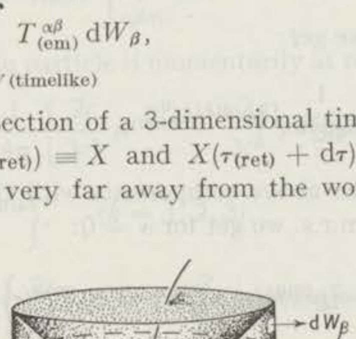

Fig. 1. Intersections of two light cones from X and X' with a timelike surface dW (cylindrical surface); the surface element of dW is given by the vector dW_β.

† The definition (A.61) differs from Rohrlich's definition by a minus sign (see eq. (5.10) of ref. 12), as a consequence of the difference in sign between his definition of $T^{\alpha\beta}_{(\mathrm{em})}$ and ours [eq. (A.58)].

between X and X'; the surface element of $\mathrm{d}W$, with spacelike unit normal pointing outward (*i.e.* increasing distance in space) is given by the vector $\mathrm{d}W_\beta$. One can prove that expression (A.61) is a four-vector.

To calculate the energy-momentum radiation rate (with respect to the eigentime at retarded value) we may appropriately choose for $\mathrm{d}W$ the timelike surface, which is a straight circular cylinder in the m.r.s. of the particle:

$$\mathrm{d}W_\beta = n_\beta r^2\, \mathrm{d}\omega\, c\mathrm{d}\tau, \tag{A.62}$$

where $\mathrm{d}\omega$ is the element of solid angle in ordinary space and n_β is the spacelike normal given in (A.56). From (A.59), (A.60) and (A.61) we get with (A.62) for the energy-momentum radiation rate:

$$\frac{\mathrm{d}P^\alpha_{(\mathrm{rad})}}{\mathrm{d}\tau} = \lim_{r\to\infty} \int T^{\alpha\beta}_{(\mathrm{em})} n_\beta r^2\, \mathrm{d}\omega = \int T^{\alpha\beta}_{(\mathrm{em})}(r^{-2})\, n_\beta r^2\, \mathrm{d}\omega =$$

$$= -\frac{1}{(4\pi c^3)^2} \int \hat{R}^\alpha \hat{R}_\beta \ddot{\mu}^{\beta\gamma} \ddot{\mu}_{\gamma\delta} \hat{R}^\delta\, \mathrm{d}\omega, \tag{A.63}$$

where the integration is over the solid angle, and use is made of (A.56).

Eq. (A.63) can now be calculated in the frame where the reference point is momentarily at rest at time $\tau_{(\mathrm{ret})}$. For $\alpha = 0$ we get:

$$\left(\frac{\mathrm{d}P^0_{(\mathrm{rad})}}{\mathrm{d}\tau}\right)^{(0)} = -\frac{1}{4\pi c^6}\{-\tfrac{2}{3}\ddot{\mu}^{(0)0k}\ddot{\mu}^{(0)}_{k0} + \tfrac{1}{3}\ddot{\mu}^{(0)kl}\ddot{\mu}^{(0)}_{lk}\}. \tag{A.64}$$

For $\alpha = k$ ($= 1, 2, 3$) we get:

$$\left(\frac{\mathrm{d}P^k_{(\mathrm{rad})}}{\mathrm{d}\tau}\right)^{(0)} = -\frac{1}{4\pi c^6}\{\tfrac{2}{3}\ddot{\mu}^{(0)kl}\ddot{\mu}^{(0)}_{l0}\}. \tag{A.65}$$

