DISPERSION RELATION CALCULATIONS FOR PION ELECTROPRODUCTION AND COMPTON SCATTERING

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PROEFSCHRIFT

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S T E L L I N G E N

I. In het geval van verstrooiing van een Diracelectron aan een uitwendige potentiaal kan voor de asymptotische conditie behalve de gebruikelijke zwakke convergentie, ook sterke convergentie en zelfs norm-convergentie worden afgeleid.

S.S. Schweber, in "The Mathematics of Physics and Chemistry", vol.II, sec.10.3; edited by H. Margenau and G.M. Murphy.

II. Het verdient aanbeveling om bij het uitvoeren van hoge-temperatuur ontwikkelingen in de statistische mechanica gebruik te maken van algebraische computerprogramma's. Als voorbeeld van een dergelijke ontwikkeling kan gedacht worden aan de berekeningen die recentelijk door Capel et al. zijn uitgevoerd aan het XY-model.

H.W. Capel, E.J. van Dongen, Th.J. Siskens; preprint Leiden, mei 1974.

III. De bewering van Bardeen en Tung, dat Hearn en Leader een verkeerd teken gebruiken bij één van de invariante amplitudes voor Comptonverstrooiing, is onjuist. Dit misverstand is terug te voeren op afwijkende definities van de bewuste amplitude bij de verschillende auteurs.

A.C.Hearn, E.Leader; Phys.Rev. 126(1962) 789.
W.A.Bardeen, Wu-Ki Tung; Phys.Rev. 173 (1968)1423.

IV. Het invoeren van een subtractie in één van de invariante amplitudes (in casu A_6) voor electroproductie, teneinde het effect van de axiale vector-vormfactor in rekening te brengen, geeft problemen voor het gedrag van de botsingsdoorsnede bij hoge energieën.

N. Dombey, B.J. Read; Nucl.Phys. B60(1973)65.

V. Bij de door Hite en Jacob voorgestelde "grens-dispersierelaties" kan in het geval van niet-elastische verstrooiing een kinematische singulariteit optreden.

G.E.Hite, R.Jacob; Phys.Rev. D5(1972)422.

VI. Bij het uitvoeren van dispersierelatie-berekeningen voor Comptonverstrooiing aan nucleonen verdient het de voorkeur om gebruik te maken van de door Bardeen en Tung geintroduceerde amplitudes, aangezien deze het juiste gedrag vertonen bij lage energieën; dit in tegenstelling tot de doorgaans gebruikte amplitudes van Hearn en Leader.

W. Pfeil, H. Rollnik, S. Stankowski;
preprint Bonn, 1973.

VII. In de afleiding door Caplin et al. van een coherentielengte voor electron-phonon verstrooiing, ter verklaring van de afwijking van de regel van Mathiessen, wordt een onjuist gebruik gemaakt van de onzekerheidsrelatie van Heisenberg.

A.D.Caplin, F.Napoli, D.Sherrington;
J. Phys. F (Metal Phys.) 3(1973)L93.

VIII. De wijze waarop het thermomagnetisch drukverschil afhangt van de hoek tussen magneetveld en temperatuurgradient, kan uit eenvoudige symmetriebeschouwingen worden gevonden.

H. Vestner; Z. Naturforsch. 28a(1973)869.

IX. Gezien de uitgangspunten voor de berekeningen door Gordon en Kim van intermoleculaire potentialen, is de uitstekende overeenstemming tussen hun resultaten en de experimentele gegevens niet geheel vanzelfsprekend.

R.G. Gordon, Y.S. Kim;
J. Chem. Phys. 56(1972)3122.

Leiden, 25 juni 1974
C.P. Louwerse

AAN MIJN OUDERS

AAN INEKE

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INTRODUCTION

In this thesis we consider two types of processes:

1) electro- and neutrinoproduction of π-mesons, i.e. inelastic scattering of electrons or neutrinos on nucleons, in which process a pion is produced:

$$e\, N \rightarrow e\, N\, \pi$$

$$\nu_e\, N \rightarrow e\, N\, \pi$$

$$\nu_\mu\, N \rightarrow \mu\, N\, \pi .$$

The first of these reactions is due to electromagnetic interactions, the other two proceed via weak interactions.

2) Compton scattering on nucleons, i.e. elastic scattering of photons on nucleons:

$$\gamma\, N \rightarrow \gamma\, N ,$$

which is again an electromagnetic process.

Although these processes are possible only through electromagnetic or weak interactions, strong interaction effects are important, because of the presence of hadrons (in this case: pions and/or nucleons). Whereas the effects of the electromagnetic and weak interactions can be calculated adequately by applying perturbation theory in lowest order, this is not the case for the strong interactions. Thus we have to take into account the complete strong interaction effects. However, since there exists no theoretical procedure to calculate the full effect of these interactions, we are not able to obtain from "first principles" numerical values for the cross-sections for pion production and Compton scattering. For this reason we must base our treatment on an other scheme. We assume that the scattering amplitudes for the processes under consideration have certain analytic properties in suitably chosen kinematical variables. This assumption gives rise to dispersion relations for these amplitudes. Together with the condition that the scattering matrix should be unitary, this forms the basis for dispersion relation theory. Within the context of this theory it is possible to relate the strong interaction effects in our scattering amplitudes to other processes where strong interactions play a role, and for which sufficient experimental data are available. In this way we can then use experimental information to compensate our lack of theoretical knowledge about the strong interactions. Thus we can use pion-nucleon scattering data in our electro- and neutrinoproduction calculations, and data from pion photoproduction in the calculations for Compton scattering. With this additional information it is then possible to evaluate the desired amplitudes and cross-sections. The results can be compared with the experimental data on these processes, as far as available, or they can serve as predictions for the outcome of future experiments. From a theoretical point of view an agreement between the theoretical and experimental results gives support to the assmumptions on which the dispersion relation theory is based. In the present work the first three chapters deal with electro- and neutrinoproduction, and the last three with Compton scattering. The treatment of these two cases is more or less parallel.

In Chapter I we give formal expressions for the cross-sections for electro- and neutrinoproduction, and we treat the kinematics. Several decompositions of the scattering matrix element in terms of different sets of amplitudes are then given, the most important of which are the Lorentz-invariant amplitudes and the multipole amplitudes. In Chapter II we first concentrate on the invariant amplitudes, and exploit their assumed analytic properties to obtain dispersion relations. These dispersion relations can then be transformed into a coupled set of integral equations for the multipole amplitudes. In Chapter III we show that in a limited energy region we can obtain the phases of the multipole amplitudes from the corresponding partial wave amplitudes for pion-nucleon scattering, which are well-known experimentally. With these experimental values as input, the multipole dispersion relations can be solved (in the energy region considered), to yield approximate numerical solutions for the multipole amplitudes. These amplitudes further have to be multiplied by the appropriate pion or nucleon formfactors. We have calculated amplitudes and cross-sections for electroproduction, using the nucleon formfactors as obtained from elastic electron-nucleon scattering. The pion formfactor is not well-known as yet, and should be obtained from a comparison between theoretical and experimental results on pion electroproduction. As can be seen in this chapter, the results are still too uncertain to fix the values of this formfactor completely. For a complete neutrino-production calculation also the axial vector formfactors of the nucleons are needed, one of which is not known very well, while again the pion formfactor must be taken from electroproduction. Moreover, the experimental information about this process is still very scarce. For these reasons we have calculated here for neutrinoproduction only the multipole amplitudes (without inserting the formfactors) and no cross-sections.

The chapters in which we treat Compton scattering are organized in essentially the same way as those on pion production. Chapter IV gives formal cross-section formulae and kinematics, while also various sets of amplitudes are introduced here. Chapter V often refers back to Chapter II, since the analytic properties of the invariant amplitudes are quite similar in both cases. In Chapter VI it is shown that in the first resonance region we can solve the dispersion relations for the Compton scattering amplitudes by using pion photoproduction amplitudes as input, via the unitarity relation. These latter amplitudes are well-known from analyses of experimental photoproduction data. We can thus calculate amplitudes and cross-sections for Compton scattering, and make a comparison with the experimental data on this process. Furthermore, by using only the unitarity relation and the usual invariance assumptions (C, P, T) for the S-matrix, the photoproduction amplitudes provide a lower bound for the Compton scattering cross-sections. This lower bound is violated by a few of the experimental points. To see if this discrepancy can be removed by dropping the requirement of T-invariance, we have generalized the formalism accordingly, and performed some calculations in which a T-violating effect is introduced in a simple way in the input photoproduction amplitudes. The results show that this may indeed change the cross-sections somewhat in the desired direction, but this change seems to be too small to remove the discrepancy.

Finally, we include some appendices, in which we summarize the conventions we used, and where some of the details of the calculations are collected.

CHAPTER I

GENERAL FORMALISM FOR ELECTRO- AND NEUTRINOPRODUCTION OF π-MESONS

I.1 S-MATRIX AND CROSS-SECTION

We consider the pion production processes

$$\begin{aligned} &ep \to ep\pi^0 \ , \quad en \to en\pi^0 \ , \\ &ep \to en\pi^+ \ , \quad en \to ep\pi^- \ , \end{aligned} \tag{1.1}$$

called electroproduction, and

$$\begin{aligned} &\nu_\ell p \to \ell^- p\pi^+ \ , \quad \bar{\nu}_\ell p \to \ell^+ p\pi^- \ , \\ &\nu_\ell n \to \ell^- n\pi^+ \ , \quad \bar{\nu}_\ell p \to \ell^+ n\pi^0 \ , \\ &\nu_\ell n \to \ell^- p\pi^0 \ , \quad \bar{\nu}_\ell n \to \ell^+ n\pi^- \ , \end{aligned} \tag{1.2}$$

called neutrinoproduction. The symbols e, μ, p, n, π denote respectively an electron, muon, proton, neutron, and a pi-meson (pion); ℓ stands for either e or μ, and ν_ℓ is the corresponding neutrino.

With the formalism that will be described here, several other processes of the same general form, $\ell_1 N_1 \to \ell_2 N_2 \pi$, could be described as well (in this case, N_i stands for p or n, and ℓ_i can be any lepton). We will however restrict our attention to the processes mentioned above, because only for these sufficient experimental information is available.

Electroproduction is caused by electromagnetic interactions, and neutrinoproduction by weak interactions, but due to the presence of hadrons, the strong interactions will play an important role in both processes. For the dynamical description of these processes we use an interaction Hamiltonian density $H_I(x)$, consisting of a strong interaction part $H_S(x)$, and a weak or electromagnetic interaction part $H_L(x)$ $(=H_W(x)$ or $H_E(x))$, i.e. $H_I(x) = H_S(x) + H_L(x)$. Using perturbation theory, we formally write for the time evolution operator in the interaction picture (cf.[Sc61], p.330 ff, and references given there), the well-known expression

$$U(t_2,t_1) = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \int_{t_1}^{t_2}\ldots\int dx_1\ldots dx_n \; T(H_I(x_1)\ldots H_I(x_n)), \tag{1.3^a}$$

and define the scattering operator in the usual way by

$$S = U(+\infty,-\infty). \tag{1.3^b}$$

Since perturbation theory does not give an adequate description of the strong interactions, in eq. (1.3) we have to keep formally all orders in $H_S(x)$. For $H_L(x)$ it is sufficient to retain only the lowest orders, so that we write, up to second order in $H_L(x)$,

$$S = S_S - i\int_{-\infty}^{+\infty} dx\; T(H_L(x)S_S) - \frac{1}{2!}\iint_{-\infty}^{+\infty} dxdy\; T(H_L(x)H_L(y)S_S)), \tag{1.4}$$

where S_S is obtained from eqs.(1.3) by replacing H_I by H_S.

