On the random-cluster model

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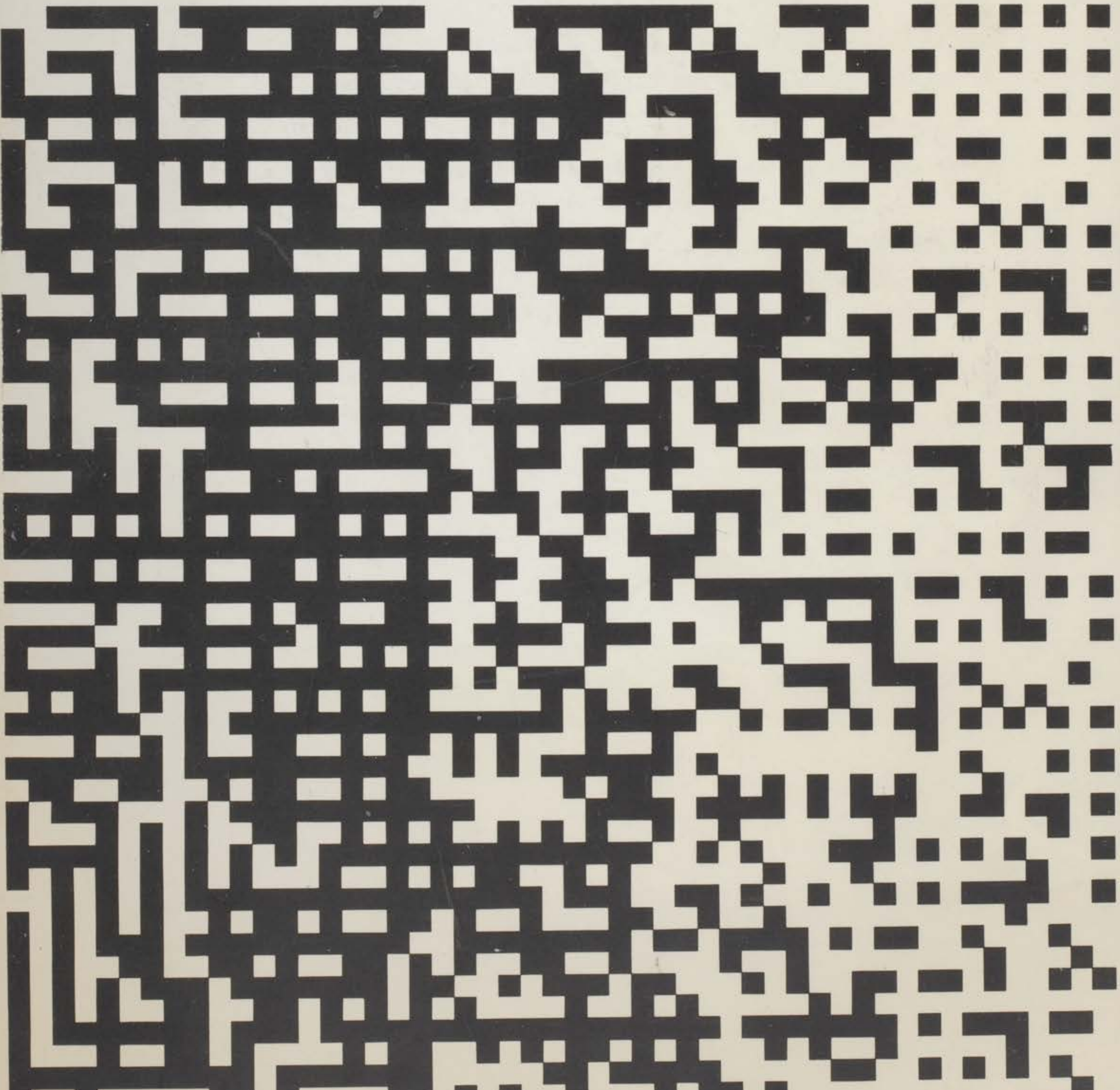

ON THE RANDOM-CLUSTER MODEL

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INHOUD - CONTENTS

SAMENVATTING

Vele fysische systemen komen in twee scherp van elkaar te onderscheiden fasen voor, bijv. een vaste en een vloeibare, een magnetische en een niet magnetische. Het verschil tussen twee van zulke fasen is, vanuit atomair standpunt, gelegen in het al of niet bestaan van een zekere ordening over grote afstanden binnen het systeem. Uit berekeningen aan eenvoudige modellen, zoals het Ising model, is gebleken dat een dergelijke ordening optreedt zodra de wisselwerking tussen de atomen een zekere kritische waarde overschrijdt. Anderzijds zijn er ook systemen waarin "kritieke" verschijnselen van schijnbaar geheel andere aard optreden. Een poreus materiaal kan waterdoorlatend worden zodra de porositeit een kritieke waarde overschrijdt. In dit geval kan niet van een ordening over grote afstand worden gesproken, maar wel van samenhang op grote schaal.

Het onderwerp van dit proefschrift is een nadere analyse van het begrip "samenhang op grote schaal", en de relatie hiervan met het begrip "ordening over grote afstanden". Daartoe wordt een eenvoudig model ingevoerd van een systeem dat bestaat uit wisselwerkende objecten, waarin de objecten worden voorgesteld door punten en de mogelijke wisselwerking tussen twee objecten door een lijn tussen de corresponderende punten. Op deze manier wordt het gehele systeem voorgesteld door een netwerk. Essentiëel voor het model is dat elk van de wisselwerkingen een zekere kans heeft om niet te functioneren. Als we alle lijnen van niet functionerende wisselwerkingen weglaten, zal het overblijvende netwerk in het algemeen uit een of meer brokken bestaan, clusters geheten, die eindig of oneindig groot kunnen zijn. Het ligt voor de hand om een oneindig cluster in verband te brengen met samenhang op grote schaal.

In het eerste deel wordt dit model, het random-cluster model, in detail gedefiniëerd en in verband gebracht met andere modellen. In het bijzonder wordt aangetoond dat het zowel een generalisatie is van het Ising model als van het percolatiemodel. Tevens wordt het model tegen een wat meer algemene wiskundige achtergrond geplaatst.

In het tweede deel worden de eigenschappen van het model onderzocht voor het eenvoudigst mogelijke geval, waarin het gelijk is aan het percolatiemodel. In het derde deel wordt deze analyse uitgebreid tot het meer algemene geval, waaronder ook het Ising model valt. Een van de voornaamste resultaten is een stelling die het verband legt tussen de samenhang op grote schaal in het percolatiemodel en de ordening over lange afstanden in het Ising model. Verder wordt het verband aangetoond tussen een aantal criteria voor samenhang op grote schaal, en de invloed nagegaan van een uitwendige invloed op die samenhang.

INTRODUCTION AND SUMMARY

In this thesis we are dealing with a model in the theory of phases. As is well known a physical system in equilibrium, which from a microscopic point of view consists of many particles, can from a macroscopic point of view exist only in a few different pure phases, the system undergoing a phase transition if it changes from the one pure phase to the other. One of the main questions is how a pure phase can be characterized and what types of phases we can have.

Classically, a phase transition can be characterized by the properties of the Helmholtz free energy (which is directly related to the Gibbs probability measure on the states of the system), in particular by singularities of the free energy. The existence of a pure phase can be characterized by a homogeneity property, or the vanishing of long-range correlations, and its nature by certain order parameters such as the magnetization or the long-range many spin expectations in the case of a ferromagnet.

Recently, a theory has been developed in which a pure phase of a system can be characterized as soon as the symmetries of the system are given. In particular, each probability measure of that system (e.g. the Gibbs measure) is a unique linear combination of probability measures associated with the pure phases of that system, i.e. a macroscopic state is a unique mixture of pure phases. Moreover, these pure phases are characterized by cluster properties which are associated with the symmetries of the system, and which are comparable to the vanishing of long-range correlations.

One should notice that in most of the characterizations of pure phases or phase transitions mentioned above, the symmetries of the system play an essential role. For example, all proofs of the existence of the free energy depend strongly on translational symmetry. Moreover, the concept of order itself suggests a certain regularity or symmetry. It is felt, however, that the concept of cooperative behaviour is of a more general nature, and should not depend so heavily on the symmetries of the system.

In this connection it is interesting to remark that there are other physical systems which show a cooperative phenomenon, e.g. porous media through which a liquid percolates, and certain cascade processes. The cooperative effect is in these cases the impregnation of the whole medium when the fraction of wide pores exceeds a certain critical value, and the avalanche effect occurring in the cascade if the probability of an individual event exceeds a certain critical value. In their simplest form both these systems can be described as special cases of the so-called percolation model. This model is to be compared with the Ising model, which is known to be a suitable model for phase transitions of the usual type. The concept of long-range order in the Ising model is in the percolation model replaced by the concept of "connectivity-at-large" or "large-range connectivity", for which a symmetry of the system is irrelevant.

The purpose of this thesis is to develop a theory of "interacting objects" in which the concept of large-range connectivity is analysed. To this end we introduce a simple model of a system consisting of infinitely many interacting objects, in which the objects are represented by points, called vertices, and the possible interactions between two objects are represented by lines, called edges, between the corresponding points. In this way, the system is represented by a graph; obviously, we are mainly interested in connected infinite graphs. The possible states of the system are obtained by allowing each of the interactions to "function" or not. The edge representing such an interaction is called a constituting edge or a dummy edge, respectively. If for a given state all dummy edges are deleted, the graph thus obtained from an infinite connected graph consists in general of one or more mutually unconnected pieces, called clusters, which may be finite or infinite. It is natural to relate the occurrence of an infinite cluster with a collective phenomenon. Finally, the probability measure on these states is obtained by associating to each edge a probability of being a constituting edge; in general, this probability may depend in a prescribed way on the state of the other edges.

In the first part of this thesis this model, the random-cluster model, is defined and related to other models. In particular it is shown that it is a generalization both of the percolation model and of the Ising model. It is further shown that the theory of the random-cluster model is intimately connected with the combinatorial theory of graphs.

In the second part the properties of this model are investigated for the case where the edges are statistically independent. In this case the model reduces to the percolation model .The main observables investigated are the probability that a given vertex belongs to an infinite cluster and the long-range limit of the probability that two vertices are connected. It is shown that these observables are related to each other as well as to a derivative of the "free energy" of the system (if this exists). Moreover, it is shown that the first observable has a clustering property which is, however, not related to a symmetry of the system.

In the third part the analysis of the second part is extended to the case where the edges are not necessarily independent; this case covers both the percolation model and the Ising model. One of the main results is the establishment of a relation between the large-range connectivity of the percolation model and the long-range ordering of the Ising model.

ON THE RANDOM-CLUSTER MODEL

I. Introduction and relation to other models

Synopsis The random-cluster model is defined as a model for phase transitions and other phenomena in lattice systems, or more generally in systems with a graph structure. The model is characterized by a (probability) measure on a graph and a real parameter κ. By specifying the value of κ to 1,2,3,... it is shown that the model covers the percolation model, the Ising model, the Ashkin-Teller-Potts model with 3,4,... states per atom, respectively, and thereby, contains information on graph-colouring problems; in the limit $\kappa \downarrow 0$ it describes linear resistance networks. It is shown that the function which for the random-cluster model plays the role of a partition function, is a generalization of the dichromatic polynomial earlier introduced by Tutte, and related polynomials.

1. INTRODUCTION

This paper is the first of a sequence of papers devoted to a model for phase transitions which was recently introduced by the authors [1)*)]. This model, to be called the random-cluster model, is actually a one-parameter family of systems, which includes among its members the spin ½ Ising model and the percolation model, but also systems representing graph colourings and certain electrical networks.

The member of the family which hitherto has been most thoroughly investigated is the Ising model, introduced by Ising in 1925 as a model for ferromagnetism upon a suggestion by Lenz [2)], and later on also applied to antiferromagnetism, ordening in binary alloys, condensation of a lattice gas and many other phenomena. In 1943 Ashkin and Teller introduced a lattice model in which each atom can be in four states, which was a direct analogue of the two-state Ising model [3)]. In 1952 Potts generalized both models to one with an arbitrary number of states per atom [4)].

A less-known member of the family is the percolation model (connectivity model) which was introduced in 1957 by Broadbent and Hammersley [5)] as a model for the percolation of a liquid through a porous medium, the spread of a disease through a community and similar phenomena. Its resemblance to the Ising model was first recognized by Hammersley [6)], and various methods developed for the Ising model were translated and applied to it by Sykes and Essam [7)], but a precise relation between the models was not established until 1968 (see ref. 1).

The problem of finding the number of ways in which the vertices of a given graph can be coloured with not more than a given number of colours n so that adjacent vertices have different colours (n-colourings) has a longer tradition than

*) A preliminary account of this work was given at the Summer School and Seminar on Critical Phenomena at Banff (August 1968).

the models mentioned above; in the form of the four-colour conjecture it has a history which goes back to the middle of the 19th century. In his research on the colouring problem, G.D. Birkhoff introduced in 1912 the chromatic polynomial, which is an extension of the number of n-colourings from integral values to arbitrary real values of n [8]. It is easy to see that the number of n-colourings of a graph is equal to the degeneracy of the ground state of the "anti-ferromagnetic" Ashkin-Teller-Potts model. This establishes the relation of the colouring problem to the models discussed above.

Finally, the oldest member of the family is the linear electrical network, investigated since the beginning of the 19th century. It was Kirchhoff who showed in 1847 that a central role in the systematic analysis of these networks is played by what nowadays is called the generating function for spanning trees [9]. In 1954 an important relation between this generating function and the chromatic polynomial was discovered by Tutte [10]. He showed that for a given graph both functions are special cases of a two-variable polynomial which he called the dichromate of the graph and which is now generally referred to as the Tutte polynomial. Another two-variable polynomial, which later on was called the dichromatic polynomial and was shown to be identical, apart from a certain factor and a shift of variables, with the Tutte polynomial, had been introduced by Tutte in 1947 [11]. This polynomial was introduced in a different way by Zykov in a study of recursive functions on graphs. [12] The generating function for spanning trees also served as a model for the partition function of a branched polymer without rings. [13]

As a final step in establishing the relations between these models and problems the random-cluster model was introduced[1], which, as we shall show in detail in this paper, embodies the entire family. The model is defined for an arbitrary graph, and associates with each edge e of the graph a real parameter p_e; if $0 \leq p_e \leq 1$ for all edges e, the model has a probabilistic interpretation. In addition, one real positive parameter κ occurs in the description of the model; it represents in a way the complexity of the model.

Different values of κ describe systems with different properties, the various systems discussed above appearing as the cases where κ is an integer ≥ 0, sometimes combined with special limiting values of the p_e; $\kappa = 0$ corresponds to the electrical network, $\kappa = 1$ to the percolation model, $\kappa = 2$ to the Ising model, $\kappa = n \geq 2$ to the Ashkin-Teller-Potts model with n states per atom and to the n-colouring problem.

After having introduced the random-cluster model we observed that if all parameters p_e are given equal values, the function which plays the central role in the theory of the model reduces, after a simple change of variables, to the dichromatic polynomial. Temperley independently made a similar observation [14], and, together with Lieb, developed a transfer-matrix approach for the case of a quadratic lattice [15]. Essam also investigated the relation between the aforesaid problems, paying particular attention to cluster expansions [16].

The aim of the present paper is the precise definition of the random-cluster model for an arbitrary countable graph. After an introductory section 2 on graph-theoretic notions we first define in section 3 the percolation model and a number of characteristic quantities related with it. We then derive a recursion relation for these quantities and a differentiation relation connecting some of them. In section 4 we show that the Ising model, and more generally, the Ashkin-Teller-Potts model, can be formulated entirely in terms of the percolation model. The same procedure is applied to the chromatic polynomial in section 5, and to a certain class of electrical networks in section 6. In all these cases the recursion relation derived in section 3 plays an essential role. It is a special case of a more general recursion relation studied by Zykov [12]. Therefore, the characteristic functions of the various models are recursive functions on graphs in the sense of Zykov. Their interrelation is discussed in section 7, in which the random-cluster model is defined and some of its properties are discussed. Finally, in section 8, the position of the random-cluster model with respect to the above-mentioned systems and problems, and with respect to the branch of combinatorial mathematics to which it belongs is briefly sketched.

2. GRAPHS AND CLUSTERS

A graph G is defined by a set V of vertices, a set E of edges and an incidence relation i between edges and vertices, associating with each edge $e \in E$ an unordered pair i(e) of vertices $v, v' \in V$, the ends of e; if $v = v'$ the edge is called a loop. The edge e is said to be incident with the vertices in i(e) and vice versa. If G is the graph thus defined, we write $G = (V,E,i)$. If more than one graph is considered, the vertex set and the edge set of G are denoted by V(G) and E(G), respectively.

We shall frequently encounter products of commuting quantities Q_a over all elements a of a set A; we shall denote them by by $Q^A \equiv \Pi_{a \in A} Q_a$. For convenience we put $Q^{\emptyset} \equiv 1$. The (cardinal) number (of elements) of a set A will be denoted by $|A|$. So the number of vertices is $|V|$, the number of edges $|E|$. A graph G is finite if both $|V(G)|$ and $|E(G)|$ are finite, and infinite otherwise. If both $|V(G)|$ and $|E(G)|$ are finite or infinite countable the graph G is countable. If the number of edges incident with a vertex v is finite for all $v \in V(G)$ the graph G is locally finite.

A subgraph of a graph $G = (V,E,i)$ is a graph $G' = (V',E',i')$

such that $V' \subseteq V$, $E' \subseteq E$ and $i'(e) = i(e)$ for all $e \in E'$. Since i' is the restriction of i to the domain E' we shall denote it simply by i. If $G' = (V',E',i)$ and $G'' = (V'',E'',i)$ are subgraphs of a graph $G = (V,E,i)$ then $(V' \cup V'',E' \cup E'',i)$ and $(V' \cap V'',E' \cap E'',i)$ are subgraphs of G, to be called the union graph and the intersection graph of G' and G'' in G, and to be denoted by $G' \cup G''$ and $G' \cap G''$. If $V' \subseteq V''$ and $E' \subseteq E''$, then G' is a subgraph of G'', and we write $G' \subseteq G''$. In particular for subgraphs G' of G we have $G' \subseteq G$. A spanning subgraph or partial graph of G is a subgraph with $V' = V$. The spanning subgraph G' with the set of edges E' will be denoted by $G_{E'}$.

A path between two vertices v and v' in a graph G is a finite sequence of alternatingly vertices and edges of G: $v=v_0$, e_1, v_1, e_2, ..., v_{n-1}, e_n, $v_n=v'$, such that $i(e_k) = \{v_{k-1},v_k\}$ for $k=1, 2, \ldots n$; it is often represented only by the edges which it contains. Two vertices v and $V' \in V(G)$ are connected in G if there is a path in G between them; if not, they are disconnected. The relation of connection between vertices is an equivalence relation. A graph G is connected if any two of its vertices are connected. A cluster (or connected component) of a graph is a maximal non-empty connected subgraph. The smallest cluster consists of one vertex and no edges (isolated vertex). The number of clusters of a graph is easily seen to equal the number of equivalence classes under the relation of connection.

A polygon in a graph G is the subgraph consisting of the vertices and edges of a path in G, containing at least one edge, between two coinciding vertices and with all vertices distinct except the first and last vertex. The smallest polygon is a vertex with a loop. A set of polygons is called dependent if the sum modulo 2 (symmetric difference) of the edge sets of a finite subset of polygons is the empty set. The number of independent polygons of a graph is called the *cyclomatic number* of that graph. A *tree* is a connected non-empty graph having no polygon as a subgraph. The smallest tree consists of one isolated vertex. A *tree in a graph* is a subgraph which is a tree. A *spanning tree* in a graph is a spanning subgraph which is a tree. A *forest* is a non-empty graph having no polygon as a subgraph. A *forest in a graph* is a subgraph which is a forest. A *maximal spanning forest* in a graph is a maximal spanning subgraph which is a forest.

We shall find it convenient to have defined the operations of deleting and contracting edges from a graph. Let $G = (V,E,i)$ be a graph and $E' \subseteq E$ a subset of edges. Then we shall say that the spanning subgraph $G_{E-E'} = (V,E-E',i)$ is obtained from G by *deleting* the edges of E' from G, and we denote it by $\mathcal{D}^{E'} G \equiv (V,E-E',i)$. Let further $\overline{V}$ be the set of equivalence classes of the set of vertices V under the relation of connection in $G_{E'}$ (in other words, let the vertices of G which are connected in $G_{E'}$ be identified), and let $\overline{i}$ be a relation on the edges of E-E' such that if the edge $e \in E-E'$ is incident in G with $i(e) = \{v,v'\}$, then $\overline{i}(e) \equiv \{\overline{v},\overline{v}'\}$, where $\overline{v} \in \overline{V}$ is the class of V containing v. We shall say that the edges of E' are *contracted from* G in order to obtain the graph $\mathcal{C}^{E'} G \equiv (\overline{V},E-E',\overline{i})$.

Obviously, if E',E" are disjoint subsets of E we have $\mathcal{D}^{E'}(\mathcal{D}^{E''} G) = \mathcal{D}^{E' \cup E''} G = \mathcal{D}^{E''}(\mathcal{D}^{E'} G)$ and $\mathcal{C}^{E'}(\mathcal{D}^{E''} G) = \mathcal{D}^{E''}(\mathcal{C}^{E'} G)$. Moreover we can prove that $\mathcal{C}^{E'}(\mathcal{C}^{E''} G) \simeq \mathcal{C}^{E' \cup E''} G \simeq \mathcal{C}^{E''}(\mathcal{C}^{E'} G)$ (see lemma 1 in section 3.2), where $G \simeq G'$ denotes that the graphs G and G' are isomorphic; we shall write = instead of ≃. It follows that we may define $\prod_{e \in E'} \mathcal{D}_e \equiv \mathcal{D}^{E'}$ with $\mathcal{D}_e \equiv \mathcal{D}^{\{e\}}$ and $\prod_{e \in E'} \mathcal{C}_e \equiv \mathcal{C}^{E'}$ with $\mathcal{C}_e \equiv \mathcal{C}^{\{e\}}$, where $\mathcal{D}_e$ is the operation of deleting the edge e from the graph, and $\mathcal{C}_e$ the operation of contracting the edge e from the graph. Observe that by definition $\mathcal{D}^{\emptyset} G = G$ and $\mathcal{C}^{\emptyset} G = G$. A graph $\mathcal{C}^{E'} \mathcal{D}^{E''} G$ with E' and $E'' \subseteq E(G)$ disjoint is called a *descendant* of G. The vertex

$\overline{v} \in \overline{V} = V(\mathcal{C}^{E'} \mathcal{D}^{E''} G)$ is called the vertex of the descendant associated with the vertex v of the graph G.

3. THE PERCOLATION MODEL

3.1. Description of the model

With an arbitrary graph we can associate various mathematical systems which serve as models for certain physical systems. One of these is the (bond) percolation model, introduced in 1957 by Broadbent and Hammersley [5] as a model for a medium with randomly distributed pores through which a liquid percolates. In this section we shall discuss this model for the case of a graph with non-directed edges. We shall successively introduce events on a graph (cf. Rényi [17] ch. I), local events, random events, probabilities of random events (cf. Rényi [17] ch. II), random variables and expectation values (cf. Zaanen [18] ch.3). Special care is to be taken in the definition of probabilities and expectation values on infinite graphs because of the occurrence of infinite products and sums and of the values $\pm\infty$ of certain expectation values.

Let G = (V,E,i) be an arbitrary graph, each edge of which can be in two different states, to be denoted by c and d. For each $e \in E$ we consider the two events: "e is in the state c" ≡ "e is a c-edge" and "e is in the state d" ≡ "e is a d-edge". These events are considered to be each other's negation; we denote them by c_e and d_e, respectively. In the literature on the percolation model the basic elements such as vertices, c-edges, d-edges, occur under various names. For convenience we give a short translation list:

vertex	atom	site	vertex	atom
edge	bond	bond	link	bond
c(onstituting)	undammed	black	active	occupied
d(ummy)	dammed	white	passive	vacant
this paper	ref. 5)	ref. 7)	ref. 19)	ref. 20)

From these events, to be called edge events, we construct more detailed events by taking (logical) products; we call these events product events and denote them as algebraic products. Thus $c_e d_{e'}$ is the event "e is a c-edge and e' is a d-edge". Using the symbolic power introduced in § 2 we can denote the general product event by $c^{E'} d^{E''}$ where $E', E'' \subseteq E$; $c_e d_e = 0 \equiv$ "the false event". For completeness we write $c^{\emptyset} = d^{\emptyset} = 1 =$ "the true event". Obviously $c_e + d_e = 1$. The most detailed (smallest) product events are those of the form $c^C d^D$ with $C \cup D = E$ and $C \cap D = \emptyset$. We call them elementary events. The set of all elementary events is called the event space, denoted by Ω. The (logical) sum of two events a and a' is denoted a+a'. From now on we shall assume complete distributivity for logical sums and products. Two events a and a', say, are called incompatible, or disjoint, if aa' = 0.

The events formed by finite sums of finite product events (that is, the events obtained by closing the collection of edge events under finite sums and finite products) are called local events. The events formed by closing the collection of local events under countable sums and countable products are called random events. The most general events are obtained by closing the collection of random events under arbitrary sums and products. By the assumption of complete distributivity each event can be written uniquely as a sum of elementary events, so that there is a one-to-one correspondence between events and subsets of the event space. This correspondence will be used extensively in the following.

We next define the probability P(a) of local events a. Firstly P(0) = 0 and P(1) = 1, secondly for the edge events $P(c_e) = p_e$ and $P(d_e) = q_e = 1-p_e$, where $0 \leq p_e \leq 1$. For finite product events $c^{E'} d^{E''}$ with $E' \cap E'' = \emptyset$ we define $P(c^{E'} d^{E''}) = p^{E'} q^{E''}$, i.e. the edge events are considered to be independent. For finite sums of disjoint finite product events we have $P(\Sigma_{i=1}^{n} a_i) = \Sigma_{i=1}^{n} P(a_i)$. Using the above-mentioned correspondence between events and subsets of the event space, we see that the probability on local events corresponds to a normed measure on the algebra of the cylinder sets corresponding to the local events.

A local variable will be a real function f on the event space Ω which assumes only a finite number of different values f_i such that for each i the sum of all elementary events with $f(c^C d^D;G) = f_i$ is a local event a_i. For brevety we shall often write $f(c^C d^D;G) = f(c^C d^D) = f(C;G) = f(C)$. The expectation value with respect to P of a local variable f is defined to be $\langle f \rangle = \Sigma_{i=1}^{n} f_i P(a_i) = \langle f;G,P \rangle$. The local variables correspond to the simple functions with respect to the algebra of cylinder sets, the expectation value corresponds to the integral with respect to P of a simple function. The functions obtained by closing the collection of non-negative local variables under the suprema and infima of countable collections (admitting the value $+\infty$) are called non-negative random variables. The difference between two non-negative random variables, not both assuming a value $\neq 0$ at the same time, is called a random variable; the non-negative random variables are called its positive and negative part.

Using the extension procedure of measures on semirings together with the Daniell integral scheme, we can, given a probability P on local events with the corresponding expectation value $\langle f \rangle$, extend these uniquely to a probability on random events and an expectation value of random variables (Zaanen [18] ch. 2, 3), for which we use again the notation P(a) and $\langle f \rangle$. If the expectation value of a random variable is finite, the random variable is said to be summable. If not both the expectation values of the positive and negative part of a random variable are $+\infty$, the random variable is said to be integrable. In the special case that the graph is finite, the expectation value of a random variable reduces to a sum: $\langle f \rangle = \Sigma_{C \subseteq E} f(C)\, p^C q^D$; here and in the following we understand by D the set E-C. In general we write $\langle f \rangle = \int_{C \subseteq E} dP(C)\, f(C)$.

A particular class of (non-negative) random variables is formed by the indicators of random events; the indicator of an event a is the function which takes the value 1 if a occurs and the value 0 if a does not occur. For convenience we use the same symbol for the event and for the indicator of that event. So c_e will represent both the edge event c_e and the indicator (of the event) that e is a c-edge. We have $\langle a \rangle = P(a)$ for random events a.

A countable graph G together with a probability P as described above we call a percolation model, to be denoted (G,P). Notice that the absence of correlations constitutes a most essential feature of the model. The probability P is completely characterized by the mapping p from E into the real interval $[0,1]$ such that $p(e) = p_e = P(c_e)$. We shall say that the measure P is generated by the mapping p.

We shall say that two vertices v and v' of G are c-connected in G, if there is a path in G between v and v' such that all edges in that path are c-edges. If G_C is the spanning subgraph of G with $E(G_C) = C \equiv$ the set of all c-edges, we may equivalently say that v and v' are c-connected in G if they are connected in G_C. Analogously we define a c-cluster, a c-polygon, the c-cyclomatic number etc.

Finally, we shall list some functions which will be considered in this or in subsequent papers. Most of these are indicators; for brevity we shall omit from their definition the words "the indicator of the event that".

γ ≡ the number of c-clusters

ω ≡ the c-cyclomatic number

$\gamma_{G'}$ ≡ G' is a c-cluster

$\gamma_{G';v}$ ≡ G' is a c-cluster containing the vertex v

$\gamma^{f}_{G';v}$ ≡ G' is a finite c-cluster containing the vertex v

$\gamma^{\infty}_{G';v}$ ≡ G' is an infinite c-cluster containing the vertex v

γ^{f}_{v} ≡ there is a finite c-cluster containing the vertex v

γ^{∞}_{v} there is an infinite c-cluster containing the vertes v; obviously $\gamma^{\infty}_{v} = 1 - \gamma^{f}_{v}$

$\gamma_{vv'}$ ≡ the vertices v and v' are c-connected = v and v' belong to the same c-cluster

γ_{e} ≡ the ends of the edge e are c-connected

$\gamma_{vV'}$ ≡ the vertex v is c-connected with at least one vertex v' of the set of vertices V'

For negations of γ indicators we use the symbol δ:

$\delta_{vv'}$ ≡ the vertices v and v' are not c-connected (are c-disconnected); $\delta_{vv'} = 1 - \gamma_{vv'}$

δ_{e} ≡ the ends of the edge e are not c-connected (are c-disconnected); $\delta_{e} = 1 - \gamma_{e}$

$\delta_{vV'}$ ≡ the vertex v is not c-connected with any of the vertices of V'; $\delta_{vV'} = 1 - \gamma_{vV'}$

$\delta^{f}_{v} = \gamma^{\infty}_{v}$

$\delta^{\infty}_{v} = \gamma^{f}_{v}$

Obviously, we have for finite graphs

$$\gamma = \sum_{G' \subseteq G} \gamma_{G'} .$$

For finite graphs the above-mentioned functions are random variables. We shall prove in a subsequent paper that for countable graphs they are also random variables.

3.2. Associated random variables on descendants. Recursion Theorem

Let f be a random variable defined on the event space of a graph G. Let E' and E" be disjoint subsets of E(G), then $\mathcal{C}^{E'}\mathcal{D}^{E''}G$ is a descendant of G. Now we associate with f a function $\overline{f}$ on the event space of $\mathcal{C}^{E'}\mathcal{D}^{E''}G$ by the definition (for disjoint sets E' and E" the union E'∪E" is alternatively denoted E' + E"):

$$\overline{f}(C;\mathcal{C}^{E'}\mathcal{D}^{E''}G) \equiv f(C+E';G) \quad \text{for all } C\subseteq E(\mathcal{C}^{E'}\mathcal{D}^{E''}G) = E(G)-E'-E''. \tag{3.1}$$

This function $\overline{f}$ is nothing but the section of f determined by the product events $c^{E'}d^{E''}$ (cf. Halmos[21] § 34), and it follows that if f is a random variable, $\overline{f}$ is a random variable. Moreover, if f is summable, $\overline{f}$ is summable. This procedure uniquely defines $\overline{f}$ on all descendants of G. In particular $\overline{f}(C;G) = f(C;G)$ and $\overline{f}(\emptyset;\mathcal{C}^{C}\mathcal{D}^{D}G) = f(C;G)$.

The state of an edge is a property of the edge alone. Therefore, if the state of all edges of G is given, the relation of c-connection is determined on all descendants of G, and also on all subgraphs of G. But this relation of c-connection on descendants is ultimately connected with the relation of c-connection in G, as is shown by the next lemma.

Lemma 1 Let G be a graph, let E',E" be disjoint subsets of E(G) and v,v' be vertices of V(G). Let $\mathcal{C}^{E'}G$ be the graph obtained by contracting the edges of E' from G, and $\overline{v},\overline{v'}$ the vertices of $V(\mathcal{C}^{E'}G)$ associated with v,v'. Then v and v' are connected in $G_{E'\cup E''}$ if and only if $\overline{v}$ and $\overline{v'}$ are connected in $(\mathcal{C}^{E'}G)_{E''}$.

Proof. Preliminary remarks: (a) if v and v' are connected in a graph G, there is, by definition, a path in G connecting them, and we may even say that there is a <u>vertex-disjoint</u> path in G connecting them, i.e. a path in which each vertex occurs only once; (b) all edges of $(\mathcal{C}^{E'}G)_{E''}$ and $G_{E'}$ are edges of $G_{E'\cup E''}$.

If there is a vertex-disjoint path in $G_{E'\cup E''}$ between v and v' we construct from it a path in $(\mathcal{C}^{E'}G)_{E''}$ between $\overline{v}$ and $\overline{v'}$ in the following way: If any edge of E' occurs in the path, we remove it together with the preceding and succeeding vertex and we replace it by the vertex of $\mathcal{C}^{E'}G$ associated with the preceding vertex.