Let us now compare the above components with those of the four-vector $F^\alpha_{(\mathrm{d})}$, eq. (60A). In the m.r.s. we get for $\alpha = 0$:

$$F^{(0)0}_{(\mathrm{d})}(\tau) = \frac{1}{4\pi c^6}\{-\tfrac{2}{3}\ddot{\mu}^{(0)0k}(\tau)\, \ddot{\mu}^{(0)}_{k0}(\tau) + \tfrac{1}{3}\ddot{\mu}^{(0)kl}(\tau)\, \ddot{\mu}^{(0)}_{lk}(\tau)\}, \tag{A.66}$$

and for $\alpha = k$ ($= 1, 2, 3$):

$$F^{(0)k}_{(\mathrm{d})}(\tau) = \frac{1}{4\pi c^6}\{\tfrac{2}{3}\ddot{\mu}^{(0)kl}(\tau)\, \ddot{\mu}^{(0)}_{l0}(\tau)\}. \tag{A.67}$$

If this result is taken for $\tau = \tau_{(\mathrm{ret})}$, it is exactly equal to the negative of (A.64) and (A.65). Physically this means that we may interpret $F^\alpha_{(\mathrm{d})}$ as the radiation reaction four-force on the atom, since it is equal to the negative of the rate at which electromagnetic radiation energy-momentum is emited by this atom (*cf.* ref. 12 pg. 149).

The angular momentum radiated in a time $\mathrm{d}\tau$ is‡:

$$\mathrm{d}J^{\alpha\beta}_{(\mathrm{rad})} = \lim_{r\to\infty} \int \{x^\alpha T^{\beta\gamma}_{(\mathrm{em})}(x) - x^\beta T^{\alpha\gamma}_{(\mathrm{em})}(x)\}\, n_\gamma r^2 \,\mathrm{d}\tau\, \mathrm{d}\omega, \tag{A.68}$$

where the integration is over the solid angle. Choosing the reference frame in such a way that its origin coincides with $X^\alpha(\tau_{(\mathrm{ret})})$, we get from (A.68) and (A.56) for the angular momentum radiated by the atomic system:

$$\frac{\mathrm{d}J^{\alpha\beta}_{(\mathrm{rad})}}{\mathrm{d}\tau} = \lim_{r\to\infty} \int \{\hat{R}^\alpha T^{\beta\gamma}_{(\mathrm{em})}(x) - \hat{R}^\beta T^{\alpha\gamma}_{(\mathrm{em})}(x)\}\, n_\gamma r^3 \,\mathrm{d}\omega. \tag{A.69}$$

It is easily seen from this equation, that only the terms $T^{\alpha\beta}_{(\mathrm{em})}(r^{-2})$, eq. (A.59), and $T^{\alpha\beta}_{(\mathrm{em})}(r^{-3})$, eq. (A.60), of the energy-momentum tensor of the electromagnetic field will contribute to $\mathrm{d}J^{\alpha\beta}_{(\mathrm{rad})}/\mathrm{d}\tau$, and not terms of higher negative order in r. It is also directly seen that $T^{\alpha\beta}_{(\mathrm{em})}(r^{-2})$ and the terms in $T^{\alpha\beta}_{(\mathrm{em})}(r^{-3})$ proportional to $\hat{R}^\alpha\hat{R}^\beta$ vanish in eq. (A.69). Substituting (A.60) into (A.69) and making use of (A.56) we obtain:

$$\begin{aligned}\frac{\mathrm{d}J^{\alpha\beta}_{(\mathrm{rad})}}{\mathrm{d}\tau} = \frac{1}{(4\pi)^2} \int \Big[&-\frac{e}{c^3}\hat{R}^\alpha \ddot{\mu}^{\beta\gamma}\hat{R}_\gamma + \frac{2}{c^5}\hat{R}^\alpha\dot{\mu}^{\beta\gamma}\ddot{\mu}_{\gamma\delta}\hat{R}^\delta \\ &+ \frac{2}{c^6}\hat{R}^\alpha\ddot{\mu}^{\beta\gamma}\hat{R}_\gamma U_\delta \dddot{\mu}^{\delta\varepsilon} n_\varepsilon + \frac{2}{c^6}\hat{R}^\alpha\ddot{\mu}^{\beta\gamma}\hat{R}_\gamma n_\delta \ddot{\mu}^{\delta\varepsilon} U_\varepsilon \\ &+ \frac{2}{c^6} n^\alpha U^\beta \hat{R}_\gamma \dot{\mu}^{\gamma\delta}\ddot{\mu}_{\delta\varepsilon}\hat{R}^\varepsilon \Big]_{-(\beta\alpha)} \mathrm{d}\omega. \end{aligned}\tag{A.70}$$