Scattering-matrix elements are now obtained by taking the matrix elements $\langle\beta|S|\alpha\rangle$ of the operator S between states $|\alpha\rangle$ and $|\beta\rangle$. We note, that $U_S(t_2,t_1)$ satisfies the group property,

$$U_S(t_2,t_1) = U_S(t_2,t)U_S(t,t_1)\ ,$$

and recall the definitions of the asymptotic in- and out-states [Le55,Le57] (Heisenberg states for the strong interactions)

$$U_S(0,-\infty)|\alpha\rangle = |\alpha\rangle_{in}$$

$$U_S(0,+\infty)|\alpha\rangle = |\alpha\rangle_{out}\ .$$

Further we need the relation $O^H(t) = U_S(0,t)O(t)U_S(t,0)$ between operators in the interaction picture and in the Heisenberg picture for the strong interactions (H). Using these relations, we obtain

$$\langle\beta|S|\alpha\rangle = {}_{out}\langle\beta|\alpha\rangle_{in} - i\int_{-\infty}^{+\infty}dx\; {}_{out}\langle\beta|H_L(x)^H|\alpha\rangle_{in} -$$

$$- \frac{1}{2!}\iint_{-\infty}^{+\infty} dxdy\; {}_{out}\langle\beta|T(H_L(x)^H H_L(y)^H)|\alpha\rangle_{in} \tag{1.5}$$

where the Heisenberg operators are of the form

$$H_{EM}(x)^H = e\, A_\mu(x) \bar{J}_\mu^{EM}(x)^H$$

and $$H_W(x)^H = \frac{G}{\sqrt{2}} \bar{J}_\mu^W(x)^H \bar{J}_\mu^W(x)^{H\dagger} .$$

From now on we will omit the index H. We use the summation convention for greek indices; four-vectors are written as $Q=(\vec{q}, iq_0)$, etc., and their scalar product is given by $P.Q=P_\mu Q_\mu=\vec{p}.\vec{q}-p_0q_0$. The electromagnetic and weak currents $\bar{J}_\mu^{EM}$ and $\bar{J}_\mu^W$ both contain a lepton and a hadron part. The electromagnetic interaction is of the form current times electromagnetic field A_μ (with a coupling constant e), whereas the weak interaction is taken to be here of the current-current type (with a coupling constant $\frac{G}{\sqrt{2}}$). Because of this, we find that for the weak processes (1.2) the second term in (1.5) contributes (i.e. first order in G), while the electromagnetic processes (1.1) are described by the third term (i.e. second order in e). The first term in (1.5), $_{out}\langle\beta|\alpha\rangle_{in}$, is zero in these cases. Using K_i, P_i and Q for the momenta of the leptons, the nucleons and the pion, respectively, and $m_{\ell i}$, m and μ for their masses (i=1,2 for initial, final particles), the final result for the matrix element can be written for both processes as

$$\langle \ell_2 N_2 \pi|S|\ell_1 N_1\rangle = -(2\pi)^{-7/2}\,\delta^{(4)}(P_1+K_1-P_2-K_2-Q)\left(\frac{m_{\ell 1}m_{\ell 2}m^2}{2k_{10}k_{20}p_{10}p_{20}q_0}\right)^{\frac{1}{2}} T_{fi}, \tag{1.6}$$

$$T_{fi} = (2\pi)^{9/2}\left(\frac{2p_{10}p_{20}q_0}{m^2}\right)^{\frac{1}{2}} \varepsilon_\mu\, {}_{out}\langle\pi N_2|\bar{J}_\mu^L(0)|N_1\rangle_{in} =$$

$$= \varepsilon_\mu\, {}_{out}\langle\pi N_2|J_\mu^L(0)|N_1\rangle, \tag{1.7}$$

where *)

$$\varepsilon_\mu = \frac{e^2}{K^2}\,\bar{u}(\vec{k}_2)\gamma_\mu\, u(\vec{k}_1) \qquad \text{(for electroproduction)} \tag{1.8^a}$$

or $$\varepsilon_\mu = \frac{G}{\sqrt{2}}\,\bar{u}(\vec{k}_2)\gamma_\mu(1+\gamma_5)u(\vec{k}_1) \qquad \text{(for neutrinoproduction).} \tag{1.8^b}$$

---

*) for conventions on spinors and γ-matrices, see Appendix A.

The factor $1/K^2$ in (1.8^a), with $K=K_1-K_2$, is the propagator of the virtual photon (cf. sec.I.2), and is due to the two factors A_μ appearing in (1.5) when $H_{EM}(x)$ is substituted in the third term. The cross-section is then given by

$$d^9\sigma = (2\pi)^{-5}\,\delta^{(4)}(K_1+P_1-K_2-P_2-Q)\,\frac{m_{\ell 1}m_{\ell 2}m^2}{2}\,[(K_1.P_1)^2-m_{\ell 1}^2m^2]^{-\frac{1}{2}}.$$

$$\cdot\,|T_{fi}|^2\,\frac{d\vec{p}_2}{p_{20}}\,\frac{d\vec{k}_2}{k_{20}}\,\frac{d\vec{q}}{q_0}\,. \tag{1.9}$$

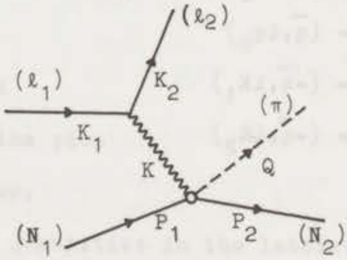
(Even in the case that ℓ_i is a neutrino the factor $m_{\ell i}$ in (1.9) does not give problems, because it cancels against an other factor $m_{\ell i}$ that appears in the denominator when $|T_{fi}|^2$ is calculated explicitly.)

Due to the approximation of taking only lowest order in H_L, T_{fi} is split up into a lepton part and a hadron part, so that the strong interaction effects contained in the matrix element ${}_{out}\langle \pi N_2|J_\mu^L(0)|N_1\rangle_{in}$, can be treated separately. In the following chapters we will be concerned with obtaining sufficient information about this hadron current matrix element.

I.2 KINEMATICS

According to the lowest order perturbation theory our processes can be represented in the conventional way by the diagram in fig.1, which we will use to illustrate the kinematics. In electroproduction the line with momentum label K indicates a virtual photon; in neutrinoproduction it can be considered as representing the weak vector boson. (Provided that the mass of this boson is large (e.g. > 2GeV), in lowest order this description is equivalent to the local current-current interaction that was implied by the weak interaction Hamiltonian we used in sec.I.1.)

Fig. 1
The labels Q,K,K_i,P_i represent the momenta



The following relations hold

$$K = K_1 - K_2$$

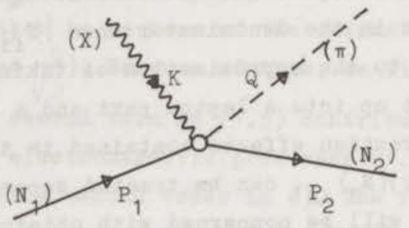
$$K + P_1 = Q + P_2$$

$$K_i^2 = - m_{\ell i}^2 \ ; \ P_i^2 = - m^2 \ , \ Q^2 = - \mu^2 \qquad (i=1,2)$$

and we define $P = \frac{1}{2}(P_1+P_2)$, $L = \frac{1}{2}(K_1+K_2)$.

Since the theoretical problem lies in the strong interaction matrix element $_{out}\langle \pi N_2 | J_\mu^L | N_1 \rangle_{in}$, it is convenient to consider the lower vertex in fig. 1 separately. We then have the process represented in fig. 2, where X indicates the virtual incoming particle with momentum K and "mass" $\sqrt{(-K^2)}$.

fig. 2



For this process we introduce the Lorentz-invariant variables

$$s = -(K + P_1)^2$$
$$t = -(K - Q)^2 \qquad (1.10)$$
$$u = -(K - P_2)^2,$$

satisfying the relation

$$s + t + u = 2m^2 + \mu^2 - K^2 \qquad (1.11)$$

Unless stated otherwise, in the following we will use the centre-of-mass frame for this process (i.e. the pion-nucleon centre-of-mass frame), where the momentum components are specified as

$$K = (\vec{k}, ik_0)$$
$$Q = (\vec{q}, iq_0) \qquad (1.12)$$
$$P_1 = (-\vec{k}, iE_1)$$
$$P_2 = (-\vec{q}, iE_2)$$

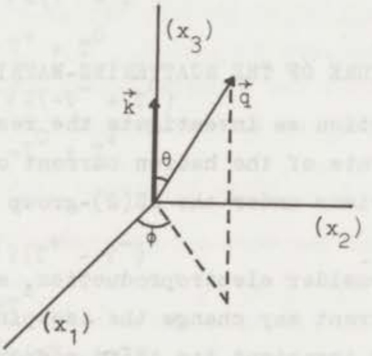
We define

$$W = k_0 + E_1 = q_0 + E_2 \quad \text{(total c.m.energy)} ,$$
$$k = |\vec{k}| , \; q = |\vec{q}| ,$$
$$\hat{k} = \vec{k}/k , \; \hat{q} = \vec{q}/q ,$$
$$x = \cos\theta = \hat{k}.\hat{q} ,$$

and find from (1.10)

$$s = W^2$$
$$t = 2\vec{k}.\vec{q} - 2k_0q_0 + \mu^2 - K^2 \tag{1.13}$$
$$u = -2\vec{k}.\vec{q} - 2k_0E_2 + m^2 - K^2 .$$

We use a coordinate frame such that the x_1x_3-plane is spanned by the lepton 3-momenta ($\vec{k}_1 \times \vec{k}_2$ along the x_2-axis) and $\vec{k}$ lies along the x_3-axis.



Thus we have

$$\vec{q} = (q\sin\theta\cos\varphi, \; q\sin\theta\sin\varphi , \; q\cos\theta)$$
$$\vec{k} = (0,0,k) .$$

The angular momenta of the particles in the πN_2-centre-of-mass system are defined as follows (angular momentum quantum number and third component),

($\frac{1}{2}$,s) initial nucleon spin

($\frac{1}{2}$,s') final nucleon spin

(L,M_L) angular momentum of X

(ℓ,M_ℓ) angular momentum of the pion

(J,M_J) total angular momentum.

In a few places we will need quantities in the laboratory system.

There we choose a coordinate frame with the same orientation of the axes as for the πN_2-centre-of-mass system. Quantities in this laboratory frame will be denoted by a superscript L. We will need especially

$$K_j^L = (\vec{k}_j^L, ik_{j0}^L) = (k_{j1}^L, 0, k_{j3}^L, ik_{j0}) \qquad \text{(for } j=1,2\text{)}.$$

Since the laboratory system moves with a velocity $v=k/E_1$ along the positive x_3-axis of the centre-of-mass frame, we have in general for an arbitrary vector a_μ in the centre-of-mass frame

$$a_1 = a_1^L \; ; \quad a_2 = a_2^L$$

$$a_3 = \frac{1}{m}(E_1 a_3^L + k a_0^L)$$

$$a_0 = \frac{1}{m}(k a_3^L + E_1 a_0^L).$$

I.3 ISOSPIN STRUCTURE OF THE SCATTERING-MATRIX ELEMENT

In this section we investigate the restrictions that are imposed on the matrix elements of the hadron current operator J_μ^L, by assumptions about its behaviour under the SU(2)-group of isospin transformations.

We first consider electroproduction, and note that the hadron electromagnetic current may change the isospin of the hadron states by one unit but leaves invariant its third component. We now assume for J_μ^{EM} the most simple form with these characteristics, i.e. it will consist only of an isoscalar part and a part that transforms as the third component of an isovector. (This form seems to be well-established for pion photoproduction; cf. [Be716]). Thus we can write in an obvious notation

$$T_{fi} = \varepsilon_\mu \; {}_{out}\langle \pi N_2 | J_\mu^{EM} | N_1 \rangle_{in} = \varepsilon_\mu \; {}_{out}\langle \pi N_2 | (J_\mu^0 + J_\mu^3) | N_1 \rangle_{in}. \qquad (1.14^a)$$

We now use explicitly the isospinors χ_i (i=1,2) for the nucleons and the isovector V_α for the pion (see Appendix A) and write the matrix elements of J_μ^0 and J_μ^3 as

$$\varepsilon_\mu \; {}_{out}\langle \pi N_2 | J_\mu^0 | N_1 \rangle_{in} = V_\alpha \, \chi_2^\dagger \, T_{EM}^0 \, \tau_\alpha \, \chi_1$$

and $$\varepsilon_\mu \; {}_{out}\langle \pi N_2 | J_\mu^3 | N_1 \rangle_{in} = V_\alpha \, \chi_2^\dagger (T_{EM}^+ \, \delta_{\alpha 3} + T_{EM}^- \, \tfrac{1}{2}[\tau_\alpha, \tau_3]) \chi_1. \qquad (1.14^b)$$

In eq.(1.14^b) the isospin structure is given explicitly; the functions $T^{0,\pm}$ contain the spin and momentum dependence which will be analyzed in sec.I.4.

In the case of neutrinoproduction the current J_μ^W changes the third component of the isospin of the hadron states by one unit, and so we assume here that J_μ^W is an isovector operator. Depending on whether we have incident neutrinos or anti-neutrinos we have to take the + or - component ($J_\mu^\pm = J_\mu^1 \pm iJ_\mu^2$), and we obtain for the matrix element

$$T_{fi} = \varepsilon_\mu \; {}_{out}\langle \pi N_2 | J_\mu^\pm | N_1 \rangle = V_\alpha \; \chi_2^\dagger (\tfrac{1}{2} T_W^+ \{\tau_\alpha, \tau_\pm\} + \tfrac{1}{2} T_W^- [\tau_\alpha, \tau_\pm]) \chi_1 \; . \tag{1.14^c}$$

From the relations (1.14) we find by explicit calculation (omitting the indices EM and W) for electroproduction

$$\begin{aligned}
\langle \pi^+ \; n \; e^- | T | p \; e^- \rangle &= \sqrt{2}(T^- + T^0) \\
\langle \pi^0 \; p \; e^- | T | p \; e^- \rangle &= T^+ + T^0 \\
\langle \pi^- \; p \; e^- | T | n \; e^- \rangle &= \sqrt{2}(-T^- + T^0) \\
\langle \pi^0 \; n \; e^- | T | n \; e^- \rangle &= -T^0 + T^+
\end{aligned} \tag{1.15}$$

and for neutrinoproduction

$$\begin{aligned}
\langle \pi^+ \; p \; \ell^- | T | \nu \; p \rangle &= \sqrt{2}(T^+ - T^-) \\
\langle \pi^0 \; p \; \ell^- | T | \nu \; n \rangle &= 2T^- \\
\langle \pi^+ \; n \; \ell^- | T | \nu \; n \rangle &= \sqrt{2}(T^+ + T^-) \\
\langle \pi^- \; n \; \ell^+ | T | \bar{\nu} \; n \rangle &= \sqrt{2}(T^+ - T^-) \\
\langle \pi^0 \; n \; \ell^+ | T | \bar{\nu} \; p \rangle &= 2T^- \\
\langle \pi^- \; p \; \ell^+ | T | \bar{\nu} \; p \rangle &= \sqrt{2}(T^+ + T^-) \; .
\end{aligned} \tag{1.16}$$

Instead of the functions $T^\pm$ used here, which are useful because of their simple crossing relations (cf. sec.II.3), it is sometimes convenient to use forms corresponding to final πN-states with definite isospin. We denote these new functions by T^1 and T^3, corresponding to final isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively. They are related to the $T^\pm$ by

$$\begin{aligned}
T^+ &= \tfrac{1}{3}(2T^3 + T^1) \qquad & T^1 &= T^+ + 2T^- \\
T^- &= \tfrac{1}{3}(T^1 - T^3) \qquad & T^3 &= T^+ - T^- \; .
\end{aligned} \tag{1.17}$$

I.4 SPACE-TIME STRUCTURE OF THE SCATTERING-MATRIX ELEMENT

To describe pion production in terms of Lorentz-invariant amplitudes we have to make an expansion of the matrix element T_{fi}, using a complete set of independent spinor matrices with the same Lorentz-transformation character as T_{fi} itself. The coefficients in this expansion will then be Lorentz-invariant functions of the momenta of the particles and will be called "invariant amplitudes". These amplitudes are defined in part A of this section. Since a solution of the present problem can best be obtained by using multipole amplitudes, we define these in part C, while a useful set of amplitudes for the πN_2-centre-of-mass frame is defined in part B. The relations between these sets of amplitudes are summarized in part D.