If there is a vertex-disjoint path in $(\mathcal{C}^{E'}G)_{E''}$ between $\overline{v}$ and $\overline{v'}$, we construct from it a path in $G_{E'\cup E''}$ between v and v' in the following way. First, if v is incident in G with the first edge e_1 of the path, let v be the first vertex of the path in $G_{E'\cup E''}$ to be constructed. If v is not incident in G with e_1, there is a vertex v" of V(G) incident with e_1 in G such that $\overline{v''} = \overline{v}$, so there is, by definition of contraction, a path in $G_{E'}$ between v and v" and this path will be taken as the first part of the path in $G_{E'\cup E''}$ from v to v' to be constructed. Secondly, e_1 is also incident in G with a vertex v"' $\neq$ v,v", so the second vertex of the path in $(\mathcal{C}^{E'}G)_{E''}$ must be $\overline{v'''}$, and we can repeat the procedure on the second vertex and the second edge of the latter path. By the finiteness of the path in $(\mathcal{C}^{E'}G)_{E''}$ we obtain in this way a path in $G_{E'\cup E''}$ between v and v'.||

Corollary For all descendants of a graph G:
$\overline{\gamma} = \gamma$, $\overline{\gamma_{vv'}} = \gamma_{\overline{v}\overline{v'}}$, $\overline{\delta_{vv'}} = \delta_{\overline{v}\overline{v'}}$, $\overline{\gamma_{vV'}} = \gamma_{\overline{v}\overline{V'}}$, $\overline{\delta_{vV}} = \delta_{\overline{v}\overline{V'}}$.
Functions having this property are called $\mathcal{CD}$-invariant. The $\mathcal{CD}$-invariance property does not hold for all random variables; e.g. $\overline{\omega} \neq \omega$, $\overline{\gamma_{G'}} \neq \gamma_{\overline{G'}}$, $\gamma^f_v \neq \gamma^f_{\overline{v}}$ in general. However, if the number of contractions is finite, then $\gamma^f_v = \gamma^f_{\overline{v}}$, $\gamma^\infty_v = \gamma^\infty_{\overline{v}}$. For convenience we shall sometimes drop the association bar over functions and vertices, so we shall write f instead of $\overline{f}$, and v instead of $\overline{v}$, where no confusion can arise.

Using the extension of f to all descendants of G, we shall prove a recursion theorem in which the expectation value $\langle f;G\rangle$ is expressed in terms of the expectations values $\langle\bar{f};\mathcal{C}_e G\rangle$ and $\langle\bar{f};\mathcal{D}_e G\rangle$ defined on smaller graphs. This property is especially useful in the case of $\mathcal{CD}$-invariant random variables.

Theorem 1 Recursion theorem: Let (G,P) be a percolation model and f an integrable random variable. Then for all edges $e \in E(G)$

$$\langle f;G\rangle = p_e \langle\bar{f};\mathcal{C}_e G\rangle + q_e \langle\bar{f};\mathcal{D}_e G\rangle \, . \tag{3.2}$$

Proof. By definition $\langle f\rangle = \int dP(C)\, f(C)$. By construction P can be regarded as a product measure i.e. $P = P^E = P^e \times P^{E-e}$, where the upper index specifies the domain of P. If f is summable we can apply Fubini's theorem, if f is non-negative it is the limit of a non-decreasing sequence of summable random variables and we may again apply Fubini's theorem:

$$\int_{C\subseteq E} dP^E(C)\, f(C) = \int_{C'\subseteq\{e\}} dP^e(C') \int_{C''\subseteq E-e} dP^{E-e}(C'')\, f(C'+C'';G) =$$

$$= p_e \int_{C\subseteq E-e} dP^{E-e}(C)\, f(C+e;G) + q_e \int_{C\subseteq E-e} dP^{E-e}(C)\, f(C;G). \tag{3.3}$$

By the definition of the extension of f to the descendants $\mathcal{C}_e$ and $\mathcal{D}_e G$ this is equal to

$$p_e \int_{C\subseteq E-e} dP(C)\, \bar{f}(C;\mathcal{C}_e G) + q_e \int_{C\subseteq E-e} dP(C)\, \bar{f}(C;\mathcal{D}_e G) = p_e\langle\bar{f};\mathcal{C}_e G\rangle + q_e\langle\bar{f};\mathcal{D}_e G\rangle. \tag{3.4}$$

Finally, if f is integrable, but not necessarily summable or non-negative, then either the positive part f^+ of f or the negative part f^- of f is summable, say f^-, without loss of generality. We may use Fubini's theorem on the positive and negative part of f and collect the terms with p_e and q_e:

$$\langle f;G\rangle \equiv \langle f^+;G\rangle - \langle f^-;G\rangle = \left\{p_e\langle\overline{f^+};\mathcal{C}_e G\rangle + q_e\langle\overline{f^+};\mathcal{D}_e G\rangle\right\} -$$

$$- \left\{p_e\langle\overline{f^-};\mathcal{C}_e G\rangle + q_e\langle\overline{f^-};\mathcal{D}_e G\rangle\right\} =$$

$$= p_e\left\{\langle\overline{f^+};\mathcal{C}_e G\rangle - \langle\overline{f^-};\mathcal{C}_e G\rangle\right\} + q_e\left\{\langle\overline{f^+};\mathcal{D}_e G\rangle - \langle\overline{f^-};\mathcal{D}_e G\rangle\right\}. \tag{3.5}$$

Because $\overline{f^+} = \overline{f}^+$ and $\overline{f^-} = \overline{f}^-$ this equals

$$p_e\left\{<\overline{f}^+;\mathcal{C}_e G> - <\overline{f}^-;\mathcal{C}_e G>\right\} + q_e\left\{<\overline{f}^+;\mathcal{D}_e G> - <\overline{f}^-;\mathcal{D}_e G>\right\} \equiv$$

$$\equiv p_e <\overline{f};\mathcal{C}_e G> + q_e <\overline{f};\mathcal{D}_e G> \;. \quad || \tag{3.6}$$

Since the summability of a function implies the summability of the sections we have the following corollary.

Corollary The expectation value of a summable random variable f is a linear function of p_e with the finite boundary values:

$$<f;G,p_e = 0> = <\overline{f};\mathcal{D}_e G> \quad \text{and} \quad <f;G,p_e = 1> = <\overline{f};\mathcal{C}_e G> \;. \tag{3.7}$$

The recursion theorem may be generalized using the extension of functions to general descendants in the case of summable or non-negative random variables:

$$<f;G> = \int_{C'\subseteq E'} dP^{E'}(C') <\overline{f};\mathcal{C}^{C'}\mathcal{D}^{D'}G> \quad \text{for all } E'\subseteq E(G),\ D'=E'-C'. \tag{3.8}$$

In particular we obtain for E' = E:

$$<f;G> = \int_{C\subseteq E(G)} dP^E(C) <\overline{f};\mathcal{C}^C\mathcal{D}^D G> = \int_{C\subseteq E(G)} dP(C)\ f(C;G). \tag{3.9}$$

3.3. A relation between γ and δ_e.

An application of the recursion theorem is the proof of a proposition relating the expectation value of the number of c-clusters, γ, with the expectation value of the indicator δ_e of the event that the ends of the edge e are in different c-clusters. To prove this we need a lemma which contains the essential feature of the relationship (cf. Berge [22] ch. 4, Th. 1).

We recall that the number of clusters of a graph G is the number of equivalence classes of the vertices of G under the relation of connection in G: if G' is a cluster of G, V(G') is an equivalence class of V(G) under connection in G.

Lemma 2 Let G be a countable graph. Then for all edges $e \in E(G)$ and all subsets $C \subseteq E(G)-e$

(3.10) $$\gamma(C;G) = \gamma(C+e;G) + \delta_e(C;G).$$

Proof. We recall that $\gamma(C;G)$ is the number of clusters of G_C, $\gamma(C+e;G)$ is the number of clusters of G_{C+e}, and $\delta_e(C;G)$ is the indicator of the event that the ends of e are disconnected in G_C. Evidently, a path in G_C between two vertices is a path in G_{C+e} between the same vertices. Let $v,v' \in V(G)$ belong to a cluster of G_{C+e}, then there is a path in G_{C+e} between them. If the cluster of G_{C+e} containing v,v' does not contain e, this path is a path in G_C, too. Therefore, a cluster of G_{C+e} which does not contain e is a cluster of G_C which does not contain an end of e. There is just one cluster of G_{C+e} containing e, with vertex set V', say, and the vertices of V' may belong to several clusters of G_C. We shall prove that "several" can be only 1 or 2. Either $\delta_e(C;G) = 0$ or $\delta_e(C;G) = 1$. First, let $\delta_e(C;G) = 0$; then there is a path in G_C between the ends of e. Consequently, a path in G_C between any two vertices of V' can be obtained as follows: by definition, there is a path connecting them in G_{C+e}; if this path contains e, replace e by the path in G_C between the ends of e. Hence, there is just one cluster of G_C containing the vertices V', and so $\gamma(C;G) = \gamma(C+e;G)$. Secondly, let $\delta_e(C;G) = 1$, then there is no path in G_C between the ends of e. In this case, each vertex v of V' is connected in G_C either with one end or with the other end of e. For, there is a path in G_{C+e} to a given end of e; the part from v to the first end of e occurring in it (which may be the given end) is a path in G_C from v to that end, not containing e. If there was also a path in G_C to the other end of e, we could construct a path in G_C between the ends of e, contrary to the hypothesis. Hence, there are just two clusters of G_C, containing together the vertices of V', and so $\gamma(C;G) = \gamma(C+e;G) + 1$. ||

Proposition 1 Differentiation relation.

Let (G,P) be a percolation model such that γ is summable. Then for all edges $e \in E(G)$, with $p_e = 1-q_e$:

$$q_e \frac{\partial}{\partial q_e} \langle\gamma\rangle = \langle\delta_e\rangle. \tag{3.11}$$

Proof. By the preceding lemma $\gamma(C;G) = \gamma(C+e;G) + \delta_e(C;G)$ for each $C \subseteq E(G)-e = E(\mathcal{D}_e G) = E(\mathcal{C}_e G)$. By the $\mathcal{C}\mathcal{D}$-invariance of γ and of δ_e this is equivalent to $\gamma(G;\mathcal{D}_e G) = \gamma(C;\mathcal{C}_e G) + \delta_e(C;\mathcal{D}_e G)$, and integrating we get

$$\langle\gamma;\mathcal{D}_e G\rangle = \langle\gamma;\mathcal{C}_e G\rangle + \langle\delta_e;\mathcal{D}_e G\rangle. \tag{3.12}$$

By the recursion theorem $\langle\gamma;G\rangle = p_e\langle\gamma;\mathcal{C}_e G\rangle + q_e\langle\gamma;\mathcal{D}_e G\rangle$, from which it follows that

$$\langle\gamma;G\rangle = \langle\gamma;\mathcal{C}_e G\rangle + q_e\langle\delta_e;\mathcal{D}_e G\rangle. \tag{3.13}$$

By the summability of γ this can be differentiated, giving:

$$q_e \frac{\partial}{\partial q_e} \langle\gamma;G\rangle = q_e\langle\delta_e;\mathcal{D}_e G\rangle. \tag{3.14}$$

The proposition follows because $\langle\delta_e;G\rangle = p_e\langle\delta_e;\mathcal{C}_e G\rangle + q_e\langle\delta_e;\mathcal{D}_e G\rangle$, again by the recursion theorem, and $\langle\delta_e;\mathcal{C}_e G\rangle = 0$ because in $\mathcal{C}_e G$ the ends of the edge e coincide so that $\delta_e = 0$. ||

Notice: obviously, for a finite graph γ is summable for any P.

4. THE ISING MODEL OF FERROMAGNETISM

4.1. Reformulation of the partition function

We consider a finite spin ½ Ising system and we represent this system by a finite graph G, so that to each spin there corresponds a vertex of the graph, and to each interaction between a pair of spins there corresponds an edge incident with the corresponding vertices. With each vertex v we associate a spin variable σ_v, which can take the value +1 or -1. With each edge e we associate a coupling constant J_e and an edge variable $\sigma_e = \sigma_v \cdot \sigma_{v'}$, where v and v' are the ends of e; i.e. $\{v,v'\} = i(e)$. The Hamiltonian of the system is taken to be

$$H = - \sum_{e \in E(G)} J_e(\sigma_e - 1), \tag{4.1}$$

where $E(G) = E$ is the set of edges of G, and the energy of the ferromagnetic ground state has been chosen to be zero.

Let σ represent a sequence of values of σ_v for all $v \in V(G)$ and Σ_σ denote the summation over all possible sequences σ. The canonical partition function Z of the model is defined as

$$Z \equiv \sum_{\sigma} \exp\{-\beta H(\sigma)\} . \tag{4.2}$$

We shall formulate the partition function in terms of the percolation model [1]. To this end we shall first show that the partition function Z(G) of a system with finite graph G satisfies a recursion relation. For any edge $e \in E(G)$

$$Z(G) = p_e Z(\mathscr{C}_e G) + q_e Z(\mathscr{D}_e G), \tag{4.3}$$

where

$$q_e \equiv \exp(-2\beta J_e), \quad p_e \equiv 1 - q_e . \tag{4.4}$$

To prove this relation one notices that the sum over all states of G can be split up into a sum over all states with $\sigma_e = +1$, and a sum over all states with $\sigma_e = -1$:

$$\sum_{\sigma} \exp\left\{\beta \sum_{e'\in E(G)} J_{e'}(\sigma_{e'}-1)\right\} = \left(\sum_{\substack{\sigma\\ \sigma_e=+1}} + \sum_{\substack{\sigma\\ \sigma_e=-1}}\right) \exp\left\{\sum_{e'\in E} \beta J_{e'}(\sigma_{e'}-1)\right\} =$$

(4.5) $$\sum_{\substack{\sigma\\ \sigma_e=+1}} \exp\left\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'}-1)\right\} + q_e \cdot \sum_{\substack{\sigma\\ \sigma_e=-1}} \exp\left\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'}-1)\right\}.$$

The spin states of the graph $\mathcal{C}_e G$, with the edge e contracted, are in one-to-one correspondence with the spin states of the graph G which have $\sigma_e = +1$, and $E(\mathcal{C}_e G) = E(G)-e$, so

(4.6) $$Z(\mathcal{C}_e G) = \sum_{\substack{\sigma\\ \sigma_e+1}} \exp\left\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'}-1)\right\}.$$

Because $V(\mathcal{D}_e G) = V(G)$, the spin states of the graph $\mathcal{D}_e G$ are just the same as those of the graph G, and the summation over all states may be split up as before. Using $E(\mathcal{D}_e G) = E(G)-e$, we obtain

(4.7) $$Z(\mathcal{D}_e G) = \sum_{\substack{\sigma\\ \sigma_e=+1}} \exp\left\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'}-1)\right\} + \sum_{\substack{\sigma\\ \sigma_e=-1}} \exp\left\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'}-1)\right\}.$$

The above-mentioned recursion relation for Z(G) follows by elimination of the partial sums from Z(G), $Z(\mathcal{C}_e G)$ and $Z(\mathcal{D}_e G)$ in eqs. (4.5), (4.6), (4.7).

The recursion relation for Z(G) implies an interpretation for $Z(\mathcal{C}_e G)$ and $Z(\mathcal{D}_e G)$:

(4.8) $$Z(\mathcal{C}_e G) = \lim_{J_e\to+\infty} Z(G), \qquad Z(\mathcal{D}_e G) = \lim_{J_e\to 0} Z(G);$$

so taking the limit of strong ferromagnetic coupling is equivalent with contracting edges, and taking the limit of weak coupling is equivalent with deleting edges as one would expect.

If we iterate the recursion relation with respect to all edges we finally get

(4.9) $$Z(G) = \sum_{C\subseteq E} p^C q^D \, Z(\mathcal{C}^C\mathcal{D}^D G),$$

which expresses the partition function of G in terms of the partition function of systems without interaction. Evidently $Z(\mathcal{C}^C\mathcal{D}^D G) = \exp\{|V(\mathcal{C}^C\mathcal{D}^D G| \ln 2\}$. Because the number of vertices of such a graph equals the number of clusters, and the latter, unlike the former, is a $\mathcal{C}\mathcal{D}$-invariant random variable, we can write $|V(\mathcal{C}^C\mathcal{D}^D G)| = \gamma(\emptyset;\mathcal{C}^C\mathcal{D}^D G) = \gamma(C;G)$, and therefore:

(4.10) $$Z(G) = \sum_{C\subseteq E} p^C q^D \, 2^{\gamma(C;G)} = \langle 2^{\gamma};G,P\rangle$$

in the terminology of the percolation model, with the probability measure P generated by (4.4). The term probability is justified only in the ferromagnetic case, i.e. when $J_e \geq 0$ for all edges $e\in E(G)$, because only then $0 \leq q_e \leq 1$. By eq. (4.10) we have rewritten the partition function of an Ising system, which by definition is a sum over spin states, as a percolation model average, i.e. as a sum over edge states.

4.2. Generalized reformulation of the Ising model.

In the theory of the Ising model one is not only interested in the partition function but also in canonical averages of spin functions $f(\sigma)$:

(4.11) $$\langle f\rangle_{can} \equiv \sum_{\sigma} f(\sigma) \exp\{-\beta H(\sigma)\} \Big/ \sum_{\sigma} \exp\{-\beta H(\sigma)\}.$$

The denominator in the right-hand side of this definition is the partition function Z, which in the previous section has been rewritten as a percolation model average. The numerator of the right-hand side can also be rewritten as a percolation-model average, but the method used in the previous section cannot be applied. Instead we shall use an alternative method, which, of course, also applies to Z.

We start by writing $1 = \frac{1}{2}(1+\sigma_e) + \frac{1}{2}(1-\sigma_e)$, and thereby

$$\sum_\sigma f(\sigma)\exp\{-\beta H(\sigma)\} =$$

$$= \sum_\sigma \tfrac{1}{2}(1+\sigma_e)\, f(\sigma)\exp\{-\beta H(\sigma;G)\} + \sum_\sigma \tfrac{1}{2}(1-\sigma_e)\, f(\sigma)\exp\{-\beta H(\sigma;G)\} =$$

$$= \sum_\sigma \tfrac{1}{2}(1+\sigma_e)\, f(\sigma)\exp\{-\beta H(\sigma;\mathscr{D}_e G)\} + q_e \sum_\sigma \tfrac{1}{2}(1-\sigma_e)\, f(\sigma)\exp\{-\beta H(\sigma;\mathscr{D}_e G)\}, \tag{4.12}$$

by the same argument that was used in section 4.1. Notice however that both terms in the last member are sums defined for the same graph $\mathscr{D}_e G$. Multiplying the first sum by $1 = p_e + q_e$ and collecting the terms with p_e and those with q_e we obtain:

$$q_e \sum_\sigma f(\sigma)\exp\{-\beta H(\sigma;\mathscr{D}_e G)\} + p_e \sum_\sigma \tfrac{1}{2}(1+\sigma_e)\, f(\sigma)\exp\{-\beta H(\sigma;\mathscr{D}_e G)\}\,. \tag{4.13}$$

Iterating this with respect to all edges we get the expansion

$$\sum_\sigma f(\sigma)\exp\{-\beta H(\sigma;G)\} = \sum_{C\subseteq E} p^C q^{E-C} \sum_\sigma \{\tfrac{1}{2}(1+\sigma)\}^C f(\sigma)\exp\{-\beta H(\sigma;\mathscr{D}^E G)\} =$$

$$= \sum_{C\subseteq E} p^C q^{E-C} \sum_\sigma \{\tfrac{1}{2}(1+\sigma)\}^C f(\sigma), \tag{4.14}$$

because the Hamiltonian of a graph without interactions is zero. If we define a function on the event space of G, thus for every $C\subseteq E(G)$, by

$$\hat{f}(C;G) \equiv \sum_\sigma \{\tfrac{1}{2}(1+\sigma)\}^C f(\sigma;G), \tag{4.15}$$

then by the definition of canonical averages we have

$$\langle f\rangle_{can} = \langle\hat{f}\rangle/\langle\hat{1}\rangle. \tag{4.16}$$

In this way we have arrived at a description of canonical averages in terms of percolation model averages. The factor $\{\frac{1}{2}(1+\sigma)\}^C$ can be interpreted as the restriction that all spins connected by c-edges must be parallel in order to give a non-zero contribution to the summation. So each c-connected component of G acts as one spin. Since the operation $\hat{}$ is linear and all spin functions $f(\sigma)$ can be linearly expressed in terms of 1, σ_v, $\sigma_v\sigma_{v'}$, etc., it is sufficient to consider the random variables related with these special functions:

$$\hat{1}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^C \cdot 1 = 2^{\gamma(C)},$$

$$\hat{\sigma}_v(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^C \cdot \sigma_v = 0,$$

$$\widehat{\sigma_v \sigma_{v'}}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^C \cdot \sigma_v \sigma_{v'} = \gamma_{vv'}(C)\, 2^{\gamma(C)},$$

and in general for $V' \subseteq V(G)$:

$$\widehat{\sigma^{V'}}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^C \sigma^{V'} = \varepsilon_{V'}(C)\, 2^{\gamma(C)}, \tag{4.17}$$

where $\varepsilon_{V'}$ is the indicator that each c-cluster contains an even number of the vertices of V'. The random variables (4.17) are all $\mathcal{CD}$-invariant and obviously satisfy the recursion relation. From eqs. (4.15)-(4.17) we find:

$$\begin{aligned} Z(G) &= \langle 2^{\gamma}; G\rangle, \\ \langle\sigma_v\rangle_{can} &= 0, \\ \langle\sigma_v\sigma_{v'}\rangle_{can} &= \langle\gamma_{vv'} 2^{\gamma}\rangle / \langle 2^{\gamma}\rangle, \\ \langle\sigma^{V'}\rangle_{can} &= \langle\varepsilon_{V'} 2^{\gamma}\rangle / \langle 2^{\gamma}\rangle. \end{aligned} \tag{4.18}$$

If in eq. (4.18) v and v' are the ends of the edge $e \in E(G)$ we have in particular $\langle\sigma_e\rangle_{can} = \langle\gamma_e 2^{\gamma}\rangle / \langle 2^{\gamma}\rangle$. On the other hand, one easily sees by differentiating the free energy, $F \equiv -\beta^{-1} \ln Z$, with respect to J_e:

$$\frac{\partial F}{\partial J_e} = Z^{-1} \sum_{\sigma} \frac{\partial H(\sigma)}{\partial J_e} \exp\{-\beta H(\sigma)\} = Z^{-1} \sum_{\sigma} (1-\sigma_e) \exp\{-\beta H(\sigma)\} = 1 - \langle\sigma_e\rangle_{can}. \tag{4.19}$$

Using eqs. (4.4), (4.10) and (4.18) we obtain

$$q_e \frac{\partial}{\partial q_e} 2 \ln \langle 2^{\gamma}\rangle = \langle\delta_e 2^{\gamma}\rangle / \langle 2^{\gamma}\rangle. \tag{4.20}$$

This equation (4.20) is analogous to eq. (3.11), and shows that in the percolation model the function $\langle\gamma\rangle$ plays the same role as the free energy in the Ising model.

Up to now we have chosen the ground-state energy for ferromagnetic interaction zero in order to normalize the measure $P:P(1)=1$, and to make it possible to interpret P as a probability: $0 \leq P(d_e)=q_e \leq 1$. In the antiferromagnetic case the above given procedure leads by eq. (4.4) to values $q_e>1$ and $p_e<0$. It is possible to retain a probabilistic interpretation of the p's and q's by replacing for "antiferromagnetic edges" the factor (σ_e-1) in eq. (4.1) by (σ_e+1) and the factors $\frac{1}{2}(1+\sigma_e)$ in eqs. (4.12)ff by $\frac{1}{2}(1-\sigma_e)$, and vice versa, but in that case the function 2^γ in eq. (4.10) is replaced by a more complicated one.

4.3 Ising model in magnetic field.

In section 4.1 and section 4.2 we have considered the spin $\frac{1}{2}$ Ising system without an external magnetic field. In the case where there is a magnetic field, which has the value B_v at the position of the spin v with magnetic moment m_v, the Hamiltonian of the system is

$$H = - \sum_{e \in E(G)} J_e(\sigma_e-1) - \sum_{v \in V(G)} m_v B_v(\sigma_v-1). \tag{4.21}$$

It is well known (cf. Griffiths [23], Suzuki [24]) that such an external magnetic field can be replaced by one supplementary "ghost spin" which interacts with any spin v with a coupling constant $m_v B_v$. We may therefore replace the Hamiltonian of the system with graph G in an external magnetic field, given in (4.21), by the Hamiltonian of the system with graph G^o, obtained from G by adding one vertex o and for each vertex v of G an edge incident with v and o:

$$H^o = - \sum_{e \in E(G^o)} J_e(\sigma_e-1). \tag{4.22}$$

The partition function Z^0 calculated from this Hamiltonian H^o

is twice the partition function Z calculated from H, which has, of course, no influence on the expectation values of spin correlations.

4.4. Ashkin-Teller-Potts model.

There is a straightforward generalization of the Ising model in which each atom can be in n different states, where n is an arbitrary number ≥ 2 [3,4]. In this so-called Ashkin-Teller-Potts model the energy between two interacting spins is taken to be zero if the atoms are in the same state, and equal to a constant if they are in different states. If the system is represented by a graph, just as in the Ising model, and if the above-mentioned constant is denoted by $2J_e$, the Hamiltonian can again be written in the form (4.1), where the edge variables σ_e have the values +1 and -1, accordingly as the atoms at the ends of e are in the same state or not. Although we cannot introduce simple spin variables for the states of the atoms (i.e. σ_e cannot be written as a simple product of two spin variables), we shall still denote the states of the system by σ. We can then apply the same procedure as was used in sections 4.1 and 4.2. We thus get a recursion relation for the partition function $Z=Z_n(G,\beta,J)$ just as in the case of 2 states per atom (see eq. 4.3).

(4.3)' $$Z(G) = p_e\, Z(\mathcal{C}_e G) + q_e\, Z(\mathcal{D}_e G),$$

where p_e and q_e are again defined by eq. (4.4). But now, after iterating eq. (4.3), we have to substitute in eq. (4.9) $Z(\mathcal{C}^C\mathcal{D}^D G) = n^{\gamma(C;G)}$, because in the graph $\mathcal{C}^C\mathcal{D}^D G$ each of the "atoms" can independently be in n states. So eq. (4.10) is generalized to

(4.23) $$Z(G) = \sum_{C\subseteq E} p^C q^D n^{\gamma(C;G)} = \langle n^\gamma;G,P\rangle.$$

In case the interaction energy $2J_e$ is positive, we have again $0 \leq P \leq 1$, i.e. P is a probability measure.

5. GRAPH COLOURINGS

5.1. Formulation of the problem

An old and well-known problem in graph theory is the following. Let $G = (V,E,i)$ be a finite graph and Q_n a set of n elements called "colours". Each mapping f of the set of vertices V into the set of colours Q_n is called a (vertex) colouring of the graph with at most n colours; colourings with the property that for each edge the ends have different colours are called n-colourings. The problem is to study the total number of n-colourings of the graph, which is denoted by $P(n;G)$, as a function of n.

In the special case that G is planar, i.e. if there exists a faithful representation of the graph as a map in a plane such that lines representing edges do not cross, the number of n-colourings of G is equal to the number of country colourings of the dual map such that neighbouring countries have different colours.

5.2. Recursion relation

As found by R.M. Foster (unpublished, see however ref. 25) there exists a recursion relation for the total number of n-colourings $P(n;G)$. It is derived by dividing the n-colourings of $\mathcal{D}_e G$ for a given edge e into those which have the property $f(v) \neq f(v')$ where v and v' are the ends of the edge e, and those with $f(v) = f(v')$ (possible because e is not an edge of $\mathcal{D}_e G$). The former ones are just the n-colourings of G, the latter ones just the n-colourings of $\mathcal{C}_e G$, because in the latter graph $v=v'$. So

$$P(n;\mathcal{D}_e G) = P(n;G) + P(n;\mathcal{C}_e G), \tag{5.1}$$

and we arrive at the recursion relation:

$$P(n;G) = P(n;\mathcal{D}_e G) - P(n;\mathcal{C}_e G). \tag{5.2}$$

This may be compared with eq. (3.2) and eq. (4.3). Iterating eq. (5.2) with respect to all edges we get G.D. Birkhoff's formula [8]

(5.3) $$P(n;G) = \sum_{C\subseteq E} (-)^C\, P(n;\mathcal{C}^C\mathcal{D}^D G) = \sum_{C\subseteq E} (-)^C\, n^{\gamma(C;G)},$$

because the graph $\mathcal{C}^C\mathcal{D}^D G$ consists of just $\gamma(C;G)$ isolated vertices which can be coloured each one independently with n colours.

We can write, with $p=1-q$, $D=E-C$:

(5.4) $$\sum_{C\subseteq E} (-)^C\, n^{\gamma(C)} = \lim_{q\to\infty} q^{-E} \sum_{C\subseteq E} p^C q^D\, n^{\gamma(C)}.$$

If we allow also negative values of measures in the percolation model, as we did in the case of the antiferromagnetic Ising model, we can write eq. (5.3), with the aid of (5.4), as

(5.5) $$P(n;G) = \lim_{q\to\infty} q^{-E(G)} \langle n^{\gamma};G,P\rangle, \qquad p = 1-q.$$

This should be compared with eq. (4.10) and eq. (4.23). For the antiferromagnetic case of the Ashkin-Teller-Potts model the probability $q\to\infty$ corresponds to temperature zero, so the number of n-colourings is equal to the degeneracy of the ground state of the antiferromagnetic Ashkin-Teller-Potts model with n states per atom.

6. LINEAR RESISTANCE NETWORKS

6.1. Formulation of Kirchhoff's problem

In this section we shall consider finite electrical networks consisting of linear resistors and generators of electromotive force; the electrical character of the network is in no way essential to what follows. We shall represent such an electrical-resistance network by a finite connected graph G = (V,E,i) where

V is the set of nodes of the network, E the set of branches of the network (resistors or generators) and i the incidence relation. Kirchhoff [9] solved in 1847 the problem of finding the currents through the branches of a finite network each of which has a resistance and an electromotive force. By virtue of the superposition principle, however, it is sufficient to solve the case where only one edge e has an electromotive force U_e, say, while every other edge $e' \neq e$ has a resistance $R_{e'}$. Moreover we shall concentrate on Kirchhoff's solution for the electric current I_e through the edge e. This solution can in our notation be written as follows. If U_e and I_e are measured in the same sense, then

(6.1) $$I_e = - U_e \sum_{T \in \mathbf{T}(\mathcal{D}_e G)} R^{E-T-e} \Big/ \sum_{T \in \mathbf{T}(\mathcal{C}_e G)} R^{E-T-e},$$

where for any graph G, $\mathbf{T}(G)$ is the collection of edge sets of all spanning trees in G.

To get this in a more usual form we multiply the numerator and denominator of (6.1) with the product S^{E-e} of all conductances $S_{e'} \equiv R_{e'}^{-1}$:

(6.2) $$I_e = -U_e \sum_{T \in \mathbf{T}(\mathcal{D}_e G)} S^T \Big/ \sum_{T \in \mathbf{T}(\mathcal{C}_e G)} S^T.$$

Eq. (6.2) expresses the current I_e as a quotient of the generating functions of spanning trees of the graphs $\mathcal{D}_e G$ and $\mathcal{C}_e G$. The effective resistance $R_e^{eff} \equiv - U_e/I_e$ "seen by" the electromotive force is, in the special case when e is parallel to some resistance, say e',

(6.3) $$R_e^{eff} = \sum_{T \in \mathbf{T}(\mathcal{C}_e G)} S^T \Big/ \sum_{T \in \mathbf{T}(\mathcal{D}_e G)} S^T = \frac{\partial}{\partial S_{e'}} \ln \sum_{T \in \mathbf{T}(\mathcal{D}_e G)} S^T; \quad i(e) = i(e').$$

Indeed, the spanning trees of $\mathcal{C}_e G$ can be made to spanning trees of $\mathcal{D}_e G$ by undoing the identification of the ends of e (which are the ends of e' too) and adding the edge e'. These spanning trees are just the spanning trees of $\mathcal{D}_e G$ containing e'; hence

(6.4) $$\frac{\partial}{\partial S_{e'}} \sum_{T\in\mathcal{T}(\mathcal{D}_e G)} S^T = \sum_{T\in\mathcal{T}(\mathcal{C}_e G)} S^T; \qquad i(e) = i(e'),$$

which proves (6.3). A comparison of eq. (6.3) with (4.20) shows that the generation function of spanning trees $\Sigma_{T\in\mathcal{T}(G)} S^T$ plays a role similar to that of the partition function of the Ising model.

6.2. Reformulation

We may observe that the generating function of spanning trees $Z_T(G,S) = Z_T(G) \equiv \Sigma_{T\in\mathcal{T}(G)} S^T$ obeys a recursion relation:

(6.5) $$Z_T(G) = Z_T(\mathcal{D}_e G) + S_e Z_T(\mathcal{C}_e G) \quad \text{if } e \text{ is not a loop in } G,$$

because we can devide the spanning trees of G into two classes according to the occurrence or non-occurrence of the edge e: if a spanning tree of G does not contain e, it is also a spanning tree of $\mathcal{D}_e G$; if it does contain e, $\mathcal{C}_e G_T$ is just a spanning tree of $\mathcal{C}_e G$. This recursion relation is to be compared with those in eq. (3.2), (4.3) or (5.1).

We can derive an expression for the generating function of spanning trees which closely resembles expression (4.10) for the partition function of the Ising model. To that end we observe that we may characterize the spanning trees G_T of a finite connected graph G by the property $\omega(T;G) = 0$, $\gamma(T;G) = 1$, or equivalently by

(6.6) $$\omega(T;G) + \gamma(T;G) = 1 = \inf_{C\subseteq E(G)} \{\omega(C;G) + \gamma(C;G)\}.$$

The last equality follows from the inequalities $\omega\geq 0$ and $\gamma\geq 1$. This characterization can be used to generate all spanning trees of a given finite connected graph G by a polynomial in x:

(6.7) $$\sum_{T\in\mathcal{T}(G)} S^T = \sum_{C\subseteq E} S^C \lim_{x\downarrow 0} x^{\{\omega(C;G)+\gamma(C;G)-1\}} = \lim_{x\downarrow 0} x^{-1} \sum_{C\subseteq E} S^C x^{\{\omega(C)+\gamma(C)\}}.$$

Now putting $x = \kappa^{\frac{1}{2}}$ for positive κ and using Euler's formula (cf. ref. 22, ch. 4, th. 2)

(6.8) $$\omega(C;G) = |C| - |V(G)| + \gamma(C;G)$$

in eq. (6.7) we get, with

(6.9) $$p \equiv \kappa^{\frac{1}{2}}S(1 + \kappa^{\frac{1}{2}}S)^{-1}, \qquad q \equiv (1 + \kappa^{\frac{1}{2}}S)^{-1},$$

the equality

(6.10) $$\sum_{T\in\mathcal{T}(G)} S^T = \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}} \sum_{C\subseteq E} S^C \kappa^{\frac{1}{2}\{|C|-|V|+2\gamma(C)\}} = \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}\{|V|+1\}} q^{-E} \sum_{C\subseteq E} \kappa^{\gamma(C)} p^C q^D.$$

Notice that $p + q = 1$ and $0 \leq p \leq 1$ for $S \geq 0$, so that we may write the generating function of spanning trees in terms of the percolation model as:

(6.11) $$Z_T(G;S) = \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}\{|V|+1\}} \langle\kappa^\gamma;G,P\rangle.$$

Eq. (6.11) is to be compared with eq. (4.10) and (5.5).