In the frame where the particle is momentarily at rest at time τ_{ret} we get:

$$\begin{aligned}\left(\frac{\mathrm{d}J^{0k}_{(\mathrm{rad})}}{\mathrm{d}\tau}\right)^{(0)} = -\frac{1}{4\pi}\Big\{ &\frac{2e}{3c^3}\ddot{\mu}^{(0)0k} + \frac{2}{3c^5}\dot{\mu}^{(0)0l}\ddot{\mu}^{(0)k}_l \\ &+ \frac{2}{3c^5}\ddot{\mu}^{(0)0l}\dot{\mu}^{(0)k}_l \Big\}, \qquad (k = 1, 2, 3), \end{aligned}\tag{A.71}$$

$$\begin{aligned}\left(\frac{\mathrm{d}J^{kl}_{(\mathrm{rad})}}{\mathrm{d}\tau}\right)^{(0)} = \frac{1}{4\pi}\Big\{ &\frac{2e}{3c^3}\ddot{\mu}^{(0)kl} + \frac{2}{3c^5}(\dot{\mu}^{(0)k0}\ddot{\mu}^{(0)l0} - \dot{\mu}^{(0)l0}\ddot{\mu}^{(0)k0}) \\ &+ \frac{2}{3c^5}(\ddot{\mu}^{(0)km}\dot{\mu}^{(0)l}_m - \ddot{\mu}^{(0)lm}\dot{\mu}^{(0)k}_m \Big\}, \qquad (k, l = 1, 2, 3). \end{aligned}\tag{A.72}$$

Now consider the four-tensor $d^{\alpha\beta}_{(-)}$, eq. (62.A). In the m.r.s. we get:

$$\begin{aligned} d^{(0)0k}_{(-)}(\tau) = \frac{1}{4\pi}\Big\{ &\frac{2}{3c^5}\dot{\mu}^{(0)0l}(\tau)\,\ddot{\mu}^{(0)k}_l(\tau) + \frac{2}{3c^5}\ddot{\mu}^{(0)0l}(\tau)\,\dot{\mu}^{(0)k}_l(\tau)\Big\}, \\ &(k = 1, 2, 3), \end{aligned}\tag{A.73}$$

‡ This formula can be derived in a completely analogous way as we have obtained (A.63) above. One can prove that expression (A.68) is a four-tensor.

$$d^{(0)kl}_{(-)}(\tau) = -\frac{1}{4\pi}\left\{\frac{2}{3c^5}\left[\dot{\mu}^{(0)k0}(\tau)\,\ddot{\mu}^{(0)l0}(\tau) - \dot{\mu}^{(0)l0}(\tau)\,\ddot{\mu}^{(0)k0}(\tau)\right]\right.$$

$$\left. + \frac{2}{3c^5}\left[\ddot{\mu}^{(0)km}(\tau)\,\dot{\mu}^{(0)l}_{m}(\tau) - \ddot{\mu}^{(0)lm}(\tau)\,\dot{\mu}^{(0)k}_{m}(\tau)\right]\right\}, \qquad (k, l = 1, 2, 3). \tag{A.74}$$