We recall here the isospin decomposition of sec.I.3, according to which we can write the matrix element for a specific process as $T_{fi} = \sum_n g_n T^n$, with $n=0,+,-$, and where the numbers g_n are the coefficients that appear in (1.15) and (1.16). A similar isospin decomposition can be made for the various amplitudes that will be introduced in this section, e.g. $A_i = \sum_n g_n A_i^n$ (with the same g_n as above). From now on we will explicitly indicate the isospin structure of the amplitudes and the matrix elements only when this is necessary for clarity.

A. Invariant amplitudes

In sec.I.1 we found that in our lowest order approximation

$$T_{fi} = \varepsilon_\mu \; {}_{out}\langle \pi N_2 | J_\mu^L | N_1 \rangle_{in} \quad ,$$

in which the matrix element of the hadron current J_μ^L is still unknown. We can write this last matrix element as $\bar{u}(\vec{p}_2)\, \Gamma_\mu \, u(\vec{p}_1)$, where $u(\vec{p}_i)$ are free nucleon spinors (see Appendix A) and Γ_μ is the most general four-vector operator that can be formed from the γ-matrices and the momenta (Q,K,P_1,P_2). One of these momentum variables is eliminated by energy-momentum conservation, and we choose as the remaining ones K, Q, and $P = \frac{1}{2}(P_1+P_2)$. We define then a set of matrices [Ba61]

$$N_{1\mu} = i\,\gamma_5\,\gamma_\mu\,(\gamma . K) \qquad N_{5\mu} = \gamma_5 \gamma_\mu$$

$$N_{2\mu} = 2\,i\,\gamma_5\,P_\mu \qquad N_{6\mu} = \gamma_5 (\gamma . K) P_\mu$$

$$N_{3\mu} = 2\,i\,\gamma_5\,Q_\mu \qquad N_{7\mu} = \gamma_5 (\gamma . K) K_\mu \tag{1.18}$$

$$N_{4\mu} = 2\,i\,\gamma_5\,K_\mu \qquad N_{8\mu} = \gamma_5 (\gamma . K) Q_\mu$$

and a corresponding set $\{Na_{i\mu}\}$, obtained from (1.18) by omitting the γ_5. These form a complete set of matrices, in the sense that all other matrices constructed from the same variables, when appearing in the form $\bar{u}\,\Gamma'_\mu\,u$, can be reduced to a linear combination of the $N_{i\mu}$ and $Na_{i\mu}$ also taken between nucleon spinors.

We now consider first electroproduction, in which case the current J_μ^{EM} is a proper Lorentz-vector since parity is conserved in electromagnetic and strong interactions. We then need only the set $\{N_{i\mu}\}$ *) and write, denoting the coefficients by B_i and using $N_i \equiv \varepsilon_\mu\, N_{i\mu}$,

$$T_{fi} = \varepsilon_\mu \; {}_{out}\langle \pi N_2 | J_\mu^{EM} | N_1 \rangle_{in} = \bar{u}(\vec{p}_2) \sum_i B_i\, N_i\, u(\vec{p}_1), \tag{1.19}$$

or with the isospin dependence indicated explicitly

$$T_{fi} = \sum_n g_n\, T^n_{EM} \qquad (n=0,+,-)$$

and $\quad T^n_{EM} = \bar{u}(\vec{p}_2) \sum_i B_i^n\, N_i\, u(\vec{p}_1).$

The coefficients B_i, the "invariant amplitudes", are Lorentz-invariant functions of scalar products of the momenta or (equivalently) of K^2 and the variables s, t and u which we defined in eq.(1.10). In the following we will not explicitly indicate the K^2-dependence of the invariant amplitudes.

Next we turn to neutrinoproduction where the current J_μ^W consists of a vector and an axial vector part, J_μ^V and J_μ^A, because the weak interactions do not conserve parity, [Le56], [Wu57]. The matrix element can then be written as

$$T_{fi} = \varepsilon_\mu \; {}_{out}\langle \pi N_2 | \, (J_\mu^V + J_\mu^A) \, | N_1 \rangle_{in} = \bar{u}(p_2) \sum_i [\, B'_i N_i + Ba_i Na_i \,] u(\vec{p}_1), \tag{1.20}$$

---

*) The fact that for the vector current matrices with a γ_5 are needed, is due to the pseudoscalar character of the pion.

where again $T_{fi} = \sum_n g_n T_W^n$, etc.

We have introduced until now three sets of eight amplitudes ($B_i^0, B_i^{\pm}$) for electroproduction, and four sets ($B'^{\pm}_i, Ba^{\pm}_i$) for neutrinoproduction. This situation is simplified by the so-called conserved vector current theory (C.V.C.), [Ge56], [Fe58], [Su58], which gives a relation between the isovector part of the electromagnetic current, J_μ^3, and that part of the (isovector) weak current that also transforms as a four-vector, i.e. $J_\mu^{\pm V}$ (cf. sec.I.3). The C.V.C.-theory states that the operators J_μ^3, J_μ^{+V} and J_μ^{-V} form an isotriplet and thus have the same space-time structure. This means in particular that for these currents we have $\partial_\mu J_\mu = 0$, since for the electromagnetic current this relation holds (due to charge conservation), and that the matrix elements of the currents are equal, apart from isospin Clebsch-Gordan coefficients. Consequently, the amplitudes B_i^n and B'^n_i, introduced in (1.19) and (1.20), are the same and will be denoted by B_i^n from now on. ($n=+,-$ or $1,3$). The relation $\partial_\mu J_\mu = 0$ leads to

$$K_\mu \, {}_{out}\langle \pi N_2 | J_\mu | N_1 \rangle_{in} = 0$$

for the vector currents, which imposes two restrictions on the amplitudes $B_i^{0,\pm}$

$$\begin{aligned} &\tfrac{1}{2}K^2 B_1 + P.K\, B_2 + Q.K\, B_3 + K^2\, B_4 = 0 \\ &B_5 + P.K\, B_6 + K^2\, B_7 + Q.K\, B_8 = 0 \ , \end{aligned} \qquad (1.21)$$

as can be seen from (1.18) and (1.19). To find a set of independent amplitudes we have to eliminate two of the B_i, usually taken as B_3 and B_5. Following Dennery [De61] we introduce therefore a new set of six amplitudes A_i via the relation

$$\bar{u}(\vec{p}_2) \sum_{i=1}^{8} B_i N_i \, u(\vec{p}_1) = \bar{u}(\vec{p}_2) \sum_{i=1}^{6} A_i M_i \, u(\vec{p}_1) \ , \qquad (1.22^a)$$

where the M_i are defined by

$$\begin{aligned} M_1 &= \tfrac{1}{2}\, i\, \gamma_5 \gamma_\mu \gamma_\nu F_{\mu\nu} & M_4 &= 2\gamma_5 \gamma_\mu P_\nu F_{\mu\nu} - 2m\, M_1 \\ M_2 &= 2\, i\, \gamma_5 P_\mu (Q - \tfrac{1}{2}K)_\nu F_{\mu\nu} & M_5 &= i\, \gamma_5 K_\mu Q_\nu F_{\mu\nu} \\ M_3 &= \gamma_5 \gamma_\mu Q_\nu F_{\mu\nu} & M_6 &= \gamma_5 K_\mu \gamma_\nu F_{\mu\nu} \end{aligned} \qquad (1.22^b)$$

and $F_{\mu\nu} = \varepsilon_\mu K_\nu - \varepsilon_\nu K_\mu$. From these definitions we can find

$$A_1 = B_1 - m\,B_6 \qquad A_4 = -\tfrac{1}{2}\,B_6$$

$$A_2 = {}^{2B_2}/(t-\mu^2) \qquad A_5 = \frac{1}{Q.K}\,(B_1 + 2B_4 - \frac{2P.K}{t-\mu^2}\,B_2) \tag{1.23}$$

$$A_3 = -\,B_8 \qquad A_6 = B_7\ .$$

A slightly different choice of amplitudes is discussed in Appendix E.

For the axial vector part of the weak current there is no such relation to reduce the number of amplitudes, so we will need a set of eight. We will use from now on a set of axial vector amplitudes $\{Aa_i\}$ that is somewhat different from the set $\{Ba_i\}$, but completely equivalent to it. By using this other set [Ad68], several relations that we need later on, will appear in a simpler form. We define the set $\{Aa_i\}$ by

$$\bar{u}(\vec{p}_2)\ \sum_{i=1}^{8} Ba_i Na_i\ u(\vec{p}_1) = \bar{u}(\vec{p}_2)\ \sum_{i=1}^{8} Aa_i Ma_i\ u(\vec{p}_1) \tag{1.24^a}$$

where

$$\begin{aligned}
Ma_1 &= \tfrac{1}{2}i[(\gamma.Q)(\gamma.\varepsilon)-(\gamma.\varepsilon)(\gamma.Q)] & Ma_5 &= -2(\gamma.K)(P.\varepsilon)\\
Ma_2 &= 2i(P.\varepsilon) & Ma_6 &= -(\gamma.K)(Q.\varepsilon)\\
Ma_3 &= i(Q.\varepsilon) & Ma_7 &= i(K.\varepsilon)\\
Ma_4 &= -m(\gamma.\varepsilon) & Ma_8 &= -(\gamma.K)(K.\varepsilon)\ .
\end{aligned} \tag{1.24^b}$$

Using these new amplitudes we write for neutrinoproduction

$$\begin{aligned}
T_{fi} &= \varepsilon_\mu\ {}_{out}\langle \pi N_2 | (J^V_\mu + J^A_\mu) | N_1 \rangle_{in}\\
&= \sum_n \varepsilon_n\ \bar{u}(\vec{p}_2)[\sum_{i=1}^{6} A^n_i(s,t,u,K^2)M_i + \sum_{i=1}^{8} Aa^n_i(s,t,u,K^2)Ma_i]\,u(\vec{p}_1),
\end{aligned} \tag{1.25}$$

while for electroproduction we have a similar formula, without the axial part. (Note that ε_μ is different for electro- and neutrinoproduction and consequently the same is true for the M_i; cf. eq. (1.8)).

B. Pion-nucleon centre-of-mass amplitudes

In the centre-of-mass frame of the pion and the (final) nucleon we can write

$$T_{fi} = \chi_2^\dagger\,[F + Fa]\,\chi_1\ , \tag{1.26^a}$$

where the χ_i are 2-component Pauli spinors for the nucleons with the z-axis (in the direction of $\vec{k}$) as the axis of quantization, and F and Fa are 2x2-matrices, constructed from the momenta (in the c.m.frame) and the Pauli matrices. We define two sets of matrices $\{\Lambda_i\}$ and $\{\Lambda a_i\}$

$$\begin{aligned}
\Lambda_1 &= i\,\vec{\sigma}.\vec{b} & \Lambda_4 &= i(\vec{\sigma}.\hat{q})(\hat{q}.\vec{b}) \\
\Lambda_2 &= (\vec{\sigma}.\hat{q})\vec{\sigma}.(\hat{k}\times\vec{b}) & \Lambda_5 &= -i(\vec{\sigma}.\hat{q})b_0 \\
\Lambda_3 &= i(\vec{\sigma}.\hat{k})(\hat{q}.\vec{b}) & \Lambda_6 &= -i(\vec{\sigma}.\hat{k})b_0
\end{aligned} \tag{1.26^b}$$

where $b_\mu = \varepsilon_\mu - \frac{\vec{\varepsilon}.\hat{k}}{k} K_\mu$, and

$$\begin{aligned}
\Lambda a_1 &= i(\vec{\sigma}.\hat{q})(\vec{\sigma}.\vec{\varepsilon}) & \Lambda a_5 &= i(\vec{\sigma}.\hat{q})(\vec{\sigma}.\hat{k})(\hat{k}.\vec{\varepsilon}) \\
\Lambda a_2 &= \vec{\sigma}.(\hat{k}\times\vec{\varepsilon}) & \Lambda a_6 &= i(\hat{k}.\vec{\varepsilon}) \\
\Lambda a_3 &= i(\vec{\sigma}.\hat{q})(\vec{\sigma}.\hat{k})(\hat{q}.\vec{\varepsilon}) & \Lambda a_7 &= -i\,\varepsilon_0 \\
\Lambda a_4 &= i(\hat{q}.\vec{\varepsilon}) & \Lambda a_8 &= -i(\vec{\sigma}.\hat{q})(\vec{\sigma}.\hat{k})\,\varepsilon_0 \,.
\end{aligned} \tag{1.26^c}$$

Centre-of-mass amplitudes are then introduced by writing

$$\chi_2^\dagger F \chi_1 = \chi_2^\dagger \sum_{i=1}^{6} F_i \Lambda_i \chi_1 \tag{1.27}$$

and $\chi_2^\dagger Fa\, \chi_1 = \chi_2^\dagger \sum_{i=1}^{8} Fa_i \,\Lambda a_i\, \chi_1$,

describing the vector and axial vector part of the matrix element. The amplitudes F_i and Fa_i are functions of scalar products of the momenta $\vec{k}$ and $\vec{q}$, and of the parameter K^2. Their isospin decomposition is of the form $F_i = \sum_n g_n F_i^n$ (similar for Fa_i) and the amplitudes F_i^n (for n=+,-) are again the same for electro- and neutrinoproduction.