7. RANDOM-CLUSTER MODEL

7.1. Description of the model

After having shown, in the preceding sections, that for a number of models and problems the functions which play a key role in the calculations can be expressed in a uniform way in terms of the percolation model, we shall in this section introduce a new model with a "key function" which includes the above-mentioned key functions as special cases.

Let first $G = (V,E,i)$ be a finite graph and P a normed measure ($P(1)=1$) on the event space of G, generated by a function p on E. We shall find it convenient to allow negative values for p, i.e. we consider P to be a signed measure. Let κ be a real number and let γ denote the number of c-clusters. We define the <u>cluster (generating) function</u> of G by $(q \equiv 1-p)$

$$Z(G;p,\kappa) \equiv \sum_{C\subseteq E} p^{C} q^{E-C} \kappa^{\gamma(C;G)}, \tag{7.1}$$

and a normed signed measure by

$$\mu(C) = \mu(C;G,p,\kappa) \equiv p^{C} q^{E-C} \kappa^{\gamma(C;G)} Z^{-1}(G,p,\kappa) \tag{7.2}$$

for all subsets $C\subseteq E(G)$ and for all (G,p,κ) such that $Z(G,p,\kappa) \neq 0$.

Next, let G be an infinite countable graph, and let G_n be an increasing sequence of finite subgraphs of G such that $\cup_{n=1}^{\infty} G_n = G$ and $Z(G_n,p,\kappa) \neq 0$ for almost all n. For any local event a on G there is an n(a) such that for $n \geq n(a)$, a is a local event on G_n. The normal signed measure $\mu(a)$ of a local event on G will be defined by

$$\mu(a) = \lim_{n\to\infty} \mu_n(a), \qquad n \geq n(a), \tag{7.3}$$

where μ_n is the signed measure defined on G_n. A necessary and sufficient condition for this limit to exist is that $\mu(c^C)$ exists for all finite subsets $C\subseteq E(G)$. This signed measure μ may be extended to random events by the procedure mentioned in section 3.1, and the corresponding expectation value to random variables f, to be denoted by $\langle f;G,\mu\rangle = \langle f;G,p,\kappa\rangle = \langle f\rangle$. Notice that, unlike P, the measure μ is not a product measure. The influence of the c-clusters makes the edge events dependent on each other, and thus introduces a global effect in the measure.

A countable graph G together with a normed signed measure μ as described above we call a <u>random-cluster model</u>, to be denoted (G,μ) or (G,p,κ).

In order to have in the random-cluster model an analogue of the magnetic field in the Ising model, we shall occasionally add to the graph G a supplementary vertex 0 and to each vertex v of V(G) one supplementary edge incident with v and o. The graph thus obtained will be called supplemented and denoted $G^o = (V^o, E^o, i)$, with $V^o = V(G) \cup \{o\}$, $E^o = E(G) \cup E_o$, where E_o is the set of supplementary edges. The probability for the edge incident with o and v to be a c-edge will be p_{ov}. Furthermore, $1 - p_{ov} = q_{ov}$. The measures generated by $p^o \equiv p \cup p_o$ are denoted p^o and μ^o.

7.2. Some properties of the random-cluster model

In the preceding sections we have shown that in the various systems considered the "key function" obeys a recursion and a differentiation relation. We shall now show that in the random-cluster model the cluster function Z obeys a recursion relation and a differentiation relation.

The cluster function Z is defined for finite graphs G, so the number of c-clusters lies between 1 and $|V(G)|$. Consequently, Z is finite for any finite graph. Let G be a finite graph and $e \in E(G) \equiv E$, then with $E - e \equiv E'$, $E - C \equiv D$, $E' - C' \equiv D'$ for $C' \subseteq E'$,

$$Z(G) = \sum_{C \subseteq E} p^C q^D \kappa^{\gamma(C;G)} = \sum_{C' \subseteq E'} p^{C'} q^{D'} \left\{ p_e \kappa^{\gamma(C'+e;G)} + q_e \kappa^{\gamma(C';G)} \right\} =$$

$$= \sum_{C' \subseteq E'} p^{C'} q^{D'} \left\{ p_e \kappa^{\gamma(C';\mathcal{C}_e G)} + q_e \kappa^{\gamma(C';\mathcal{D}_e G)} \right\}, \tag{7.5}$$

because γ is $\mathcal{C}\mathcal{D}$-invariant, i.e. $\gamma(C';\mathcal{C}_e G) = \gamma(C'+e;G)$ and $\gamma(C';\mathcal{D}_e G) = \gamma(C';G)$ for all e and all $C' \subseteq E'$. From eq. (7.5) we obtain the following:

Proposition 2 Recursion relation.

Let (G,p,κ) be a finite random-cluster model. Then for all edges $e \in E(G)$:

(7.6) $$Z(G) = p_e Z(\mathcal{C}_e G) + q_e Z(\mathcal{D}_e G).$$

Notice that for $0 \leq p \leq 1$, eq. (7.6) is a particular case of theorem 1.

In order to obtain a differentiation relation for Z, we make use of lemma 2 and the same type of argument as was used in the proof of proposition 1. Thus we easily obtain the following:

Proposition 3 Differentiation relation.

Let (G,p,κ) be a finite random-cluster model with $p = 1 - q$ and $Z \neq 0$. Then for all edges $e \in E(G)$:

(7.7) $$q_e \frac{\partial}{\partial q_e} \ln Z(G,p,\kappa) = (1-\kappa^{-1}) \langle \delta_e ; G,p,\kappa \rangle .$$

Finally, we mention the following, almost trivial, property of the cluster functions.

Proposition 4 Product relation.

Let (G,p,κ) be a finite random-cluster model and G' and G" disjoint subgraphs of G. Then

(7.8) $$Z(G' \cup G'') = Z(G') \cdot Z(G'').$$

Indeed, if $C \subseteq E(G') \cup E(G'')$, $C' \equiv C \cap E'$, $C'' \equiv C \cap E''$, $D' \equiv D \cap E'$ and $D'' \equiv D \cap E''$, then $C = C' \cup C''$, $D = D' \cup D''$, $\gamma(C;G) = \gamma(C';G') + \gamma(C'';G'')$, and therefore

(7.9) $$p^C q^D \kappa^{\gamma(C;G' \cup G'')} = p^{C'} q^{D'} \kappa^{\gamma(C';G')} \cdot p^{C''} q^{D''} \kappa^{\gamma(C'';G'')} .$$

Since summation over all $C \subset E' \cup E''$ is equivalent to a repeated summation over all $C' \subset E'$ and all $C'' \subset E''$, eq. (7.8) follows by summing (7.9).

7.3. Special cases of the random-cluster model

In this section we regard only finite graphs. We show that the random-cluster model generalizes the systems discussed in previous sections. The percolation problem is regained by putting $\kappa = 1$.

$$Z(G,p,1) = 1 \text{ and } \mu(C;G,p,1) = p^C q^D, \tag{7.10}$$

i.e. μ reduces to the original measure P.

As shown by the equations (4.10) and (4.23), together with (4.4), expressing the partition function of the Ising and Ashkin-Teller-Potts model in terms of the percolation model, we have for $\kappa = n \geq 2$:

$$Z_n(G,\beta,J) = Z(G,1-\exp(-2\beta J),n). \tag{7.11}$$

As shown by equation (5.5) for the chromatic polynomial we have further

$$P(n;G) = \lim_{q\to\infty} q^{-E(G)} Z(G,1-q,n). \tag{7.12}$$

Finally, the generating function of spanning trees in connected graphs, occurring in the theory of linear resistance networks, can, by eq. (6.9) and (6.11), be written as

$$Z_T(G,S) = \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}\{|V(G)|+1\}} Z(G,\kappa^{\frac{1}{2}}S/(1+\kappa^{\frac{1}{2}}S),\kappa). \tag{7.13}$$

The differentiation relations for the various systems follow from the single differentiation relation (7.7) for the random-cluster model. The equations (3.11), restricted to finite graphs, (4.20) and (6.3) are obtained from (7.7) using the same procedure as for the cluster function.

To begin with, for $\kappa = 2$ and probabilities (4.4), eq. (7.7) reduces to eq. (4.20). Next we observe that for $\kappa = 1$ both sides of (7.7) vanish. In order to obtain eq. (3.11) we first divide both sides of (7.7) by $(\kappa-1)$ for $\kappa \neq 1$, and then take the limit $\kappa\to 1$. From the left-hand side of (7.7) we get, after having interchanged limit and derivative,

$$q_e\frac{\partial}{\partial q_e}\lim_{\kappa\to 1}\frac{\ln Z}{\kappa-1} = q_e\frac{\partial}{\partial q_e}\lim_{\kappa\to 1}\frac{\frac{\partial}{\partial\kappa}\ln Z}{\frac{\partial}{\partial\kappa}(\kappa-1)}\text{, by l'Hôpital's rule,}$$

$$(7.14)\qquad = q_e\frac{\partial}{\partial q_e}\lim_{\kappa\to 1} Z^{-1}\langle\gamma\kappa^{\gamma-1};G,P\rangle = q_e\frac{\partial}{\partial q_e}\langle\gamma;G,P\rangle.$$

From the right-hand side of (7.7) we get

$$(7.15)\qquad \lim_{\kappa\to 1}\kappa^{-1}\langle\delta_e;G,p,\kappa\rangle = \langle\delta_e;G,p,1\rangle = \langle\delta_e;G,P\rangle.$$

So eqs. (7.14) and (7.15), together with (7.7), give eq. (3.11).

Finally, we have seen that at least one quantity of linear resistance networks is also obtained in an asymptotic way, namely by putting $p = \kappa^{\frac{1}{2}}S/(1+\kappa^{\frac{1}{2}}S)$, (see eq. (6.9)), and taking the limit $\kappa\downarrow 0$. Because for $\kappa\downarrow 0$ both sides of (7.7) tend to $-\infty$, they have first to be multiplied by $-\kappa^{\frac{1}{2}}$, as we shall show, in order to obtain eq. (6.3). The procedure used to derive eq. (6.11) is now applied in the reversed direction to $Z = \langle\kappa^{\gamma};G,P\rangle$ and $\langle\delta_e;G,p,\kappa\rangle = \langle\delta_e\kappa^{\gamma};G,P\rangle/Z$. We then obtain

$$(7.16)\qquad \langle\kappa^{\gamma};G,P(\kappa,S)\rangle = \kappa^{\frac{1}{2}(|V|+1)}(1+\kappa^{\frac{1}{2}}S)^{-E}\sum_{C\subseteq E} S^{C}\,\kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}},$$

$$(7.17)\qquad \langle\delta_e\kappa^{\gamma};G,P(\kappa,S)\rangle = \kappa^{\frac{1}{2}(|V|+2)}(1+\kappa^{\frac{1}{2}}S)^{-E}\sum_{C\subseteq E} S^{C}\delta_e(C)\kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-2\}}.$$

The reason for splitting off the factor $\kappa^{\frac{1}{2}(|V|+2)}$ in eq. (7.17) is that we want the power of κ in the summation to be non-negative. Under the constraint $\delta_e(C) = 1$, which gives the non-vanishing terms, the ends of e are in different c-clusters, so there are at least two c-clusters. Since $\omega \geq 0$, it follows that $\omega(C) + \gamma(C) \geq 2$ in this case. The infimum is reached for those sets $T \subseteq E$ for which $\omega(T) = 0$ and $\gamma(T) = 2$. Evidently, T+e is just the edge set of a spanning tree of G containing e. By the considerations leading to (6.5) we may equivalently say that $T \in \mathcal{T}(\mathcal{C}_e G)$, i.e., on the analogy of eq. (6.6), we may define a spanning tree of $\mathcal{C}_e G$ with edge set T by

(7.18) $$\omega(T;G) + \gamma(T,G) = 2 = \inf_{C:\ \delta_e(C)=1} \{\omega(C) + \gamma(C)\}\ , \quad T \in \mathcal{T}(\mathcal{C}_e G).$$

For the left-hand side of (7.7) we get, after multiplying by $-\kappa^{\frac{1}{2}}$ and changing the differentiation variable to S_e (eq. (6.9)),

$$\lim_{\kappa \downarrow 0} -\kappa^{\frac{1}{2}} \frac{(1+\kappa^{\frac{1}{2}} S_e)}{-\kappa^{\frac{1}{2}}} \frac{\partial}{\partial S_e} \ln Z = \lim_{\kappa \downarrow 0} \frac{\partial}{\partial S_e} \ln Z =$$

$$= \lim_{\kappa \downarrow 0} \frac{-\kappa^{\frac{1}{2}}}{(1+\kappa^{\frac{1}{2}} S_e)} + \lim_{\kappa \downarrow 0} \frac{\partial}{\partial S_e} \ln \sum_{C \subseteq E} S^C \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}}, \text{ by (7.16)},$$

(7.19) $$= \frac{\partial}{\partial S_e} \ln \sum_{C \subseteq E} S^C \lim_{\kappa \downarrow 0} \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}} = \frac{\partial}{\partial S_e} \ln Z_T(G,S), \text{ by (6.6)}.$$

For the right-hand side of (7.7) we get after multiplying by $-\kappa^{\frac{1}{2}}$ and substituting (7.16) and (7.17)

$$\lim_{\kappa \downarrow 0} -\kappa^{\frac{1}{2}} (1-\kappa^{-1}) \kappa^{\frac{1}{2}} \sum_{C \subseteq E} S^C \delta_e(C) \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-2\}} \Big/ \sum_{C \subseteq E} S^C \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}} =$$

$$= \sum_{C \subseteq E} S^C \lim_{\kappa \downarrow 0} \delta_e(C) \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-2\}} \Big/ \sum_{C \subseteq E} S^C \lim_{\kappa \downarrow 0} \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}} =$$

(7.20) $$= Z_T(\mathcal{C}_e G,S) \big/ Z_T(G,S),$$

by (6.6) and (7.18). From (7.7), (7.19) and (7.20) we obtain

(7.21) $$\frac{\partial}{\partial S_e} \ln Z_T(G,S) = Z_T(\mathcal{C}_e G,S) \big/ Z_T(G,S).$$

In order to obtain eq. (6.3), we have to apply (7.21) to an edge e' parallel to the given edge e, and to the graph $\mathcal{D}_e G$.

Notice that $\mathcal{C}_e, \mathcal{D}_e G$ and $\mathcal{C}_e G$ differ only by a loop.

7.4. The cluster generating function and other polynomials

In this section we shall derive relations between the cluster generating function and other graphs polynomials which in the course of time have been introduced by several authors, and which will be defined explicitly below. First we mention the two-variable polynomial Q, introduced for arbitrary graphs by Tutte in 1947 [11]. In 1954, Tutte [10], in a study of graph-colouring problems, introduced another two-variable polynomial for finite graphs, the dichromate χ. It was not until 1967 that it was explicitly stated, again by Tutte [27], that the polynomials Q and χ are identical apart from a factor and a shift of variables. A somewhat different line of research was pursued by Zykov [12], who in 1962, in a study of recursive functions on graphs, introduced a two-variable polynomial ψ, and showed that the four-variable polynomial ψ', also introduced by him, the two-variable polynomial P, which is identical, up to a factor, to Tutte's Q, and the dichromate χ are all particular cases of the polynomial ψ apart from factors and changes of variables.

Finally, we mention the two-variable polynomial ρ, introduced by Crapo [28] for finite pregeometries (matroids), which he showed to be identical in the above sense to the generalization of the dichromate to matroids. We shall show that the cluster-generating function Z, which is a $(|E|+1)$-variable polynomial, is a generalization of the above-mentioned polynomials, in the sense that different edges can have different "weights".

Before showing the connection between Z and the polynomials Q, χ, ψ, ψ' and ρ, we shall introduce a slightly generalized polynomial Z' and a corresponding measure μ'. Let $G = (V,E,i)$ be a finite graph, x and y be two mappings of E into the set of real numbers and let ξ, η be two real numbers. Then we define ($D \equiv E-C$)

(7.22) $$Z'(G,x,y,\xi,\eta) \equiv \sum_{C\subseteq E} x^C y^D \xi^{\gamma(C;G)-\gamma(E;G)} \eta^{\omega(C;G)},$$

(7.23) $$\mu'(C;G,x,y,\xi,\eta) \equiv x^C y^D \xi^{\gamma(C;G)-\gamma(E;G)} \eta^{\omega(C;G)} \Big/ Z'(G,x,y,\xi,\eta).$$

This polynomial Z' and measure μ' are related to Z and μ through Euler's formula

(7.24) $$|V(G)| + \omega(C;G) = |C| + \gamma(C;G).$$

Eliminating ω from Z' and μ' by (7.24), we deduce

(7.25) $$Z'(G,x,y,\xi,\eta) = (x\eta + y)^E \xi^{-\gamma(E;G)} \eta^{-|V(G)|} Z(G,x\eta \big/ (x\eta + y),\ \xi\eta)$$

(7.26) $$\mu'(C;G,x,y,\xi,\eta) = \mu(C;G,x\eta \big/ (x\eta + y),\ \xi\eta).$$

One observes that apart from factors there is no loss of generality in going from the $2(|E|+1)$-variable polynomial Z' to the $(|E|+1)$-variable polynomial Z.

The polynomial Q, now called the dichromatic polynomial, is defined for finite graphs by $Q(G,t,z) \equiv \Sigma_{C\subseteq E} t^{\gamma(C;G)} z^{\omega(C;G)}$. We have immediately:

(7.27) $$Q(G,t,z) = (z+1)^{|E(G)|} z^{-|V(G)|} Z(G,z/(z+1),tz).$$

The original definition of the dichromate χ, now called the Tutte polynomial, is rather complicated and will be omitted here. As Zykov has shown [12], the Tutte polynomial is uniquely determined by the following properties, which were deduced by Tutte. If all edges of G are loops or isthmi, $\chi(G,x,y) = x^{|E|-\omega(E;G)} y^{\omega(E;G)}$. If G has an edge, e say, which is neither a loop nor an isthmus, χ satisfies the recursion relation $\chi(G) = \chi(\mathcal{C}_e G) + \chi(\mathcal{D}_e G)$. One readily verifies that $Z'(G,1,1,x-1,y-1)$ obeys these conditions, so

(7.28) $$\chi(G,x,y) = y^{|E(G)|} (x-1)^{-\gamma(E;G)} (y-1)^{-|V(G)|} Z\left(G,1-y^{-1},(x-1)(y-1)\right).$$

The polynomials ψ and ψ' introduced by Zykov are defined in the following way for finite graphs. If all edges of G are loops, $\psi(G,\alpha,\beta) = 1$, $\psi'(G,\alpha,\beta,u,v) = u^{|V(G)|} v^{|E(G)|}$. If G has an edge, e say, which is not a loop, ψ and ψ' are defined recursively by the recursion relations $\psi(G) = \alpha\psi(\mathcal{D}_e G) + \beta\psi(\mathcal{C}_e G)$, $\psi'(G) = \alpha\psi'(\mathcal{D}_e G) + \beta\psi'(\mathcal{C}_e G)$. One readily verifies that the functions $\{\beta/(1-\alpha)\}^{|V|} Z(G,1-\alpha,(1-\alpha)/\beta)$ and $u^{\gamma(E;G)} \times$ $\times\, Z'(G,\beta,\alpha,u,(v-\alpha)/\beta)$ obey these conditions for ψ and ψ' respectively, so

$$(7.29)\qquad \psi(G,\alpha,\beta) = \left(\frac{\beta}{1-\alpha}\right)^{|V(G)|} Z\left(G,1-\alpha,\frac{1-\alpha}{\beta}\right)$$

$$(7.30)\qquad \psi'(G,\alpha,\beta,u,v) = \left(\frac{\beta}{v-\alpha}\right)^{|V(G)|} v^{|E(G)|}\, Z\left(G,\frac{v-\alpha}{v},\frac{u(v-\alpha)}{\beta}\right).$$

The rank generating function ρ was defined for matroids by Crapo. A <u>matroid</u>, or <u>(combinatorial)</u> <u>pregeometry</u>, is the Boolean algebra of all subsets of a finite set X together with an integral-valued rank function r on this algebra, satisfying the following relations. (1) $r(\emptyset)=0$, (2) for all $x\in X$ and $X'\subseteq X-x$ we have $r(X'+x)-r(X')$ is 0 or 1, (3) for all $x,x'\in X$ and $X'\subseteq X-x-x'$ we have $r(X'+x+x')-r(X'+x)-r(X'+x')+r(X')$ is 0 or -1. If X is the edge set E(G) of a finite graph G, and if the rank function is the function $|C|-\omega(C;G) = |V(G)|-\gamma(C;G)$, ρ has the following form:

$$\rho(G,\xi,\eta) \equiv \sum_{C\subseteq E} \xi^{\gamma(C;G)-\gamma(E;G)}\, \eta^{\omega(C;G)} . \quad \text{So,}$$

$$(7.31)\qquad \rho(G,\xi,\eta) = (\eta+1)^{|E(G)|}\, \xi^{-\gamma(E;G)}\, \eta^{-|V(G)|}\, Z(G,\eta/(\eta+1),\xi\eta).$$

8. DISCUSSION

From the foregoing analysis one can draw two main conclusions. a) A number of seemingly unrelated physical systems, such as the linear resistance network, the percolation model and the Ising model, can be considered as special cases of one single model, the random-cluster model. This model has the advantage over the Ashkin-Teller-Potts model, which constitutes another generalization of the Ising model, that the parameter κ characterizing the various special cases can take all real values, including the remaining non-negative integral values 0 and 1. This fact enables one to study the properties of the model as a function of a continuously varying additional parameter. If, e.g., the system exhibits a phase transition in the thermodynamic limit, one can investigate how its critical behaviour changes with κ.

In this connection it might be of interest to study those quantities which form the generalization of the thermodynamic quantities and spin correlations of the Ising model. As such we mention the <u>generalized free energy</u>

(8.1) $$F(G^o,p^o,\kappa) \equiv \ln Z(G^o,p^o,\kappa)$$

and its derivatives, $q_e\frac{\partial}{\partial q_e} F(G^o)$, $q_{e'}\frac{\partial}{\partial q_{e'}} q_e\frac{\partial}{\partial q_e} F(G^o)$ etc., the first of which can, by proposition 2, be written as

(8.2) $$q_e\frac{\partial}{\partial q_e} F(G^o) = (1-\kappa^{-1})<\delta_e;G^o,\mu^o>,$$

and in addition the quantities $<\gamma_{vv'}>$, $<\gamma_{vv'v''}>$, etc.

Of particular interest are the <u>(generalized) local magnetization</u> M and <u>local susceptibility χ</u>:

(8.3) $$M(G^o,v) \equiv (1-\kappa^{-1}) - q_{ov}\frac{\partial}{\partial q_{ov}} F(G^o) = (1-\kappa^{-1})<\gamma_{ov}>,$$

(8.4) $$\chi(G^o,v,v') \equiv q_{ov}\frac{\partial}{\partial q_{ov}} q_{ov'}\frac{\partial}{\partial q_{ov'}} F(G^o),$$

and the corresponding "global" quantities, obtained by summing over all vertices and vertex pairs, respectively.

b) The cluster generating function $Z(G,p,\kappa)$ which takes a central place in the theory of the random-cluster model, is a straightforward generalization of a polynomial in two variables, the dichromatic polynomial, which is playing a more and more important role in the theory of graphs and its extension, the theory of matroids. The dichromatic polynomial of a given graph G is the generating function for the number of spanning graphs of G with a given number of clusters and a given cyclomatic number; the cluster generating function generates all individual spanning subgraphs in such a form that the number of clusters and the cyclomatic number can be read off immediately.

The dichromatic polynomial has recently been put in a wider mathematical perspective by Brylawski[29] in an interesting study on what he calls the Tutte-Grothendieck ring. The main idea of Brylawski's work goes back to Tutte's paper [11] in which he introduced the polynomial Q. One might expect that a combination of the ideas developed in this branch of mathematics, which deals almost exclusively with finite sets, with those developed in the theoretical treatment of the thermodynamic limit in translationally invariant systems will lead to a deeper understanding of phase transitions.

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ON THE RANDOM-CLUSTER MODEL

II. The percolation model

Synopsis The relationship between several criteria for large-range connectivity in an infinite percolation model is investigated. In particular, we establish the equivalence, under a non-trivial condition, between weak and strong large-range connectivity, related, respectively, with the probability of a vertex to belong to an infinite cluster and the probability of a vertex to be connected with vertices "very far away". Furthermore, it is shown that the role of infinity can, in a certain sense, be taken over by a supplementary vertex.

1. INTRODUCTION

This paper is the second one in a sequence of papers on the random-cluster model. In the first paper [1], to be referred to as I, the random-cluster model was defined and shown to include as special or limiting cases the linear resistance network, the percolation model, the Ising model and the Ashkin-Teller-Potts model.

The main reason why the Ising model has been extensively investigated lies in the fact that it exhibits a phase transition, existing in the occurrence of a certain type of long-range ordering of the spins under certain conditions of temperature and magnetic field strength, and the absence of such an ordering under different conditions. A sharp transition between the two regimes occurs only if one takes the thermodynamic limit of a monotone increasing sequence of finite systems. All quantities of interest related with ordering, such as the free energy, the spontaneous magnetization, the magnetic susceptibility (all taken per vertex), are thereby functions on an infinite system which is the limit of finite systems.

The question arises whether the random-cluster model also exhibits a phase transition of some sort. Before studying this question in its generality it is interesting to focus attention on the special case of the percolation model. In the first place, this model can be defined directly for an infinite countable graph, without the intervention of finite graphs. Secondly, for an infinite countable graph the model shows a "phase transition", the probability that a given vertex belongs to an infinite c-cluster being zero for certain choices of p and positive for different choices (see Broadbent and Hammersly, and Hammersley [2]). Moreover, the percolation model is not only a special case, but, in a sense, also the basis of the random-cluster model. One may, therefore, expect that many properties of the percolation model will be typical for the random-cluster model as a whole.

In § 2 we show that the functions listed in I § 3.1 are random variables. In § 3 a basic theorem on covariances of random variables of a given type is derived. On the one hand, this theorem is a generalization of an inequality derived and used by Harris in a paper on the percolation model [3]. On the other hand, it is closely related to the well-known second inequality of Griffiths [4], as will be shown in a subsequent paper. A further generalization of the theorem to measures on finite distributive lattices will be given in another paper by Ginibre, Kasteleyn and the author [5].

Section 4 deals with various criteria for large-range connectivity in the percolation model. These criteria are shown to be strongly related, and independent of local disturbances of the graph. The covariance inequality turns out to be crucial in the analysis.

In § 5, it is shown that for locally finite graphs there is a connection between functions expressing the large-range connectivity of the percolation model and certain functions in the supplemented percolation model, introduced in I § 7.1. The supplementary vertex turns out to play the role of a "point at infinity".

Finally, in § 6, the results of the paper are discussed.

To conclude this introduction we make a few remarks concerning notation and other conventions.

In the proof of various propositions we shall need increasing sequences of finite graphs having a given infinite countable graph as a limit. We choose such a sequence once and for all, denoting the sequence by G_1, G_2, G_3, ... Then, by definition, (1) $G_1 \subset G_2 \subset G_3$, ... (2) $G = \cup_{n=1}^{\infty} G_n$. We shall further write $V(G_n) = V_n$, $E(G_n) = E_n$, $B(G_n) = B_n$, where $B(G_n)$ is the vertex boundary of G_n in G, to be defined in the next section. If V',V" are subsets of V then the distance between them (in G) is denoted d(V',V"), i.e. d(V',V") is the infimum of the lengths over all paths (in G) between all pairs of vertices $v' \in V'$ and

$v'' \in V''$.

We shall consider many functions on an infinite countable set, viz. the vertex set V of the infinite countable graph G, or, more generally, a subset V' of V. Of special interest are the limit points of these functions, among which the limes inferior and the limes superior. Usually, one introduces a natural number, say n, to order the elements of such a set, and one write lim $\inf_n$ or lim $\inf_{n\to\infty}$. Since the lim inf and the lim sup do not depend on the ordering of the elements of a set, this intermediate step is not necessary, and, in fact, it might complicate the analysis in this paper unnecessarily. We shall, therefore, denote the two limits by lim $\inf_{v \in V'}$ and lim $\sup_{v\in V'}$. If the two are equal, we shall write $\lim_{v\in V'}$. If an integer n arrives in a natural way, as in the sequence G_1, G_2, ..., G_n, ... , we shall not only write lim $\inf_n$, but also $\lim_n$, rather than $\lim_{n\to\infty}$. Furthermore, we shall use the usual convention for ordering and convergence of functions. If f and g map a set X into a set Y, then $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in X$, and $f \to g$ means that $f(x) \to g(x)$ for all $x \in X$.

Moreover, to make the lemmas, propositions and theorems meaningful, we restrict ourselves in each section to graphs of a certain type. Section 2 applies to infinite countable graphs, i.e. $|V \cup E|$ is infinite countable. Section 3 applies to arbitrary countable graphs, i.e. $|V \cup E|$ is finite or infinite countable. Section 4 applies to infinite countable graphs with an infinite countable vertex set, i.e. $|V|$ is infinite countable. Section 5 applies to infinite countable graphs with an infinite countable vertex set and which are locally finite, i.e. $|V|$ is infinite countable, and the number of edges incident with a given vertex is finite for all vertices of the graph.

Finally, we recall that we shall use the same symbol for an event and its indicator, as we did in I. Consequently, a sentence like "If v and v' are c-connected and v' and v" are c-connected, then v and v" are c-connected" will be written either in the form "If $\gamma_{vv'}$ and $\gamma_{v'v''}$, then $\gamma_{vv''}$ " or in the form "$\gamma_{vv'}\gamma_{v'v''}\leq\gamma_{vv''}$ ".

2. RANDOM VARIABLES ON INFINITE COUNTABLE GRAPHS

In this section we show that for a percolation model (G,P), where G = (V,E,i) is an infinite countable graph, the functions on the event space of G which were listed in I § 3.1 (end of the section) are all non-negative random variables, and, thereby, integrable. We recall that a non-negative random variable is obtained by closing the collection of non-negative local variables under the suprema and infima of countable subcollections. A local variable f is a function which assumes only a finite number of (finite) real values f_i such that the event $f = f_i$ is a local event. A local event, finally, is an event obtained by closing the collection of edge events under finite sums and finite products. It is essentially an event on a finite subgraph of G. It will be sufficient to prove that the listed functions can be obtained as limits of increasing sequences of non-negative local variables. In the proofs we shall use the monotone sequence of finite subgraphs of G introduced in section 1, $G_1 \subset G_2 \subset G_3 \ldots$ with $\cup_{n=1}^{\infty} G_n = G$.

We shall say that two vertices $v,v'\in V$ are c-connected in a subgraph $G'\subseteq G$, and denote this event (and its indicator) by $\gamma_{vv'}^{G'}$, or, $\gamma_{vv'}$ in G', if there is a c-path in G' between v and v'.

Lemma 1 Let $\gamma(V';G')$ be the number of equivalence classes of the set of vertices $V'\subseteq V$ under the relation of c-connection in the graph $G'\subseteq G$. Then,

(2.1) $\gamma(V;G) = \sup_n \inf_{n'} \gamma(V_n;G_{n'}) = \lim_n \lim_{n'} \gamma(V_n;G_{n'})$,

(2.2) $\langle\gamma;G\rangle = \sup_n \inf_{n'} \langle\gamma(V_n;G_{n'});\ G_n \cup G_{n'}\rangle$.

Proof. We first prove that $\gamma(V_n;G) = \inf_{n'}\gamma(V_n;G_{n'})$, and then that $\gamma(V;G) = \sup_n\gamma(V_n;G)$. Observe that for $v,v'\in V_n$, $\gamma_{vv'}$ in $G_{n'}$ implies $\gamma_{vv'}$ in $G_{n'+1}$, which implies $\gamma_{vv'}$ in G, because $G_{n'}\subset G_{n'+1}\subset G$. Therefore, $\gamma(V_n;G_{n'}) \geq \gamma(V_n;G_{n'+1}) \geq \gamma(V_n;G)$. Conversely, if $\gamma_{vv'}$ in G, there is an index $n' = n'(v,v')$ such that $\gamma_{vv'}$ in $G_{n'}$, because there is a (finite) c-path in G between v and v', so there is some $G_{n'}$ containing that c-path. Because V_n is finite there is an n' such that for all $v,v'\in V_n$ with $\gamma_{vv'}$ in G also $\gamma_{vv'}$ in $G_{n'}$. Therefore, for that n', $\gamma(V_n;G) = \gamma(V_n;G_{n'})$. This implies that $\gamma(V_n;G) = \inf_{n'}\gamma(V_n;G_{n'})$. To prove that $\gamma(V;G) = \sup_n \gamma(V_n;G)$, we observe that evidently $\gamma(V_n;G) \leq \gamma(V_{n+1};G) \leq \gamma(V;G)$. Furthermore, a given equivalence class of the vertices of V under c-connection in G contains at least one vertex, v say, and there is an $n = n(v)$ such that $v\in V_n$. Therefore, $\gamma(V;G) \leq \sup_n\gamma(V_n;G)$ and it follows that $\gamma(V;G) = \sup_n\gamma(V_n;G)$. This completes the proof of (2.1). From the integration theorem on monotone sequences eq. (2.2) follows (cf. Zaanen [6] Ch. 3 § 13 Th. 3,4). ||

Lemma 2 Let $\omega(G')$ be the c-cyclomatic number of a graph $G'\subseteq G$. Then

(2.3) $$\omega(G) = \sup_n\omega(G_n) = \lim_n\omega(G_n),$$

(2.4) $$\langle\omega;G\rangle = \sup_n\langle\omega;G_n\rangle = \lim_n\langle\omega;G_n\rangle.$$

Proof. The c-cyclomatic number of a graph G equals the number of c-edges which are not in a maximal spanning c-forest in G (cf. König [7] Ch. IX Th. 2 and 4). Let F_n be a maximal spanning c-forest in G_n and $G_{n+1}\supset G_n$, then we can extend F_n to a maximal spanning c-forest in G_{n+1} such that $F_{n+1}\supseteq F_n$. This we do in the following way. Let $G^{(1)}$ be the spanning subgraph of G_{n+1} with as edges $E(F_n)$ and the c-edges of G_{n+1} not in G_n. If $G^{(1)}$ does not contain a c-polygon, it is a maximal spanning c-forest of G_{n+1}. If $G^{(1)}$ contains a c-polygon, that polygon contains a c-edge, e say, with

$e \in E_{n+1} - E_n$, because otherwise the c-polygon should consist of edges of F_n and it was a c-polygon in F_n, contrary to the hypothesis. Let $G^{(2)}$ be $\mathcal{D}_e G^{(1)}$, then $G^{(2)}$ is a spanning subgraph of G_{n+1}, and we may repeat the procedure given above on $G^{(2)}$. By the finiteness of G_{n+1} we finally obtain a $G^{(i)}$ containing no c-polygon. Then $G^{(i)}$ is a maximal spanning c-forest in G_{n+1}, containing F_n. So choose F_{n+1} to be $G^{(i)}$. The maximality of $G^{(i)}$ follows, because upon adding a c-edge of G_{n+1} which is not in $G^{(i)}$ to $G^{(i)}$, the obtained graph will contain a c-polygon containing that edge, e say. Indeed, the ends of e are c-connected in G_{n+1}, so in $G^{(1)}$, and by construction also c-connected in $G^{(i)}$. So between the ends of e there is a c-path in $G^{(i)}$ which does not contain e, and which, together with e, gives a c-polygon in $G^{(i)}$. Moreover, the graph $F = \cup_{n=1}^{\infty} F_n$ is a spanning c-forest in $G = \cup_{n=1}^{\infty} G_n$. F is obviously a spanning c-forest in G. F is maximal, because if we add a c-edge, e say, not in F, to F, this graph will contain a c-polygon, because the ends of e are c-connected in G, so in some G_n, in F_n, and in F, and we can construct a c-polygon as before. Thus, by a theorem mentioned before, $\omega(C;G) = |C-E(F)| = \left|\cup_{n=1}^{\infty}\left(C\cap E_n - E(F_n)\right)\right| = \sup_n |C\cap E_n - E(F_n)| = \sup_n \omega(C\cap E_n; G_n)$, where obviously $\omega(G_n)$ is increasing. The last property together with the integration theorem on monotone sequences gives (2.3) and (2.4). ||

Lemma 3 Let G' be a cluster of G, with $G' = (V',E',i)$, and let E" be the set of all edges of G incident with V' and not in E'. Then

(2.5) $$\gamma_{G'} = \inf_n c^{E'\cap E_n} d^{E''\cap E_n} = \lim_n c^{E'\cap E_n} d^{E''\cap E_n}.$$

(2.6) $$\langle\gamma_{G'};G\rangle = \inf_n \langle c^{E'\cap E_n} d^{E''\cap E_n};G_n\rangle = \lim_n \langle c^{E'\cap E_n} d^{E''\cap E_n};G_n\rangle.$$

Proof. Obviously, $\gamma_{G'} = c^{E'} d^{E''}$, because in order that G' is a c-cluster the edges of E' must be c-edges and the edges of D" must be d-edges. The rest of the proof is immediate. ||

By $\Sigma^f_{G'\subseteq G}$ we shall understand the summation over all finite subgraphs G' of G.