When comparing the right-hand sides of (A.71) and (A.72) with those of (A.73) and (A.74), taken at $\tau_{(\mathrm{ret})}$, we find, in contrast with what has been found above (eqs. (A.64)–(A.67)), that they are not equal and opposite of sign, but we have from (A.71) and (A.73) for $\tau = \tau_{(\mathrm{ret})}$:

$$\left(\frac{\mathrm{d}J^{0k}_{(\mathrm{rad})}}{\mathrm{d}\tau}\right)^{(0)} = -\,d^{(0)0k}_{(-)} - \frac{1}{4\pi}\left(\frac{2e}{3c^3}\,\ddot{\mu}^{(0)0k}\right), \qquad k(=1, 2, 3), \tag{A.75}$$

and from eqs. (A.72) and (A.74):

$$\left(\frac{\mathrm{d}J^{kl}_{(\mathrm{rad})}}{\mathrm{d}\tau}\right)^{(0)} = -\,d^{(0)kl}_{(-)} + \frac{1}{4\pi}\left(\frac{2e}{3c^3}\,\ddot{\mu}^{(0)kl}\right), \qquad (k, l = 1, 2, 3). \tag{A.76}$$

The last terms of (A.75) and (A.76) are the components of an antisymmetric tensor written in the m.r.s. If we introduce the notations:

$$\ddot{\bar{\mu}}^{(0)0k} \equiv -\,\ddot{\mu}^{(0)0k}; \qquad \ddot{\bar{\mu}}^{(0)kl} \equiv \ddot{\mu}^{(0)kl}, \tag{A.77}$$

we can write (A.75) and (A.76) in an arbitrary reference frame as:

$$\dot{J}^{\alpha\beta}_{(\mathrm{rad})} = -\,d^{\alpha\beta}_{(-)} + \frac{1}{4\pi}\left(\frac{2e}{3c^3}\,\mathscr{L}\ddot{\bar{\mu}}^{(0)\alpha\beta}\right), \tag{A.78}$$

where $d^{\alpha\beta}_{(-)}$ is given by eq. (62.A) and the tensor $\mathscr{L}\ddot{\bar{\mu}}^{(0)\alpha\beta}$ is obtained from (A.77) by a Lorentz-transformation of an antisymmetric tensor from the m.r.s. to an arbitrary frame, which transformation depends on the velocity of the particle. In the approximation that we neglect all time derivatives of this velocity we obtain:

$$\dot{J}^{\alpha\beta}_{(\mathrm{rad})} = -\,d^{\alpha\beta}_{(-)} + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{\frac{1}{4\pi}\,\frac{2e}{3c^3}\,\bar{\mu}^{\alpha\beta}\right\}, \tag{A.79}$$

wheie $\bar{\mu}^{\alpha\beta} = \mathscr{L}\bar{\mu}^{(0)\alpha\beta}$.

REFERENCES

1) Vlieger, J., Physica **37** (1967) 165.
2) Møller, C., Ann Inst. H. Poincaré **11** (1949–50) 251.
3) Vlieger, J. and Emid, S., Physica **41** (1969) 368.
4) Vlieger, J. and Emid, S., Physica **42** (1969) 12.
5) de Groot, S. R. and Suttorp, L. G., Physica **37** (1968) 284, 297.
6) Dirac, P. A. M., Proc. Roy. Soc. (London) A **167** (1938) 148.

7) Bhabha, H. J. and Harish-Chandra, Proc. Roy. Soc. (London) A **185** (1946) 250, Harish-Chandra, Proc. Roy. Soc. (London) A **185** (1946) 269.
8) Møller, C., The theory of relativty, (Oxford, 1951).
9) de Groot, S. R. and Suttorp, L. G., Physica **39** (1968) 84.
10) Dirac, P. A. M., The principles of quantum mechanics, third edition, (Oxford, 1947).
11) Landau, L. D. and Lifshitz, E. M., The classical theory of fields (London, 1959).
12) Rohrlich, F., Classical charged particles, Addison-Wesley Publ. Comp. (Reading, 1965).
13) Cohn, J., J. Math. Phys. **5** (1969) 799.
14) Kolsrud, M. and Leer, E., Phys. Norveg. **2** (1967) 181.
15) Synge, J. L., Relativity: The special theory, second edition, North-Holland Publ. Comp. (Amsterdam, 1965).
16) Schild, A., J. Math. Anal. and Appl. **1** (1960) 127.