Note that we have already incorporated current conservation for the vector current by using the above definitions of F_i. To see this, consider the more general set of eight matrices $\{\Lambda_i^!\}$ for the vector part, without current conservation. These can formally be obtained from $\{\Lambda a_i\}$ by multiplication from the left with $(\vec{\sigma}.\hat{q})$. Define then the corresponding amplitudes $F_i^!$ by

$$\chi_2^\dagger F \chi_1 = \chi_2^\dagger \sum_{i=1}^{8} F_i^! \Lambda_i \chi_1 \,.$$

Two of these $F_i^!$ can now be eliminated since current conservation gives the relations

$$F_1^! + \hat{k}.\hat{q}\, F_3^! + F_5^! - (k_0/k)F_8^! = 0$$

and $\hat{k}.\hat{q}\, F_4^! + F_6^! - (k_0/k)F_7^! = 0\,.$

This elimination is effected here simply by replacing ε_μ in $\{\Lambda_i'\}$ by b_μ, which is allowed since $K_\mu J^V_\mu=0$, so that the matrix element remains the same. Now Λ_5' and Λ_6' become identically zero, and a renumbering gives our set (1.26[b]).

This choice, where F_5' and F_6' are eliminated, results in a formalism for the calculation in which the virtual photon (or the vector part of the weak boson) is described in terms of transverse and scalar components. Alternatively, we could have chosen for transverse and longitudinal components by eliminating F_7' and F_8', which can be done by using $a_\mu = \varepsilon_\mu - \frac{\varepsilon_0}{k_0} K_\mu$ instead of b_μ. Our choice is motivated by the fact that it results in a simpler relation between $\{F_i\}$ and the multipole amplitudes that are to be introduced in part C.

C. Multipole amplitudes

We introduce an other way of describing the matrix element T_{fi} in the πN_2-centre-of-mass frame by using a decomposition with respect to angular momentum. The multipole amplitudes needed here are defined such that their properties are analogous to those of the familiar electric and magnetic multipoles that can be used in processes with real photons.

In Appendix B it is shown that this can be done by writing

$$T_{fi} = \chi_2^\dagger \sum_{\ell=0}^{\infty} \left[\sum_{i=1}^{6} M_{\ell i}\, Z_{\ell i} + \sum_{i=1}^{8} Ma_{\ell i}\, Za_{\ell i} \right] \chi_1 \tag{1.28}$$

where the $Z_{\ell i}$ ($Za_{\ell i}$) are 2x2-matrices that are defined in the same appendix. Current conservation is ensured in the same way as in eq. (1.26[b]), by using b_μ instead of ε_μ in the definitions of the $Z_{\ell i}$. For the multipole amplitudes we use the notation (see Appendix B for details)

$$\begin{aligned} &M_{\ell i} = (E_{\ell+}, E_{\ell-}, M_{\ell+}, M_{\ell-}, S_{\ell+}, S_{\ell-}) \ , \quad \text{with } i=1,\ldots 6 \\ \text{and}\quad &Ma_{\ell i} = (Ea_{\ell+}, Ea_{\ell-}, Ma_{\ell+}, Ma_{\ell-}, La_{\ell+}, La_{\ell-}, Sa_{\ell+}, Sa_{\ell-}) \ , \quad \text{with } i=1,\ldots 8 \end{aligned} \tag{1.29}$$

For these amplitudes we have again an isospin decomposition of the form $M_{\ell i} = \sum_n g_n M^n_{\ell i}$, with the same notation as before.

D. R e l a t i o n s   b e t w e e n   t h e   v a r i o u s   a m p l i t u d e s

Using the amplitudes defined in this chapter we can write for the matrix element for pion production

$$
\begin{aligned}
T_{fi} &= \varepsilon_\mu \; {}_{out}\langle \pi N_2 | (J_\mu^V + J_\mu^A) | N_1 \rangle_{in} = \\
&= \bar{u}(\vec{p}_2) \left[\Sigma_{i=1}^{6} A_i M_i + \Sigma_{i=1}^{8} Aa_i Ma_i \right] u(\vec{p}_1) = \\
&= \chi_2^\dagger \left[\Sigma_{i=1}^{6} F_i \Lambda_i + \Sigma_{i=1}^{8} Fa_i \Lambda a_i \right] \chi_1 = \\
&= \Sigma_{\ell=0}^{\infty} \chi_2^\dagger \left[\Sigma_{i=1}^{6} M_{\ell i} Z_{\ell i} + \Sigma_{i=1}^{8} Ma_{\ell i} Za_{\ell i} \right] \chi_1 \, . \qquad (1.30)
\end{aligned}
$$

The relations between these different sets of amplitudes can be obtained by straightforward calculation, and we will summarize here only the results, given in matrix notation. Square brackets [...] will denote a matrix (6x6 or 8x8) and a 6- or 8-component vector will be indicated by a tilde ~. Thus we write for the vector amplitudes (with K^2-dependence understood everywhere)

$$
\begin{aligned}
\tilde{A}(s,t) &= [B^{-1}(s,t)][C(s)]\tilde{F}(s,t) \\
\tilde{F}(s,t) &= \Sigma_{\ell=0}^{\infty} [G_\ell(x)]\tilde{M}_\ell(s)
\end{aligned} \qquad (1.31^a)
$$

and the reciprocal relations

$$
\begin{aligned}
\tilde{F}(s,t) &= [C^{-1}(s)][B(s,t)]\tilde{A}(s,t) \\
\tilde{M}_\ell(s,t) &= \int_{-1}^{1} dx\, [D_\ell(x)]\tilde{F}(s,t) \, .
\end{aligned} \qquad (1.32^a)
$$

Explicit forms of the matrices B, C, D, G are given in Appendix C. Some kinematical factors, depending only on s and K^2, have been collected in the matrix C and its inverse, in order to simplify the form of [B] and $[B^{-1}]$.

For the axial vector amplitudes similar relations hold, using now matrices [Ba(s,t)], etc. These formulas will be referred to as (1.31^b), (1.32^b).

I.5 GENERAL OUTLINE OF THE CALCULATIONS

Our aim will be to find numerical values for the cross-sections for pion production processes and to compare these with available experimental results. Via the scattering-matrix elements the cross-sections can be expressed in terms of the various amplitudes that were introduced in sec.I.4, and so the problem is to obtain values for these amplitudes.

Due to the presence of the strong interaction we can not find directly a solvable set of equations for the amplitudes. We can make however some general assumptions about properties of the S-matrix elements. These lead to a set of integral equations for the invariant amplitudes, the so-called dispersion relations. These relations can then be transformed into a set of coupled integral equations for the multipole amplitudes. This programme is carried out in Chapter II.

For a solution of the integral equations we still need information about the strong interaction. In our case this can be extracted from pion-nucleon scattering, since under certain conditions (described in Chapter III) we can relate the multipole amplitudes for pion production to the partial wave amplitudes for pion-nucleon scattering, which are well-known from experiments. (For this reason we have introduced the multipoles in this chapter.) By further making a few approximations we are then able to find a numerical solution for the multipole equations. In Chapter III this method of solution is described and the results for the cross-sections are discussed and compared with the experimental values.

CHAPTER II

ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDES FOR PION PRODUCTION

II.1 INTRODUCTION

In Chapter I we noted that the S-matrix elements for pion production are partly determined by the strong interaction. Since there is no straightforward method to calculate the effects of this interaction, we have to rely on general properties of the S-matrix elements to obtain information about the unknown strong interaction part (see e.g. [Ed66]). Of these general requirements the most important one is the postulate that the invariant amplitudes, in terms of which the S-matrix elements can be expressed (sec.I.4), have certain analytic properties in suitably chosen kinematical variables. In sec.II.2 we state this postulate and derive from it the general form of the dispersion relations that have to be satisfied by the amplitudes. Useful symmetry relations for the amplitudes ("crossing symmetry") are obtained in sec.II.3. In sec.II.4 we find more information about the analytic structure of the amplitudes from the unitarity of the S-matrix, and from the assumption of what is called "extended unitarity". The pole terms occurring in the dispersion relations are derived in sec.II.5, while sec.II.6 gives the final form in which the dispersion relations can be written, using previous results. From these relations finally the integral equations for the multipole amplitudes are derived.

II.2 ANALYTICITY

We have seen in Chapter I that for a scattering process $XN \to \pi N$

(cf. fig. 2) we can express the scattering-matrix element in terms of a set of invariant amplitudes, as defined in sec.I.4, which are functions of the variables s, t and u. (Only two of these variables are independent, due to the relation $s+t+u=\sum_i m_i^2$ (1.11).) For such a process s equals the square of the total energy in the centre-of-mass frame.

Together with this process we consider two others, obtained from the first one by replacing an incoming particle by an outgoing antiparticle and vice versa, i.e. $X\bar{\pi}\to N\bar{N}$ and $X\bar{N}\to\pi\bar{N}$. For these two processes we can again define similar sets of amplitudes, depending on s, t and u. (These variables are still defined by (1.10) where it is understood that the particles retain their labels in the "crossing".) In the first of these two cases the square of the total centre-of-mass energy is now given by the variable t, while in the second it is given by u. Using this as a distinction between the three processes these are often called s-, t- and u-channel processes respectively. We note that for the different channels the regions in which the variables s, t and u have physical values (the "physical regions") are disjoint.

It is clear that in general a different set of amplitudes is needed for each channel. The analyticity postulate states that only one set of amplitudes is needed for a description of all three channels and that these amplitudes are meromorphic functions of two variables (e.g. s and t) with only those singularities that have to be present because of other assumptions about the S-matrix. (The origin and nature of these singularities will be discussed in sec.II.4.) This means that given the set of amplitudes in one physical region, the amplitudes for the two other processes can be obtained by analytic continuation from this first set.

We will first consider the analytic properties of the amplitudes as a function of complex s, for fixed real t; so we need to know the singularities of the amplitudes in the complex s-plane. We will find (sec.II.4) that these consist of branch cuts along the positive and negative real axis (from $-\infty$ to b and from a to $+\infty$ (see fig.3)), together with two poles at $s=m^2$ and at $u=m^2$ (s=c and s=d in fig. 3). In sec.II.4 we also find that the amplitudes are real analytic functions, i.e. $B_j(s^*,t) = B_j^*(s,t)$. We now take the contour C in fig. 3 and apply Cauchy's theorem. This leads to

$$\int_C \frac{B_j(s',t)}{s'-s}\, ds' = 2\pi i \left\{ B_j(s,t) + \frac{R^j_s}{m^2-s} + \frac{R^j_u}{m^2-u} \right\}, \qquad (2.1)$$

where s is a point within C, and R^j_s and R^j_u are the residues of the poles at c and d respectively. When we take the limit where the radius of the

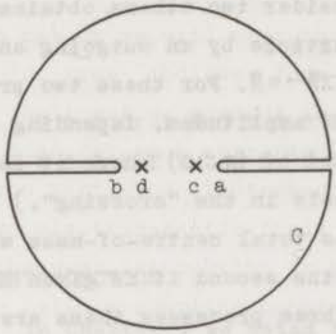


fig. 3

$a = (m+\mu)^2$
$b = m(m-2\mu)-t-K^2$
$c = m^2$
$d = m^2+\mu^2-t-K^2$

circle in fig. 3 becomes infinite, the contribution of the circle to the integral in (2.1) vanishes, provided that the amplitudes approach zero sufficiently fast as $s \to \infty$. If this is indeed the case, as we will assume for the moment, we are left with

$$B_j(s,t) = \frac{R^j_s}{s-m^2} + \frac{R^j_u}{u-m^2} + \frac{1}{2\pi i}\int_{-\infty}^{b} ds' \frac{\mathrm{disc}[B_j]_u}{s'-s} + \frac{1}{2\pi i}\int_{a}^{\infty} ds' \frac{\mathrm{disc}[B_j]_s}{s'-s}, \qquad (2.2)$$

where disc $[B_j]_{s,u}$ are the discontinuities of the amplitudes across the branch cuts on the positive and negative real s-axis (i.e. in the s- and u-channel physical region). In (2.1) and (2.2) s is complex and in order to obtain the physical amplitudes it has to approach the real axis. To specify how this limit is to be taken, we define in the conventional way the physical amplitude in the s-channel (s real; $s > (m+\mu)^2 = a$) as $B_j(s,t) = \lim_{\varepsilon \downarrow 0} B_j(s+i\varepsilon,t)$. In the u-channel ($u > (m+\mu)^2$, i.e. $s < -t-K^2+m(m-2\mu) = b$) we then have to take the limit $u+i\varepsilon$ with $\varepsilon \downarrow 0$, which means that on the left-hand cut in the s-plane we have $B_j(s,t) = \lim_{\varepsilon \downarrow 0} B_j(s-i\varepsilon,t)$.