Lemma 4 If v is a vertex of G, then

(2.7) $\gamma^f_v = \sum^f_{G'\subseteq G} \gamma_{G';v}$,

(2.8) $\langle\gamma^f_v\rangle = \sum^f_{G'\subseteq G} \langle\gamma_{G';v}\rangle$.

Proof. Firstly, we notice that the number of finite subgraphs of a countable graph is countable, so the summation over all subgraphs restricted to the finite ones makes sense. Furthermore, by definition, $\gamma^f_v = \sup_{G'\subseteq G}\gamma^f_{G';v}$, and all $\gamma^f_{G';v}$ are incompatible, from which the lemma follows. ||

Lemma 5 If v, v' are vertices of G, then

(2.9) $\gamma^G_{vv'} = \sup_n \gamma^{G_n}_{vv'} = \lim_n \gamma^{G_n}_{vv'}$,

(2.10) $\langle\gamma_{vv'};G\rangle = \sup_n \langle\gamma_{vv'};G_n\rangle = \lim_n \langle\gamma_{vv'};G_n\rangle$.

Proof. Because $G\supset G_{n+1}\supset G_n$, if v and v' are c-connected in G_n, they are c-connected in G_{n+1} and in G. Thus $\gamma_{vv'}$ in $G_n \leq \gamma_{vv'}$ in $G_{n+1} \leq$ $\leq \gamma_{vv'}$ in G. On the other hand, if v and v' are c-connected in G, there is a (finite) c-path between them in G, and there is a n such that G_n contains that path, from which it follows that v and v' are c-connected in G_n. Therefore, $\gamma_{vv'}$ in G $\leq \sup_n \gamma_{vv'}$ in G_n. It follows that $\gamma^G_{vv'} = \sup_n(\gamma_{vv'}$ in $G_n)$. ||

In the special case that the graph G is locally finite, i.e. the number of edges incident with a given vertex is finite for all vertices of the graph, we have in addition a useful lemma concerning the event γ_v^f that v belongs to a finite c-cluster. This is mainly Th. 5 of Broadbent and Hammersley [2].

We define the <u>vertex boundary in G</u> of a subgraph G' of G as the set of vertices of G' which are incident with edges of G not in G', and denote it by B(G'). Furthermore, we shall write $B(G_n) \equiv B_n$.

Lemma 6 Let v be a vertex of a locally finite graph G. Then

(2.11) $\gamma_v^f = \lim\sup_n \delta_{vB_n}^{G_n} = \lim_n \delta_{vB_n}^{G_n}$,

(2.12) $\langle\gamma_v^f;G\rangle = \lim\sup_n \langle\delta_{vB_n};G_n\rangle = \lim_n \langle\delta_{vB_n};G_n\rangle$.

Proof. For convenience we shall prove the negation of the assertion i.e. $\gamma_v^\infty = \lim\inf_n \gamma_{vB_n}$ in G_n. First of all we notice that in a locally finite connected infinite graph G there is for any vertex v an infinite vertex-disjoint path with initial vertex v, i.e. an infinite sequence of alternatingly vertices and edges of G: $v_0 = v, e_1, v_1, e_2, \ldots$ such that $i(e_k) = \{v_{k+1}, v_k\}$ for $k = 1,2, \ldots$ (cf. König [7], Ch. VI, theorem 3). So if v belongs to an infinite c-cluster, there is an infinite vertex-disjoint c-path in G with initial vertex v. There is an n such that v belongs to G_n, and we may construct a c-path in G_n from v to some vertex $v' \in B_n$: let v' be the last vertex in the infinite c-path in G such that all preceding edges belong to G_n. In the same way one may construct from a c-path in G_{n+1} between v and some vertex $v'' \in B_{n+1}$, a c-path in G_n between v and some $v' \in B_n$ (as long as v belongs to G_n). Therefore, γ_v^∞ in G $\leq$ $\leq \gamma_{vB_{n+1}}$ in $G_{n+1} \leq \gamma_{vB_n}$ in G_n, or, γ_v^∞ in G $\leq \lim\inf_n \gamma_{vB_n}$ in G_n. On the other hand, if for all $n' \geq n$ we have $\gamma_{vB_{n'}}$ in $G_{n'}$, the number of vertices in the c-cluster of G containing v is at least $\sup_{n' \geq n} d_{n'}(v,B_{n'})$, where $d_{n'}(v,B_{n'})$ is the distance in G_n between v and $B_{n'}$, i.e. the length of the shortest path in $G_{n'}$ connecting v and $B_{n'}$. Since $\cup_n G_n = G$ and G is locally finite, $\lim \sup_n B_n$ is empty: for every $v' \in V$ there is an n such that v' and all edges incident with it in G belong to G_n, so $v' \notin B_{n'}$ for $n' \geq n$. Therefore, $\lim \sup_{n' \geq n} d_{n'}(v, B_{n'}) = \infty$ and it follows that v belongs to an infinite c-cluster. Consequentely γ_v^∞ in G = $\lim \inf_n \gamma_{vB_n}$ in G_n, and the lemma follows.||

For graphs that are not locally finite, the random variables γ_v^f can, in general, not be approached via the local variable γ_{vB_n} in G_n. In that case, it is sometimes useful to approach them via other, not local, variables, provided the graph is bilocally finite. We shall say that a graph is <u>bilocally finite</u>, if for all pairs of vertices v, $v' \in V$ the number of edges incident with v and v' is finite and thus, in particular, the number of loops incident with a given vertex v = v' is finite. The complement of V_n in V will be denoted by U_n, i.e. $U_n \equiv V - V_n$.

Lemma 7 Let G be a bilocally finite graph and $v \in V$. Then

(2.13) $$\gamma_v^f = \lim_n \delta_{vU_n} = \sup_n \delta_{vU_n},$$

(2.14) $$\langle \gamma_v^f \rangle = \lim_n \langle \delta_{vU_n} \rangle = \sup_n \langle \delta_{vU_n} \rangle .$$

Proof. First, because V_n is non-decreasing in n, U_n is non-increasing in n, and therefore, γ_{vU_n} is non-increasing in n. Secondly, if γ_v^∞, v belongs to an infinite c-cluster of G. Moreover, the number of vertices c-connected with v is infinite, by the assumption that G is bilocally finite, because otherwise the number of c-edges in the c-cluster containing v, and hence the c-cluster itself, should also be finite, contrary to the hypothesis. So, each set U_n contains an infinite number of vertices c-connected with v, and it follows that γ_{vU_n} for all n. On the other hand, assuming γ_v^f, there is an n such that all the vertices c-connected with v belong to V_n, so in U_n there is no vertex c-connected with v, hence δ_{vU_n}, or equivalently:

not for all n γ_{vU_n}. Consequently, by the last two implications $\gamma_v^\infty = \inf_n \gamma_{vU_n}$ and $\gamma_v^f = \sup_n \delta_{vU_n}$. The first remark, together with the integration theorem on monotone sequences, completes the proof of the lemma. ||

3. COVARIANCE INEQUALITY

In I § 3.2 we proved a recursion theorem for integrable random variables. In this section we shall present a second theorem on a subclass of these functions, which is a generalization of an inequality on "combinations of links" derived by Harris in a paper on the percolation model (ref. 3, lemma 4.1). The proof given here is an example of the use of the recursion theorem. Before stating the inequality, we shall define the type of functions to which the theorem applies. They are characterized by the property that for each edge $e \in E(G)$ and all subsets $C \subseteq E(G)-e$, the function f defined on the event space of G obeys $f(C+e;G) \geq f(C;G)$; these functions will be called <u>locally increasing functions</u>. A function satisfying the reversed inequality $f(C+e;G) \leq f(C;G)$ will be called a <u>locally decreasing function</u>. In terms of the associated functions on the descendants $\mathcal{C}_e G$ and $\mathcal{D}_e G$, as defined in I § 3.2, we can write $\overline{f}(C;\mathcal{C}_e G) \geq \overline{f}(C;\mathcal{D}_e G)$ for locally increasing functions. Evidently, if f is locally increasing, the positive part f^+ of f is locally increasing and the negative part f^- of f is locally decreasing.

For two summable random variables f and g in a percolation model (G,P) we define their <u>covariance</u> as follows:

$$\mathrm{cov}(f,g;G,P) \equiv \langle f,g;G,P\rangle - \langle f;G,P\rangle\langle g;G,P\rangle .$$

Theorem 1 Covariance inequality: Let (G,P) be a percolation model, and let f and g be summable, or non-negative, locally increasing random variables. Then

(3.1) $$\langle fg;G,P\rangle \geq \langle f;G,P\rangle\langle g;G,P\rangle .$$

Proof. First, let f, g and fg be summable. By the recursion theorem on an edge $e \in E(G)$, say, we get, omitting the association bar over f and g:

$$\mathrm{cov}(f,g;G) = p_e\langle fg;\mathcal{C}_e G\rangle + q_e\langle fg;\mathcal{D}_e G\rangle - p_e^2\langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{C}_e G\rangle -$$

$$- q_e^2\langle f;\mathcal{D}_e G\rangle\langle g;\mathcal{D}_e G\rangle - p_e q_e(\langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{D}_e G\rangle + \langle f;\mathcal{D}_e G\rangle\langle g;\mathcal{C}_e G\rangle) =$$

$$= p_e(\langle fg;\mathcal{C}_e G\rangle - \langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{C}_e G\rangle) + q_e(\langle fg;\mathcal{D}_e G\rangle - \langle f;\mathcal{D}_e G\rangle\langle g,\mathcal{D}_e G\rangle) +$$

$$+ (p_e - p_e^2)\langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{C}_e G\rangle + (q_e - q_e^2)\langle f;\mathcal{D}_e G\rangle\langle g;\mathcal{D}_e G\rangle -$$

(3.2) $$p_e q_e(\langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{D}_e G\rangle + \langle f;\mathcal{D}_e G\rangle\langle g;\mathcal{C}_e G\rangle).$$

Because $p_e - p_e^2 = p_e q_e = q_e - q_e^2$, we get from eq. (3.2)

$$\mathrm{cov}(f,g;G) = p_e\,\mathrm{cov}(f,g;\mathcal{C}_e G) + q_e\,\mathrm{cov}(f,g;\mathcal{D}_e G) + p_e q_e(\langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{C}_e G\rangle +$$

$$+ \langle f;\mathcal{D}_e G\rangle\langle g;\mathcal{D}_e G\rangle - \langle f;\mathcal{C}_e G\rangle\langle g;\mathcal{D}_e G\rangle - \langle f;\mathcal{D}_e G\rangle\langle g;\mathcal{C}_e G\rangle) = p_e\,\mathrm{cov}(f,g;\mathcal{C}_e G) +$$

(3.3) $$+ q_e\,\mathrm{cov}(f,g;\mathcal{D}_e G) + p_e q_e(\langle f;\mathcal{C}_e G\rangle - \langle f;\mathcal{D}_e G\rangle)(\langle g;\mathcal{C}_e G\rangle - \langle g;\mathcal{D}_e G\rangle) .$$

By the definition of locally increasing functions, $f(C;\mathcal{C}_e G) \geq f(C;\mathcal{D}_e G)$, so $\langle f;\mathcal{C}_e G\rangle \geq \langle f;\mathcal{D}_e G\rangle$. Hence we get for the covariance the inequality

(3.4) $$\mathrm{cov}(f,g;G) \geq p_e\,\mathrm{cov}(f,g;\mathcal{C}_e G) + q_e\,\mathrm{cov}(f,g;\mathcal{D}_e G).$$

Iterating this inequality for a finite number of edges $e \in E' \subseteq E(G)$, we get

(3.5) $$\mathrm{cov}(f,g;G) \geq \sum_{C' \subseteq E'} p^{C'} q^{D'}\,\mathrm{cov}(f,g;\mathcal{C}^{C'}\mathcal{D}^{D'} G), \qquad C' + D' = E'.$$

In case G is finite, we can choose $E' = E(G)$ and obtain the covariances for the smallest descendants of G

$$\mathrm{cov}(f,g;\mathcal{C}^C\mathcal{D}^D G) = \langle fg;\mathcal{C}^C\mathcal{D}^D G\rangle - \langle f;\mathcal{C}^C\mathcal{D}^D G\rangle\langle g;\mathcal{C}^C\mathcal{D}^D G\rangle =$$

(3.6) $$= \{fg(C)\} - \{f(C)\}\{g(C)\} = 0,$$

from which the theorem follows. In case G is an infinite countable graph, we define for each finite subset $E'\subset E(G)$ local variables $h_{E'}$ by $h_{E'}(C) \equiv \mathrm{cov}(f,g;\mathcal{C}^{C\cap E'}\mathcal{D}^{D\cap E'}G)$. One observes that these functions only depend on the states of the edges in E'. Furthermore, the $h_{E'}$ are summable and have the property

$$\langle h_{E'};G\rangle = \sum_{C'\subseteq E'} p^{C'} q^{D'} \mathrm{cov}(f,g;\mathcal{C}^{C'}\mathcal{D}^{D'}G). \tag{3.7}$$

For the increasing sequence $\emptyset\subset E_1\subset E_2\subset\ldots$ of finite subsets $E_n\subset E(G)$ with $\cup_n E_n = E(G)$ we get by a repeated use of eqs. (3.5) and (3.7), writing $h_{E_n} \equiv h_n$,

$$\mathrm{cov}(f,g) = \langle h_\emptyset\rangle \geq \langle h_1\rangle \geq \langle h_2\rangle \geq \ldots \geq \inf_n \langle h_n\rangle. \tag{3.8}$$

In fact, the functions h_n will converge to zero almost everywhere (a.e.), because by Lévy's theorem on bounded sequences of conditional expectations (cf. Doob [8] Ch. VII Th. 4.3 corollary 1), which in this case reads:

$$\text{if } f \text{ is summable, } \lim_n \langle f;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G\rangle = f(C;G) \text{ a.e.}, \tag{3.9}$$

we have for the limit function of the sequence of summable functions h_n

$$\lim_n h_n(C;G) = \lim_n \mathrm{cov}(f,g;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G) = \lim_n \langle fg;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G\rangle -$$

$$- (\lim_n \langle f;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G\rangle)(\lim_n \langle g;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G\rangle) =$$

$$= \{fg(C)\} - \{f(C)\}\{g(C)\} = 0 \text{ a.e.} .$$

Notice that f,g and fg are finite a.e. by their summability. By the convergence of the sequence h_n to zero a.e., and the finiteness of the measure ($P(1) = 1$), we conclude that also the expectation values $\langle h_n\rangle$ wil converge to zero. In fact, by eq. (3.8), this convergence is monotone.

Secondly, let f and g be locally increasing non-negative random variables. Then we may define functions f and g by: $f_n(C) = \min\{f(C),n\}$, and analogously for g_n. Observe that if f is locally increasing, so is f_n: if $f(C+e) < n$, then $f(C) \leq f(C+e) < n$ and hence $f_n(C+e) \geq f_n(C)$; if $f(C+e) \geq n$, either $f(C) < n$ and hence $n = f_n(C+e) \geq f_n(C)$, or $f(C) \geq n$ and hence $f_n(C+e) = n \geq n = f_n(C)$. Moreover, f_n is summable, because it is non-negative and bounded by n. Therefore, the functions f_n form a non-decreasing sequence of locally increasing summable functions converging to f as n tends to infinity. Applying the covariance inequality just proved to f_n and $g_{n'}$, we have $\langle f_n g_{n'}\rangle \geq \langle f_n\rangle\langle g_{n'}\rangle$, because f_n, $g_{n'}$ and $f_n g_{n'}$ are all summable locally increasing functions. By the integration theorem on monotone sequences, we get $\langle fg\rangle = \sup_n \sup_{n'} \langle f_n g_{n'}\rangle \geq$ $\geq (\sup_n \langle f_n\rangle)(\sup_{n'}\langle g_{n'}\rangle) = \langle f\rangle\langle g\rangle$. That proves the theorem for non-negative locally increasing random variables. Analogously one proves eq.(3.1) for non-positive locally increasing random variables.

Finally, we remove the summability condition on fg in the first part of the proof by observing that if $\langle fg\rangle = +\infty$, the inequality (3.1) holds trivially, whereas $\langle fg\rangle = -\infty$ can be excluded by the following argument. If $\langle fg\rangle = -\infty$, then for the negative part $(fg)^-$ of fg we should have $\langle (fg)^-\rangle = +\infty$. But, $\langle (fg)^-\rangle = \langle (f^+g^- + f^-g^+)\rangle =$ $= -\langle f^+(-g^-)\rangle - \langle(-f^-)g^+\rangle \leq - \langle f^+\rangle\langle -g^-\rangle - \langle -f^-\rangle\langle g^+\rangle = \langle f^+\rangle\langle g^-\rangle +$ $+ \langle f^-\rangle\langle g^+\rangle$, because f^+, g^+, $-f^-$ and $-g^-$ are all locally increasing. Because f and g are summable by assumption, f^+, g^+, f^- and g^- are summable. Thus we get $\langle (fg)^-\rangle < \infty$. ||

Corollary Harris' lemma 4.1. [3]. Let $A_1, A_2, \ldots, A_m$ be a finite number of finite subsets of E(G), let a_i be the event c^{A_i} and let a be the event $a_1 + a_2 + \ldots + a_m$. Let $B_1, B_2, \ldots, B_n$ be a finite number of finite subsets of E(G) and define the events b_i and b analogously. Then $P(ab) \geq P(a)P(b)$.

Proof. Let f be the indicator of the event a, and let g be the indicator of the event b. Then f and g are locally increasing and non-negative (and summable). To show that f is locally increasing, we first show that a_i is locally increasing. This follows because for any $C\subseteq E(G)-e$, $a_i(C) = 1$ implies $A_i\subseteq C$, hence $A_i\subseteq C+e$, from which $a_i(C+e) = 1$. Because $a = \Sigma_i a_i$, $f(C) = a(C) \leq a(C+e) = f(C+e)$. Analogously, one proves that g is locally increasing. Hence, the covariance inequality can be applied and the corollary follows. Notice that Harris' lemma remains true if we allow a countable number of countable subsets of E(G). ||

In a subsequent paper, to be referred to as III, a generalization of the covariance inequality to the random-cluster model will be derived. It will be shown that for spin-systems with only pair interactions the second Griffiths-Kelly-Sherman inequality [4] is a corollary of this generalized covariance inequality.

Finally, we give another, typical, property of locally increasing functions by comparing the expectation values with respect to two different probabilities P. The relation of this proposition with the covariance inequality will be discussed in III.

Proposition 1 Let (G,P) and (G,P') be the percolation models generated by the mappings p and p', and let f be a locally increasing random variable. If $p' \leq p$, and if f is non-negative or P-summable, then

(3.10) $$\langle f;G,P'\rangle \leq \langle f;G,P\rangle.$$

Proof. First, suppose that f is non-negative. If $\langle f;G,P\rangle = \infty$, (3.10) is trivially true. Therefore, suppose that $\langle f;G,P\rangle < \infty$, i.e. f is non-negative and P-summable. By the recursion theorem, for any $e\in E(G)$,

(3.11) $$\langle f;G,P\rangle = p_e\langle f;\mathcal{C}_eG,P\rangle + q_e\langle f;\mathcal{D}_eG,P\rangle = \langle f;\mathcal{D}_eG,P\rangle + p_e(\langle f;\mathcal{C}_eG,P\rangle - \langle f;\mathcal{D}_eG,P\rangle).$$

Because f is locally increasing, $(\langle f;\mathcal{C}_eG,P\rangle - \langle f;\mathcal{D}_eG,P\rangle) \geq 0$, and

$p_e \geq p'_e$, by assumption, so from eq. (3.11) we get

$<f;G,P> \geq <f;\mathcal{D}_e G,P> + p'_e(<f;\mathcal{C}_e G,P> - <f;\mathcal{D}_e G,P>) =$

(3.12) $= p'_e<f;\mathcal{C}_e G,P> + q'_e<f;\mathcal{D}_e G,P>.$

If we write the measure P as a product measure, $P = P^E$, eq. (3.12) may be written as $<f;G,P^E> \geq <f;G,P^{E-e} \times (P')^e>$, or iterated for the finite number of edges $e \in E_n$,

(3.13) $<f;G,P^E> \geq <f;G,P^{E-E_n} \times (P')^{E_n}> .$

Analogously to the procedure used in the proof of the covariance inequality, we proceed by defining the functions $f_n(C;G) \equiv \equiv <f;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G,P>$, which have the property $<f_n;G,P'> = =<f;G,P^{E-E_n} \times (P')^{E_n}>$. By the repeated use of eq. (3.13) we obtain, with $E_0 = \emptyset$.

(3.14) $<f;G,P> = <f_0;G,P'> \geq <f_1;G,P'> \geq <f_2;G,P'> \geq \ldots \geq \lim\inf_n <f_n;G,P'>.$

By Fatou's lemma on non-negative sequences, $\lim\inf_n < f_n;G,P'> \geq \geq <\lim\inf_n f_n;G,P'> \geq 0$, and by Levy's theorem on bounded sequences of conditional expectations, $\lim_n f_n(C;G) = = \lim_n <f;\mathcal{C}^{C\cap E_n}\mathcal{D}^{D\cap E_n}G,P> = f(C;G)$ a.e.. Consequently, from eq. (3.14) we obtain eq. (3.10), $<f;G,P> \geq <f;G,P'>$, which completes the proof for non-negative locally increasing functions.

If f is P-summable, its positive part f^+ is locally increasing and P-summable, and its negative part f^- is locally decreasing and P-summable. For the positive part f^+ it follows at once from the preceding considerations that $<f^+;G,P> \geq <f^+;G,P'>$, and f^+ is P'-summable, too. As f^- is locally decreasing, $(-f^-)$ is locally increasing, so instead of eq. (3.14) we have $<f^-;G,P> \leq \lim\sup_n <f_n^-;G,P'>$. If $<f^-;G,P'> < \infty$, by Lebesque's theorem and Levy's theorem it will follow that $<f^-;G,P> \leq <f^-;G,P'>$, and consequently $<f;G,P> = = <f^+;G,P> - <f^-;G,P> \geq <f^+;G,P'> - <f^-;G,P'> = <f;G,P'>$, whereas in the case $<f^-;G,P'> = \infty$, eq. (3.10) is trivially satisfied. ||

4. LARGE-RANGE CONNECTIVITY IN INFINITE COUNTABLE GRAPHS.

On the analogy of the concept of long-range order which is an important element in the theory of the Ising model, we shall introduce the concept of large-range connectivity in the percolation model to describe the extent to which the vertices of a graph are connected on the average. As long as we do not restrict ourselves to locally finite graphs, the fact that a vertex is connected to a large number of other vertices does not imply that it is connected to vertices at "long distance". Therefore, the term "large range" is preferred to "long range"; for locally finite graphs, however, the terms are equivalent. Just as in the Ising model one distinguishes various criteria for long-range order, we shall distinguish various criteria for large-range connectivity.

A first criterion for large-range connectivity is based on the value (zero or positive) of the quantities $<\gamma_v^\infty>$. In order to have a criterion that does not depend on the choice of a special vertex, we consider in particular the limes inferior of these quantities. We say that a percolation model has <u>weak large-range connectivity</u>, to be denoted by W, if $\lim\inf_v <\gamma_v^\infty> > 0$. The property $<\gamma_v^\infty> > 0$ is denoted by W_v. In addition we shall later on discuss the quantity $\lim\inf_v |V_n|^{-1}\Sigma_{v\in V_n} <\gamma_v^\infty>$ for the increasing sequence of finite subgraphs G_n of G. If $\lim\inf_n |V_n|^{-1}\Sigma_{v\in V_n} <\gamma_v^\infty> > 0$ we say that the percolation model has <u>global large-range connectivity.</u>

Next, we consider the quantities $\lim\inf_{v'} <\gamma_{vv'}>$ and $\lim\inf_v \lim\inf_{v'} <\gamma_{vv'}>$. If the latter quantity is >0 we say that the percolation model has <u>strong large-range connectivity</u>, and we denote this property by S. The property $\lim\inf_{v'} <\gamma_{vv'}> > 0$ is denoted by S_v. A justification for the terms "weak" and "strong" will be given later on in this section.

In order to be able to have somewhat more specified criteria for large-range connectivity, we shall often consider the limes inferior, not over the whole set of vertices V of G, but rather over an arbitrarily given infinite subset V' of V. The corresponding criteria for large-range connectivity will be denoted by primed symbols: W', S'_v and S'. Taking in particular V' = V one gets back

$W' = W$, etc.

For convenience we shall list the several types of large-range connectivity:

W_v : $\langle \gamma_v^\infty \rangle > 0$,

W : $\liminf_{v \in V} \langle \gamma_v^\infty \rangle > 0$, W' : $\liminf_{v \in V'} \langle \gamma_v^\infty \rangle$ 0

S_v : $\liminf_{v' \in V} \langle \gamma_{vv'} \rangle > 0$, S'_v : $\liminf_{v' \in V'} \langle \gamma_{vv'} \rangle > 0$

S : $\liminf_{v \in V} \liminf_{v' \in V} \langle \gamma_{vv'} \rangle > 0$, S' : $\liminf_{v \in V'} \liminf_{v' \in V'} \langle \gamma_{vv'} \rangle > 0$.

In this section we shall always suppose that the set of vertices V of G is infinite countable.

Lemma 8 The relation ψ between vertices of V defined by $v \psi v'$ if and only if $\langle \gamma_{vv'} \rangle > 0$, is an equivalence relation.

Proof. Evidently, ψ is reflexive and commutative. Transitivity follows, because if $\langle \gamma_{vv'} \rangle > 0$ and $\langle \gamma_{v'v''} \rangle > 0$, then by the transitivity of connection

(4.1) $\langle \gamma_{vv''} \rangle \geq \langle \gamma_{vv'} \gamma_{v'v''} \rangle \geq \langle \gamma_{vv'} \rangle \langle \gamma_{v'v''} \rangle$,

by the covariance inequality (Th.1), so $\langle \gamma_{vv''} \rangle > 0$. ||

Lemma 9 For any two vertices $v, v' \in V$, $\langle \gamma_{vv'} \rangle > 0$ if and only if there is a path in G between v and v' such that for all edges e in the path $p_e > 0$.

Proof. Because G is countable, the number of paths is countable. If there is no path between v and v' in G such that $p^{E'} = \langle c^{E'} \rangle > 0$, where E' is the set of edges in the path, then $\gamma_{vv'}$ is the sum of a countable number of events of probability zero, and hence an event of probability zero itself, so that $\langle\gamma_{vv'}\rangle=0$. If there is a path in G with $\langle c^{E'}\rangle > 0$, then $\langle\gamma_{vv'}\rangle \geq \langle c^{E'}\rangle > 0.||$

By Lemma 8 we can divide the vertices of G into equivalence classes of ψ. If two vertices are in the same class, there is, by Lemma 9, a path between them such that for each edge in the path $p_e>0$. So the equivalence classes of ψ are equal to the equivalence classes of the vertices of the graph obtained from G by deleting all edges with $p_e=0$. The connection defined by $\langle\gamma_{vv'}\rangle > 0$ will be called P-connection, i.e. two vertices v and v' are P-connected in G if $\langle\gamma_{vv'};G,P\rangle > 0$. A cluster defined by P-connection will be called a P-cluster. Analogously one defines a P-path, a P-polygon, etc.

Proposition 2 Let $v,v'\in V$ belong to the same P-cluster. Then

(a) W_v if and only if $W_{v'}$,

(b) S'_v if and only if $S'_{v'}$.

Proof. Evidently, $\gamma_v^\infty \geq \gamma_v^\infty\gamma_{vv'} = \gamma_{vv'}^\infty = \gamma_{vv'}\gamma_{v'}^\infty$, so $\langle\gamma_v^\infty\rangle \geq \langle\gamma_{vv'}\gamma_{v'}^\infty\rangle$. Both functions $\gamma_{vv'}$ and γ_v^∞ are locally increasing, so by the covariance inequality and the last inequality

(4.2) $$\langle\gamma_v^\infty\rangle \geq \langle\gamma_{vv'}\rangle\langle\gamma_{v'}^\infty\rangle.$$

By definition, if $W_{v'}$ then $\langle\gamma_{v'}^\infty\rangle > 0$, and by assumption $\langle\gamma_{vv'}\rangle > 0$, so it follows from eq. (4.2) that $\langle\gamma_v^\infty\rangle > 0$, so W_v. By the symmetry between v and v' the converse statement is also true which proves (a).

Taking the limes inferior over v" in eq. (4.1), we get

(4.3) $$\liminf_{v''\in V'} \langle\gamma_{vv''}\rangle \geq \langle\gamma_{vv'}\rangle \liminf_{v''\in V'} \langle\gamma_{v'v''}\rangle ,$$

from which we prove (b) in the same way as we proved (a) from (4.2).||

Part (a) of Proposition 2 is formally identical to Th. 2 of Broadbent and Hammersley [2]. These authors, however, consider oriented graphs which are locally finite and have a high degree of regularity, and in which all p_e have the same value. In their proof they make an implicit use of the covariance inequality.

Proposition 3 Let $v \in V$. Then

(4.4) $$\liminf_{v' \in V'} \langle \gamma_{vv'} \rangle = \liminf_{v' \in V'} \langle \gamma^{\infty}_{vv'} \rangle .$$

Proof. By Lebesque's theorem on a bounded sequence of functions (cf. Zaanen [6] Ch. 3 § 14 Th. 3) applied to the set of functions $\gamma^{f}_{vv'}$ (the indicator that v and v' belong to the same finite c-cluster), ordered in an arbitrary way, we have:

(4.5) $$0 \leq \langle \liminf_{v' \in V'} \gamma^{f}_{vv'} \rangle \leq \liminf_{v' \in V'} \langle \gamma^{f}_{vv'} \rangle \leq \limsup_{v' \in V'} \langle \gamma^{f}_{vv'} \rangle \leq \langle \limsup_{v' \in V'} \gamma^{f}_{vv'} \rangle \leq 1.$$

Now for any $C \subseteq E$, $\gamma^{f}_{vv'}(C) = 1$ only for a finite number of vertices v', so $\limsup_{v' \in V'} \gamma^{f}_{vv'}(C) = 0$, and consequently $\langle \limsup_{v' \in V'} \gamma^{f}_{vv'} \rangle = 0$. Therefore, it follows from eq. (4.5) that

(4.6) $$\lim_{v' \in V'} \langle \gamma^{f}_{vv'} \rangle = 0.$$

The proposition follows from the relation $\gamma_{vv'} = \gamma^{\infty}_{vv'} + \gamma^{f}_{vv'}$, and eq. (4.6).

Proposition 4 Let $v \in V$. Then
(a) S'_v implies W_v, (b) S'_v implies S', (c) S' implies W'.

Proof. Evidently, for any $v' \in V'$: $\gamma^{\infty}_{v} \geq \gamma^{\infty}_{vv'}$, so $\langle \gamma^{\infty}_{v} \rangle \geq \langle \gamma^{\infty}_{vv'} \rangle$. Taking the limes inferior over $v' \in V'$, and using Proposition 3, we get

(4.7) $$\langle \gamma^{\infty}_{v} \rangle \geq \liminf_{v' \in V'} \langle \gamma_{vv'} \rangle .$$

If S'_v, $\liminf_{v' \in V'} \langle \gamma_{vv'} \rangle > 0$, by definition, and hence $\langle \gamma^{\infty}_{v} \rangle > 0$

by eq. (4.7), so W_v. That proves (a). Taking the lim inf over v in eq. (4.7), we obtain

(4.8) $$\liminf_{v\in V'} \langle\gamma_v^\infty\rangle \geq \liminf_{v\in V'} \liminf_{v'\in V'} \langle\gamma_{vv'}\rangle,$$

from which (c) follows. In order to prove (b), interchange v and v' in eq. (4.1), and take the lim inf over v' and over v", obtaining

(4.9) $$\liminf_{v\in V'} \liminf_{v'\in V'} \langle\gamma_{vv'}\rangle \geq \left(\liminf_{v'\in V'} \langle\gamma_{vv'}\rangle\right)^2 ,$$

from which (b) follows. ||

A somewhat sharper result can be obtained by using a simple condition on the vertex v with respect to the set of vertices V'.