SAMENVATTING

In dit proefschrift worden de klassieke relativistische bewegingsvergelijkingen van atomen en moleculen in een electromagnetisch veld behandeld. Als uitgangspunt is de theorie van Møller genomen voor de relativistische dynamica van systemen met een inwendig impulsmoment in een uitwendig veld. De bewegingsvergelijkingen van Møller zijn eerder door Vlieger toegepast op een klassiek model van geladen atomen en moleculen met een electrisch en een magnetisch dipoolmoment. De aldus verkregen atomaire vergelijkingen werden door hem gebruikt voor de afleiding van de energie-impulstensor van een systeem bestaande uit N van zulke atomen of moleculen.

De bewegingsvergelijkingen van Møller bevatten echter een soort "Zitterbewegung" (een trillende of beter nog snel roterende beweging met een kleine amplitude en een zeer hoge frequentie), gesuperponeerd op de gewone beweging. Het elimineren van de onphysische beweging is het eerste onderwerp van dit proefschrift.

Vervolgens worden de vergelijkingen uitgebreid voor het geval, waarbij de atomen (of moleculen) behalve dipoolmomenten ook een electrisch quadrupoolmoment hebben. Dit is van dezelfde orde van grootte als het magnetisch dipoolmoment.

In het laatste hoofdstuk hebben we de klassieke theorie voor stralende atomen en moleculen behandeld.

We hebben hetzelfde klassieke model van atomen en moleculen genomen als in het werk van Vlieger. Voor alle bovengenoemde gevallen zijn atomaire bewegingsvergelijkingen afgeleid uit de sub-atomaire wetten voor energie-impuls en impulsmoment. De vergelijkingen worden vervolgens gebruikt om de atomaire energie-impulstensor voor een systeem bestaande uit een groot aantal dipool- en quadrupool-atomen (of moleculen) af te leiden.

Na een samenvatting van de door Vlieger verkregen resultaten in hoofdstuk I hebben we in hoofdstuk II met behulp van een iteratie-procedure laten zien, dat in het geval van atomen en moleculen de termen die de trillende beweging in de bewegingsvergelijkingen beschrijven, verwaarloosbaar klein zijn. Dit maakt het mogelijk om bewegingsvergelijkingen op te schrijven van het gebruikelijke tweede orde type en een balansvergelijking voor het inwendige impulsmoment van de eerste orde.

In hoofdstuk III is de theorie toegepast op atomen en moleculen die naast een electrisch en een magnetisch dipoolmoment ook een electrisch quadrupoolmoment hebben. Het blijkt, dat het veld-gedeelte van de atomaire energie-impulstensor niet van dezelfde vorm is als in het geval van alleen dipolen en het is ook niet meer uit te drukken alleen in termen van grootheden die in de atomaire veldvergelijkingen voorkomen.

Stralingseffecten zijn verwaarloosd in de hoofdstukken I–III, maar renormalisatie-effecten zijn covariant in beschouwing genomen zonder expliciete uitdrukkingen ervoor te geven. Als gevolg van de verwaarlozing van de stralingseffecten zijn de atomaire energie-impulstensoren die in deze hoofdstukken gevonden zijn, symmetrisch.