From the fact that the amplitudes are real analytic functions, $B_j(s^*,t) = B^*_j(s,t)$ (cf. sec.II.4 and ref.[0162]) it follows that for real values of s

$$\text{disc}\,[B_j] = 2i\,\text{Im}\,B_j\,,$$

and that the residues of the poles are real. We can then rewrite (2.2) for real s as

$$B_j(s,t) = \frac{R_s^j}{s-m^2} + \frac{R_u^j}{u-m^2} + \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{b} ds'\,\frac{\text{Im}B_j(s',t)}{s'-s\pm i\varepsilon} + \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{a}^{\infty} ds'\,\frac{\text{Im}B_j(s',t)}{s'-s\pm i\varepsilon}\,, \tag{2.3}$$

where we have to use $+i\varepsilon$ if s lies on the left-hand cut, and $-i\varepsilon$ for the right-hand cut. Taking the real part of (2.3) and using

$$\text{Re}\,\frac{1}{s'-s\pm i\varepsilon} = P\,\frac{1}{s'-s}$$

(P denotes the principal value) gives a dispersion relation

$$\text{Re}\,B_j(s,t) = \frac{R_s^j}{s-m^2} + \frac{R_u^j}{u-m^2} + \frac{P}{\pi}\int_{-\infty}^{b} ds'\,\frac{\text{Im}B_j(s',t)}{s'-s} + \frac{P}{\pi}\int_{a}^{\infty} ds'\,\frac{\text{Im}B_j(s',t)}{s'-s}\,,$$

or, defining u' by $s'+t+u' = \sum_i m_i^2$,

$$\text{Re}\,B_j(s,t) = \frac{R_s^j}{s-m^2} + \frac{R_u^j}{u-m^2} + \frac{P}{\pi}\int_{a}^{\infty} \frac{ds'}{s'-s}\,\text{Im}[B_j(s',t)+B_j(u',t)]\,. \tag{2.4}$$

In the same way we can consider the analytic properties of $B_j(s,t)$ for complex t and fixed real s. Then we obtain the result (as shown in sec.II.4) that the expression for the amplitude $B_j(s,t)$ contains a pole term $\frac{R_t^j}{t-\mu^2}$, analogous to the s- and u-channel poles. To obtain a correct behaviour for $B_j(s,t)$ at $s\to\infty$, this term has to be included in eqs.(2.1)..(2.4), so that finally we find the dispersion relation for fixed t

$$\text{Re}\,B_j(s,t) = \frac{R_s^j}{s-m^2}+\frac{R_t^j}{t-\mu^2}+\frac{R_u^j}{u-m^2}+\frac{P}{\pi}\int_{a}^{\infty}\frac{ds'}{s'-s}\,\text{Im}[B_j(s',t)+B_j(u',t)]\,. \tag{2.5}$$

We will not discuss possible convergence problems for the dispersion relation (2.5), although in sec.II.5 some remarks are made concerning subtractions that are necessary to avoid evident divergencies.

In deriving these results the argument was based on the postulated analyticity properties of the amplitudes as functions of one complex variable, while the other had a fixed real value. It is also possible to consider the analytic structure of the amplitudes as functions of two complex variables. This led Mandelstam [Ma58] to postulate the double spectral representation for the invariant amplitudes, from which the simple (one-dimensional) dispersion relation can again be derived. This double dispersion relation or Mandelstam representation for the amplitudes B_j reads as follows

$$B_j(s,t) = \frac{R_s^j}{s-m^2} + \frac{R_t^j}{t-\mu^2} + \frac{R_u^j}{u-m^2} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{\rho_s^j(s')}{s'-s} +$$

$$+ \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\rho_t^j(t')}{t'-t} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du' \frac{\rho_u^j(u')}{u'-u} +$$

$$+ \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \frac{b_{st}^j(s',t')}{(s'-s)(t'-t)} +$$

$$+ \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{(m+\mu)^2}^{\infty} du' \frac{b_{su}^j(s',u')}{(s'-s)(u'-u)} +$$

$$+ \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} dt' \int_{(m+\mu)^2}^{\infty} du' \frac{b_{tu}^j(t',u')}{(t'-t)(u'-u)} . \qquad (2.6)$$

This representation can be made plausible by taking (2.2) (including the t-channel pole) and considering disc$[B_j]_{s,u}$ as an analytic function in t, and writing a dispersion relation for the discontinuity in that variable. Evidently we need several assumptions about convergence properties of the integrals in (2.6), i.e. about the behaviour of the double and single spectral functions for large s and t.

In this section we have used only the amplitudes B_j, but completely similar relations can be obtained for the axial vector amplitudes $\{Ba_i\}$ or $\{Aa_i\}$ as defined in sec.I.4. It can be shown [Ba61], [He61] that these amplitudes do not contain any kinematical singularities, so that analytic properties could be postulated for them. In the set of vector amplitudes $\{A_i\}$ however, a kinematical singularity appears in A_2 and A_5 due to current conservation relations (1.21). Except for the effects of these extra singularities the A_i have still the same analytic properties as the B_i. Since in practical calculations it is more convenient to have current conservation built in already, in the rest of this chapter we will use mainly the amplitudes A_i (and Aa_i). The complications due to the kinematical singularities are discussed in sec.II.5.

II.3 CROSSING SYMMETRY

As a consequence of the analyticity postulate the matrix elements in all three channels can be expressed in terms of the amplitudes B_i, or equivalently in terms of A_i. For the matrix elements of the (electromagnetic or weak) vector hadron current J_μ^V, this substitution rule reads

$$\begin{aligned}
{}_{out}\langle N_2(P_2),\pi(Q)|\,J_\mu^V|\,N_1(P_1)\rangle_{in} &= \bar{u}(\vec{p}_2)[\textstyle\sum_i A_i(s,t,u)M_{i\mu}]\,u(\vec{p}_1) \\
-{}_{out}\langle N_2(P_2),\bar{N}_1(-P_1)|\,J_\mu^V|\bar{\pi}(-Q)\rangle_{in} &= \bar{u}(\vec{p}_2)[\textstyle\sum_i A_i(s,t,u)M_{i\mu}]\,v(-\vec{p}_1) \\
{}_{out}\langle \bar{N}_1(-P_1),\pi(Q)|\,J_\mu^V|\,\bar{N}_2(-P_2)\rangle_{in} &= \bar{v}(-\vec{p}_2)[\textstyle\sum_i A_i(s,t,u)M_{i\mu}]\,v(-\vec{p}_1)
\end{aligned} \tag{2.7}$$

for the s-, t-, and u-channel matrix elements, respectively. Analogous expressions can be given for the matrix elements of the weak axial vector hadron current J_μ^A. To derive "crossing" symmetry relations between the amplitudes A_i for different values of their arguments, we combine eqs.(2.7) with the behaviour of the matrix elements under charge conjugation. The corresponding operator C_{st} (in the following denoted as C) transforms a particle into its anti-particle; it is defined such that it is conserved by the strong interaction (see Appendix A).

We need the transformation properties

$$\begin{aligned}
C\,J_\mu^V\,C^{-1} &= -J_\mu^V \\
C\,J_\mu^A\,C^{-1} &= J_\mu^A \\
C\,\pi^\alpha\,C^{-1} &= -(-1)^\alpha\,\pi^\alpha\,,
\end{aligned} \tag{2.8}$$

where we have assumed that the currents $J_\mu^{V,A}$ are both first class currents [We58] (for second class currents we would get the opposite sign in (2.8)), and for the pion we have used eigenstates π^α of C, π^α denoting the α-component of the pion field (α=1,2,3), i.e. a linear combination of physical pion states. $(\pi^1=\frac{1}{\sqrt{2}}(\pi^+ +\pi^-);\pi^2=-\frac{i}{\sqrt{2}}(\pi^+-\pi^-);\pi^3=\pi^0.)$ Using the transformation properties we can write

$$ {}_{\text{out}}\langle N_2(P_2),\pi^\alpha(Q)|J^V_\mu|N_1(P_1)\rangle_{\text{in}} = (-1)^\alpha\ {}_{\text{out}}\langle \bar N_2(P_2),\pi^\alpha(Q)|J^V_\mu|\bar N_1(P_1)\rangle_{\text{in}}, \tag{2.9}$$

and a similar relation for the J^A_μ matrix element, where an extra minus has to be included. Apart from different signs for P_i the right-hand side of (2.9) contains just the u-channel matrix element. Indicating again explicitly the isospin dependence of the amplitudes and the momentum dependence of the matrices $M_{i\mu}$ (using the notation from sec.I.3) we find from eqs.(2.7) and (2.9) for the vector current in electroproduction

$$\begin{aligned}
&\chi_2^\dagger\ \bar u(\vec p_2)[\textstyle\sum_i(A_i^0(s,t,u)\tau_\alpha + A_i^+(s,t,u)\delta_{\alpha 3} +\\
&+ A_i^-(s,t,u).\tfrac{1}{2}[\tau_\alpha,\tau_3])M_{i\mu}(P,K,Q)]u(\vec p_1)\chi_1 =\\
&= -(-1)^\alpha\ \chi_1^\dagger\ \bar v(\vec p_1)[\textstyle\sum_i(A_i^0(u,t,s)\tau_\alpha + A_i^+(u,t,s)\delta_{\alpha 3} +\\
&+ A_i^-(u,t,s).\tfrac{1}{2}[\tau_\alpha,\tau_3])M_{i\mu}(-P,K,Q)v(\vec p_2)\chi_2 =\\
&= -(-1)^\alpha\ \chi_1^\dagger\ (-u^T(\vec p_1)C^{-1})[.(\ldots).](C\ \bar u^T(\vec p_2))\chi_2 =\\
&= -(-1)^\alpha\ \chi_1^\dagger\ (-u^T(\vec p_1))[\textstyle\sum_i(\ldots)C^{-1}M_{i\mu}(-P,K,Q)C]\bar u^T(\vec p_2)\chi_2 =\\
&= -(-1)^\alpha\ \chi_1^\dagger\ u^T(\vec p_1)[\textstyle\sum_i(\ldots)\eta_i\ M^T_{i\mu}(P,K,Q)]\bar u^T(\vec p_2)\chi_2 =\\
&= \chi_2^\dagger\ \bar u(\vec p_2)[\textstyle\sum_i(A_i^0(u,t,s)\tau_\alpha + A_i^+(u,t,s)\delta_{\alpha 3} -\\
&- A_i^-(u,t,s).\tfrac{1}{2}[\tau_\alpha,\tau_3])\eta_i\ M_{i\mu}(P,K,Q)]u(\vec p_1)\chi_1\ .
\end{aligned}\tag{2.10}$$

The χ_i are isospinors, and we have defined η_i by means of the relation

$$-C^{-1}\ M_{i\mu}(-P,K,Q)C = \eta_i\ M^T_{i\mu}(P,K,Q)\ .$$

From the explicit form of the $M_{i\mu}$ (1.22^b) we find $\{\eta_i\}$ to be (+1,+1,-1, +1,-1,-1).

In neutrinoproduction we have to omit the term with $A_i^0\tau_\alpha$, and we must replace $\delta_{\alpha 3}$ by $\frac{1}{2}\{\tau_\alpha,\tau_\pm\}$, and $[\tau_\alpha,\tau_3]$ by $[\tau_\alpha,\tau_\pm]$. For the weak vector current only these changes are necessary in eq.(2.10), and consequently $\{\eta_i\}$ is the same as for electroproduction. For the weak axial vector current we have to use Aa_i, η_{ai} and $Ma_{i\mu}$, while we also get an additional minus sign. (Note that here i=1,..8.) The result for $\{\eta_{ai}\}$ is then (-1,-1,+1,-1,+1,-1,+1,-1).

We now define ξ=1 for amplitudes with + or 0 isospin index and

$\xi=-1$ for - isospin index. The matrix $[\xi]$ is defined by $[\xi]_{ij}=\xi\delta_{ij}\eta_i$. The crossing relations that follow from (2.10) can then be summarized by

$$\tilde{A}(s,t,u) = [\xi]\tilde{A}(u,t,s)$$

and

$$\tilde{A}a(s,t,u) = [\xi a]\tilde{A}a(u,t,s) .$$

II.4 UNITARITY

Defining T by $S=1+iT$, the unitarity relation $S^\dagger S=SS^\dagger=1$ for the S-operator can be rewritten as $(T-T^\dagger)=iT^\dagger T$, or in terms of matrix elements between initial and final states i and f

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle , \qquad (2.11)$$

where the summation is over a complete set of states n. For the matrix element $\langle f|T|i\rangle$ we have ($\varepsilon\downarrow 0$ understood)

$$\langle f|T|i\rangle \equiv T_{fi} = \bar{u}(\vec{p}_2)\sum_j A_j(s+i\varepsilon,t)M_j\, u(\vec{p}_1) , \qquad (2.12)$$

while it can be shown [O162] that if the A_j are analytic functions in s

$$\langle f|T^\dagger|i\rangle = \langle i|T|f\rangle^* = \bar{u}(\vec{p}_2)\sum_j A_j(s-i\varepsilon,t)M_j\, u(\vec{p}_1) , \qquad (2.13)$$

so that opposite boundary values of the amplitudes appear in these two expressions. Substituting this in the unitarity relation (2.11) we find

$$\bar{u}(\vec{p}_2)\sum_j\{A_j(s+i\varepsilon,t)-A_j(s-i\varepsilon,t)\}M_j\, u(\vec{p}_1) = i\sum_n\langle f|T^\dagger|n\rangle\langle n|T|i\rangle ,$$

or

$$\bar{u}(\vec{p}_2)\sum_j \mathrm{disc}[A_j]_{s,u}\, M_j\, u(\vec{p}_1) = i\sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle . \qquad (2.14)$$

The right-hand side of this equation is non-zero when a state n exists that gives non-vanishing matrix elements. In that case at least for some of the amplitudes A_j there must be a discontinuity corresponding to a branch cut singularity or a pole on the real s-axis (see e.g. [Ed66]). For low energies in the physical region with s or u between $(m+\mu)^2$ and $(m+2\mu)^2$ the only state n that is possible will be a pion-nucleon state. Consequently there will be cuts on the real s-axis from $-\infty$ to b and from a to $+\infty$ (cf.fig.3). A new cut will start at each value of s or u where a new state becomes possbile; these cuts can be taken along the real axis as well. In the t-channel the first cuts arise from two- or three-pion states, starting at $t=4\mu^2$ or $9\mu^2$, respectively. (In this channel, due to G-parity [Le56A], states with an even number of pions contribute only to isoscalar 4-vector amplitudes and to isovector axial 4-vector amplitudes; those with an odd number of pions only to isovector 4-vector amplitudes.) The effects of these t-channel cuts will be neglected in the following calculations.