Proposition 5 Let $v\in V$ belong to a P-cluster containing an infinite number of vertices of V'. Then
(a) W' implies W_v, (b) S'_v is equivalent with S'.

Proof. To prove (a), let not W_v, i.e. $\langle\gamma_v^\infty\rangle = 0$. Then by Prop. 2(a) for each vertex v' in the P-cluster containing v, we also have $\langle\gamma_{v'}^\infty\rangle = 0$. In particular $\langle\gamma_{v'}^\infty\rangle = 0$ for the infinite number of vertices of V' belonging to that P-cluster by assumption, from which it follows that $\liminf_{v'\in V'}\langle\gamma_{v'}^\infty\rangle = 0$, i.e. not W'. This proves (a).

We prove (b) by first proving, analogously to (a), that from the assumption not S'_v, i.e. $\liminf_{v'\in V'}\langle\gamma_{vv'}\rangle = 0$, it follows that $\liminf_{v\in V'} \liminf_{v'\in V'}\langle\gamma_{vv'}\rangle = 0$, i.e. not S' by Prop. 2(b) and the assumption. The converse of this is true by Prop. 4(b). Therefore (b) follows. ||

A strong result on the equivalence of strong large-range connectivity and weak large-range connectivity can by obtained by imposing a much stronger condition on the vertex v, the set V', and the system.

Theorem 2 Weak Strong equivalence. Let $v \in V$ belong to a P-cluster containing an infinite number of vertices of V' and let $\lim_{v' \in V'} \langle \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty \rangle = 0$. Then W' is equivalent with S', is equivalent with S_v'.

Proof. Let $v \in V'$, then evidently, $\gamma_{vv'}^\infty = \gamma_v^\infty \gamma_{vv'} \gamma_{v'}^\infty = \gamma_v^\infty \gamma_{v'}^\infty - \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty$. Taking expectation values this gives

$$\langle \gamma_{vv'}^\infty \rangle = \langle \gamma_v^\infty \gamma_{v'}^\infty \rangle - \langle \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty \rangle. \tag{4.10}$$

By Proposition 3, $\lim\inf_{v' \in V'} \langle \gamma_{vv'}^\infty \rangle = \lim\inf_{v' \in V'} \langle \gamma_{vv'} \rangle$, and by assumption $\lim_{v' \in V'} \langle \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty \rangle = 0$, so from eq. (4.10) we obtain by taking the limes inferior over v',

$$\liminf_{v' \in V'} \langle \gamma_{vv'} \rangle = \liminf_{v' \in V'} \langle \gamma_v^\infty \gamma_{v'}^\infty \rangle. \tag{4.11}$$

By the covariance inequality on the locally increasing functions γ_v^∞ and $\gamma_{v'}^\infty$, $\langle \gamma_v^\infty \gamma_{v'}^\infty \rangle \geq \langle \gamma_v^\infty \rangle \langle \gamma_{v'}^\infty \rangle$, and therefore we obtain from (4.11) that

$$\liminf_{v' \in V'} \langle \gamma_{vv'} \rangle \geq \langle \gamma_v^\infty \rangle \liminf_{v' \in V'} \langle \gamma_{v'}^\infty \rangle. \tag{4.12}$$

If W', then by Proposition 5(a) also W_v, and consequently $\langle \gamma_v^\infty \rangle \lim\inf_{v' \in V'} \langle \gamma_{v'}^\infty \rangle > 0$. So by eq. (4.12) also $\lim\inf_{v' \in V'} \langle \gamma_{vv'} \rangle > 0$, i.e. S_v'. By Proposition 4(b), S_v' implies S', and by Proposition 4(c), S' implies W'. Therefore, W' ultimately implies S', which implies W', which proves the theorem. ||

Harris [3] proved that for all $v, v' \in V$ of the quadratic lattice, with all $p_e = p$, $\langle \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty \rangle = 0$. For an extension to other planar lattices the reader is referred to Fisher [9]. On the other hand, we shall give an example in which W' and S' are not equivalent, and indeed $\lim_{v' \in V'} \langle \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty \rangle \neq 0$. Let G be a Bethe lattice with coordination number n, i.e. an infinite countable connected tree in which each vertex is incident with n edges, and let for all edges $p_e=p$. It can be derived from Fisher's [10] analysis of this case that $<\gamma_v^\infty> = 0$ for $p \leq (n-1)^{-1}$, and $<\gamma_v^\infty> > 0$ for $p > (n-1)^{-1}$. Furthermore, $<\gamma_{vv'}> = p^{d(v,v')}$, and $\lim_{v'} <\gamma_{vv'}> = p^\infty$, because $\lim_{v'} d(v,v') = \infty$ by the locally finiteness. So if $p = 1$, then $\lim_{v'} <\gamma_{vv'}> = 1$, and if $p < 1$, then $\lim_{v'} <\gamma_{vv'}> = 0$. If follows that in the open interval $(n-1)^{-1} < p < 1$ we have W' and not S'. It can further be shown that in that interval

$\lim_{v'} <\gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty> = \lim_{v'} (1-<\gamma_v^f>^{1/n})^{2(n-1)} (1-p^{d(v,v')}) =$
$= (1-<\gamma_v^f>^{1/n})^{2(n-1)} \neq 0$. Finally, one notices that $\lim_{v'} <\gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty> = 0$ for $p = 1$ and for $0 \leq p \leq (n-1)^{-1}$. In the former case, we have both W' and S' and in the latter case neither W' nor S'. We see that in this example the equivalence of W' and S' just depends on the value of $\lim_{v'} <\gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty>$.

If we restrict ourselves to bilocally finite graphs, we are able to give Theorem 2 in a sharper form. This will follow directly from the clustering property of $\gamma_v^\infty \gamma_{v'}^\infty$, or equivalently from the clustering property of $\gamma_v^f \gamma_{v'}^f$, in bilocally finite graphs, which is the subject of the next theorem. We recall that a graph is bilocally finite of for all pairs of vertices $v,v' \in V$ the number of edges incident with both vertices is finite.

Theorem 3 Clustering property. Let G be a bilocally finite graph and $v \in V$. Then

(4.13) $$\lim_{v' \in V'} \operatorname{cov}(\gamma_v^\infty, \gamma_{v'}^\infty) = \lim_{v' \in V'} \operatorname{cov}(\gamma_v^f, \gamma_{v'}^f) = 0.$$

Proof. Let G_1, G_2, G_3, ... be the increasing sequence of finite subgraphs of G introduced in § 1, and let $U_n = V-V_n$ be the set of vertices of G not in G_n. Evidently, $\gamma_v^f = \gamma_v^f \delta_{vU_n} + \gamma_v^f \gamma_{vU_n}$ for any $v \in V$, so

(4.14) $$<\gamma_v^f \gamma_{v'}^f> \leq <\gamma_v^f \delta_{vU_n} \gamma_{v'}^f> + <\gamma_v^f \gamma_{vU_n}> \leq <\gamma_v^f \delta_{vU_n} \gamma_{v'}^f \delta_{v'U_n}> + <\gamma_{v'}^f \gamma_{v'U_n}> + <\gamma_v^f \gamma_{vU_n}>,$$

(4.15) and $\langle\gamma_v^f\rangle\langle\gamma_{v'}^f\rangle \geq \langle\gamma_v^f\delta_{vU_n}\rangle\langle\gamma_{v'}^f\delta_{v'U_n}\rangle$.

By definition, if $\gamma_{v'U_n}$, v' is c-connected with at least one of the vertices of U_n; consequently, because U_n is countable,

(4.16) $$\langle\gamma_{v'}^f\gamma_{v'U_n}\rangle \leq \sum_{v''\in U_n} \langle\gamma_{v'}^f\gamma_{v'v''}\rangle = \sum_{v''\in U_n} \langle\gamma_{v'v''}^f\rangle .$$

Combining the eqs. (4.14), (4.15) and (4.16), we obtain

(4.17) $$\mathrm{cov}(\gamma_v^f,\gamma_{v'}^f) \leq \mathrm{cov}(\gamma_v^f\delta_{vU_n},\gamma_{v'}^f\delta_{v'U_n}) + \langle\gamma_v^f\gamma_{vU_n}\rangle + \sum_{v''\in U_n} \langle\gamma_{v'v''}^f\rangle .$$

Taking successively the lim sup over v' and the lim inf over n in eq. (4.17), and using the fact that lim sup $\Sigma a \leq \Sigma$ lim sup a for a≥0, and that lim inf (a+b) ≤ lim sup a + lim inf b, we get

$$\limsup_{v'\in V'} \mathrm{cov}(\gamma_v^f, \gamma_{v'}^f) \leq \limsup_n \limsup_{v'\in V'} \mathrm{cov}(\gamma_v^f\delta_{vU_n},\gamma_{v'}^f\delta_{v'U_n}) +$$

(4.18) $$+ \limsup_n \langle\gamma_v^f\gamma_{vU_n}\rangle + \liminf_n \sum_{v''\in U_n} \limsup_{v'\in V'} \langle\gamma_{v'v''}^f\rangle.$$

We shall show that each of the terms in the right-hand side of eq. (4.18) is zero. First, we can choose n so large that $v\in V_n$. Because V_n is finite, except for a finite number of vertices of V', $v'\notin V_n$. Consequently, $\delta_{v'U_n} = 0$ for these vertices, and thus

(4.19) $$\limsup_n \limsup_{v'\in V'} \mathrm{cov}(\gamma_v^f\delta_{vU_n}, \gamma_{v'}^f\delta_{v'U_n}) = 0.$$

Secondly, by the considerations in the proof of Lemma 7, γ_{vU_n} tends monotonically to γ_v^∞. So, by the integration theorem for monotone sequences,

(4.20) $$\limsup_n \langle\gamma_v^f\gamma_{vU_n}\rangle = \langle\gamma_v^f \limsup_n\gamma_{vU_n}\rangle = \langle\gamma_v^f\gamma_v^\infty\rangle = 0.$$

Finally, by eq. (4.6), $\limsup_{v'\in V'}\langle\gamma_{v'v''}^f\rangle = 0$. Hence, from the eqs. (4.6), (4.18), (4.19) and (4.20), it follows that

(4.21) $$\limsup_{v'\in V'} \mathrm{cov}(\gamma_v^f,\gamma_{v'}^f) \leq 0.$$

By the covariance inequality on the locally decreasing functions γ_v^f and $\gamma_{v'}^f$, $\mathrm{cov}(\gamma_v^f,\gamma_{v'}^f) \geq 0$, so we conclude that $\lim_{v'\in V}\mathrm{cov}(\gamma_v^f, \gamma_{v'}^f) = 0$. Substituting $\gamma_v^f = 1-\gamma_v^\infty$ we obtain the other equality in (4.13). ||

Corollary of Th. 3 Let G be a bilocally graph and let $v\in V$ belong to a P-cluster containing an infinite number of vertices of V'. If $\lim_{v'\in V'} \langle\gamma_v^\infty \delta_{vv'}, \gamma_{v'}^\infty\rangle = 0$, then

(4.22) $$\liminf_{v'\in V'} \langle\gamma_{vv'}\rangle = \langle\gamma_v^\infty\rangle \liminf_{v'\in V'} \langle\gamma_{v'}^\infty\rangle .$$

Proof. Eq. (4.22) follows directly from eq. (4.11) and Th. 3. ||

Notice that the clustering property of Th. 3 does not require any property of the percolation model except the bilocally finiteness of the graph, which is essential for the proof of eq. (4.20). In the special case that the percolation model is such that all vertices are equivalent, the corollary of Th. 3 states that $\liminf_{v'\in V'} \langle\gamma_{vv'}\rangle = \langle\gamma_v^\infty\rangle^2$, independently of v and the set V'(!)

Up to now, we have seen that the types of large-range connectivity which we have considered, even if defined with respect to a certain vertex, do not depend essentially on the vertex to be chosen. We shall now show that these types of large-range connectivity are not changed if we contract or delete a finite number of edges, if chosen appropriately. Consequently, we may change the values of p_e for a finite number of edges, if not chosen to bad, without changing the large-range connectivity.

Proposition 6 Let $v\in V$, and $e\in E$ be an edge which is not a P-isthmus of (G,P). Then contracting or deleting the edge e does not change the large-range connectivities of the types W_v, W', S_v' and S'.

Proof. Let the ends of e be $i(e) = \{v'',v'''\}$. If $v''=v'''$, e is a loop, and the state of the edge e does not influence γ_v^∞ or $\gamma_{vv'}$, so the proposition is trivial. Therefore, let $v'' \neq v'''$. Because the functions γ_v^∞ and $\gamma_{vv'}$ are locally increasing, we have by Proposition 1

(4.23)
$$\langle\gamma_v^\infty; \mathcal{D}_e G\rangle \leq \langle\gamma_v^\infty; G\rangle \leq \langle\gamma_v^\infty; \mathcal{C}_e G\rangle,$$
$$\langle\gamma_{vv'}; \mathcal{D}_e G\rangle \leq \langle\gamma_{vv'}; G\rangle \leq \langle\gamma_{vv'}; \mathcal{C}_e G\rangle.$$

It follows from eq. (4.23) and the definitions of W_v, W', S'_v and S' that in order to prove the proposition it is sufficient to prove that the large-range connectivities in the graph $\mathcal{C}_e G$ imply the corresponding large-range connectivities in the graph $\mathcal{D}_e G$. We shall make use of the following properties of γ_v^∞ and $\gamma_{vv'}$. For all $C \subseteq E-e$,

(4.24)
$$\gamma_v^\infty(C+e) = \gamma_v^\infty(C) + \gamma_{vv''}^f(C)\gamma_{v'''}^\infty(C) + \gamma_{vv'''}^f(C)\gamma_{v''}^\infty(C),$$
$$\gamma_{vv'}(C+e) = \gamma_{vv'}(C) + \delta_{v''v'''}(C)\left(\gamma_{vv''}(C)\gamma_{v'''v'}(C) + \gamma_{vv'''}(C)\gamma_{v''v'}(C)\right)$$

To verify (4.24), observe that $\gamma_v^\infty(C+e) = \gamma_v^\infty(C)\gamma_v^\infty(C+e) + \gamma_v^f(C)\gamma_v^\infty(C+e) = \gamma_v^\infty(C) + \gamma_v^f(C)\gamma_v^\infty(C+e)$ and $\gamma_{vv'}(C+e) = \gamma_{vv'}(C)\gamma_{vv'}(C+e) + \delta_{vv'}(C)\gamma_{vv'}(C+e) = \gamma_{vv'}(C) + \delta_{vv'}(C)\gamma_{vv'}(C+e)$. If $\gamma_v^f(C)\gamma_v^\infty(C+e)=1$, v belongs to a finite cluster of G_C, however to an infinite cluster of G_{C+e}. It follows that one end of e must be in the same finite cluster of G_C as v is, whereas the other end of e must be in an infinite cluster of G_C, so in another one, i.e. $\gamma_v^f(C)\gamma_v^\infty(C+e) = \gamma_{vv''}^f(C)\gamma_{v'''}^\infty(C) + \gamma_{vv'''}^f(C)\gamma_{v''}^\infty(C)$. If $\delta_{vv'}(C)\gamma_{vv'}(C+e) = 1$, v and v' belong to different clusters of G_C, whereas they belong to the same cluster in G_{C+e}. If follows that one end of e must be in the same cluster of G_C as v is, whereas the other end of e must be in another cluster of G_C containing v', i.e. $\delta_{vv'}(C)\gamma_{vv'}(C+e) = \gamma_{vv''}(C)\delta_{v''v'''}(C)\gamma_{v'''v'}(C) + \gamma_{vv'''}(C)\delta_{v'''v''}(C)\gamma_{v''v'}(C)$. That completes the proof of eq. (4.24).

Using $\gamma_{vv''}^f \leq \gamma_{vv''}$ and $\delta_{v''v'''} = \delta_e \leq 1$, and taking expectation values in eq. (2.24) we obtain

$$\langle\gamma_v^\infty;\mathcal{C}_e G\rangle \leq \langle\gamma_v^\infty;\mathcal{D}_e G\rangle + \langle\gamma_{vv''}\gamma_{v'''}^\infty;\mathcal{D}_e G\rangle + \langle\gamma_{vv'''}\gamma_{v''}^\infty;\mathcal{D}_e G\rangle,$$

$$\langle\gamma_{vv'};\mathcal{C}_e G\rangle \leq \langle\gamma_{vv'};\mathcal{D}_e G\rangle + \langle\gamma_{vv''}\gamma_{v'''v'};\mathcal{D}_e G\rangle + \langle\gamma_{vv'''}\gamma_{v''v'};\mathcal{D}_e G\rangle . \tag{4.25}$$

By the reasoning leading to the eqs. (4.2) and (4.1), we get

$$\langle\gamma_v^\infty\rangle \geq \langle\gamma_{vv''}\gamma_{v''v'''}\gamma_{v'''}^\infty\rangle \geq \langle\gamma_{v''v'''}\rangle\langle\gamma_{vv''}\gamma_{v'''}^\infty\rangle,$$

$$\langle\gamma_{vv'}\rangle \geq \langle\gamma_{vv''}\gamma_{v''v'''}\gamma_{v'''v'}\rangle \geq \langle\gamma_{v''v'''}\rangle\langle\gamma_{vv''}\gamma_{v'''v'}\rangle. \tag{4.26}$$

We observe that $\langle\gamma_{v''v'''};\mathcal{D}_e G\rangle = \langle\gamma_e;\mathcal{D}_e G\rangle > 0$, by the assumption that e is not a P-isthmus. Using this fact we obtain from eqs. (4.25), (4.26)

$$\langle\gamma_v^\infty;\mathcal{C}_e G\rangle \leq \langle\gamma_v^\infty;\mathcal{D}_e G\rangle(1 + 2/\langle\gamma_e;\mathcal{D}_e G\rangle),$$

$$\langle\gamma_{vv'};\mathcal{C}_e G\rangle \leq \langle\gamma_{vv'};\mathcal{D}_e G\rangle(1 + 2/\langle\gamma_e;\mathcal{D}_e G\rangle). \tag{4.27}$$

Taking the limes inferior over $v\in V'$ in the first part of eq. (4.27) we obtain a similar equation for $\liminf_{v\in V'}\langle\gamma_v^\infty\rangle$. Taking the $\liminf_{v'\in V'}$ and $\liminf_{v\in V'}\liminf_{v'\in V'}$ in the second part of eq. (4.27) we obtain similar equations for $\liminf_{v'\in V'}\langle\gamma_{vv'}\rangle$ and $\liminf_{v\in V'}\liminf_{v'\in V'}\langle\gamma_{vv'}\rangle$. Using eq. (4.27) and the last obtained similar ones, it is clear that some large-range connectivity in $\mathcal{C}_e G$ implies the corresponding large-range connectivity in $\mathcal{D}_e G$. That completes the proof of the proposition. ||

5. THE SUPPLEMENTARY VERTEX AND LARGE-RANGE CONNECTIVITY

We shall show in this section that the concept of weak large-range connectivity in a locally finite percolation model is related with the accessibility of the supplementary vertex in the supplemented percolation model, as introduced in I § 7.

We recall the relevant definitions. If (G,P) is a percolation model, the supplemented percolation model (G^o,P^o) is obtained by adding the supplementary vertex o and the set of supplementary edges E_o, consisting of just one edge incident with o and v for each vertex $v \in V(G)$, to the graph G, and the map p_o from E_o into the interval $[0,1]$. So with $G = (V,E,i)$ we obtain $G^o = (V \cup o, E \cup E_o, i \cup i_o)$ and $P^o = P \times P_o$, where P_o is generated by p_o, i.e. P^o is generated by $p \cup p_o = p^o$. If the graph G is locally finite, we say that (G,P) is a locally finite percolation model and that (G^o,P^o) is a supplemented locally finite percolation model. In this section we consider percolation models with a variable supplementary measure P_o. In particular, we are interested in the limit where the function p_o goes to zero; this is the analogue of the limit of a vanishing magnetic field for the Ising model. We shall then call (G^o,P^o) a <u>variable supplementation</u> of (G,P).

Lemma 10 Let (G^o,P^o) be a supplemented percolation model such that $\liminf_{v \in V} p_{ov} > 0$, and let $v \in V$. Then $\langle \gamma_{ov}\gamma^f_v; G^o,P^o \rangle = 0$.

Proof. If $\gamma_{ov}\gamma^f_v$, then both o and v belong to the same finite c-cluster of G^o, and by lemma 4 we may write

(5.1) $$\langle \gamma_{ov}\gamma^f_v;G^o,P^o \rangle = \sum^f_{G' \subset G^o} \langle \gamma_{G';ov};G^o,P^o \rangle .$$

If o belongs to a finite c-cluster G', with finite set of vertices V', evidently all supplementary edges from o to the vertices of V-V' are d-edges, so $\gamma_{G';ov} \leq d^{E''}$, where E" is the set of supplementary edges incident with vertices of V-V'. Because V' is finite, V-V' is infinite, so E" is infinite. Consequently, $\langle \gamma_{G';ov};G^o,P^o \rangle \leq q^{E''} = 0$, by the assumption that $\liminf_{v \in V} p_{ov} > 0$. Therefore, $\langle \gamma_{G',ov};G^o,P^o \rangle = 0$ and by eq. (5.1) also $\langle \gamma_{ov}\gamma^f_v;G^o,P^o \rangle = 0$. ||

The following proposition contains the essential idea of the above-mentioned relationship between γ^∞_v and γ_{ov}.

Proposition 7 Let (G^o,P^o) be a supplemented locally finite percolation model such that $\lim\inf_{v\in V} p_{ov} > 0$. Then $\gamma_v^\infty = \gamma_{ov}$ a.e. .

Proof. In order to prove that $\gamma_v^\infty = \gamma_{ov}$ a.e., we have to prove that $\gamma_v^\infty \delta_{ov} = 0$ a.e. and that $\gamma_{ov}\gamma_v^f = 0$ a.e. Lemma 10 is equivalent to the latter equation, so it remains to prove that $<\gamma_v^\infty \delta_{ov};G^o,P^o> = 0$. Using the Lemma's 5,6, we obtain

$$<\gamma_v^\infty \delta_{ov};G^o,P^o> = \lim_n <\gamma_{vB_n}\delta_{ov};G_n^o,P^o>, \tag{5.2}$$

where we used the fact that if δ_{ov}, γ_v^∞ in $G^o = \gamma_v^\infty$ in G, because the c-cluster containing v cannot contain the vertex o or any of the supplementary edges. If $\gamma_{vB_n}\delta_{ov}$ in G_n^o, we can say equivalently that there is a c-cluster G' in G_n such that it contains v and at least one of the vertices of B_n, i.e. $\gamma_{G';vB_n}$, and such that all supplementary edges incident with it are $\bar{d}$-edges. Therefore,

$$<\gamma_{vB_n}\delta_{ov};G_n^o,P^o> = \sum_{G'\subseteq G_n} <\gamma^{G_n}_{G';vB_n} d^{E'};G_n^o,P^o> = \sum_{G'\subseteq G_n} q_o^{V(G')}<\gamma_{G';vB_n};G_n,P>, \tag{5.3}$$

where E' is the set of supplementary edges incident with V(G'). If for all $C:\gamma_{G';vB_n}(C) = 0$, G' does not contribute to the sum in the right-hand side of eq. (5.3). If $\gamma_{G';vB_n}$ can be 1, i.e. G' contains v and at least one vertex of B_n and is connected, V(G') contains at least $d(v,B_n)+1$ vertices. By the assumption $\lim\inf_v p_{ov} > 0$, we have, except for a finite number of vertices v, $q_{ov} \leq 1-\lim\inf_v p_{ov} \equiv a < 1$ and hence $q_o^{V(G')} \leq b\, a^{|V(G')|} \leq b\, a^{d(v,B_n)}$ for the contributing c-clusters, where a and b are constants. Therefore, we obtain from eq. (5.3)

$$<\gamma_{vB_n}\delta_{ov};G_n^o,P^o> \leq b\, a^{d(v,B_n)} \sum_{G'\subseteq G_n} <\gamma_{G';vB_n};G_n,P> = b\, a^{d(v,B_n)}<\gamma_{vB_n};G_n,P> . \tag{5.4}$$

So, using eq. (5.2) and (5.4), and the property of locally finite graphs that $\lim_n d(v,B_n) = \infty$, we have

$$<\gamma_v^\infty \delta_{ov};G^o,P^o> \leq b \lim_n a^{d(v,B_n)} = 0, \tag{5.5}$$

and it follows that $<\gamma_v^\infty \delta_{ov};G^o,P^o> = 0$. ||

From this proposition we immediately deduce a relation between weak large-range connectivity and the accessibility of the supplementary vertex. Furthermore, the role of the supplementary vertex with respect to strong large-range connectivity is clarified by the following theorem.

Theorem 4 Let (G,P) be a locally finite percolation model and let $v,v' \in V$. If (G^o,P^o) is a variable supplementation of (G,P) such that $\liminf_{v\in V} p_{ov} > 0$, then

$$\lim_{p_o \downarrow 0} \langle \gamma_{ov}; G^o, P^o \rangle = \langle \gamma_v^\infty; G, P \rangle, \tag{5.6}$$

$$\lim_{p_o \downarrow 0} \langle \gamma_{vv'}; G^o, P^o \rangle = \langle \gamma_{vv'}; G, P \rangle + \langle \gamma_v^\infty \delta_{vv'} \gamma_{v'}^\infty; G, P \rangle. \tag{5.7}$$

Proof. Evidently, $\gamma_{ov} = \gamma_v^\infty = \gamma_v^\infty \gamma_v^{G\infty} + \gamma_v^\infty \gamma_v^{Gf} = \gamma_v^{G\infty} + \gamma_v^\infty \gamma_v^{Gf}$ a.e., by Proposition 7 and the assumption. Hence,

$$\lim_{p_o \downarrow 0} \langle \gamma_{ov}; G^o, P^o \rangle = \langle \gamma_v^\infty; G, P \rangle + \lim_{p_o \downarrow 0} \langle \gamma_v^\infty \gamma_v^{Gf}; G^o, P^o \rangle . \tag{5.8}$$

Evidently, for any n, $\gamma_v^\infty \gamma_v^{Gf} = \gamma_v^\infty \gamma_v^{Gf} \delta^G_{vB_n} + \gamma_v^\infty \gamma_v^{Gf} \gamma^G_{vB_n} = \gamma_v^\infty \delta^G_{vB_n} + \gamma_v^\infty \gamma_v^{Gf} \gamma^G_{vB_n}$. The number of supplementary edges incident with V_n is finite, so $\lim_{p_o \downarrow 0} \langle \gamma_v^\infty \delta^G_{vB_n}; G^o, P^o \rangle = 0$, because it is sufficient to let go to zero the p_{ov} with $v \in V_n$ in order to get $\gamma_{ov} = 0$. So it follows that

$$\lim_{p_o \downarrow 0} \langle \gamma_v^\infty \gamma_v^{Gf}; G^o, P^o \rangle = \lim_{p_o \downarrow 0} \langle \gamma_v^\infty \gamma_v^{Gf} \gamma^G_{vB_n}; G^o, P^o \rangle. \tag{5.9}$$

Obviously, $\gamma_v^\infty \leq 1$, and thus,

$$\lim_{p_o \downarrow 0} \langle \gamma_v^\infty \gamma_v^{Gf} \gamma^G_{vB_n}; G^o, P^o \rangle \leq \liminf_n \langle \gamma_v^f \gamma_{vB_n}; G, P \rangle . \tag{5.10}$$

By Lemma 6, $\gamma_v^\infty = \lim_n \gamma_{vB_n}$, and the limit is obtained monotonically, so by the integration theorem on monotone sequences and Lemma 6 we have

(5.11) $\liminf_n \langle\gamma_v^f \gamma_{vB_n};G,P\rangle = \langle\gamma_v^f \lim_n \gamma_{vB_n};G,P\rangle = \langle\gamma_v^f\gamma_v^\infty;G,P\rangle = 0.$

Using eqs. (5.10), (5.11) and the non-negativeness of indicators it follows that

(5.12) $\lim_{p_o \downarrow 0} \langle\gamma_v^\infty\gamma_v^{Gf}\gamma_{vB_n}^G;G^o,P^o\rangle = 0.$

From eqs. (5.8), (5.9) and (5.12), eq. (5.6) follows.

To prove eq. (5.7), we first observe that $\gamma_{vv'} = \gamma_{vv'}\gamma_{vv'}^G + \gamma_{vv'}\delta_{vv'}^G = \gamma_{vv'}^G + \gamma_{ov}\delta_{vv'}^G\gamma_{ov'} = \gamma_{vv'}^G + \gamma_v^\infty\delta_{vv'}^G\gamma_{v'}^\infty$ a.e. Hence,

(5.13) $\lim_{p_o \downarrow 0} \langle\gamma_{vv'};G^o,P^o\rangle = \langle\gamma_{vv'};G,P\rangle + \lim_{p_o \downarrow 0} \langle\gamma_v^\infty\delta_{vv'}^G\gamma_{v'}^\infty;G^o,P^o\rangle.$

Let f be a bounded random variable, then

(5.14) $\lim_{p_o \downarrow 0} (\langle f\gamma_v^\infty;G^oP^o\rangle - \langle f\gamma_v^{G\infty};G^o,P^o\rangle) = \lim_{p_o \downarrow 0} \langle f\gamma_v^\infty\gamma_v^{Gf};G^o,P^o\rangle = 0,$

because by (5.9) and (5.12) $\lim_{p_o\downarrow 0}\langle\gamma_v^\infty\gamma_v^{Gf};G^o,P^o\rangle = 0$, whereas f is bounded. Using (5.14) repeatedly in the right-hand side of eq. (5.13), we obtain eq. (5.7). ||

Corollary Let (G,P) be a locally finite percolation model and $v \in V$. If (G^o,P^o) is a variable supplementation of (G,P) such that $\liminf_{v\in V} p_{ov} > 0$, then

(5.15) $\liminf_{v'\in V'} \lim_{p_o \downarrow 0} \langle\gamma_{vv'};G^o,P^o\rangle = \lim_{p_o \downarrow 0} \langle\gamma_{ov};G^o,P^o\rangle \liminf_{v'\in V'} \lim_{p_o \downarrow 0} \langle\gamma_{ov'};G^o,P^o\rangle.$

Proof. By eq. (4.10), $\langle\gamma_{vv'}\rangle + \langle\gamma_v^\infty\delta_{vv'}\gamma_{v'}^\infty\rangle = \langle\gamma_v^\infty\gamma_{v'}^\infty\rangle + \langle\gamma_{vv'}^f\rangle$ for (G,P). By (5.7), the left-hand side can be replaced by $\lim_{p_o\downarrow 0}\langle\gamma_{vv'};G^o,P^o\rangle$. By Th. 3 and eq. (4.6) the right-hand side, after taking the limes inferior over v', becomes $\langle\gamma_v^\infty\rangle \liminf_v \langle\gamma_v^\infty\rangle$, which by (5.6) equals $\lim_{p_o\downarrow 0}\langle\gamma_{ov};G^o,P^o\rangle \liminf_v \lim_{p_o\downarrow 0}\langle\gamma_{ov};G^o,P^o\rangle$.

Collecting these results the corollary follows. ||

Another consequence of Theorem 4 is obtained by using the differentiation relation, I Prop. 1, together with Lemma 5, thus relating $\langle\gamma_v^f\rangle$ with a derivative of the average number of clusters.

Corollary Let (G,P) be a locally finite percolation model and $v \in V$. If (G^o,P^o) is a variable supplementation of (G,P) such that $\liminf_{v \in V} p_{ov} > 0$, then

$$\langle\gamma_v^f;G,P\rangle = \lim_{p_o \downarrow 0} \lim_n q_{ov} \frac{\partial}{\partial q_{ov}} \langle\gamma;G_n^o,P^o\rangle. \tag{5.16}$$

Proof. By I Prop. 1, applied to the supplementary edge between o and v, and the finite graph G_n^o,

$$\langle\delta_{ov};G_n^o,P^o\rangle = q_{ov} \frac{\partial}{\partial q_{ov}} \langle\gamma;G_n^o,P^o\rangle. \tag{5.17}$$

Furthermore, by Lemma 5, eq. (2.10) applied to the ends of the same edge,

$$\langle\gamma_{ov};G^o,P^o\rangle = \lim_n \langle\gamma_{ov};G_n^o,P^o\rangle . \tag{5.18}$$

By eq. (5.17), eq. (5.18) and eq. (5.6), with $\delta_{ov} = 1-\gamma_{ov}$, the corollary follows. ||

Along the line of the last corollary of Theorem 4 we can give a characterization of the global large-range connectivity $\liminf_n |V_n|^{-1} \Sigma_{v \in V_n} \langle\gamma_v^\infty\rangle$, or rather of a quantity closely related to it, $\limsup_n |V_n|^{-1} \Sigma_{v \in V_n} \langle\gamma_v^f\rangle$. For convenience we shall use a variable supplementation of the percolation model with $p_{ov} = p_o$ for all $v \in V$ in the following proposition. Before giving the proposition we shall derive a lemma.

Lemma 11 Let (G^o, P^o) be a variable supplementation of a locally finite percolation model such that $p_{ov'} = p_o$ for all $v' \in V$, let $v \in V$ and k be an integer ≥ 0. Then, with $p_o = 1-q_o$,

(5.19) $$\left(q_o \frac{\partial}{\partial q_o}\right)^k <\delta_{ov}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o> = <\delta_{ov}\gamma_v^f\left(\sum_{v'\in V}\gamma_{vv'}\right)^{k-1};G^o,P^o> ,$$

where both members of (5.19) are continuously decreasing and bounded in $0 \leq p_o \leq 1$.