In hoofdstuk IV is de theorie die in de hoofdstukken I en II is behandeld, uitgebreid tot het geval van stralende dipool-atomen en -moleculen. We hebben gebruik gemaakt van Dirac's covariante splitsing van geretardeerde electromagnetische velden in een plus- (of "zelf") gedeelte (de halve som van geretardeerde en geavanceerde velden) en een min- (of "straling") gedeelte (het halve verschil van geretardeerde en geavanceerde velden). Het blijkt dan voor een willekeurige (eindige) lading-stroomverdeling mogelijk te zijn om een sub-atomaire energie-impulstensor van het stralende atoom (of molecule) te definiëren. Hierin is de bijdrage van de intra-atomaire velden expliciet afgeleid met behulp van het plus-gedeelte van deze velden. De gevonden tensor heeft de belangrijke eigenschap, dat hij eindig is, zoals vereist is in de theorie van Møller. Het blijkt bovendien, dat het min-veld behandeld kan worden als een uitwendig veld bij de afleiding van de bewegingsvergelijkingen en van de balansvergelijking voor het inwendige impulsmoment.

De atomaire massa en het atomaire impulsmoment die met de bovengenoemde sub-atomaire energie-impulstensor zijn gedefinieerd, zijn op een manifest covariante wijze gerenormaliseerd voor de effecten van de intra-atomaire velden. Uit de algemene uitdrukkingen voor deze grootheden hebben we in appendix II van hoofdstuk IV de bijdragen van de intra-atomaire velden expliciet uitgerekend tot de orde c^{-2} en we vinden overeenstemming met de literatuur.

Vervolgens zijn de bewegingsvergelijkingen en de balansvergelijking voor het inwendige impulsmoment afgeleid met de methode van de voorgaande hoofdstukken. In appendix III van hoofdstuk IV is een procedure gegeven voor de berekening van de min-velden die in deze vergelijkingen voorkomen.

Om de nogal gecompliceerde vergelijkingen te vereenvoudigen, hebben we aangenomen, dat alle termen afkomstig van deze min-velden die tijdsafgeleiden van de vier-snelheid bevatten, verwaarloosd kunnen worden. (Physisch betekent dit, dat we in de bewegingsvergelijkingen de stralingseffecten tengevolge van de barycentrische versnellingen van de atomen ("Bre~~h~~msstrahlung") verwaarlozen en alleen de demping tengevolge van de vibraties van hun dipoolmomenten in beschouwing nemen.) Het blijkt, dat de bijdrage van de min-velden tot de bewegingsvergelijkingen gesplitst kan worden in een gedeelte, dat geïnterpreteerd kan worden als de stralingsdempingskracht en een gedeelte, dat een totale tijdsafgeleide is. Een analoge splitsing in een stralingsdempingskoppel en een totale tijdsafgeleide is uitgevoerd in de balansvergelijking voor het inwendige impulsmoment. In appendix IV is tenslotte een rechtvaardiging gegeven voor de interpretatie van deze dempingskracht en het dempingskoppel, door deze grootheden in verband te brengen met respectievelijk de straling van energie-impuls en van impulsmoment.

In aanwezigheid van straling is de atomaire energie-impulstensor niet langer symmetrisch, wat ook niet het geval behoeft te zijn voor niet-gesloten systemen. De behandeling van stralende atomen die ook een electrisch quadrupoolmoment hebben, is aangegeven aan het eind van hoofdstuk IV.

Op verzoek van de Faculteit der Wiskunde en Natuurwetenschappen volgen hier enige gegevens over mijn studie.

In 1960 legde ik het eindexamen A.M.S.-B af te Paramaribo. In hetzelfde jaar begon ik mijn studie aan de Universiteit te Leiden en behaalde in oktober 1963 het candidaatsexamen natuur- en wiskunde (A'). In oktober 1967 legde ik het doctoraalexamen af met als hoofdvak theoretische natuurkunde en als bijvakken klassieke mechanica en meteorologie. Gedurende de studie hiervoor volgde ik natuurkundecolleges o.a. van Prof. Dr. P. Mazur, Prof. Dr. P. W. Kasteleyn en Prof. Dr. J. A. M. Cox, en meteorologie van Prof. Dr. P. Groen aan de Vrije Universiteit te Amsterdam. Naast de doctoraalstudie volgde ik van september 1965 tot en met februari 1966 de opleiding synoptische meteorologie bij het K.N.M.I. te de Bilt.