Since the unitarity relation holds only in the physical region, we obtain no information about the part of the real s-axis between b and a. Therefore we introduce here an extra assumption, usually called "extended unitarity", which states that eq.(2.14) holds also for unphysical values of s and t. From this we conclude that the right-hand side of eq.(2.14) vanishes on the real s-axis between b and a, except in the points $c=m^2$ and $d=m^2+\mu^2-K^2-t$, which correspond to n being a one-nucleon state. In the t-channel there will be a singularity at $t=\mu^2$, due to the one-pion state (again contributing only to isovector 4-vector amplitudes). The singularities in these three points are single poles, which are discussed further in the next section.

Using the behaviour of the amplitudes under charge conjugation, it can finally be shown [O162] that A_j (and B_j) are real analytic functions, i.e. $A^*_j(z^*,t)=A_j(z,t)$. This means in particular that we have

$$A^*_j(s+i\varepsilon,t) = A_j(s-i\varepsilon,t) ,$$

and thus

$$\mathrm{disc}[A_j] = \lim_{\varepsilon\downarrow 0} [A_j(s+i\varepsilon,t)-A_j(s-i\varepsilon,t)] = 2i \lim_{\varepsilon\downarrow 0} \mathrm{Im}\, A_j(s+i\varepsilon,t)$$
$$= 2i\, \mathrm{Im}\, A_j(s,t) .$$

An other consequence of this is of course the reality of the pole residues which follows immediately from the fact that the poles lie on the real axis.

From the assumptions of unitarity and extended unitarity we thus find several singularities for the amplitudes on the real s-axis. The analyticity postulate for the amplitudes is that these are the only singularities in the whole complex s-plane ("maximal analyticity").

II.5 POLE TERMS

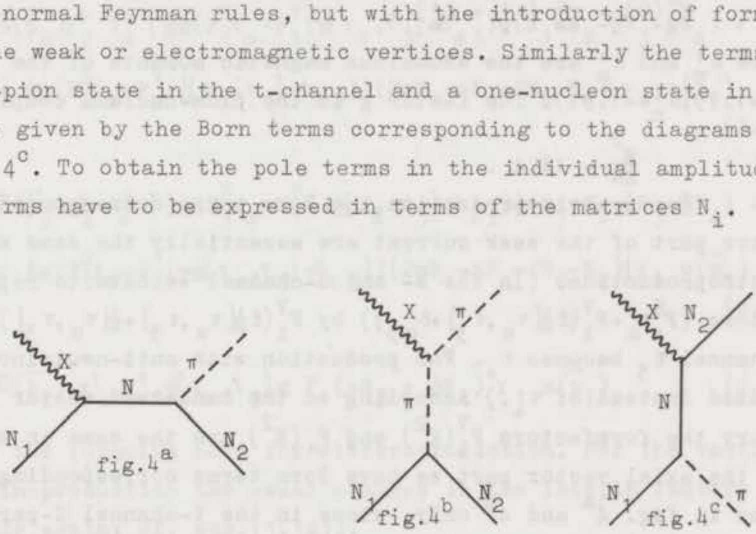
To obtain the explicit form of the pole terms in the dispersion relations we use again the amplitudes B_j. (This is done to avoid at this stage problems connected with kinematical singularities in the A_j.)

Thus eq.(2.14) is written as

$$\bar{u}(\vec{p}_2)\sum_j[\,\mathrm{Im}\;B_j(s,t)]N_j\;u(\vec{p}_1)=\tfrac{1}{2}\sum_n\langle f|T^\dagger|n\rangle\langle n|T|i\rangle\quad . \tag{2.16}$$

On the right-hand side of this equation, in the term where n is a one-nucleon state in the s-channel, a factor $\delta(s-m^2)$ will appear because the nucleon has to be on the mass-shell. The contribution of this term to the form $\bar{u}(\vec{p}_2)\sum_j B_j N_j u(\vec{p}_1)$ will then contain a factor $\frac{1}{s-m^2}$. It can be shown (cf. [Ma70],p.284ff) that this contribution is just the renormalized Born term, corresponding to the diagram in fig. 4^a, calculated with normal Feynman rules, but with the introduction of form-factors at the weak or electromagnetic vertices. Similarly the terms due to a one-pion state in the t-channel and a one-nucleon state in the u-channel are given by the Born terms corresponding to the diagrams in figs. 4^b and 4^c. To obtain the pole terms in the individual amplitudes these Born terms have to be expressed in terms of the matrices N_i.

fig. 4



For the electroproduction process the Born terms read as follows

s-channel

$$g\;\bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger\gamma_5\;\frac{i\gamma\cdot(K+P_1)-m}{(K+P_1)^2+m^2}\;\{\tfrac{1}{2}(F_1^S\tau_\alpha+F_1^V(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))\gamma_\mu\;+$$

$$+\;\tfrac{1}{2}(F_2^S\tau_\alpha+F_2^V(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))\sigma_{\mu\nu}K_\nu\}\varepsilon_\mu\;\chi_1\;u(\vec{p}_1) \tag{2.17}$$

u-channel

$$g\;\bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger\{\tfrac{1}{2}(F_1^S\tau_\alpha+F_1^V(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))\gamma_\mu\;+$$

$$+\;\tfrac{1}{2}(F_2^S\tau_\alpha+F_2^V(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))\sigma_{\mu\nu}K_\nu\}\varepsilon_\mu\;\frac{i\gamma\cdot(P_2-K)-m}{(P_2-K)^2+m^2}\gamma_5\chi_1 u(\vec{p}_1) \tag{2.18}$$

t-channel

$$i\;g\;\bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger\gamma_5\cdot\tfrac{1}{2}[\tau_\alpha,\tau_3]\frac{1}{(Q-K)^2+\mu^2}\;\{2(\varepsilon\cdot Q)-(\varepsilon\cdot K)\}\,F_\pi\;\chi_1\;u(\vec{p}_1) \tag{2.19}$$

The isospin notation is the same as in sec.I.3. The formfactors $F_i(K^2)$ of the nucleons (i=1,2 for Dirac- and Pauli-formfactos) have been decomposed into an isoscalar and an isovector part ($F_i^V=F_i^p-F_i^n$; $F_i^S=F_i^p+F_i^n$), and $F_\pi(K^2)$ is the electromagnetic formfactor of the pion. The normalization is given by *)

$$F_1^S(0) = F_1^V(0) = F_\pi(0) = 1$$

$$F_2^V(0) = \frac{1}{2m}(\mu'_p - \mu'_n)$$

$$F_2^S(0) = \frac{1}{2m}(\mu'_p + \mu'_n) \; ,$$

where μ'_p and μ'_n are the anomalous magnetic moments of the nucleons ($\mu'_p=1.79;\mu'_n=-1.91$). The factor g is the pion-nucleon coupling constant,

$$\frac{g^2}{4\pi} = 14.4 \; .$$

For neutrinoproduction, the Born terms corresponding to the vector part of the weak current are essentially the same as for electroproduction. (In the s- and u-channel we have to replace the factors $(F_i^S\tau_\alpha+F_i^V(\pm\frac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))$ by $F_i^V(\pm\frac{1}{2}[\tau_\alpha,\tau_3]+\frac{1}{2}\{\tau_\alpha,\tau_3\})$, and in the t-channel τ_3 becomes τ_+. For production with anti-neutrinos τ_- should be used instead of τ_+.) According to the conserved vector current theory the formfactors $F_i^V(K^2)$ and $F_\pi(K^2)$ are the same in both cases. For the axial vector part we have Born terms corresponding to the diagrams in fig. 4[a] and 4[c] only, since in the t-channel G-parity [Le56A] allows only intermediate states with an even number of pions in this case (cf. sec.II.4). The axial vector Born terms are

s-channel

$$\tfrac{1}{2}\, g\, \bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger \gamma_5 \frac{i\gamma\cdot(K+P_1)-m}{(K+P_1)^2+m^2}\{G_A(\tfrac{1}{2}[\tau_\alpha,\tau_\pm] + \tfrac{1}{2}\{\tau_\alpha,\tau_\pm\})\gamma_\mu\gamma_5 -$$

$$- iH_A(\tfrac{1}{2}[\tau_\alpha,\tau_\pm] + \tfrac{1}{2}\{\tau_\alpha,\tau_\pm\})\gamma_5 K_\mu\}\varepsilon_\mu\, \chi_1\, u(\vec{p}_1) \qquad (2.20)$$

u-channel

$$\tfrac{1}{2}\, g\, \bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger\, \{G_A(-\tfrac{1}{2}[\tau_\alpha,\tau_\pm] + \tfrac{1}{2}\{\tau_\alpha,\tau_\pm\})\gamma_\mu\gamma_5 -$$

$$- iH_A(-\tfrac{1}{2}[\tau_\alpha,\tau_\pm]+\tfrac{1}{2}\{\tau_\alpha,\tau_\pm\})\gamma_5 K_\mu\}\varepsilon_\mu \frac{i\gamma\cdot(P_2-K)-m}{(P_2-K)^2+m^2}\gamma_5\chi_1\, u(\vec{p}_1) \qquad (2.21)$$

---

*) a factor e that would appear normally in eqs.(2.17,18,19) has been absorbed in ε_μ (eq.(1.8[a])) in order to obtain a simpler correspondence between electromagnetic and weak amplitudes.

The axial vector formfactor $G_A(K^2)$ is normalized by $G_A(0)=g_A=1.23\pm0.01$ (experimental value; cf. [Ch67]); $H_A(K^2)$ is the induced pseudoscalar formfactor (see e.g. [Be68]). If we may assume that the axial vector current is "partially conserved" (PCAC-theory; for a review, see e.g. Ma69), then we have $H_A(K^2)=2mG_A(K^2)/(K^2+\mu^2)$.

To obtain the contributions of the Born terms to the amplitudes B_j we express them in terms of the matrices N_j (sec.I.4, eqs.(1.18,19)) and find the following results.

Vector part, s-channel

$$\frac{1}{s-m^2}\,\bar{u}(\vec{p}_2)\chi_2^\dagger\, V_\alpha^\dagger\,\{\tfrac{1}{2}g(F_1^S\tau_\alpha+F_1^V(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))(N_1-N_2-\tfrac{1}{2}N_3-\tfrac{1}{2}N_4)+$$
$$+\tfrac{1}{2}g(F_2^S\tau_\alpha+F_2^V(\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))(2mN_1-mN_4+2N_6+N_8)\}\chi_1\, u(\vec{p}_1) \tag{2.22}$$

u-channel

$$\frac{1}{u-m^2}\,\bar{u}(\vec{p}_2)\chi_2^\dagger\, V_\alpha^\dagger\,\{\tfrac{1}{2}g(F_1^S\tau_\alpha+F_1^V(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))(N_1-N_2+\tfrac{1}{2}N_3-\tfrac{1}{2}N_4)+$$
$$+\tfrac{1}{2}g(F_2^S\tau_\alpha+F_2^V(-\tfrac{1}{2}[\tau_\alpha,\tau_3]+\delta_{\alpha 3}))(2mN_1-mN_4+2N_6-N_8)\}\chi_1\, u(\vec{p}_1) \tag{2.23}$$

t-channel

$$\frac{1}{t-\mu^2}\,\bar{u}(\vec{p}_2)\,\chi_2^\dagger\, V_\alpha^\dagger\cdot\tfrac{1}{2}[\tau_\alpha,\tau_3]\,g\,F_\pi(-N_3+\tfrac{1}{2}N_4)\,\chi_1\, u(\vec{p}_1) \tag{2.24}$$

(In this form the formulae hold for electroproduction. For the vector part in neutrinoproduction the usual changes in the isospin factors have to be made again; cf. eqs.(1.14)).