Proof. First of all we notice that by Proposition 7 for $p_o > 0$, $\delta_{ov}\gamma_v^f = \delta_{ov}$ a.e. For k=0, we have $<\delta_{ov}(\Sigma_{v'\in V}\gamma_{vv'})^{-1};G^o,P^o> =$ $= <\delta_{ov}\gamma_v^f(\Sigma_{v'\in V}\gamma_{vv'})^{-1};G^o,P^o>$ because even for $p_o=0$, if γ_v^∞, $\Sigma_{v'\in V}\gamma_{vv'} = \infty$ by the locally finiteness, and hence $(\Sigma_{v'\in V}\gamma_{vv'})^{-1} = 0$. Furthermore, if $\delta_{ov}\gamma_v^f$, v belongs to a finite c-cluster of G^o which does not contain o, so v belongs to a finite c-cluster of G and all supplementary edges incident with this cluster are d-edges. So it follows, using Lemma 4, that

(5.20) $$<\delta_{ov}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o> = \sum_{G'\subset G}^f |V(G')|^{-1} q_o^{|V(G')|} <\gamma_{G';v};G,P>.$$

By inspection, the right-hand side of eq. (5.20) is a power series in q_o with non-negative coefficients, bounded by 1, so the radius of convergence is larger than 1. Therefore, the series converges uniformly, as well as its derivatives, and we may interchange derivative and summation, even without changing the radius of convergence. So we also have

(5.21) $$\left(q_o \frac{\partial}{\partial q_o}\right)^k <\delta_{ov}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o> = \sum_{G'\subset G}^f |V(G')|^{k-1} q_o^{|V(G')|} <\gamma_{G';v};G,P>$$

and both sides of eq. (5.21) are continuously decreasing and bounded in $0 \leq p_o \leq 1$. It is seen that the right-hand side of eq. (5.21) is equal to the right-hand side of eq. (5.19) by the same reasoning as was used to deduce eq. (5.20), so the lemma follows. ||

We are now in the position to give a proposition about the global large-range connectivity for a subclass of the locally finite percolation models. We shall say that a graph is <u>locally bounded</u>, if there exists a finite number n such that for all vertices v of the graph the number of edges incident with the given vertex is not larger than n, i.e. there is a uniform bound for the number of edges incident with the same vertex.

Proposition 8 Let (G,P) be a locally bounded percolation model and let the increasing sequence of finite subgraphs G_n of G be such that $\lim_n |V_n|^{-1}|B_n| = 0$ and $\cup_n G_n = G$. If (G^o,P^o) is a variable supplementation of (G,P) such that $p_{ov} = p_o$ for all $v\in V$, then

(5.22) $$\lim_n |V_n|^{-1}\{<\gamma;G^o_n,P^o> - \sum_{v\in V_n} <\delta_{ov}\Big(\sum_{v'\in V}\gamma_{vv'}\Big)^{-1};G^o,P^o>\} = 0.$$

If $\lim_n |V_n|^{-1}<\gamma;G^o_n,P^o>$ exists and $\sup_{v\in V}<\gamma^f_v(\Sigma_{v'\in V}\gamma_{vv'})^{k-1};G,P> < \infty$, where k is an integer ≥ 0, then the following limits exist, are finite and equal:

(5.23) $$\lim_{p_o\downarrow 0}\left(q_o\frac{\partial}{\partial q_o}\right)^k \lim_n |V_n|^{-1}<\gamma;G^o_n,P^o> = \lim_n |V_n|^{-1}\sum_{v\in V_n}<\gamma^f_v\Big(\sum_{v'\in V}\gamma_{vv'}\Big)^{k-1};G,P>.$$

Proof. The c-clusters of G^o_n may be divided in the c-cluster containing the supplementary vertex, and those which do not contain o. Furthermore, as we count for every vertex $v\in V_n$, c-disconnected with o, the inverse of the number of vertices of V_n c-connected with v, i.e. $(\Sigma_{v'}\gamma_{vv'})^{-1}$, we just count each c-cluster not containing o once. So we have

(5.24) $$<\gamma;G^o_n,P^o> = 1 + \sum_{v\in V_n}<\delta_{ov}\Big(\sum_{v'\in V_n}\gamma_{vv'}\Big)^{-1};G^o_n,P^o>.$$

Obviously, we can write $<\delta_{ov}(\Sigma_{v'\in V_n}\gamma_{vv'})^{-1};G^o_n,P^o> = <\delta^{G^o_n}_{ov}(\Sigma_{v'\in V}\gamma^{G_n}_{vv'})^{-1};G^o,P^o>$. Because δ_{ov} in G^o_n is decreasing and $\gamma_{vv'}$ in G_n is increasing in n, it follows from Lemma 5 that the limit

(5.25) $$\lim_n <\delta_{ov}\Big(\sum_{v'\in V_n}\gamma_{vv'}\Big)^{-1};G^o_n,P^o> = <\delta_{ov}\Big(\sum_{v'\in V}\gamma_{vv'}\Big)^{-1};G^o,P^o>$$

exists as the limit of a monotonically non-increasing sequence. Hence,

$$0 \le \left\langle \delta_{ov}\left(\sum_{v'\in V_n}\gamma_{vv'}\right)^{-1};G_n^o,P^o\right\rangle - \left\langle \delta_{ov}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o\right\rangle. \tag{5.26}$$

Evidently, $\delta_{ov}\delta_{vB_n}(\Sigma_{v'\in V}\gamma_{vv'})^{-1}$ only depends on the states of the edges in G_n^o, so the right-hand side of eq. (5.26) is equal to

$$\left\langle \delta_{ov}\gamma_{vB_n}\left(\sum_{v'\in V_n}\gamma_{vv'}\right)^{-1};G_n^o,P^o\right\rangle - \left\langle \delta_{ov}\gamma_{vB_n}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o\right\rangle \le$$

$$\le \left\langle \delta_{ov}\gamma_{vB_n}\left(\sum_{v'\in V_n}\gamma_{vv'}\right)^{-1};G^o,P^o\right\rangle. \tag{5.27}$$

An upper bound for the right-hand side of inequality (5.27) is found as follows. If v is c-connected with the boundary B_n of G_n, and not with o, at least $d(v,B_n)+1$ vertices are c-connected with v, where $d(v,B_n)$ is the distance between v and B_n. From (5.26) and (5.27) we obtain, therefore,

$$0 \le \left\langle \delta_{ov}\left(\sum_{v'\in V_n}\gamma_{vv'}\right)^{-1};G_n^o,P^o\right\rangle - \left\langle \delta_{ov}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o\right\rangle \le \{d(v,B_n)+1\}^{-1}. \tag{5.28}$$

From the eqs. (5.24), (5.25) and (5.28) we may conclude that

$$0 \le \lim_n |V_n|^{-1}\left\{\langle\gamma;G_n^o,P^o\rangle - \sum_{v\in V_n}\left\langle \delta_{ov}\left(\sum_{v'\in V}\gamma_{vv'}\right)^{-1};G^o,P^o\right\rangle\right\} \le$$

$$\le \lim_n |V_n|^{-1}\sum_{v\in V_n}\{d(v,B_n)+1\}^{-1}. \tag{5.29}$$

In order to show that the utmost right part of ineq. (5.29) is zero, we divide the vertices of V_n into two parts. One part, B_{n1}, consists of the vertices of V_n within a distance n' from B_n. So,

$$\lim_n |V_n|^{-1}\sum_{v\in V_n}\{d(v,B_n)+1\}^{-1} \le \lim_n |V_n|^{-1}\left\{\sum_{v\in B_{n1}}\cdot 1 + \sum_{v\in V_n - B_{n1}}(n')^{-1}\right\}. \tag{5.30}$$

Because the number of edges incident with any vertex is bounded by n", say, by assumption, we may approximate the number of vertices of B_{n1}. The vertices of B_{n1} may be divided into those of the boundary B_n, those at a distance 1 from B_n, etc., and, finally, the vertices at a distance $(n'-1)$ from B_n. Each vertex of B_n has at most $(n''-1)$ edges of G_n incident with it, because by definition there is at least one edge incident with it which is not in G_n. Therefore, the number of vertices at a distance 1 from B_n is at most $(n''-1)|B_n|$. Repeating this argument, one finds that

$$|B_{n1}| \leq |B_n| + (n''-1)|B_n| + (n''-1)^2|B_n| + \ldots + (n''-1)^{n'-1}|B_n| = C|B_n|, \tag{5.31}$$

where C is a number depending on n' and n". From eq. (5.31) and the assumption $\lim_n |V_n|^{-1}|B_n| = 0$ it follows that

$$\lim_n |V_n|^{-1}\{\sum_{v\in B_{n1}} 1 + \sum_{v\in V_n - B_{n1}} (n')^{-1}\} \leq (n')^{-1}. \tag{5.32}$$

Because for n' we may choose any number, this limit is zero, and by eq. (5.30) it follows from (5.29) that (5.22) holds.

To prove the second part of the proposition, we notice that by Lemma 11 the functions $\langle\delta_{ov}\gamma_v^f(\Sigma_{v'}\gamma_{vv'})^{k-1};G^o,P^o\rangle$ are convex functions in $\ln q_o$ for all $v\in V$ and all $k\geq 0$, because they are finite and have a non-negative second derivative with respect to $\ln q_o$. Now for a sequence f_n of convex functions, if the limit exists and is finite, the limit function is convex and continuous and, moreover, piecewise differentiable, with the property that $(d/dx)\lim_n f_n(x) = \lim_n (d/dx) f_n(x)$ (cf. Fisher [10], Lemma III). If $\sup_v \langle\gamma_v^f(\Sigma_{v'}\gamma_{vv'})^{k-1};G,P\rangle < \infty$, the functions $\langle\delta_{ov}\gamma_v^f(\Sigma_{v'}\gamma_{vv'})^{k'-1};G^o,P^o\rangle$ are uniformly bounded in p_o, v and $k'\leq k$, because they are decreasing in p_o and increasing in k', by Lemma 11. Therefore, from the existence, by assumption, of $\lim_n|V_n|^{-1}\langle\gamma;G_n^o,P^o\rangle$, so of $\lim_n|V_n|^{-1}\Sigma_{v\in V_n}\langle\delta_{ov}(\Sigma_{v'\in V}\gamma_{vv'})^{-1};G_n^o,P^o\rangle$ we conclude that $q_o(\partial/\partial q_o)\lim_n|V_n|^{-1}\langle\gamma;G_n^o,P^o\rangle$ exists piecewise and is equal, by Lemma 11, to $\lim_n|V_n|^{-1}\Sigma_{v\in V_n}\langle\delta_{ov}\gamma_v^f(\Sigma_{v'\in V}\gamma_{vv'})^0;G^o,P^o\rangle$. But, from the convexity of this last limit we know that the limit function is continuous, and, consequently, eq. (5.23) is valid for k=1 for all p_o. By the repeated use of this argument, the existence, finiteness and equality of the right-hand side and left-hand side of eq. (5.23) follows. ||

6. DISCUSSION

In this second paper on the random-cluster model our attention was focussed on the percolation model. In § 2 we proved that in infinite countable graphs a number of functions are random variables, and we derived in § 3 an inequality, important in the analysis of §§ 4 and 5. In the last sections the main body of this paper is contained, from which we take two main points.

First, in § 4 the relation between weak and strong large-range connectivity was established, in particular their equivalence under a non-trivial condition. This equivalence is also related with a clustering property. In the theory of phase transitions such a clustering property is related with the translational invariance, or, more generally, with the existence of a group of automorphisms of the system in question. In the present case, however, the existence of such automorphisms is not required.

Secondly, in § 5 it is shown that the supplementary vertex plays the role of infinity, in such a way that instead of investigating the connections with infinity, we can investigate the connections with the supplementary vertex. Since the supplementary vertex was introduced to provide an analogue for the random-cluster model of the magnetic field in an Ising system, the established relationship paves the way to a relation between large-range connectivity (i.e. the occurrence of infinite clusters) and spontaneous magnetization. Such a relation will be derived in a subsequent paper (III), in which the present analysis will be extended to the random-cluster model.

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ON THE RANDOM-CLUSTER MODEL

III. The simple random-cluster model

Synopsis The possibility of the occurrence of a phase transition in an infinite simple random-cluster model ($0 < p < 1$, $\kappa > 1$), which includes the percolation model and the ferromagnetic Ising and Ashkin-Teller-Potts model, is studied by means of several criteria for large-range connectivity. It is shown that graphs which contain the square lattice exhibit a phase transition in this sense. The large-range connectivities in the simple random-cluster model turn out to have the same properties as those in the percolation model. Furthermore, it is shown that in graphs with a lattice structure the generalized spontaneous magnetization is strongly related to global large-range connectivity.

1. INTRODUCTION

This paper is the third one in a sequence of papers on the random-cluster model. In the first paper [1], to be referred to as I, the random-cluster model was defined and shown to include as special cases the percolation model, the Ising model and the Ashkin-Teller-Potts model. In the second paper [2], to be referred to as II, we investigated the relationship between several criteria of large-range connectivity and the role of the supplementary vertex in the percolation model, in order to get some insight into the possible occurrence of phase transitions in the randomcluster model.

In this paper we are concerned with the same questions regarding phase transitions as in II, generalized to what we shall call the simple random-cluster model, which includes as special cases the percolation model and the ferromagnetic Ising and Ashkin-Teller-Potts model. The expectation, formulated in II § 1, that many properties of the percolation model will be typical for the random-cluster model, will in this paper be confirmed for the simple random-cluster model.

An important feature of the simple random-cluster model will be the fact that all functions of interest can be defined on, or related with functions on infinite countable graphs. This enables us to study the questions regarding phase transitions on infinite countable graphs without the intervention of the "thermodynamic limit". Moreover, the properties which where crucial in the analysis of the criteria for large-range connectivity in the percolation model can be extended to the simple random-cluster model. So the whole analysis of the criteria for large-range connectivity, as given in II § 4, can be extended to the simple random-cluster model, thus generalizing some of the methods developed for the percolation model or for the Ising model.

The possible occurrence of a phase transition will again be studied by means of the various criteria of large-range connectivity, introduced in II. We shall show the simple random-cluster model shows a phase transition in this sense if the graph is, or contains, the square lattice. The argument used to establish the non-occurrence of weak large-range connectivity is an extension of an argument used by Hammersley[3] for the percolation model and by Fisher for the Ising model[4]. The argument for the occurrence of weak large-range connectivity is an extension of the celebrated Peierls[5,6,7] argument, used in the analysis of the Ising model and of an argument used by Hammersley for the percolation model[8]. In § 2 we extend some of the properties of the percolation model to the simple random-cluster model. In particular we show that the recursion theorem, the covariance inequality, and a typical property of increasing functions can be extended to the simple random-cluster model, which is shown to exist for countable graphs. In § 3 we give the main theorem, stating that all properties of large-range connectivity which in II were shown to hold for the percolation model, also hold for the simple random-cluster model. Furthermore, we give conditions for the occurrence or non-occurrence of large-range connectivity, and apply them to the square lattice. Finally, in § 4, we show that, again, the supplementary vertex in the simple random-cluster model plays the role of infinity. Moreover, for locally finite random-cluster models with a lattice structure we show that the generalized spontaneous magnetization is related with global large-range connectivity in a model which is the limit of the simple random-cluster model with vanishing supplementary edges.

We conclude this introduction with some remarks about notation. Expectation values in the percolation model will be denoted by $<f;G,P>$, where f is a random variable, G a countable graph and P the measure generated by the mapping p (see I § 3.1). In the (simple) random-cluster model we shall denote this expectation value by $<f;G,\mu>$ or, more explicitly, as $<f;G,p,\kappa>$ (I § 7.1). By $<f>$ or $<f;G>$ or $<f;\mu>$, we always understand the expectation value in a random-cluster model. If the model is supplemented (I § 7.1) we replace G, P, p, μ by G^o, P^o, p^o, μ^o.

2. RECURSION THEOREM AND COVARIANCE INEQUALITY

In this section we prove that the recursion theorem I Th. 1 and the covariance inequality II Th. 1 for the percolation model can be extended to the random-cluster model under suitable conditions. We shall require that the measure P is a probability measure, i.e. P is generated by a mapping such that $0 \leq p \leq 1$. Furthermore, the values of κ are restricted to $\kappa \geq 1$. So, for $\kappa = 1$ the percolation model is covered, and for $\kappa = 2,3,\ldots$ the ferromagnetic Ising and Ashkin-Teller-Potts model. We shall call a random-cluster model (G,p,κ) with $0 \leq p \leq 1$ and $\kappa \geq 1$ a simple random-cluster model. In this paper, we consider only simple random-cluster models.

In order to prove the existence, and properties, of the simple random-cluster model, we first prove some properties of a finite simple random-cluster model.

Lemma 1 Let (G,p,κ) be a finite simple random-cluster model, $e \in E$ and f a local variable on G. Then

(2.1) $$\langle f;G\rangle = \langle c_e;G\rangle\langle \overline{f};\mathcal{C}_e G\rangle + \langle d_e;G\rangle\langle \overline{f};\mathcal{D}_e G\rangle.$$

Proof. By definition (see I § 7), $\langle f;G\rangle \equiv \langle f\kappa^{\gamma};G,P\rangle/Z(G)$. By the recursion property I Th. 1, $\langle f\kappa^{\gamma};G,P\rangle = p_e\langle \overline{f}\kappa^{\gamma};\mathcal{C}_e G,P\rangle + q_e\langle \overline{f}\kappa^{\gamma};\mathcal{D}_e G,P\rangle$, and it follows that

(2.2) $$\langle f;G\rangle = \{p_e Z(\mathcal{C}_e G)/Z(G)\}\langle \overline{f};\mathcal{C}_e G\rangle + \{q_e Z(\mathcal{D}_e G)/Z(G)\}\langle \overline{f};\mathcal{D}_e G\rangle.$$

Applying eq. (2.2) to c_e and d_e, and using the fact that $\overline{c}_e = 1$ on $\mathcal{C}_e G$ and $\overline{c}_e = 0$ on $\mathcal{D}_e G$ we obtain

(2.3) $$\langle c_e;G\rangle = p_e Z(\mathcal{C}_e G)/Z(G),$$

(2.4) $$\langle d_e;G\rangle = q_e Z(\mathcal{D}_e G)/Z(G).$$

From the eqs. (2.2), (2.3) and (2.4), eq. (2.1) follows. ||

Corollary If $p_e \neq 0$, then

(2.5) $$<\overline{f};\mathcal{C}_e G,\mu> = <fc_e;G,\mu>/<c_e;G,\mu>.$$

If $p_e \neq 1$, then

(2.6) $$<\overline{f};\mathcal{D}_e G,\mu> = <fd_e;G,\mu>/<d_e;G,\mu>.$$

Proof. First we notice from eq. (2.3) that $<c_e> = 0$ if and only if $p_e = 0$, because $Z \geq \kappa$, and analogously that $<d_e> = 0$ if and only if $p_e = 1$. Applying Lemma 1 to the functions fc_e and fd_e we obtain the corollary. ||

Before proving the covariance inequality for finite graphs, we recall a few definitions. <u>The covariance</u> of two summable functions is defined as $\mathrm{cov}(f,g) \equiv <fg> - <f><g>$. A function is called <u>locally increasing</u> if for all $e \in E$ and $C \subseteq E-e$ we have $f(C) \leq f(C+e)$. A function will be called <u>increasing</u> if for all $C \subseteq C' \subseteq E$ we have $f(C) \leq f(C')$. Evidently, if $|E|$ is finite, there is no difference between locally increasing and increasing functions. If $-f$ is (locally) increasing, f is called <u>(locally) decreasing</u>.

Lemma 2 Let (G,p,κ) be a finite simple random-cluster model and f,g increasing local variables on G, then

(2.7) $$\mathrm{cov}(f,g) \geq 0.$$

Proof. We prove Lemma 2 by induction on the number of elements of $|E(G)|$. If $|E(G)| = 0$,

$$\mathrm{cov}(f,g) \equiv \left\{\frac{f(\emptyset)g(\emptyset)\kappa^{\gamma(\emptyset)}}{\kappa^{\gamma(\emptyset)}}\right\} - \left\{\frac{f(\emptyset)\kappa^{\gamma(\emptyset)}}{\kappa^{\gamma(\emptyset)}}\right\}\left\{\frac{g(\emptyset)\kappa^{\gamma(\emptyset)}}{\kappa^{\gamma(\emptyset)}}\right\} = 0,$$

so eq. (2.7) holds with the equality sign. Suppose the lemma is true for $|E(G)| < n$, and let $|E(G)| = n$. In the same way as in the proof of II Th. 1, we obtain for any $e \in E(G)$, by the recursion property Lemma 1,

$$\mathrm{cov}(f,g;G) = \langle c_e;G\rangle \mathrm{cov}(\bar{f},\bar{g};\mathcal{C}_e G) + \langle d_e;G\rangle \mathrm{cov}(\bar{f},\bar{g};\mathcal{D}_e G) +$$

(2.8) $$+ \langle c_e;G\rangle\langle d_e;G\rangle(\langle\bar{f};\mathcal{C}_e G\rangle - \langle\bar{f};\mathcal{D}_e G\rangle)(\langle\bar{g};\mathcal{C}_e G\rangle - \langle\bar{g};\mathcal{D}_e G\rangle).$$

By the induction hypothesis, observing that $|E(\mathcal{C}_e G)| = |E(\mathcal{D}_e G)| = n-1$, we obtain from eq. (2.8)

(2.9) $$\mathrm{cov}(f,g;G) \geq \langle c_e;G\rangle\langle d_e;G\rangle(\langle\bar{f};\mathcal{C}_e G\rangle - \langle\bar{f};\mathcal{D}_e G\rangle)(\langle\bar{g};\mathcal{C}_e G\rangle - \langle\bar{g};\mathcal{D}_e G\rangle).$$

By definition,

$$\langle\bar{f};\mathcal{C}_e G\rangle = \frac{\langle\bar{f}\kappa^\gamma;\mathcal{C}_e G,P\rangle}{\langle\kappa^\gamma;\mathcal{C}_e G,P\rangle} = \frac{\sum_{C\subseteq E-e} f(C+e)\kappa^{\gamma(C+e)}p^C q^D}{\sum_{C\subseteq E-e} \kappa^{\gamma(C+e)}p^C q^D}, \quad D = E-e-C,$$

and by using I Lemma 2, $\gamma(C+e) = \gamma(C) - \delta_e(C)$, and the assumption $f(C+e) \geq f(C)$, this can be written as

$$\langle\bar{f};\mathcal{C}_e G\rangle \geq \frac{\sum_{C\subseteq E-e} f(C)\kappa^{-\delta_e(C)}\kappa^{\gamma(C)}p^C q^D}{\sum_{C\subseteq E-e}\kappa^{\gamma(C)}p^C q^D} \bigg/ \frac{\sum_{C\subseteq E-e}\kappa^{-\delta_e(C)}\kappa^{\gamma(C)}p^C q^D}{\sum_{C\subseteq E-e}\kappa^{\gamma(C)}p^C q^D} =$$

(2.10) $$= \langle\bar{f}\kappa^{-\delta_e};\mathcal{D}_e G\rangle / \langle\kappa^{-\delta_e};\mathcal{D}_e G\rangle.$$

The function δ_e is decreasing, so $\kappa^{-\delta_e}$ is increasing, by the assumption $\kappa \geq 1$, and by the induction hypothesis the lemma holds for $\mathcal{D}_e G$, so $\langle\bar{f}\kappa^{-\delta_e};\mathcal{D}_e G\rangle/\langle\kappa^{-\delta_e};\mathcal{D}_e G\rangle \geq \langle\bar{f};\mathcal{D}_e G\rangle$, and it follows from (2.10) and (2.7) that

(2.11) $$\langle\bar{f};\mathcal{C}_e G\rangle \geq \langle\bar{f};\mathcal{D}_e G\rangle.$$

Consequently, $(\langle\bar{f};\mathcal{C}_e G\rangle - \langle\bar{f};\mathcal{D}_e G\rangle)(\langle\bar{g};\mathcal{C}_e G\rangle - \langle\bar{g};\mathcal{D}_e G\rangle) \geq 0$, and hence from eq. (2.9) we obtain eq. (2.7), which completes the proof. ||

Using the relation between the Ising model and the random-cluster model, as described in I § 4.2, we easily obtain the second Griffiths-Kelly-Sherman inequality for a ferromagnetic Ising model with pair interactions only. For $\kappa=2$, and $p=1-\exp(-2\beta J)$, by eq. I (4.4), we have for $V'\subseteq V$ that $\langle\sigma^{V'}\rangle_{can} = \langle\varepsilon_{V'}\rangle$, where $\varepsilon_{V'}$ is the event that each c-cluster contains an even number of vertices of V' (including zero, possibly). So the GKS inequality $\langle\sigma^{V'}\sigma^{V''}\rangle_{can} \geq \langle\sigma^{V'}\rangle_{can}\langle\sigma^{V''}\rangle_{can}$, reads in terms of the random-cluster model $\langle\varepsilon_{V'\Delta V''}\rangle \geq \langle\varepsilon_{V'}\rangle\langle\varepsilon_{V''}\rangle$, where $V'\Delta V'' = (V'-V'')+(V''-V')$. It is sufficient to take disjoint sets V', V''.

Corollary Let $V',V''\subseteq V(G)$, then $\langle\varepsilon_{V'\Delta V''}\rangle \geq \langle\varepsilon_{V'}\rangle\langle\varepsilon_{V''}\rangle$.

Proof. If $\varepsilon_{V'}$ and $\varepsilon_{V''}$, obviously $\varepsilon_{V'\Delta V''}$. Thus $\langle\varepsilon_{V'\Delta V''}\rangle \geq \langle\varepsilon_{V'}\varepsilon_{V''}\rangle$. The functions $\varepsilon_{V'}$ are increasing, because if $\varepsilon_{V'}(C) = 1$, each cluster of G_C contains an even number of vertices of V', and therefore each cluster of G_{C+e} contains an even number of vertices of V', so $\varepsilon_{V'}(C+e) = 1$. Consequently, by Lemma 2, $\langle\varepsilon_{V'}\varepsilon_{V''}\rangle \geq \langle\varepsilon_{V'}\rangle\langle\varepsilon_{V''}\rangle$ and the corollary follows. ||

We have shown that the restriction of the range of p to $0 \leq p \leq 1$ and of the values of κ to $\kappa \geq 1$ are sufficient to guarantee the covariance inequality for all increasing functions on all finite graphs. We shall now show that there is no weaker condition on p and κ, independent of G, under which the covariance inequality holds for all increasing functions on all finite graphs. First, let G have one edge e with $p_e = p$. Taking $f = g = c_e$, we obtain $Z = \langle(c_e+d_e)\kappa^\gamma;P\rangle$ and $Z\langle c_e;\mu\rangle = \langle c_e\kappa^\gamma;P\rangle$. Zo $Z^2\mathrm{cov}(c_e,c_e) =$
$= \langle(c_e+d_e)\kappa^\gamma;P\rangle\langle c_e\kappa^\gamma;P\rangle - \langle c_e\kappa^\gamma;P\rangle^2 = \langle d_e\kappa^\gamma;P\rangle\langle c_e\kappa^\gamma;P\rangle = qp\kappa^{\gamma(\emptyset)+\gamma(e)}$.
If the covariance should be non-negative, it follows that (qp) and $\kappa^{\gamma(e)-\gamma(\emptyset)}$ must have the same sign. If e is a loop, $\gamma(e)-\gamma(\emptyset) = 0$, so it follows that this sign must be positive, hence $0 \leq p \leq 1$. If e is not a loop, $\gamma(e)-\gamma(\emptyset) = 1$, and therefore $\kappa \geq 0$. Secondly, let G have two edges, e and e', with $p_e=p$ and $p_{e'}=p'$. Taking $f=c_e$ and $g=c_{e'}$, we obtain $Z = \langle(c_e+d_e)(c_{e'}+d_{e'})\kappa^\gamma;P\rangle$, $Z\langle c_e;\mu\rangle = \langle c_e(c_{e'}+d_{e'})\kappa^\gamma;P\rangle$, $Z\langle c_{e'};\mu\rangle = \langle(c_e+d_e)c_{e'}\kappa^\gamma;P\rangle$ and $Z\langle c_ec_{e'};\mu\rangle = \langle c_ec_{e'}\kappa^\gamma;P\rangle$. So $Z^2\mathrm{cov}(c_e,c_{e'}) =$
$\langle(c_e+d_e)(c_{e'}+d_{e'})\kappa^\gamma;P\rangle\langle c_ec_{e'}\kappa^\gamma;P\rangle - \langle c_e(c_{e'}+d_{e'})\kappa^\gamma;P\rangle\langle(c_e+d_e)c_{e'}\kappa^\gamma;P\rangle =$

$= \langle d_e(c_{e'}+d_{e'})\kappa^\gamma;P\rangle\langle c_e c_{e'}\kappa^\gamma;P\rangle - \langle c_e(c_{e'}+d_{e'})\kappa^\gamma;P\rangle\langle d_e c_{e'}\kappa^\gamma;P\rangle =$
$= \langle d_e d_{e'}\kappa^\gamma;P\rangle\langle c_e c_{e'}\kappa^\gamma;P\rangle - \langle c_e d_{e'}\kappa^\gamma;P\rangle\langle d_e c_{e'}\kappa^\gamma;P\rangle =$
$= qq'pp'\kappa^{\gamma(\emptyset)+\gamma(e+e')} - pq'qp'\kappa^{\gamma(e)+\gamma(e')} =$
$= qq'pp'\kappa^{\gamma(e)+\gamma(e')}\left\{\kappa^{\gamma(e+e')+\gamma(\emptyset)-\gamma(e)-\gamma(e')} - 1\right\}$. In order that this covariance should be non-negative, taking into account that $0 \leq p \leq 1$ and $\kappa \geq 0$, we must have that $\kappa^{\gamma(e+e')+\gamma(\emptyset)-\gamma(e)-\gamma(e')} \geq 1$. If e and e' are parallel edges, i.e. both incident with the same different vertices, $\gamma(e+e') + \gamma(\emptyset) - \gamma(e) - \gamma(e') = 1$, and it follows that $\kappa \geq 1$. Consequently, we see that $0 \leq p \leq 1$ and $\kappa \geq 1$ is necessary and sufficient for the covariance inequality.

At this point it is convenient to introduce, in addition to p, another mapping from E to the real numbers, related to p and κ, namely $p_\kappa \equiv p/(p+q\kappa)$. We shall write $1 - p_\kappa \equiv q_\kappa$ and denote by P_κ the measure generated by p_κ. Obviously, p_κ is an increasing function of p for $\kappa > 0$, and a non-increasing function of κ for $0 \leq p \leq 1$. Further, if $0 \leq p \leq 1$ and $\kappa \geq 0$, then also $0 \leq p_\kappa \leq 1$.

Lemma 3 Let G be a finite graph and f a local variable on G. Then f is increasing if and only if for any two simple random-cluster models (G,p,κ) and (G,p',κ') with $p' \leq p$ and $p'_{\kappa'} \leq p_\kappa$,

(2.12) $\langle f;G,p',\kappa'\rangle \leq \langle f;G,p,\kappa\rangle$.

Proof. First, suppose that f is increasing, $p' \leq p$ and $p'_{\kappa'} \leq p_\kappa$. If $\kappa' \geq \kappa$, it follows from $p' \leq p$ that $p'_{\kappa'} \leq p_\kappa$, and if $\kappa' \leq \kappa$, it follows from $p'_{\kappa'} \leq p_\kappa$ that $p' \leq p$. So let first $\kappa' \geq \kappa$ and $p' \leq p$. Then $\langle f\rangle$ is non-increasing in κ because $\langle f\rangle = \langle f\kappa^\gamma;G,P\rangle/\langle\kappa^\gamma;G,P\rangle$, and thus

(2.13) $$\kappa\left(\frac{\partial}{\partial\kappa}\right)_p \langle f;G,p,\kappa\rangle = \frac{\langle f\gamma\kappa^\gamma;P\rangle}{\langle\kappa^\gamma;P\rangle} - \frac{\langle f\kappa^\gamma;P\rangle\langle\gamma\kappa^\gamma;P\rangle}{\langle\kappa^\gamma;P\rangle^2} = \mathrm{cov}(f,\gamma;G,p,\kappa) \leq 0,$$

by Lemma 2, f being increasing and γ decreasing. Furthermore, $\langle f\rangle$ is non-decreasing in p, because it is non-decreasing in each p_e. Indeed, by Lemma 1,

$$<f;G> = <c_e;G><\overline{f};\mathcal{C}_e G> + <d_e;G><\overline{f};\mathcal{D}_e G> =$$

$$= <\overline{f};\mathcal{D}_e G> + <c_e;G>(<\overline{f};\mathcal{C}_e G> - <\overline{f};\mathcal{D}_e G>) , \tag{2.14}$$

where the only p_e dependence is in $<c_e;G>$. Furthermore, $<c_e;G>$ is increasing in p_e, because by eq. (2.3) it is equal to $p_e Z(\mathcal{C}_e G)/Z(G)$ and $Z(G)$ is decreasing in p_e by II Prop. 1, and $(<\overline{f};\mathcal{C}_e G> - <\overline{f};\mathcal{D}_e G>)$ is non-negative by the proof of Lemma 2, in particular eq. (2.11). It follows that if $\kappa' \geq \kappa$ and $p' \leq p$, eq. (2.12) holds. In case $\kappa' \leq \kappa$ and $p'_{\kappa'} \leq p_\kappa$, we observe that $<f>$ is non-decreasing in κ at constant p_κ. Indeed,

$$<f;G,p,\kappa> = \frac{<f\kappa^\gamma;G,P>}{<\kappa^\gamma;G,P>} = \frac{<f\kappa^\omega;G,P_\kappa>}{<\kappa^\omega;G,P_\kappa>} , \tag{2.15}$$

by eq. I(7.26) with $x = p_\kappa$, $y = q_\kappa$, $\xi = 1$ and $\eta = \kappa$. It follows that

$$\kappa\left(\frac{\partial}{\partial\kappa}\right)_{p_\kappa} <f;G,p,\kappa> = \text{cov}(f,\omega;G,p,\kappa) \geq 0, \tag{2.16}$$

where the inequality holds because ω is increasing. So, if $p''_{\kappa'} = p_\kappa$, we have $<f;G,p,\kappa> \geq <f;G,p'',\kappa'>$, which is larger than $<f;G,p',\kappa'>$ because $p''_{\kappa'} = p_\kappa \geq p'_{\kappa'}$, so $p'' \geq p'$, and by the preceding part of the proof. It follows that if $\kappa' \leq \kappa$ and $p'_{\kappa'} \leq p_\kappa$, eq. (2.12) holds. Consequently, if $p' \leq p$ and $p'_{\kappa'} \leq p_\kappa$, then eq. (2.12) holds. On the other hand, suppose that for $p' \leq p$ and $p'_{\kappa'} \leq p_\kappa$ we have eq. (2.12). For any two sets C and C' such that $C' \subseteq C \subseteq E$ we consider the particular mappings p' and p defined by $p'_e = 1$ for $e \in C'$, $p'_e = 0$ for $e \notin C'$, $p_e = 1$ for $e \in C$ and $p_e = 0$ for $e \notin C$. Applying eq. (2.12) we obtain $f(C') = f(C')\kappa^{\gamma(C')}/\kappa^{\gamma(C')} =$ $= <f;G,p',\kappa> \leq <f;G,p,\kappa> = f(C)\kappa^{\gamma(C)}/\kappa^{\gamma(C)} = f(C)$, i.e. f is increasing. This completes the proof of the lemma. ||

In Lemma 3 we have shown that the bounds on p and κ, namely $0 \leq p \leq 1$ and $\kappa \geq 1$, are sufficient to guarantee that the expectation value of an increasing local variable is non-decreasing in p and is non-increasing in κ. We shall now show that again these bounds are necessary in order that eq. (2.12) holds for all increasing functions on all finite graphs.