Van november 1967 tot februari 1970 was ik werkzaam op het Instituut-Lorentz voor theoretische natuurkunde, eerst als doctoraal-assistent en vanaf januari 1969 als wetenschappelijk medewerker. In deze periode verrichtte ik onderzoek op het gebied van de relativistische dynamica van gepolariseerde systemen, in het bijzonder over de relativistische beweging van atomaire systemen in een electromagnetisch veld. Dit onderzoek, waarvan de resultaten de inhoud van dit proefschrift vormen, vond plaats onder leiding van Dr. J. Vlieger.

Vanaf februari 1970 ben ik in dienst van de Technische Hogeschool te Delft, waar ik op het Laboratorium voor Technische Fysica bij de werkgroep Magnetische Resonantie onder leiding van Prof. Dr. Ir. J. Smidt onderzoek verricht op het gebied van de magnetische resonantie.

PUBLICATIES

On the relativistic dynamics of polarized systems II, J. Vlieger en S. Emid, Physica **41** (1969) 368.

On the relativistic dynamics of polarized systems III, J. Vlieger en S. Emid, Physica **42** (1969) 12.

On the relativistic dynamics of polarized systems IV, S. Emid en J. Vlieger, te publiceren in Physica.

A method for selective measurements of T_2 in liquid h.r. nmr-spectra, M. F. Augusteijn, S. Emid, A. F. Mehlkopf en J. Smidt, wordt gepubliceerd.

Op verzoek van de Faculteit der Wiskunde en Natuurwetenschappen volgen hier enige gegevens over mijn studie.

In 1962 legde ik het eindexamen A.M.S.-B af te Paramaribo. In hetzelfde jaar begon ik mijn studie aan de Universiteit te Leiden en behaalde in oktober 1965 het candidaatsexamen natuur- en wiskunde (A'). In oktober 1967 legde ik het doctoraalexamen af met als hoofdvak theoretische natuurkunde en als bijvakken klassieke mechanica en meteorologie. Gedurende de studie hiervoor volgde ik natuurkundecolleges o.a. van Prof. Dr. P. Mazur, Prof. Dr. P. W. Kasteleyn en Prof. Dr. J. A. M. Cox, en meteorologie van Prof. Dr. P. Groen aan de Vrije Universiteit te Amsterdam. Naast mijn doctoraalstudie volgde ik van september 1965 tot en met februari 1967 de opleiding synoptische meteorologie bij het K.N.M.I. te de Bilt.

Van november 1967 tot februari 1970 was ik werkzaam op het Instituut-Lorentz voor theoretische natuurkunde, eerst als doctoraal-assistent en vanaf januari 1969 als wetenschappelijk medewerker. In deze periode verrichtte ik onderzoek op het gebied van de relativistische dynamica van gepolariseerde systemen, in het bijzonder over de relativistische beweging van atomaire systemen in een electromagnetisch veld. Dit onderzoek, waarvan de resultaten de inhoud van dit proefschrift vormen, vond plaats onder leiding van Dr. J. Vlieger.

Vanaf februari 1970 ben ik in dienst van de Technische Hogeschool te Delft, waar ik op het Laboratorium voor Technische Fysica bij de werkgroep Magnetische Resonantie onder leiding van Prof. Dr. Ir. J. Smidt onderzoek verricht op het gebied van de magnetische resonantie.

PUBLICATIES

On the relativistic dynamics of polarized systems I, J. Vlieger en S. Emid, Physica 41 (1969) 368.

On the relativistic dynamics of polarized systems II, J. Vlieger en S. Emid, Physica 42 (1969) 12.

On the relativistic dynamics of polarized systems III, S. Emid en J. Vlieger, te publiceren in Physica.

A method for relative measurements of T_1 in liquid ³He, M. Z. Arogyaswamy, S. Emid, A. F. Mehlkopf en J. Smidt, wordt gepubliceerd.