Axial vector part, s-channel

$$\frac{1}{s-m^2}\,\bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger\{-\tfrac{1}{2}g(\tfrac{1}{2}[\tau_\alpha,\tau_\pm]+\tfrac{1}{2}\{\tau_\alpha,\tau_\pm\})(G_A(Ma_1+Ma_3)+H_A Ma_8)\}\chi_1\, u(\vec{p}_1) \tag{2.25}$$

u-channel

$$\frac{1}{u-m^2}\,\bar{u}(\vec{p}_2)\chi_2^\dagger V_\alpha^\dagger\{\tfrac{1}{2}g(-\tfrac{1}{2}[\tau_\alpha,\tau_\pm]+\tfrac{1}{2}\{\tau_\alpha,\tau_\pm\})(G_A(Ma_1-Ma_3)+H_A Ma_8)\}\chi_1\, u(\vec{p}_1) \tag{2.26}$$

From these equations we can obtain the residues. For the isoscalar 4-vector amplitudes B_j^0 we have

$$R^1_{s,u}=\tfrac{1}{2}g\,(F_1^S+2mF_2^S) \qquad R^4_{s,u}=-\tfrac{1}{4}g\,(F_1^S+2mF_2^S)$$
$$R^2_{s,u}=-\tfrac{1}{2}g\,F_1^S \qquad R^6_{s,u}=g\,F_2^S \tag{2.27^a}$$
$$R^3_{s,u}=\mp\tfrac{1}{4}g\,F_1^S \qquad R^8_{s,u}=\pm\tfrac{1}{2}g\,F_2^S$$

(upper sign for s-channel, lower for u-channel residues). For isovector (+ or -) amplitudes $B_j^{\pm}$ the residues are obtained from (2.27) by changing F_i^S to F_i^V and, in the case of (-) amplitudes, adding an extra - sign for the u-channel residues. For the (-) amplitudes only, we also have residues

$$R_t^3 = - 2R_t^4 = - g F_\pi . \tag{2.27^b}$$

For the isovector axial 4-vector amplitudes $Aa_j^{\pm}$ we have

$$\begin{aligned} Ra_s^1 &= - g G_A & Ra_u^1 &= \pm g G_A \\ Ra_s^3 &= - g G_A & Ra_u^3 &= \mp g G_A \\ Ra_s^8 &= - g H_A & Ra_u^8 &= \pm g H_A \end{aligned} \tag{2.28}$$

(upper or lower sign for (+) or (-) amplitudes). All other residues are zero.

Next we have to consider again the question of subtractions. Therefore we first recall the two current conservation restrictions (1.21) for the amplitudes B_j

$$\tfrac{1}{2} K^2 B_1 + P.K B_2 + Q.K B_3 + K^2 B_4 = 0$$

$$B_5 + P.K B_6 + K^2 B_7 + Q.K B_8 = 0 .$$

Substituting here for the B_j only the pole contributions, we find that for the isovector (-) amplitudes the left-hand side adds up to a non-zero constant. To ensure compatibility between the dispersion relations and the current conservation conditions, we have to include a subtraction constant in the dispersion relations for two of the amplitudes B_j^- to cancel these constant terms. This can be done conveniently by adding a term $C_4^- = \frac{1}{2}g(F_\pi - F_1^V)/K^2$ to the dispersion relations for B_4^-, and $C_5^- = gF_2^V$ to the one for B_5^-. (This is allowed since B_4 and B_5 are even under crossing ($s \rightleftarrows u$).) Further subtractions in the fixed-t dispersion relations might follow from the one-dimensional spectral functions ρ^j in the Mandelstam representation (2.6). For the vector current however, the compatibility with current conservation requires all ρ^j to be zero. We will assume that this is also the case for the axial vector current.

The fixed-t dispersion relations for the amplitudes A_i are now obtained from those for the B_j by using eqs.(1.23). With this step we clearly introduce a kinematical singularity in A_2 and A_5. In principle this should not cause problems, since in all physical quantities the amplitudes appear in the combination $\sum_j A_j M_j$, in which the singularity is cancelled. In approximate calculations, and in numerical work, care should be taken to avoid spurious effects of these singularities (cf. Appendix E). With the results of sec.II.3 the pole contributions to the amplitudes $\tilde{A}$ (using again vector notation) can now be given in the form

$$\{ \frac{1}{s-m^2} + [\xi] \frac{1}{u-m^2} \}\tilde{\Gamma}(t) + \tfrac{1}{2}(1-\xi) \frac{\tilde{\Gamma}_t}{t-\mu^2} , \tag{2.29}$$

and similarly (using $\tilde{\Gamma}a(t)$ and omitting the $\tilde{\Gamma}_t$-term) for the amplitudes $\tilde{A}a$. We have absorbed the two subtraction constants in the $\tilde{\Gamma}$, which are defined by

$$\begin{aligned}
&\Gamma^{\pm}_1 = \tfrac{1}{2}g\, F^V_1(K^2) && \Gamma a^{\pm}_1 = -g\, G_A(K^2) \\
&\Gamma^{\pm}_2 = -\frac{g}{t-\mu^2}\, F^V_1(K^2) && \Gamma a^{\pm}_2 = 0 \\
&\Gamma^{\pm}_3 = \Gamma^{\pm}_4 = -\tfrac{1}{2}g\, F^V_2(K^2) && \Gamma a^{\pm}_3 = -g\, G_A(K^2) \\
&\Gamma^{\pm}_5 = -\tfrac{1}{2}\frac{g}{t-\mu^2}\, F^V_1(K^2) && \Gamma a^{\pm}_4 = \Gamma a^{\pm}_5 = \Gamma a^{\pm}_6 = \Gamma a^{\pm}_7 = 0 \\
&\Gamma^{\pm}_6 = 0 && \Gamma a^{\pm}_8 = -g\, H_A(K^2) \\
&\Gamma_{t,j} = \delta_{j5}\, \frac{2g}{K^2}[\, F_\pi(K^2) - F^V_1(K^2)\,] .
\end{aligned} \tag{2.30}$$

The Γ^0_j are obtained from the $\Gamma^{\pm}_j$ by replacing F^V_i by F^S_i. These relations simply follow from eqs.(2.27) and (2.28), while for the vector amplitudes we also have to use eqs.(1.23), giving the connection between $\{A_i\}$ and $\{B_i\}$.

II.6 DISPERSION RELATIONS AND MULTIPOLE EQUATIONS

Combining the results of the previous sections, we can transform the dispersion relation (2.5) for the amplitudes B_j to the relation for the set $\{A_j\}$ (j=1,...6), written as

$$\mathrm{Re}\,\widetilde{A}(s,t) = \left\{\frac{1}{s-m^2} + [\xi]\,\frac{1}{u-m^2}\right\}\widetilde{\Gamma}(t) + \tfrac{1}{2}(1-\xi)\,\frac{\widetilde{\Gamma}_t}{t-\mu^2} +$$

$$+ \frac{P}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\left\{\frac{1}{s'-s} + [\xi]\,\frac{1}{s'-u}\right\}\mathrm{Im}\,A(s',t) \qquad (2.31)$$

and a similar relation (without the t-channel pole) for $\{Aa_j\}$ $(j=1,..8)$. The matrix notation was introduced in sec.I.4, the isospin variable ξ and the matrix $[\xi]$ were defined in sec.II.3, and the residues $\widetilde{\Gamma}(t)$ and $\widetilde{\Gamma}_t$ were obtained in sec.II.5.

Using the relations from sec.I.4 we transform this set of dispersion relations into a set of coupled integral equations for multipole amplitudes. Formally at least, it is straightforward to rewrite (2.31) as

$$\mathrm{Re}\,\widetilde{M}_\ell(s) = \int_{-1}^{1} dx[D_\ell(x)][C^{-1}(s)][B(s,t)]\left\{\frac{\widetilde{\Gamma}(t)}{s-m^2}+[\xi]\frac{\widetilde{\Gamma}(t)}{u-m^2}+\tfrac{1}{2}(1-\xi)\frac{\widetilde{\Gamma}_t}{t-\mu^2}\right\} +$$

$$+ \frac{P}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\int_{-1}^{1} dx[D_\ell(x)][C^{-1}(s)][B(s,t)]\left\{\frac{1}{s'-s} + [\xi]\,\frac{1}{s'-u}\right\}$$

$$\cdot\,[B^{-1}(s',t)][C(s')]\sum_{\ell'=0}^{\infty}[G_{\ell'}(x')]\ \mathrm{Im}\,\widetilde{M}_{\ell'}(s')\ , \qquad (2.32)$$

and similarly for $\widetilde{Ma}_\ell(s)$. The various matrices were defined in sec.I.4, and we use $x'=(kqx+(k_0'q_0'-k_0q_0))/k'q'$. Changing variables from $s=W^2$ to W and using a shorter notation we write

$$\mathrm{Re}\,\widetilde{M}_\ell(W) = \widetilde{M}_\ell^B(W)+\frac{P}{\pi}\int_{m+\mu}^{\infty} dW'\frac{\mathrm{Im}\widetilde{M}_\ell(W')}{W'-W}+\frac{1}{\pi}\int_{m+\mu}^{\infty} dW'\sum_{\ell'=0}^{\infty}[K_{\ell\ell'}(W,W')]\,\mathrm{Im}\widetilde{M}_{\ell'}(W') \qquad (2.33)$$

The first term in (2.32), i.e. the multipole projection of the pole terms, has been denoted here as $\widetilde{M}_\ell^B$. The integrand of the second term in (2.32) has been split into a term $\frac{\mathrm{Im}\widetilde{M}_\ell(W')}{W'-W}$, giving rise to a principal value integral, and the sum $\sum_{\ell'}[K_{\ell\ell'}(W,W')]\,\mathrm{Im}\widetilde{M}_{\ell'}(W')$. It can be shown that with this definition the kernel $[K_{\ell\ell'}]$ is non-singular in W and W'. Both $\widetilde{M}_\ell^B$ and $[K_{\ell\ell'}]$ can be calculated explicitly by performing the matrix multiplications in (2.32). The results can be found in Appendix F.

We note that in the last two equations we encounter an other convergence problem since in the multipole expansion of $\mathrm{Im}\widetilde{A}(s',t)$ under the integral unphysical values of x' occur ($|x'|>1$ in some parts of the s'-range). It can be argued however that for low energies W the expansion will converge properly (see e.g. [Do72], [Ad68]).

In the next chapter we show how a solution for this set of equations can be found and which approximations and additional information we need.

CHAPTER III

SOLUTION OF THE DISPERSION RELATIONS AND CROSS-SECTION CALCULATIONS FOR PION PRODUCTION

III.1 CONNECTION BETWEEN PION PRODUCTION AND PION-NUCLEON SCATTERING

Making use of the postulated analyticity properties of the invariant amplitudes we obtained in the second chapter an infinite set of equations (2.33) for the multipole amplitudes that describe pion production. No essential approximations were involved until then, except for the restriction to lowest order in electromagnetic or weak interactions. In the derivation we assumed however, that no subtractions other than those mentioned in sec.II.5 were needed to ensure convergence, and that the multipole expansion was valid even under the dispersion integral. For the energy range we will consider, i.e. from threshold (1080 MeV) to about 1320 MeV (in the πN_2-centre-of-mass system), these assumptions may indeed be made. In this chapter we will obtain numerical values for the pion production multipole amplitudes in this energy region and we use these amplitudes in a calculation of the differential cross-sections for electroproduction.

Equations (2.33) as they stand can not provide a complete solution for the amplitudes and we will need additional information concerning the strong interaction dynamics. This can be obtained from the experimental results on pion-nucleon scattering, since a link between this process and pion production can be made via a theorem due to Fermi and Watson [Wa54]. This theorem is based on the unitarity relation (2.11), rewritten as

$$\mathrm{Im} \langle f|T|i\rangle = \tfrac{1}{2} \sum_n \langle f|T^{\dagger}|n\rangle\langle n|T|i\rangle \ . \qquad (3.1)$$

For energies below the two-pion threshold ($\approx$1220 MeV) the intermediate states n that give non-vanishing contributions to the s- or u-channel matrix elements can only be pion-nucleon states, if we consider only lowest order in weak or electromagnetic interactions. For these states we have on the right-hand side in eq.(3.1) the product of a pion production matrix element $\langle n|T|i\rangle$ and the complex conjugate of a pion-nucleon scattering matrix element $\langle n|T|f\rangle$. By making an angular momentum decomposition it can then be shown (as is sketched in Appendix D), that a multipole amplitude for pion production with definite values of the angular momentum and isospin in the final πN-state has the same phase as the πN-scattering partial wave amplitude with these same quantum numbers. That is, writing for the latter amplitude (see e.g. [Do67])

$$f^{2I}_{\ell\pm} = \frac{1}{q} \exp\,(i\,\delta^{2I}_{\ell\pm})\,\sin\delta^{2I}_{\ell\pm} \quad , \tag{3.2}$$

where I is the total isospin, ℓ the pion angular momentum (the total angular momentum being given by $J=\ell\pm\frac{1}{2}$), q the absolute value of the centre-of-mass 3-momentum and $\delta^{2I}_{\ell\pm}$ the phaseshift, we finally obtain for the multipole amplitudes with isospin 1 or 3 ($I=\frac{1}{2}$ or $\frac{3}{2}$; cf. sec.I.3, eq.(1.17))

$$M^{2I}_{\ell i'} = |M^{2I}_{\ell i'}|\,\exp\,(i\,\delta^{2I}_{\ell\pm} + in\pi\,) \quad , \tag{3.3}$$

where n is an integer, and the + or - sign occurs for i' odd or even, respectively. For the isoscalar amplitudes, which are denoted by an index 0, we have to use the $\delta^{1}_{\ell\pm}$ phaseshift.

Although this theorem holds only for energies between the one-pion and the two-pion threshold, it can be extended to somewhat higher energies, as long as the inelasticity for the πN-scattering partial waves is small. This means that for the inelasticity parameter η, defined by the expression

$$f^{2I}_{\ell\pm} = -\frac{i}{2q}\,[\eta^{2I}_{\ell\pm}\,\exp\,(2i\,\delta^{2I}_{\ell\pm}) - 1\,] \tag{3.4}$$

(a generalization of (3.2)), we have $0.9 \lesssim \eta \leqq 1$. Inspection of the parameters for the lower πN-partial waves (table I) shows that the inelasticity becomes important only outside the energy region that we are considering, with a possible exception for the P_{11}-phaseshift. (We use the notation $L_{2I,2J}$ where L indicates the spectroscopic notation (S,P,D,..) for the angular momentum of the pion.) Due to this small inelasticity we can apply the theorem even though part of the energy region lies above the two-pion threshold.