First, let G have one edge e, which is not a loop. Taking $f = c_e$, we obtain $\langle c_e\kappa^{\gamma};G,P\rangle = p_e\kappa^{\gamma(e)}$ and $Z(G,P) = \langle (c_e+d_e)\kappa^{\gamma};G,P\rangle =$ $= p_e\kappa^{\gamma(e)} + q_e\kappa^{\gamma(\emptyset)} = p_e\kappa^{\gamma(e)} + q_e\kappa^{\gamma(e)+1}$, because e is not a loop. Therefore, $\langle c_e;G,p,\kappa\rangle = (1 + q_e p_e^{-1}\kappa)^{-1} = \{1 + (p_e^{-1}-1)\kappa\}^{-1}$. In order that $\langle c_e\rangle$ is non-decreasing in p, we must have $\kappa \geq 0$, and in order that $\langle c_e\rangle$ is non-increasing in κ, we must have $(p_e^{-1}-1) \geq 0$, so $0 \leq p_e \leq 1$.

Secondly, let G have two edges e,e', which are both incident with the same two vertices, i.e. e and e' are parallel edges. Taking $f = c_e$ again, we obtain $\langle c_e\kappa^{\gamma};G,P\rangle = \langle c_e(c_{e'} + d_{e'})\kappa^{\gamma};G,P\rangle =$ $p_e p_{e'}\kappa^{\gamma(e+e')} + p_e q_{e'}\kappa^{\gamma(e)} = p_e\kappa^{\gamma(e+e')}$, because e and e' are parallel, and $Z(G,P) = \langle (c_e + d_e)(c_{e'} + d_{e'})\kappa^{\gamma}\rangle = \kappa^{\gamma(e+e')} +$ $+ q_e q_{e'}(\kappa^{\gamma(\emptyset)} - \kappa^{\gamma(e+e')}) = \kappa^{\gamma(e+e')}\{1 + q_e q_{e'}(\kappa-1)\}$, thus $\langle c_e\rangle = p_e/\{1 + q_e q_{e'}(\kappa-1)\}$. In order that c_e is non-decreasing in p_e, we must have $\kappa-1 \geq 0$, so $\kappa \geq 1$, taking into account that $0 \leq p \leq 1$. Consequently, we see that we must have the bounds $0 \leq p \leq 1$ and $\kappa \geq 1$ on p and κ in order that eq. (2.12) holds.

Observe that if f is a local variable on a countable graph G, its values depend only on the states of a finite number of edges, forming the finite set E', say, i.e. $f(C;G) = f(C\cap E';G)$. So it follows that if we define a local variable f' on a subgraph $G'\subseteq G$ by $f'(C';G') \equiv f(C'\cap E';G)$, that then $f' = f$ for $E(G') \geq E'$. We shall make a frequent use of this extension of f, defined on G, to subgraphs of G, and hence, by the association procedure, to descendants of G.

Proposition 1 Let $(G,\mu) = (G,p,\kappa)$ be a simple random-cluster model, (G countable, $0 \leq p \leq 1$ and $\kappa \geq 1$) and let $G_1\subset G_2\subset G_3\ldots$ be an increasing sequence of finite subgraphs of G converging to G. Then the measure $\mu=(p,\kappa)$ exists, is a probability measure, and is independent of the sequence.

Proof. Let E' be a finite subset of $E(G)$ and let $f = c^{E'}$. Then f is an increasing local variable, and by Lemma 3, $\langle f';G_{n+1},p,\kappa\rangle \geq \langle \overline{f'};\mathcal{D}^{E_{n+1}-E_n}G_{n+1},p,\kappa\rangle$. By the definition of association and extension, if $C_n \subseteq E_n = E(G_n) = E(\mathcal{D}^{E_{n+1}-E_n}G_{n+1})$, $\overline{f'}(C_n;\mathcal{D}^{E_{n+1}-E_n}G_{n+1}) = f'(C_n;G_{n+1}) = f(C_n \cap E';G) = f'(C_n;G_n)$. Furthermore, the measures on $\mathcal{D}^{E_{n+1}-E_n}G_{n+1}$ and G_n are equal, because $\gamma(C_n;\mathcal{D}^{E_{n+1}-E_n}G_{n+1}) = \gamma(C_n;G_n) + |V(G_{n+1}) - V(G_n)|$, as is evident from the fact that these graphs only differ by the isolated vertices of $V(G_{n+1}) - V(G_n)$, and the definition of μ(I § 7.1). It follows that $\langle f';G_{n+1},p,\kappa\rangle \geq \langle f';G_n,p,\kappa\rangle$, so $\langle f';G_n,p,\kappa\rangle$ is a non-decreasing sequence in n, obviously bounded, so the limit exists. Moreover, if G'_n is another increasing sequence converging to G, we can construct an increasing sequence G''_n consisting alternatingly of subgraphs G_n and subgraphs G'_n, and hence converging to G. Therefore, $\lim_n\langle f;G''_n\rangle = \lim_n\langle f;G_n\rangle = \lim_n\langle f;G'_n\rangle \equiv \langle f;G\rangle$, by definition, i.e. $\langle c^{E'};G\rangle$ exists for all finite $E' \subset E(G)$ and is independent of the sequence. Consequently, the measure μ exists for the events $c^{E'}$, so for local events, and hence for all random events, and is independent of the sequence. Finally, because μ_n is a probability, μ is obviously a probability measure. ||

Now we are in a position to extend the recursion property and the covariance inequality, as well as other properties, to infinite simple random-cluster models.

Theorem 1 Recursion theorem: Let (G,p,κ) be a simple random-cluster model, $e \in E$ and f an integrable random variable. Then

$$\langle f;G\rangle = \langle c_e;G\rangle\langle f;\mathcal{C}_e G\rangle + \langle d_e;G\rangle\langle f;\mathcal{D}_e G\rangle. \tag{2.17}$$

Proof. First, let f be a local variable. Then, by definition, $\langle f\rangle = \Sigma_{i=1}^n f_i\langle a_i\rangle$, where a_i is the local event $f = f_i$. Hence, by Prop. 1, $\langle a_i;G\rangle = \lim_n\langle a_i;G_n\rangle$, and consequently, $\langle f;G\rangle = \lim_n\langle f;G_n\rangle$. By Lemma 1, $\langle f;G_n\rangle = \langle c_e;G_n\rangle\langle f;\mathcal{C}_e G_n\rangle + \langle d_e;G_n\rangle\langle f;\mathcal{D}_e G_n\rangle$, so $\langle f;G\rangle = \lim_n\langle f;G_n\rangle = \lim_n\langle c_e;G_n\rangle\lim_n\langle f;\mathcal{C}_eG_n\rangle +$ $+ \lim_n\langle d_e;G_n\rangle\lim_n\langle f;\mathcal{D}_eG_n\rangle = \langle c_e;G\rangle\langle f;\mathcal{C}_eG\rangle + \langle d_e;G\rangle\langle f;\mathcal{D}_eG\rangle$, by Prop. 1 and the preceding remark. Notice that $\lim_n\mathcal{D}_eG_n = \mathcal{D}_eG$ and $\lim_n\mathcal{C}_eG_n = \mathcal{C}_eG$.

Let L_+ be the collection of all non-negative local variables, let $\overline{L_+}^1$ be the collection of all non-negative functions which are the infimum or the supremum of a countable subcollection of L_+, and let $\overline{L_+}^{k+1}$ be the collection of all suprema and infima of countable subcollections of $\overline{L_+}^k$, with $k = 1,2,\ldots$. Then it follows that $\overline{L_+}^k$ is non-decreasing in k, converging to the collection of non-negative random variables, to be denoted $\overline{L_+}$. Moreover, one verifies that $\overline{L_+}^k$ is closed under suprema and infima of finite subcollections. We have just proved that eq. (2.17) applies to $f\in L_+$. We shall prove that eq. (2.17) applies to $f\in\overline{L_+}$ by proving that it applies to $f\in\overline{L_+}^{k+1}$, assuming that it applies to $\overline{L_+}^k$.

By definition, if $f\in\overline{L_+}^{k+1}$, either $f = \sup_n f_n$ or $f = \inf_n f_n$ with $f_n\in\overline{L_+}^k$. Furthermore, $\sup_n f_n = \lim_{n'}\sup_{n<n'}f_n$ and $\inf_n f_n =$ $= \inf_{n'}\inf_{n\leq n'}f_n$, and these limits are obtained monotonically (non-decreasing and non-increasing respectively). By the preceding remarks, $\sup_{n\leq n'}f_n$ and $\inf_{n\leq n'}f_n$ belong to $\overline{L_+}^k$, and hence, by the integration theorem on monotone sequences and the assumption, $\langle\sup_n f_n;G\rangle = \lim_{n'}\langle\sup_{n\leq n'}f_n;G\rangle = \lim_{n'}(\langle c_e;G\rangle\langle\sup_{n\leq n'}f_n;\mathcal{C}_eG\rangle +$ $+ \langle d_e;G\rangle\langle\sup_{n\leq n'}f_n;\mathcal{D}_eG\rangle) = \langle c_e;G\rangle\langle\sup_n f_n;\mathcal{C}_eG\rangle + \langle d_e;G\rangle\langle\sup_n f_n;\mathcal{D}_eG\rangle$. Analogously we derive eq. (2.17) for $\inf_n f_n$ if at least one of the f_n is bounded. If all f_n are not bounded, $\inf_n f_n$ can be obtained as the supremum of the monotonically increasing sequence of functions $g_n = \inf\{f,n\}$, which obviously belong to $\overline{L_+}^{k+1}$ and are all bounded, and therefore satisfy eq. (2.17). Repeating the previous argument for suprema we prove eq. (2.17) for $\inf_n f_n = f =$ $\sup_n g_n$, for the case where all f_n are not bounded. Consequently, if $f\in\overline{L_+}^{k+1}$, eq. (2.17) applies to f, and hence if $f\in\overline{L_+}$, eq. (2.17) applies to f.

Finally, if f is an integrable random variable, $f = f^+ - f^-$ where not both f^+ and f^- are not summable, $f^+,f^-\in\overline{L_+}$. Hence, $\langle f^+;G\rangle =$ $\langle c_e\rangle\langle f^+;\mathcal{C}_eG\rangle + \langle d_e\rangle\langle f^+;\mathcal{D}_eG\rangle$ and $\langle f^-;G\rangle = \langle c_e\rangle\langle f^-;\mathcal{C}_eG\rangle + \langle d_e\rangle\langle f^-;\mathcal{D}_eG\rangle$, and, without loss of generality, $<f^-;G>$ is finite, say. So we may subtract them and collect the terms: $<f;G> = <f^+;G> - <f^-;G> = = <c_e><f^+;\mathcal{C}_eG> + <d_e><f^+;\mathcal{D}_eG> - <c_e><f^-;\mathcal{C}_eG> - <d_e><f^-;\mathcal{D}_eG> = = <c_e><f;\mathcal{C}_eG> + <d_e><f;\mathcal{D}_eG>$. It follows that eq. (2.17) applies to all integrable random variables. ||

Notice that in the same way one may prove for a finite set $E'\subseteq E$ and integrable random variables f that

(2.18) $$<f;G> = \sum_{C'\subseteq E'} <c^{C'}d^{D'};G><f;\mathcal{C}^{C'}\mathcal{D}^{D'}G> , \quad C' + D' = E'.$$

Functions obtained by closing the collection of non-negative increasing (decreasing) local variables are called <u>non-negative random increasing (decreasing) variables</u>. The difference between a non-negative random increasing variable and a non-negative random decreasing variable, not both assuming a value $\neq 0$ at the same time, is called a <u>random increasing variable</u>. Notice that a random increasing variable is an increasing random variable, but that the converse may not be true.

Theorem 2 <u>Covariance inequality</u>: Let (G,μ) be a simple random-cluster model, f and g random increasing (or decreasing) variables which are non-negative or μ-summable. Then

(2.19) $$<fg;G,\mu> \geq <f;G,\mu><g;G,\mu>.$$

Proof. The proof of Theorem 2, starting from Lemma 2, is quite analogous to the proof of Theorem 1, starting from Lemma 1. First one proves that for increasing (or decreasing) local variables eq. (2.19) applies, by Prop. 1 and Lemma 2. Secondly, we start with the collection of non-negative increasing (or decreasing) local variables, i.e. the collection $L_+\cap I$ (or $L_+\cap D$) where I (or D) is the collection of increasing (or decreasing) functions. Notice that I(or D) is closed under countable suprema and infima, and that if $f\in I$(or D) also inf{f,n} (or sup{f,n}) is an element of I(orD).

In the same way as in the proof of Th. 1 we can prove that eq. (2.19) applies to $f,g \in \overline{L_+ \cap I}$ (or $\overline{L_+ \cap D}$), i.e. to non-negative random increasing (or decreasing) variables. Finally, if f and g are summable random increasing variables, $f = f^+ - f^-$ and $g = g^+ - g^-$, where $f^+, g^+ \in \overline{L_+ \cap I}$ and $f^-, g^- \in \overline{L_+ \cap D}$. Therefore, $<(fg)^+> = <(f^+g^+ + f^-g^-)> \geq <f^+><g^+> + <f^-><g^->$, and $-<(fg)^-> = <f^+(-g^-) + (-f^-)g^+> \geq - <f^+><g^-> - <f^-><g^+>$, where $<(fg)^->$ is finite by the summability of f and g. Hence we may add the two inequalities and obtain $<fg> = <(fg)^+> - <(fg)^-> \geq (<f^+>-<f^->)(<g^+>-<g^->) = <f><g>$, so it follows that eq. (2.19) applies to summable random increasing (or decreasing) variables. ||

Proposition 2 Let $(G,\mu) = (G,p,\kappa)$ and $(G,\mu') = (G,p',\kappa')$ be simple random-cluster models and f a random increasing variable on G. If $p' \leq p$, $p'_{\kappa'} \leq p_\kappa$ and either f is non-negative or f^+ is μ-summable or f^- is μ'-summable, then

(2.20) $$<f;G,\mu'> \leq <f;G,\mu>.$$

Proof. The proof for non-negative f is quite analogous to the proofs of Th. 1,2, in this case starting with Lemma 3. If f is a random increasing variable, $f = f^+ - f^-$, where $f^+, -f^-$ are increasing. Hence, $<f^+;\mu'> \leq <f^+;\mu>$ and $-<f^-;\mu'> \leq -<f^-;\mu>$. By assumption, either f^+ is μ-summable, so f^+ is also μ'-summable, or f^- is μ'-summable, so f^- is also μ-summable. So we may add the inequalitities to $<f;\mu'> = <f^+;\mu'> - <f^-;\mu'> \leq <f^+;\mu> - <f^-;\mu> = <f;\mu>$. ||

Finally, we observe that by the proof of Prop. 1, if f is an increasing local variable, $<f;G> = \lim_n <f;G_n> = \sup_n <f;G_n>$. If $f = \sup_n f_n$, where the f_n are increasing local variables, we can choose the f_n in such a way that f is the limit of a monotonically increasing sequence of increasing local variables, and thus we obtain by the integration theorem on monotone sequences that $<f;G> = <\sup_n f_n;G> = \sup_n <f_n;G> = \sup_n \sup_{n'} <f'_n;G_{n'}> = \sup_n <f'_n;G_n> =$

$= \sup_n \sup_{n'} \langle f'_{n'};G_n\rangle = \sup_n \langle f';G_n\rangle$. Using this it follows that most of the lemmas in II § 2 apply to the simple random-cluster model.

Lemma 4 Let (G,p,κ) be a simple random-cluster model. Then the Lemmas 1,2,4,5 and 7 in II § 2 apply to it.

Corollary $\langle \gamma_{vv'};G,p,\kappa\rangle = \lim_n \langle \gamma_{vv'};G_n,p,\kappa\rangle$.

One notices that for the indicators $\varepsilon_{V'}$, where V' is a finite subset of V, we have by the same reasoning as applied in II lemma 5 to $\gamma_{vv'} = \varepsilon_{vv'}$, that $\varepsilon_{V'} = \sup_n \varepsilon_{V'}^{G_n}$. Hence, we have analogously to eq. II (2.10)

$$\langle \varepsilon_{V'};G,p,\kappa\rangle = \lim_n \langle \varepsilon_{V'};G_n,p,\kappa\rangle, \tag{2.21}$$

which in terms of the ferromagnetic Ising model means that $\langle \sigma^{V'};G\rangle_{can} = \lim_n \langle \sigma^{V'};G_n\rangle_{can}$.

3. LARGE-RANGE CONNECTIVITY

In the previous section we have extended some of the basic properties of the percolation model to the simple random-cluster model. We are now in a position to extend the results on large-range connectivity in the percolation model, as given in II § 4, to the simple random-cluster model.

First we notice that $\gamma_{vv'} = \lim_n (\gamma_{vv'}$ in $G_n)$, by II Lemma 5, and it follows that $\gamma_{vv'}$ is a random increasing variable, because $(\gamma_{vv'}$ in $G_n)$ is an increasing local variable. Secondly, we have in bilocally finite graphs that $\gamma_v^\infty = \inf_n \gamma_{vU_n}$ with $U_n = V-V_n$, by II Lemma 7, and analogously to II Lemma 5 we also have $\gamma_{vU_n} = \lim_{n'} (\gamma_{v(V_{n'}-V_n)}$ in $G_{n'})$. Therefore, in bilocally finite graphs γ_v^∞ is a random increasing variable, and hence γ_v^f is a random decreasing variable. We recall that a bilocally finite graph is a graph such that for all pairs of vertices $v,v' \in V(G)$ the number of edges incident with both v and v' is finite. In case the graph is not bilocally finite, we can only prove that γ_v^∞ is an increasing random variable, and the covariance inequality may not be applied. (See however the Appendix)

By inspection of the proofs in II §4 one will see that, except for the proofs of II Lemma 9 and II Prop. 6, we only need the covariance inequality, apart from general measure and integration theorems, in order to prove the lemmas, propositions and theorems in II §4. But we have just shown that the covariance inequality holds in a simple random-cluster model and can be applied to $\gamma_{vv'}$ and γ_v^∞, provided the graph is bilocally finite. In the proof of II Lemma 9, we need the assertion that for a finite subset $E' \subseteq E$ we have $\langle c^{E'} \rangle > 0$ if and only if $p^{E'} > 0$.

In order to complete the proof of II Lemma 9 in the case of the simple random-cluster model, we shall deduce the following lemma, in which again $p_\kappa = p/(p+q\kappa)$ and $q_\kappa = 1-p_\kappa = q/(q+p\kappa^{-1})$.

Lemma 5 Let (G,p,κ) be a simple random-cluster model and E' a subset of $E(G)$. Then,

(3.1) $p_\kappa^{E'} \leq \langle c^{E'};G,p,\kappa\rangle \leq p^{E'}$,

(3.2) $q^{E'} \leq \langle d^{E'};G,p,\kappa\rangle \leq q_\kappa^{E'}$.

Proof. If E' is infinite, $\langle c^{E'}\rangle = \lim_n \langle c^{E'\cap E_n}\rangle$. Therefore, it is sufficient to prove the lemma for finite E'. By Prop. 1, $\langle c^{E'};G\rangle = \lim_n \langle c^{E'};G_n\rangle$, so it is sufficient to prove the lemma holds in a finite graph G_n such that $E' \subseteq E(G_n)$. So let G be finite. By the recursion relation and the definition of $\langle c^{E'}\rangle$, we have

(3.3) $$\langle c^{E'};G,p,\kappa\rangle = \frac{\sum_{C\subseteq E} c^{E'}(C)\,p^C q^D \kappa^{\gamma(C)}}{\sum_{C\subseteq E} p^C q^D \kappa^{\gamma(C)}} = \frac{p^{E'} \sum_{C''\subseteq E''} p^{C''} q^{D''} \kappa^{\gamma(C''+E')}}{\sum_{C'\subseteq E'} p^{C'} q^{D'} \sum_{C''\subseteq E''} p^{C''} q^{D''} \kappa^{\gamma(C''+C')}},$$

where C+D=E, E'+E"=E, C'+D'=E', C"+D"=E". By I Lemma 2, $\gamma(C+e) = \gamma(C) - \delta_e(C)$, and hence $\gamma(C+e) \leq \gamma(C) \leq \gamma(C+e) + 1$. Repeating the argument we obtain $\gamma(C''+E') \leq \gamma(C''+C') \leq \gamma(C''+E') + |D'|$. Substituting this in eq. (3.3), we obtain the inequalities

$$p^{E'} \left(\sum_{C' \subseteq E'} p^{C'} q^{D'} \kappa^{D'} \right)^{-1} = p_\kappa^{E'} \leq \langle c^{E'} \rangle \leq p^{E'}, \tag{3.4}$$

from which eq. (3.1) follows. Analogously one proves eq. (3.2), observing that $\gamma(C) - 1 \leq \gamma(C+e) \leq \gamma(C)$ and hence $\gamma(C'') - |C'| \leq$ $\leq \gamma(C''+C') \leq \gamma(C'')$. ||

Corollary If E' is finite, then $\langle c^{E'} \rangle > 0$ if and only if for all $e \in E'$ we have $p_e > 0$. ||In order to complete the proof of II Prop. 6 in the case of the simple random-cluster model, we use the following lemma which forms the link between the eqs. II (4.24) and II (4.25).

Lemma 6 Let (G,μ) be a simple random-cluster model $e \in E(G)$, and let the induced measures on the event spaces of $\mathcal{C}_e G$ and $\mathcal{D}_e G$ be denoted by $\mu(\mathcal{C}_e G)$ and $\mu(\mathcal{D}_e G)$. Then

$$\kappa^{-1} \mu(\mathcal{D}_e G) \leq \mu(\mathcal{C}_e G) \leq \kappa \mu(\mathcal{D}_e G). \tag{3.5}$$

Proof. First, let G be finite. By definition, for any $C \subseteq E-e$, $D = E-e-C$, we have

$\mu(C;\mathcal{C}_e G) \equiv p^C q^D \kappa^{\gamma(C;\mathcal{C}_e G)} / Z(\mathcal{C}_e G) = p^C q^D \kappa^{\gamma(C+e;G)} / Z(\mathcal{C}_e G)$, and

$\mu(C;\mathcal{D}_e G) \equiv p^C q^D \kappa^{\gamma(C;\mathcal{D}_e G)} / Z(\mathcal{D}_e G) = p^C q^D \kappa^{\gamma(C;G)} / Z(\mathcal{D}_e G)$.

Furthermore, from II Lemma 2, $\gamma(C;G) - 1 \leq \gamma(C+e;G) \leq \gamma(C;G)$, and consequently, for $\kappa \geq 1$, $\kappa^{-1} \kappa^{\gamma(C;G)} \leq \kappa^{\gamma(C+e;G)} \leq \kappa^{\gamma(C;G)}$ and thus $\kappa^{-1} Z(\mathcal{D}_e G) \leq Z(\mathcal{C}_e G) \leq Z(\mathcal{D}_e G)$. It follows that $\kappa^{-1} \mu(C;\mathcal{D}_e G) \leq$ $\leq \mu(C;\mathcal{C}_e G) \leq \kappa \mu(C;\mathcal{D}_e G)$, and, consequently, that for all events on $\mathcal{C}_e G(\mathcal{D}_e G)$ eq. (3.5) holds. If G is infinite countable, the measure of a local event is the limit of the measures on finite graphs, and therefore eq. (3.5) also holds for local events on infinite countable graphs, and consequently eq. (3.5) holds for all random events on $\mathcal{C}_e G$ ($\mathcal{D}_e G$). ||

Notice that for $\kappa = 1$ the equalities hold, as is obvious.

From the preceding considerations, and by inspection of the proofs in II § 4, we obtain the following theorem, embodying the results on large-range connectivity in the simple random-cluster model.

Theorem 3 Let (G,μ) be a bilocally finite simple random-cluster model (G bilocally finite, $0 \leq p \leq 1$, $\kappa \geq 1$). Then all the lemmas, propositions and theorems of II § 4 hold for (G,μ). ||

One should notice that the condition that the graph should be bilocally finite, is not severe. It only excludes an infinite number of loops at the same vertex, and an infinite number of parallel edges between the same pair of vertices. So if V is infinite countable, it does not exclude the complete graph (V,E,i) where there is one edge between each pair of vertices.

The last theorem, together with Prop. 2, enables us to study the question whether the simple random-cluster model exhibits a phase transition or not. To that end, we first observe that if we change μ continuously in such a way that both p and p_κ do not decrease, the functions $<\gamma_v^\infty;\mu>$ and $<\gamma_{vv'};\mu>$ do not decrease, by Prop. 2. Furthermore, at p=0, both $<\gamma_v^\infty>$ and $<\gamma_{vv'}>$ are zero, so we have not W_v, and at p=1, both $<\gamma_v^\infty>$ and $<\gamma_{vv'}>$ are 1, so we have S'. It follows from Prop. 2 and the propositions II Prop. 2,3,4,5, that if the measure μ changes from p=0 to p=1 in the way described above, we will successively have the following types of large-range connectivity: (1) not W_v, (2) W_v but not W', (3) W' but not S' and (4) S'. We may regard these four incompatible types of large-range connectivity as four phases of the model. As is pointed out, at least the first and last phase are always present in an infinite bilocally finite simple random-cluster model. On the other hand, if the model has a high degree of regularity, e.g. all vertices are equivalent, W_v is equivalent to W, i.e. we do not have the phase of type (2). Moreover, if $\lim_{v'\in V'}\langle\gamma_v^\infty\delta_{vv'},\gamma_{v'}^\infty\rangle = 0$, W' and S' are equivalent, by II Th. 2, and we do not have the phase of type (3). We shall now extend to the random-cluster model the arguments used in the analysis of the percolation model and in the Ising model to show that in certain graphs there is, or is not, large-range connectivity. These arguments are of three types. First, we have the argument showing that there is a $p \neq 0$ such that for $p' \leq p$ we have not W_v, i.e. $\langle\gamma_v^\infty\rangle = 0$. This argument was first used by Hammersley [3] for the percolation model and later on by Fisher [4] for the Ising model, and essentially uses minimal connecting sets (self-avoiding walks). Secondly, we have the argument using isolating sets (boundaries) showing that there is a $p \neq 1$ such that for $p' \geq p$ we have W, i.e. $\langle\gamma_v^\infty\rangle > 0$. This argument was first used by Peierls [5] for the Ising model, and made rigorous by Griffiths [6] and Dobrushin [7], and later on, independently, by Hammersley [8] for the percolation model. Finally, we shall give an argument, which is related with both preceding ones and essentially uses parts of disconnecting sets, which shows that there is a $p \neq 1$ such that for $p' \geq p$ $\langle\gamma_v^\infty\delta_{vv'},\gamma_{v'}^\infty\rangle = 0$, and hence, using the preceding arguments, shows that there is strong large-range connectivity S'. Obviously, the values of p to be found depend on the graph under consideration, and can only be established in graphs with a regular structure. We shall only give the general arguments, and a calculation for the square lattice (for which the calculations are simple).

Before going on we shall define a few types of edge sets and collections of edge sets. First, let G' be a connected subgraph of G, then we shall call the set of edges of G' the <u>connecting set</u> of G'. The collection of all minimal connecting sets of connected subgraphs of G containing the vertices v and v' is denoted by $\mathcal{C}_{vv'}(G)$. Obviously, a set of edges belongs to $\mathcal{C}_{vv'}(G)$ if and only if it is the set of edges of a minimal path between v and v' in G (or a vertex-disjoint path, or a self-avoiding walk). The collection of connecting sets $\cup_{v'\in V}\mathfrak{C}_{vv'}(G)$ is denoted by $\mathfrak{C}_v(G)$. Further, a set of edges E' is called a <u>disconnecting set</u> of G if the number of clusters of $\mathfrak{D}^{E'}G$ is larger than the number of clusters of G, more precisely, if there is an equivalence class of the vertices of G under the relation of connection in G which is not an equivalence class in $\mathfrak{D}^{E'}G$. Obviously, a disconnecting set is minimal if and only if there are two equivalence classes of vertices of $\mathfrak{D}^{E'}G$ such that all edges of E' are incident with a vertex of both equivalence classes. We shall call the set of edges of G which are incident both with a vertex of G' and with a vertex not in G' the <u>isolating set</u> of G'. An isolating set is either a disconnecting set or the empty set. The collection of all isolating sets of finite connected subgraphs of G containing v is denoted by $\mathfrak{D}_v(G)$. Finally, if G_n is a finite subgraph of G, we shall call a disconnecting set E' of G_n such that v and v' belong to different clusters of $\mathfrak{D}^{E'}G_n$, both containing a vertex of the vertex-boundary B_n of G_n in G, a <u>separating set</u> between v and v' of G_n in G. The collection of minimal separating sets between v and v' of G_n in G is denoted by $\mathfrak{D}_{vv'}(G_n,G)$.

Proposition 3 Let (G,p,κ) be a bilocally finite simple random-cluster model and $v,v'\in V$.

(a) If $\sum_{E'\in\mathfrak{C}_v(G)} p^{E'}<\infty$, then $\langle\gamma_v^\infty;G,p',\kappa'\rangle = 0$ for $p'\leq p$ and $p'_{\kappa'}\leq p_\kappa$,

(b) If $q^{E'}<1$ for all $E'\in D_v(G)$ and $\sum_{E'\in\mathfrak{D}_v(G)} q_\kappa^{E'}<\infty$,

then $\langle\gamma_v^\infty;G,p',\kappa'\rangle > 0$ for $p'\geq p$ and $p'_{\kappa'}\geq p_\kappa$,

(c) If $\lim_n \sum_{E'\in\mathfrak{D}_{vv'}(G_n,G)} q_\kappa^{E'} = 0$,

then $\langle\gamma_v^\infty\delta_{vv'},\gamma_{v'}^\infty;G,p',\kappa'\rangle = 0$ for $p'\geq p$ and $p'_{\kappa'}\geq p_\kappa$.

Proof. (a) By II Lemma 7, $\gamma_v^\infty=\inf_n\gamma_{vU_n}$, so obviously $\gamma_v^\infty \leq \inf_n\Sigma_{v'\in U_n}\gamma_{vv'}$. By definition, $\gamma_{vv'} = \sup_{E'\in\mathfrak{C}_{vv'}} c^{E'}$, and so we obtain, using Lemma 5,

(3.6) $$\langle\gamma_v^\infty\rangle \le \inf_n \sum_{v'\in U_n} \sum_{E'\in\mathfrak{C}_{vv'}} p^{E'} .$$

By assumption, $\Sigma_{E'\in\mathfrak{C}_v} p^{E'} < \infty$, so by the definition of $\mathfrak{C}_v$ we have $\Sigma_{v'\in V}\Sigma_{E'\in\mathfrak{C}_{vv'}} p^{E'} < \infty$. Hence, $\inf_n \Sigma_{v'\in U_n}\Sigma_{E'\in\mathfrak{C}_{vv'}} p^{E'} = 0$, because $U_n = V-V_n$ decreases to the empty set, and it follows that $\langle\gamma_v^\infty\rangle = 0$ from eq. (3.6). So by Prop. 2 and $\langle\gamma_v^\infty\rangle \ge 0$, part (a) follows.

(b) We notice that $\gamma_v^f \le \sup_{E'\in\mathfrak{D}_v} d^{E'}$, by the definition of $\mathfrak{D}_v$. Let $\mathfrak{D}'$ be a finite subset of $\mathfrak{D}_v$, and denote $\mathfrak{D}_v-\mathfrak{D}'$ by $\mathfrak{D}''$. Obviously, $(1-d^{E'})^{\mathfrak{D}'} = \Pi_{E'\in\mathfrak{D}'}(1-d^{E'})$ and $(1-d^{E'})^{\mathfrak{D}''}$ are random increasing variables, and thus by a repeated use of the covariance inequality

(3.7) $$\langle(1-d^{E'})^{\mathfrak{D}'}(1-d^{E'})^{\mathfrak{D}''}\rangle \ge \langle(1-d^{E'})\rangle^{\mathfrak{D}'}\langle(1-d^{E'})^{\mathfrak{D}''}\rangle .$$

Evidently, $(1-d^{E'})^{\mathfrak{D}'}(1-d^{E'})^{\mathfrak{D}''} = (1-d^{E'})^{\mathfrak{D}'\cup\mathfrak{D}''} = (1-d^{E'})^{\mathfrak{D}_v} = \inf_{E'\in\mathfrak{D}_v}(1-d^{E'}) = 1-\sup_{E'\in\mathfrak{D}_v} d^{E'} \le \gamma_v^\infty$, by the first remark, and thus we obtain from eq. (3.7)

(3.8) $$\langle\gamma_v^\infty\rangle \ge (1-\langle d^{E'}\rangle)^{\mathfrak{D}'}(1-\langle\sup_{E'\in\mathfrak{D}''} d^{E'}\rangle).$$

By assumption, $q^{E'} < 1$ for $E'\in\mathfrak{D}_v$, so $\langle d^{E'}\rangle < 1$ for $E'\in\mathfrak{D}_v$ and by the finiteness of $\mathfrak{D}'$ we have $(1-\langle d^{E'}\rangle)^{\mathfrak{D}'} > 0$. Obviously, $1-\langle\sup_{E'\in\mathfrak{D}''} d^{E'}\rangle \ge 1-\Sigma_{E'\in\mathfrak{D}''} q_\kappa^{E'}$, and this can be chosen to be >0, because $\Sigma_{E'\in\mathfrak{D}_v} q_\kappa^{E'} < \infty$, by assumption, and hence for $\mathfrak{D}'$ large enough $\Sigma_{E'\in\mathfrak{D}''} q_\kappa^{E} < 1$. It follows that we then have $\langle\gamma_v^\infty\rangle > 0$, by eq. (3.8), which proves (b).

(c) We observe that $\delta_{vv'} = \inf_n(\delta_{vv'} \text{ in } G_n)$ by II Lemma 5. Furthermore, $\gamma_v^\infty \le \liminf_v(\gamma_{vB_n} \text{ in } G_n)$, because if γ_v^∞, the c-cluster containing v is infinite and hence contains a vertex of the boundary of G_n in G, as soon as G_n contains v. Consequently, $\gamma_v^\infty\delta_{vv'}\gamma_{v'}^\infty \le \liminf_n(\gamma_{vB_n}\delta_{vv'}\gamma_{v'B_n} \text{ in } G_n)$, which by the definition of $\mathfrak{D}_{vv'}(G_n,G)$ is $\le \liminf_n \sup_{E'\in\mathfrak{D}_{vv'}(G_n,G)} d^{E'}$. Obviously, we therefore have

(3.9) $$\langle\gamma_v^\infty\delta_{vv'}\gamma_{v'}^\infty\rangle \le \liminf_n \sum_{E'\in\mathfrak{D}_{vv'}(G_n,G)} q_\kappa^{E'} = 0,$$

by assumption. ||

We shall show that in the square lattice we can find mappings p such that the conditions in (a), (b) and (c) are satisfied. For (a) and (b) this amounts to nothing but reproducing, in a generalized sense, the proofs in the percolation, or Ising model. First we choose all $p_e = p$. An upperbound for $\Sigma_{E'\in\mathcal{C}_V(G)} p^{E'}$ is provided by taking the summation over all, not necessarily vertex-disjoint, paths with initial vertex v. Hence, $\Sigma_{E'\in\mathcal{C}_V(G)} p^{E'} \leq \Sigma_{n=0}^{\infty}(4p)^n = (1-4p)^n$, which is $<\infty$ provided $p<\frac{1}{4}$.

In order to bound $\Sigma_{E'\in\mathfrak{D}_V(G)} q_\kappa^{E'}$, one notices that to each isolating edge set of $\mathfrak{D}_V$ in the square lattice there corresponds in the dual lattice (which is again a square lattice) a path between coinciding vertices, enclosing the face corresponding to v. If such a closed path has length n, the number of enclosed faces cannot be larger than $(\frac{1}{4}n)^2$, and hence the number of closed paths of length n with the same shape enclosing the given face cannot exceed $n^2/16$. Consequently, $\Sigma_{E'\in\mathfrak{D}_V} q_\kappa^{E'}$ can be bound by all paths with a given initial vertex, each one with a suitable multiplicity. So, $\Sigma_{E'\in\mathfrak{D}_V} q_\kappa^{E'} \leq \Sigma_{n=0}^{\infty} \frac{1}{16} n^2 (4q_\kappa)^n$ which is finite for $q_\kappa < \frac{1}{4}$, and hence for $p > (1+(3\kappa)^{-1})^{-1}$.

For an upperbound of $\Sigma_{E'\in\mathfrak{D}_{VV'}(G_n,G)} q_\kappa^{E'}$, we observe that each element of $\mathfrak{D}_{VV'}(G_n,G)$ corresponds to a path between two vertices of the vertex boundary of the dual of G_n, such that it separates the faces corresponding to v and v'. We can choose the sequence G_n as an increasing sequence of rectangular subgraphs of the square lattice containing v and v'. In that case, if a(n) is the smallest of the distances between v or v' and the vertex boundary B_n of G_n in G, that path contains at least 2 a(n) edges. Consequently, the union over the vertices of B_n of the paths with initial vertex v' and length at least 2 a(n) is larger than $\mathfrak{D}_{VV'}(G_n,G)$, and we have $\Sigma_{E'\in\mathfrak{D}_{VV'}(G_n)} q_\kappa^{E'} \leq |B_n| \Sigma_{k=2\cdot a(n)}^{\infty} (4q_\kappa)^k = |B_n|(4q_\kappa)^{2a(n)}(1-4q_\kappa)^{-1}$, provided $4q_\kappa<1$. We can choose the sequence G_n such that $|B_n|\leq b\cdot a(n)$, where b is a constant, and it follows that $\inf_n \Sigma_{E'\in\mathfrak{D}_{VV'}(G_n,G)} q_\kappa^{E'} < b' \inf_n a(n)(4q_\kappa)^{2a(n)} = 0$ if $4q_\kappa < 1$ or $p > (1+(3\kappa)^{-1})^{-1}$.

Using II Prop.'s 3 and 4 and II Th. 3, we can summarize our results for the square lattice as follows: (a) if $p<\frac{1}{4}$, then we have not

W_v, and thus the phase of type (1); (b) if $p > (1+(3\kappa)^{-1})^{-1}$, then W_v and $\langle\gamma_v^\infty\delta_{vv'},\gamma_{v'}^\infty\rangle = 0$, and hence we have S', and thus the phase of type (4).

4. THE SUPPLEMENTARY VERTEX

In this section we shall extend part of the results of II § 5 to the simple random-cluster model. In this case, however, as contrasted with the case discussed in § 3, the proofs given in II § 5 do not hold as such for the random-cluster model. In particular, one notices that the expectation value of a random variable which does not depend on the state of the supplementary edges, is in general not equal to the expectation value of that random variable for $p_o = 0$, i.e. when there is no supplementary edge. This fact will for example break down the proofs of II Prop. 7 and II Th. 4. Furthermore, the proofs of II Lemma 11 and II Prop. 8 depend strongly on the fact that the measure in the percolation model is a product measure, and this is not the case in the random-cluster model. We can for the simple random-cluster model derive a much weaker proposition, which is sufficient, however, for establishing a relation between the global large-range connectivity and the generalized free energy.

Lemma 7 Let (G^o,μ^o) be a simple supplemented random-cluster model such that $\liminf_{v\in V} p_{ov} \neq 0$, and let $v\in V$. Then $\langle\gamma_{ov}\gamma_v^f;G^o,\mu^o\rangle = 0$.

Proof. The proof is analogous to that of II Lemma 10. It requires two additional steps, namely Lemma 5 and the fact that q_κ is a monotone function of q, such that $q_\kappa = 1$ for $q = 1$, from which it follows that $\liminf_{v\in V}(p_{ov})_\kappa \neq 0$ if and only if $\liminf_{v\in V} p_{ov} \neq 0$. ||

If G' is a connected subgraph of G then we shall call the set of edges of G not in G' which are incident with G' (so with the vertex boundary of G' in G) the edge boundary of G' in G.

Lemma 8 Let (G,p,κ) be a simple random-cluster model and G' a finite connected subgraph of G with finite edge boundary and set of edges E'. If $p_e \leq p'_e$ for $e \in E'$ and $p_e \geq p'_e$ for $e \notin E'$, then

$$\langle \gamma_{G'};G,p,\kappa\rangle \leq \langle \gamma_{G'};G,p',\kappa\rangle. \tag{4.1}$$

Proof. First we notice that G', together with its edge boundary E" is contained in the finite subgraphs G_n, for n large enough, because E' and E" are finite. Thus, $\gamma_{G'}$ in G equals $\gamma_{G'}$ in G_n for n large enough. Furthermore, $\gamma_{G'}$ is a local event, so $\langle\gamma_{G'};G\rangle = \lim_n \langle \gamma_{G'};G_n\rangle$, by definition, and it follows that it suffices to prove eq. (4.1) for G finite. So let G be finite, and put $\gamma_{G'} = c^{E'} d^{E''}$. Then

$$\langle\gamma_{G'};G,p,\kappa\rangle = \langle\gamma_{G'}\kappa^{\gamma};G,P\rangle/Z(G,p,\kappa) = p^{E'} q^{E''} Z(\mathcal{C}^{E'}\mathcal{D}^{E''}G)/Z(G). \tag{4.2}$$

Because G' is a cluster, $Z(\mathcal{C}^{E'}\mathcal{D}^{E''}G) = \kappa^{1-V(G')} Z(\mathcal{D}^{E'}\mathcal{D}^{E''}G)$, and consequently we obtain, with $V' \equiv V(G')$,

$$\langle\gamma_{G'};G,p,\kappa\rangle = p^{E'} q^{E''} \kappa^{1-V'} Z(\mathcal{D}^{E'+E''}G)/Z(G) =$$

$$= \left(\frac{p}{q}\right)^{E'} \kappa^{1-V'} q^{E'+E''} Z(\mathcal{D}^{E'+E''}G)/Z(G) = \left(\frac{p}{q}\right)^{E'} \kappa^{1-V'} \langle d^{E'+E''};G,p,\kappa\rangle. \tag{4.3}$$

From eq. (4.2) and II Prop. 1, it follows that $\langle\gamma_{G'}\rangle$ is increasing in p_e for $e \in E'$. From eq. (4.3) and Lemma 3, it follows that $\langle\gamma_{G'}\rangle$ is decreasing in p_e for $e \notin E'$. Consequently, eq. (4.1) holds, and the lemma follows. ||

From the Lemmas 7 and 8 we deduce the relation between γ_v^∞ and the accessibility of the supplementary vertex o from v, i.e. γ_{ov}.

Proposition 4 Let (G^o,μ^o) be a simple supplemented locally finite random-cluster model such that $\liminf_{v\in V} p_{ov} > 0$, and let $v \in V$. Then $\gamma_v^\infty = \gamma_{ov}$ a.e.

Proof. The proof is analogous to the proof of II Prop. 7. By II Lemmas 5 and 6 it is sufficient to prove, analogously to eq. II (5.2), that

(4.4) $$\lim_n \langle \gamma_{vB_n} \delta_{ov}; G^o, \mu^o \rangle = \lim_n \lim_{n' > n} \langle \gamma_{vB_n} \delta_{ov}; G^o_{n'}, \mu^o \rangle = 0.$$

Now, instead of eq. II (5.3), we obtain

(4.5) $$\langle \gamma_{vB_n} \delta_{ov}; G^o_{n'} \rangle = \sum_{G' \subseteq G_{n'}} \langle \gamma_{G';vB_n} d^{E'}; G^o_{n'} \rangle,$$

where E' is the set of supplementary edges incident with G'. If G' contributes to the summation, i.e. G' contains v and at least one vertex of B_n and is connected, then, by the recursion theorem, and Lemma 8,

(4.6) $$\langle \gamma_{G';vB_n} d^{E'}; G^o_{n'} \rangle = \langle d^{E'} \rangle \langle \gamma_{G';vB_n}; \not\supset^{E'} G^o_{n'} \rangle \leq \langle d^{E'} \rangle \langle \gamma_{G';vB_n}; G_{n'} \rangle.$$

By the assumption $\liminf_{v \in V} p_{ov} > 0$, it will follow from Lemma 5 that for the contributing c-clusters there are constants b and a < 1 such that $\langle d^{E'} \rangle \leq (q_{o\kappa})^{E'} \leq b\, a^{d(v,B_n)}$, and therefore, by eqs. (4.5) and (4.6) that

(4.7) $$\langle \gamma_{vB_n} \delta_{ov}; G^o_{n'} \rangle \leq b\, a^{d(v,B_n)} \langle \gamma_{vB_n}; G_{n'} \rangle \leq b\, a^{d(v,B_n)}.$$

Because G is locally finite, it follows from eq. (4.7) that $\lim_n \lim_{n'} \langle \gamma_{vB_n} \delta_{ov}; G^o_{n'}, \mu^o \rangle = b \lim_n a^{d(v,B_n)} = 0$, and hence we obtain eq. (4.4), which proves the proposition. ||

From Proposition 4 we obtain a relation between weak large-range connectivity and the accessibility of the supplementary vertex. However, it concerns the large-range connectivity of a model with a measure which is the limit of supplemented random-cluster model measures. If μ^o is the measure of a random event on G^o in a variable supplementation of a simple random-cluster model (G,μ), we denote by $\mu^\wedge(a)$ the limit function $\lim_{p_o \downarrow 0} \mu^o(a) = \lim_{p_o \downarrow 0} \langle a; G^o, \mu^o \rangle$ which is defined on the random events a on G; these events are independent of the state of the supplementary edges. Because we do not know whether or not $\hat{\mu}$ is a measure, we define the measure $\bar{\mu}$ as the extension to the random events on G of the limit function $\hat{\mu}$ on the local events of G. In the percolation model, for $\kappa = 1$, obviously $\hat{\mu} = \bar{\mu} = \mu$, because in that case μ is a product measure. However, for $\kappa > 1$, in particular for $\kappa = 2$, where we obtain the Ising model, we do not know whether or not $\hat{\mu} = \bar{\mu}$, and if they equal μ or not.

The relation between weak and strong large-range connectivity and the supplementary vertex is provided by the following analogue of II Th. 4.

Theorem 4 Let $(G,\mu) = (G,p,\kappa)$ be a locally finite simple random-cluster model and $v,v' \in V$. If (G^o,μ^o) is a variable supplementation of (G,μ) such that $\liminf_{v\in V} p_{ov} > 0$, then

$$\lim_{p_o \downarrow 0} \langle \gamma_{ov};G^o,\mu^o\rangle = \langle \gamma_v^\infty;G,\bar{\mu}\rangle, \tag{4.9}$$

$$\lim_{p_o \downarrow 0} \langle \gamma_{vv'};G^o,\mu^o\rangle = \langle \gamma_{vv'};G,\bar{\mu}\rangle + \langle \gamma_v^\infty \delta_{vv'}\gamma_{v'}^\infty;G,\bar{\mu}\rangle. \tag{4.10}$$

Proof. The first part of the proof is analogous to the proof of II Th. 4. Obviously, we have to replace the expectation values $\langle \gamma_v^\infty;P\rangle$ etc. by the limit functions $\hat{\mu}(\gamma_v^\infty)$ etc. in the eqs. II (5.8), (5.10), (5.11) and (5.13). To see that $\hat{\mu}$ and $\bar{\mu}$ coincide on the event $\gamma_v^{G\infty}$, we first observe that $\gamma_v^{G\infty}$ is a random increasing variable, and hence by Prop. 2, we have

$$\hat{\mu}(\gamma_v^\infty;G) \equiv \lim_{p_o \downarrow 0} \mu^o(\gamma_v^{G\infty};G^o) = \inf_{p_o > 0} \langle \gamma_v^{G\infty};G^o,\mu^o\rangle . \tag{4.11}$$

Moreover, by II Lemma 6, $\gamma_v^{G\infty} = \liminf_n (\gamma_{vB_n} \text{ in } G_n)$, where the limit is ultimately reached monotonically from above, and the indicators $(\gamma_{vB_n} \text{ in } G_n)$ are local variables (in fact increasing). So we obtain

$$\inf_{p_o > 0} \langle \gamma_v^{G\infty};G^o,\mu^o\rangle = \inf_{p_o} \liminf_n \langle \gamma_{vB_n}^{G_n};G^o,\mu^o\rangle =$$

$$= \liminf_n \inf_{p_o} \langle \gamma_{vB_n}^{G_n};G^o,\mu^o\rangle = \liminf_n \langle \gamma_{vB_n}^{G_n};G,\bar{\mu}\rangle = \langle \gamma_v^\infty;G,\bar{\mu}\rangle \equiv \bar{\mu}(\gamma_v^\infty;G), \tag{4.12}$$

by the definition of $\bar{\mu}$. From the eqs. (4.11) and (4.12) the eq. (4.9) follows. An extension of this argument shows that $\hat{\mu}$ and $\bar{\mu}$ coincide on the event $\gamma_v^f\gamma_{vB_n}$, from which the analogue of eq. II (5.12) follows. To see that $\hat{\mu}$ and $\bar{\mu}$ coincide on the event $\gamma_{vv'} + \gamma_v^\infty\delta_{vv'}\gamma_{v'}^\infty$ we show that its indicator is a random increasing variable, and, in particular, is the limit of a monotonically non-increasing sequence of increasing local variables. Indeed, by II Lemmas 5 and 6, $\gamma_{vv'} + \gamma_v^\infty\delta_{vv'}\gamma_{v'}^\infty = \lim\inf_n(\gamma_{vv'}^{G_n} + \gamma_{vB_n}\delta_{vv'}^{G_n}\gamma_{vB_n})$, the limit is monotonically non-increasing as soon as v and v' belong to G_n, because if $\gamma_{vv'}$ in G_{n+1}, either $\gamma_{vv'}$ in G_n or $\gamma_{vB_n}\gamma_{v'B_n}$, by the definition of vertex boundary, and if $\gamma_{vB_{n+1}}\gamma_{v'B_{n+1}}$ then $\gamma_{vB_n}\gamma_{v'B_n}$. Observe that $\sup\{\gamma_{vv'}, \gamma_{vB_n}\gamma_{v'B_n}\}$ equals the function $\gamma_{vv'} + \gamma_{vB_n}\delta_{vv'}\gamma_{v'B_n}$. Because both $\gamma_{vv'}$ in G_n and γ_{vB_n} in G_n are increasing local variables, $\gamma_{vv'} + \gamma_{vB_n}\delta_{vv'}\gamma_{v'B_n}$ in G_n is an increasing local variable. Hence, by the same reasoning as we used in the first part of the proof, we obtain $\hat{\mu} = \bar{\mu}$ on the event $\gamma_{vv'} + \gamma_v^\infty\delta_{vv'}\gamma_{v'}^\infty$, from which eq. (4.10) follows. ||

Proposition 5 Let (G,μ) be a bilocally finite simple random-cluster model, then the clustering property II Th. 3 holds for $(G,\bar{\mu})$.

Proof. Analogous to the proof of II Th. 3. To see that the covariance inequality for $\gamma_v^{G\infty}\gamma_{v'}^{G\infty}$ with the measure $\bar{\mu}$ holds, observe first that it holds with the measure μ^o, by Th. 2, and therefore in the limit $\mu^o \to \hat{\mu}$. Further, $\hat{\mu}$ coincides with $\bar{\mu}$ on the events $\gamma_v^{G\infty}$, $\gamma_{v'}^{G\infty}$ and $\gamma_v^{G\infty}\gamma_{v'}^{G\infty}$ by the argument used in the proof of Th. 4. ||

Corollary Let (G,μ) be a locally finite simple random-cluster model, and $v \in V$. If (G^o,μ^o) is a variable supplementation of (G,μ) such that $\lim\inf_{v\in V} p_{ov} > 0$, then

(4.13) $\lim\inf_{v'\in V}\lim_{p_o\downarrow 0} \langle\gamma_{vv'}\rangle = \lim_{p_o\downarrow 0}\langle\gamma_{ov}\rangle \lim\inf_{v'\in V}\lim_{p_o\downarrow 0}\langle\gamma_{ov'}\rangle.$

Proof. Analogous to the proof of the corollary of II Th. 4. ||

Applying the corollary to the ferromagnetic Ising model with nagmetic field B (which need not be homogenous), and using I § 4.3 and I § 4.2, we get

$$\liminf_{v'\in V'} \lim_{B\downarrow 0} \lim_n \langle\sigma_v\sigma_{v'};G_n,B\rangle_{can} =$$

(4.14) $$= \lim_{B\downarrow 0}\lim_n \langle\sigma_v;G_n,B\rangle_{can} \liminf_{v'\in V'} \lim_{B\downarrow 0}\lim_n \langle\sigma_{v'};G_n,B\rangle_{can} .$$

In case all vertices are equivalent, this reduces to

(4.15) $$\liminf_{v'\in V'} \lim_{B\downarrow 0}\lim_n \langle\sigma_v\sigma_{v'};G_n,B\rangle_{can} = \left(\lim_{B\downarrow 0}\lim_n \langle\sigma_v;G_n,B\rangle_{can}\right)^2 ,$$

which is independent of the vertex v and the set V'.

Finally, we give a proposition relating the generalized spontaneous magnetization with the global large-range connectivity. However, we shall give this proposition under rather strong conditions on the system, compared with those used in II Proposition 8. We shall require that the system be very "regular", so that there is in fact no longer a difference between global large-range connectivity and weak large-range connectivity.

An <u>automorphism of a graph</u> G = (V,E,i) is a one-to-one mapping ψ of vertices to vertices and edges to edges such that the incidence relation is preserved, i.e. $\psi(V) = V$, $\psi(E) = E$ and for all $e\in E$, if $i(e) = \{v,v'\}$ then $i(\psi(e)) = \{\psi(v),\psi(v')\}$. Two vertices $v,v'\in V$ are called <u>equivalent in a graph</u> if there is an automorphism ψ of the graph such that $\psi(v) = v'$. If the number of equivalence classes of vertices under the relation of equivalence in the graph is finite, we say that an infinite graph has a <u>lattice structure</u>. An <u>automorphism</u> of a random-cluster model is an automorphism of the graph such that $p(\psi(e)) = p(e)$ for all $e\in E$. Two vertices $v,v'\in V$ are called <u>equivalent</u> in a random-cluster model is there is an automorphism ψ of the random-cluster model such that $\psi(v) = v'$. If the number of equivalence classes of vertices of an infinite random-cluster model under the relation of equivalence in the random-cluster model is finite, we say that the random-cluster model has a lattice structure.

Proposition 6 Let $(G,\mu) = (G,p,\kappa)$ be a locally finite simple random-cluster model with lattice structure and with an increasing sequence of finite subgraphs G_n such that $\lim_n |V_n|^{-1}|B_n| = 0$ and $\cup_n G_n = G$. If (G^o,μ^o) is a simple variable supplementation of (G,μ) such that $p_{ov} = p_o$ for all $v \in V$, then the following limits exist and are equal:

$$\lim_{p_o \downarrow 0} \left(q_o \frac{\partial}{\partial q_o}\right) \lim_n |V_n|^{-1} \ln Z(G_n^o,\mu^o) = (1-\kappa^{-1}) \lim_n |V_n|^{-1} \sum_{v \in V_n} \langle \gamma_v^f ; G, \bar{\mu} \rangle . \tag{4.16}$$

Proof. First we show that $\lim_n |V_n|^{-1} \ln Z(G_n^o,\mu^o)$ exists and in independent of the sequence G_n. This will follows from the subadditivity of ln Z, which is a direct consequence of II Prop. 1 applied to the decreasing function κ^γ and the product property of the cluster function Z, mentioned in I § 7.2 So, if G_1 and G_2 are two finite disjoint subgraphs of G^o, and G_3 is a finite subgraph of G^o obtained from G_1 and G_2 by adding edges of G^o to their union, we have $\ln Z(G_3) \leq \ln Z(G_1) + \ln Z(G_2)$. Because (G,μ) has a lattice structure and is locally finite, and hence is locally bounded, and since on the other hand $\lim_n |V_n|^{-1}|B_n| = 0$, it follows that $\lim_n |V_n|^{-1} \ln Z(G_n^o,\mu^o)$ exists and is independent of the sequence G_n (cf. Fisher [9], and Hammersley [10]). Secondly, $\ln Z(G_n^o,\mu^o)$ is a convex function of $\ln q_o$, by I Prop. 2 and Prop. 2, and hence we may interchange derivative with respect to $\ln q_o$ and limit with respect to n except for a countable number of points q_o. Because we are interested in the limit of p_o decreasing to zero, we may even neglect those points of discontinuity, if present, without changing the limit value, provided we interpret the derivative as a twovalued function consisting of the lefthand-derivative and the righthand-derivative, the last of which is continuous from the right. Thus we have

$$q_o \frac{\partial}{\partial q_o} \lim_n |V_n|^{-1} \ln Z(G_n^o,\mu^o) = (1-\kappa^{-1}) \lim_n |V_n|^{-1} \sum_{v \in V_n} \langle \delta_{ov} ; G_n^o,\mu^o \rangle , \tag{4.17}$$

by the convexity and I Prop. 2. Because (G,μ) has a lattice structure, by assumption, we can choose a finite number of vertices, one for each equivalence class, the vertices $v_1, v_2, \ldots, v_i$, say.

Let $G_{n''}$ be so large that all $v_1, \ldots, v_i$ belong to $G_{n''}$. Let n' be the largest of the distances of the vertices $v_1, \ldots, v_i$, to vertices of the vertex boundary $B_{n''}$ of $G_{n''}$ in G, i.e. $n' = \sup_i \sup_{v \in B_{n''}} d(v_i, v)$. It follows, for $n > n''$, that if B_{n1} is the set of vertices of V_n within a distance n' of the boundary B_n, for each vertex $v \in V_n - B_{n1}$ which belongs to the class of v_i we have $\langle\delta_{ov};G^o_n,\mu^o\rangle \geq \langle\delta_{ov_i};G^o_{n''},\mu^o\rangle$, because there is an automorphism ψ such that $\psi v_i = v$ and such that the graph $\psi G_{n''}$ is contained in G_n, by the construction of n' and $n > n''$, so $\langle\delta_{ov_i};G^o_{n''},\mu^o\rangle = \langle\delta_{ov};\psi G^o_{n''},\mu^o\rangle$ by the definition of equivalence classes, and this is larger than $\langle\delta_{ov};G^o_n,\mu^o\rangle$ by Prop. 2 and $\psi G_{n''} \subset G_n$. It follows that $\langle\delta_{ov};G^o_n\rangle - \langle\delta_{ov};G^o\rangle = \langle\delta_{ov};G^o_n\rangle - \langle\delta_{ov_i};G^o\rangle$ has for all $v \in V_n - B_{n1}$ a bound $a(n'') \equiv \sup_i(\langle\delta_{ov_i};G^o_{n''}\rangle - \langle\delta_{ov_i};G^o\rangle)$, which is independent of n. Moreover, by eq. II (5.31), $|B_{n1}| < c|B_n|$, where c is a constant, and hence $\lim_n |V_n|^{-1}|B_{n1}| = 0$, by the assumption on the G_n. Consequently,

$$(4.18) \qquad \lim_n |V_n|^{-1} \sum_{v \in V_n} (\langle\delta_{ov};G^o_n,\mu^o\rangle - \langle\delta_{ov};G^o,\mu^o\rangle) \leq a(n''),$$

and hence the lefthand member of eq. (4.18) equals zero because $a(n'')$ tends to zero for $n'' \to \infty$. From the eqs. (4.17) and (4.18) we so obtain

$$(4.19) \qquad q_o \frac{\partial}{\partial q_o} \lim_n |V_n|^{-1} \ln Z(G^o_n,\mu^o) = (1-\kappa^{-1}) \lim_n |V_n|^{-1} \sum_{v \in V_n} \langle\delta_{ov};G^o,\mu^o\rangle .$$

By eq. (4.9), $\lim_{p_o \downarrow 0} \langle\delta_{ov};G^o,\mu^o\rangle = \langle\gamma^f_v;G,\bar{\mu}\rangle$, and by the assumption that (G,μ) has a lattice structure, there is only a fixed number of different terms $\langle\delta_{ov};G^o,\mu^o\rangle$. Hence, the sum $|V_n|^{-1}\Sigma_{v \in V_n}\langle\delta_{ov}\rangle$ converges uniformly in n as a function of p_o, and we obtain from eq. (4.19) the eq. (4.16), which completes the proof. ||

One notices that the requirement in Prop. 6 that the sequence G_n be such that $\lim_n |V_n|^{-1}|B_n| = 0$, restricts the class of graphs to which Prop. 6 is applicable. In particular it is not applicable to Bethe lattices with coordination number $n \geq 3$, where for all finite subgraphs G' we have $|V(G')|^{-1}|B(G')| \geq (n-2)/(n-1) \geq \frac{1}{2}$.

5. DISCUSSION

The main point in this paper on the simple random-cluster model is, that the simple random-cluster model has mainly the same properties with respect to large-range connectivity as were derived for the percolation model in a preceding paper. We mention again the close relationship between weak and strong large-range connectivity, in particular their equivalence under a non-trivial condition. This equivalence is related with a clustering property, which does not depend on the occurrence of a group of automorphisms, but holds for any bilocally finite simple random-cluster model.

In § 3 of this paper we restricted ourselves to bilocally finite graphs in order to obtain the same properties with respect to large-range connectivity as were obtained in II § 4. This restriction, however, is unnecessary, as will be shown in an appendix, and it turns out that the properties mentioned in II § 4 for the percolation model hold for all simple random-cluster models.

Another interesting point is the relation between the generalized spontaneous magnetization and global large-range connectivity, as well as the relation between the generalized local spontaneous magnetization and weak large-range connectivity. However, the large-range connectivities are defined in this case in graphs with a measure $\overline{\mu}$ which has not been shown to be equal to the random-cluster model measure μ. The question whether or not $\overline{\mu} = \mu$ is an intriguing open question.

APPENDIX

We shall first show that γ_v^∞ is a random increasing variable on any countable graph. This will be a direct consequence of the following lemmas. We shall denote the set of edges $E-E_n$ by F_n. The event that there is a c-cluster containing the vertex v and an edge of F_n is denoted γ_{vF_n}. Notice that F_n is not a set of vertices, contrary to B_n and U_n. The reader is warned against confusing γ_{vF_n} with γ_{vB_n} or γ_{vU_n}.

Lemma A1 If v is a vertex of G, then

(A.1) $\gamma_v^\infty = \inf_n \gamma_{vF_n} = \lim_n \gamma_{vF_n}$,

(A.2) $\langle\gamma_v^\infty\rangle = \inf_n \langle\gamma_{vF_n}\rangle = \lim_n \langle\gamma_{vF_n}\rangle$.

Proof. First we notice that if γ_v^∞, then the c-cluster containing v has an infinite number of edges, because otherwise the number of edges, and therefore the number of vertices, should be finite, and hence the cluster should be finite contrary to the assumption. Because E_n is finite, it follows that for any n an infinite number of edges of the c-cluster containing v is not in E_n, so in F_n, so γ_{vF_n}, and thus $\gamma_v^\infty \le \inf_n \gamma_{vF_n}$. On the other hand, if γ_v^f, then the number of c-edges in the c-cluster containing v is finite, so there is an n such that all these edges are contained in E_n, so not γ_{vF_n}, and thus $\gamma_v^f \le \sup(1-\gamma_{vF_n})$. Consequently, $\gamma_v^\infty = \inf_n \gamma_{vF_n}$. Obviously, γ_{vF_n} is non-increasing in n, because E_n is increasing in n, so F_n is decreasing in n, which completes the proof of the first part of the lemma. The second part follows from the integration theorem on monotone sequences. ||

Lemma A2 Let v be a vertex of G, then

(A.3) $\gamma_{vF_n} = \sup_{n'}(\gamma_{vF_n} \text{ in } G_{n'}) = \lim_{n'}(\gamma_{vF_n} \text{ in } G_{n'})$,

(A.4) $\langle\gamma_{vF_n}\rangle = \sup_{n'}\langle\gamma_{vF_n};G_{n'}\rangle = \lim_{n'}\langle\gamma_{vF_n};G_{n'}\rangle$.

Proof. If γ_{vF_n}, there is an edge of F_n in the c-cluster containing v. It follows that there is a c-path with initial vertex v containing an edge of F_n, so there is an n' such that this path is contained in $G_{n'}$, i.e. $\gamma_{vF_n} \le \sup_{n'}(\gamma_{vF_n} \text{ in } G_{n'})$. On the other hand, if there is an n' such that $(\gamma_{vF_n} \text{ in } G_{n'})$, obviously γ_{vF_n} in G, so $\sup_{n'}(\gamma_{vF_n} \text{ in } G_{n'}) \le \gamma_{vF_n}$, and consequently $\gamma_{vF_n} = \sup_{n'}(\gamma_{vF_n} \text{ in } G_{n'})$. Evidently, $(\gamma_{vF_n} \text{ in } G_{n'})$ is non-decreasing in n', which completes the proof of the first part of the lemma. The second part follows from the integration theorem on monotone sequences. ||

Obviously, (γ_{vF_n} in $G_{n'}$) is an increasing local variable, so by Lemma A2 it follows that γ_{vF_n} is a random increasing variable, and by Lemma A1, that γ_v^∞ is a random increasing variable. By the considerations in § 3 we so have

Theorem A1 Let (G,μ) be a simple random-cluster model, then all the lemmas, propositions and theorems of II § 4 hold for (G,μ). ||

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the proof of the first part of the lemma. The second part follows from the integration theorem on monotone sequences. ||

Obviously, $(\gamma_{vF_n}$ in $G_n)$ is an increasing local variable, so by Lemma A2 it follows that $\gamma^w_{vF_n}$ is a random increasing variable, and by Lemma A1, that γ^w_v is a random increasing variable. By the considerations in § 3 we have

Theorem A1. Let (G,υ) be a simple random-cluster model, then all the lemmas, propositions and theorems of II § 4 hold for (G,υ). ||

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INDEX OF LEMMAS, PROPOSITIONS AND THEOREMS

COLOPHON

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STELLINGEN

van C.M. Fortuin

1 De reeds lang bekende ster-driehoekstransformaties in elektrische netwerken en de in het Ising model bekende ster-driehoekstransformaties zijn beide een bijzonder geval van een ster-driehoekstransformatie in het random-cluster model, die echter alleen in de genoemde twee gevallen algemeen toepasbaar is.

2 De reeksontwikkelingen bij hoge en lage temperaturen in het Ising model zijn bijzondere gevallen van reeksontwikkelingen in het random-cluster model; de coëfficiënten van dergelijke reeksen kunnen worden geïnterpreteerd als elementen van de incidentie-algebra op bepaalde tralies.
Vgl. G.C. Rota, Z.Wahrsch.Th. 2 (1964), 340.

3 De invloed van massacommunicatiemiddelen op het massapsychologische gedrag kan worden toegelicht aan de hand van het percolatiemodel. Daartoe worden individuen voorgesteld door punten, massacommunicatiemiddelen door supplementaire punten en communicatiekanalen door lijnen.

4 Het verdient aanbeveling om in de experimenten aan verdunde oplossingen van magnetische atomen in niet-magnetische materialen te trachten enerzijds de lokale magnetische en anderzijds de spin-spincorrelaties op lange afstand te meten. Een eventueel verschil in overgangstemperatuur zou mogelijk aan de hand van het random-cluster model kunnen worden geïnterpreteerd.

5 Het verdient aanbeveling om de algebra's van observabelen, zoals die in gebruik zijn in de algebraische aanpak van de statistische mechanica, uit te breiden tot algebra's met onbegrensde operatoren, teneinde het mogelijk te maken b.v. ook de soortelijke warmte als observabele te beschouwen.

6 In het algemeen wordt in de fysica te weinig aandacht besteed aan het asymptotisch gedrag van systemen bestaande uit veel deeltjes. In het bijzonder geldt dit bij het gebruik van periodieke-randvoorwaarden en de grote-volumelimiet.

7 Het verdient aanbeveling om het begrip samenhangend van een graaf als volgt te definiëren: een niet-lege graaf is samenhangend als er geen twee niet-lege disjuncte subgrafen van die graaf zijn waarvan de som gelijk is aan de graaf.
Vgl. J. Edmonds, Can.J.Math. 17 (1965) 449-467.

8 Als z een atoom van een tralie L is, L' de verzameling bestaande uit z en alle elementen van L die boven z liggen en L" de verzameling L-L', dan zijn L' en L" tralies dan en slechts dan als voor alle x en y uit L geldt dat $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

9 Als x, y en z elementen van een tralie zijn, dan zijn onder de de conditie $x \vee z = y \vee z$ de relaties $x=y$ en $x \wedge (y \vee z) = x \wedge y$ equivalent. Als tevens geldt dat $x \wedge z = y \wedge z$, dan zijn de relaties $x=y$ en $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ equivalent.
Vgl. G. Birkhoff-Lattice Theory, 3rd edition 1967, Corollary of II Th. 13.