Table I

Inelasticity in the lower πN partial wave amplitudes [Do67]

	W_1	W_2		W_1	W_2
S_{11}	1320	1415	S_{31}	1390	1510
P_{11}	1280	1360	P_{31}	1570	1670
P_{13}	1470	1570	P_{33}	1480	1670
D_{13}	1360	1470	D_{33}	1360	1510
D_{15}	1500	1570	D_{35}	1570	1670
F_{15}	1500	1600	F_{35}	1600	1715
F_{17}	1630	2980	F_{37}	1600	2980

W_1: lowest πN-centre-of-mass energy where $1-\eta \geqq 0.0001$

W_2: lowest πN-centre-of-mass energy where $\eta \leqq 0.9$

Two-pion threshold is at 1220 MeV

III.2 TRUNCATED DISPERSION RELATIONS

Because of Watsons theorem (sec.III.1) a resonance in pion-nucleon scattering can be expected to have much influence on pion production. In the relevant energy interval there is one resonance at an energy 1236 MeV in the P_{33} partial wave; this resonance will play an important role. To verify this in more detail we note first that in eq.(2.33), which reads

$$\mathrm{Re}\ \widetilde{M}_\ell(W) = \widetilde{M}^B_\ell(W) + \frac{P}{\pi}\int_{m+\mu}^{\infty} dW'\ \mathrm{Im}\ \widetilde{M}_\ell(W')/(W'-W) +$$

$$+ \frac{1}{\pi}\int_{m+\mu}^{\infty} dW'\ \Sigma_{\ell'}[K_{\ell\ell'}(W,W')]\ \mathrm{Im}\ \widetilde{M}_{\ell'}(W')\ ,$$

the multipole Born terms $M^B_{\ell i}$ and the kernels $[K_{\ell\ell'}]$ can be evaluated exactly (see Appendix F), while for $\mathrm{Im}\ M_{\ell i}(W)$ we obtain from (3.3)

$$\mathrm{Im}\, M^{2I}_{\ell i}(W) = \mathrm{tg}\, \delta^{2I}_{\ell\pm}(W)\, \mathrm{Re}\, M^{2I}_{\ell i}(W) \quad . \tag{3.5}$$

For $W \lesssim 1320$ MeV (the "low energy region") only the P_{33}-phaseshift is large (passing through 90^o at the resonance energy, W=1236 MeV), while the other phaseshifts are generally smaller than 10^o (only for the S-waves $|\delta^3_{0+}| \lesssim 20^o$). Assuming that the Born contribution $M^B_{\ell i}$ (see Appendix F) gives a rough estimate for the magnitude of the real part of the amplitude, we find from eq.(3.5) that compared to the imaginary part of the P_{33}-multipole amplitudes, all other Im $M_{\ell i}$ are small. At somewhat higher energies the situation becomes more complicated, because both in the P_{11} and D_{13} waves there is a resonance at an energy of about 1600 MeV, causing the corresponding phaseshifts to rise steeply through 90^o. Also the S-wave phaseshifts are larger in this region ($\approx 30^o$). From a numerical evaluation of the kernels we see however, that these fall off rapidly if W and W' move apart, so that for W in the low energy range these effects at higher energies will not make a very important contribution to the last term in eq.(2.33). Thus it seems a reasonable approximation (for $W \lesssim 1320$ MeV) to neglect all multipole amplitudes except those for the P_{33}-waves in this integral, and to cut off the integration at an energy W_0 = 1540 MeV. Further we find that for $\ell \geqq 2$ (i.e. D,F and higher partial waves) the corresponding phaseshifts and the kernels connecting with the P_{33}-multipoles are so small that we can neglect the whole term, while for isoscalar multipoles these kernels are identically zero. We will neglect the second term in eq.(2.33) for all multipoles other than P_{33}, P_{11} and D_{13}, and cut off this integral at W_0 = 1540 MeV for P_{33}, and at W_0' = 1700 MeV for P_{11} and D_{13} multipoles. Some remarks concerning the errors introduced by these approximations are made in sec.III.5.

This leaves us with the following relations:

for the P_{33}-multipoles

$$\mathrm{Re}\widetilde{M}_{P33}(W) \approx \widetilde{M}^B_{P33}(W) + \frac{P}{\pi}\int_{m+\mu}^{W_0} dW' \frac{\mathrm{Im}M_{P33}(W')}{W'-W} + \frac{1}{\pi}\int_{m+\mu}^{W_0} dW'[\,K_{11}(W,W')\,]\,\mathrm{Im}\widetilde{M}_{P33}(W') \tag{3.6}$$

for the P_{11}-multipoles

$$\mathrm{Re}\widetilde{M}_{P11}(W) \approx \widetilde{M}^B_{P11}(W) + \frac{P}{\pi}\int_{m+\mu}^{W_0'} dW' \frac{\mathrm{Im}M_{P11}(W')}{W'-W} + \frac{1}{\pi}\int_{m+\mu}^{W_0} dW'[\,K_{11}(W,W')\,]\,\mathrm{Im}\widetilde{M}_{P33}(W') \tag{3.7}$$

for the other S- and P-multipoles (ℓ=0 or 1)

$$\mathrm{Re}\,\widetilde{M}_{SP}(W) \approx \widetilde{M}^B_{SP}(W) + \frac{1}{\pi}\int_{m+\mu}^{W_0} dW'[K_{\ell 1}(W,W')]\,\mathrm{Im}\,\widetilde{M}_{P33}(W') \tag{3.8}$$

for the D_{13}-multipoles

$$\mathrm{Re}\,\widetilde{M}_{D13}(W) \approx \widetilde{M}^B_{D13}(W) + \frac{P}{\pi}\int_{m+\mu}^{W'_0} dW'\,\frac{\mathrm{Im}\widetilde{M}_{D13}(W')}{W'-W}, \tag{3.9}$$

while for all other multipoles we take $\widetilde{M}_\ell \approx \widetilde{M}^B_\ell$. We used the notation

$$\widetilde{M}_{P33} = (E^3_{1+},0,M^3_{1+},0,S^3_{1+},0),\ \text{etc.}$$

Finally, we can make these equations more suitable for numerical calculations by taking into account explicitly the behaviour of the multipole amplitudes for small values of the momenta k and q. Assuming that this behaviour is the same as for the Born terms, we find from the explicit forms for $M^B_{\ell i}$

$$M_{\ell i} \sim q^\ell \quad \text{for } q \to 0,$$

while for k→ 0 we obtain

$E_{\ell -} \sim k^{\ell-2}$	$Ma_{\ell +} \sim k^{\ell+1}$
$S_{\ell\pm} \sim k^{\ell\pm 1} \quad (\ell \geq 2)$	$La_{0+} \sim k$
$S_{1-} \sim k^2$	$Sa_{\ell\pm} \sim k$
other vector amplitudes $\sim k^\ell$	other axial vector ampltides $\sim k^{\ell-1}$

Defining $M'_{\ell i}(W)$ by the relation $M_{\ell i}(W)=q^\ell k^{\ell'} M'_{\ell i}(W)$, with the appropriate choice for ℓ', we can rewrite eqs.(3.6)..(3.9) in terms of $M'_{\ell i}(W)$. When the various threshold factors $q^\ell k^{\ell'}$ are then absorbed in the Born terms and the kernels, the form of these equations remains unchanged. The advantage of extracting the threshold factors is that the new amplitudes $M'_{\ell i}$ do not have very strong variations even near the threshold, which is important for numerical calculations. (We see that in this context especially the small q behaviour has to be taken into account, since q=0 at threshold, whereas the point k=0 does not lie in the physical region and so has less influence.) In the following we will use the amplitudes $M'_{\ell i}$, but we will omit the prime from now on.

III.3 METHOD OF SOLUTION FOR THE MULTIPOLE AMPLITUDES

Several methods have been applied in the literature to solve the dispersion relations for the multipole amplitudes in pion production processes (cf. sec.III.5). Most of these methods are based on the solution given by Chew et al. ([Ch57], further referred to as CGLN) for photoproduction in the static model (i.e. in the limit of infinite nucleon mass). In this treatment the M^3_{1+} multipole emerges as the dominant one in the low-energy region, where it is given in the form

$$M^3_{1+} = \frac{M^{3,SB}_{1+}}{f^{3,SB}_{1+}} f^3_{1+} = \frac{k}{q^2} \frac{e}{2m} \left(\frac{\mu_p - \mu_n}{2f} \right) \exp (i\delta^3_{1+}) \sin \delta^3_{1+} \ . \tag{3.10}$$

Here $M^{3,SB}_{1+}$ and $f^{3,SB}_{1+}$ are the Born terms in the static model for the M^3_{1+} multipole amplitude in pion photoproduction and for the P_{33} partial wave amplitude (f^3_{1+}) in pion-nucleon scattering, respectively; $f = g\mu/2m$, where g is the πN-coupling constant; μ_p and μ_n are the total magnetic moments of the nucleons. Since the other P_{33}-multipole in photoproduction (E^3_{1+}) gives a negligible contribution, the other multipole amplitudes are then calculated from the Born terms plus the contribution (via dispersion relations) from the M^3_{1+} amplitude.

Although the simple model gives quite reasonable results, a relativistic treatment is necessary for a good description of the experimental data, especially in the case of electro- or neutrinoproduction. The situation is more complicated here, because the E^3_{1+} amplitude can not be neglected, while also other P_{33}-multipoles are important here, that are absent in the case of photoproduction. To be precise, for the vector part we have three P_{33}-multipoles, E^3_{1+}, M^3_{1+} and S^3_{1+}, while for the axial vector part (in neutrinoproduction) there are four, Ea^3_{1+}, Ma^3_{1+}, La^3_{1+} and Sa^3_{1+}. For these multipole amplitudes eq. (3.6) gives a set of three (resp. four) coupled integral equations. We have tried to obtain a simple approximate solution of these equations, by using as an "Ansatz" a form for the multipoles, given by a relativistic generalization of the CGLN-formula (3.10). In the CGLN-model the M^3_{1+} amplitude satisfies a dispersion relation which is similar to the one for f^3_{1+}, and their solution gives a simple proportionality of the amplitudes M^3_{1+} and f^3_{1+}, with as a factor the ratio between the "forces" in the dispersion relations for the two amplitudes (i.e. in this case only the ratio between the static model Born terms for the two amplitudes). Although in a relativistic treatment the simple correspondence between the two dispersion relations is lost, one can try, as a first generalization of the model, to use the full relativistic Born terms for f^3_{1+} and M^3_{1+} in the first part of (3.10).

Our solution for the P_{33}-multipoles is obtained by using a further generalization of this model, where instead of the Born term $M^{3,B}_{1+}$, we use the full "force" $M^{3,F}_{1+}$ from the dispersion relation (3.6) (i.e. those terms on the right-hand side of the equation that do not contain Im M^3_{1+}), defined by

$$M^{3,F}_{1+} = M^{3,B}_{1+} + \frac{1}{\pi}\int_{m+\mu}^{W_0} dW'\{[K_{11}(W,W')]_{31}\mathrm{Im}E^3_{1+}(W')+[K_{11}(W,W')]_{35}\mathrm{Im}S^3_{1+}(W')\}, \tag{3.11}$$

or the forces $E^{3,F}_{1+}$, $S^{3,F}_{1+}$, etc., defined in a similar way. This form is used (with an adjustable factor λ) in an iteration procedure in which we find numerical solutions for the multipole amplitudes, that satisfy the dispersion relations (3.6) to within reasonable errors.

Before we describe this procedure in detail, we consider the form of the multipole Born terms as obtained in sec.II.5 (and given more explicitly in Appendix F). These Born terms can be written as a sum of three terms

$$M^B_{\ell i} = \sum_{n=1}^{3} M^B_{\ell i;n}\ ,$$

where $M^B_{\ell i;n}$ is that part of $M^B_{\ell i}$ that contains the formfactor $F_n(K^2)$. (F_1 and F_2 are the nucleon formfactors; $F_3 \equiv F_\pi$ is the pion formfactor; cf. sec.II.5.) Since the dispersion relations are linear in the formfactors, the multipole amplitudes can be split up in the same way, i.e.

$$M_{\ell i;n} = \sum_{n=1}^{3} M_{\ell i;n}\ .$$

The Ansatz for the P_{33}-multipoles is then written as

$$\widetilde{M}_{P33} = \sum_{n=1}^{3} \lambda_n \frac{\widetilde{M}^F_{P33;n}}{f^{3,B}_{1+}} \exp\,(i\delta^3_{1+})\sin\delta^3_{1+}\ , \tag{3.12}$$

where $f^{3,B}_{1+}$ is the (relativistic) Born term in the P_{33} partial wave amplitude for πN-scattering. To start the iteration procedure a first approximation for $\frac{1}{\lambda_n}$ Re $M^3_{1+;n}$ and $\frac{1}{\lambda_n}$ Im $M^3_{1+;n}$ is calculated by using (3.12) for M^3_{1+}, with as force only the Born term $M^{3,B}_{1+}$, (i.e. we start with calculating $M^{3,B}_{1+;n}$ exp $(i\delta^3_{1+})$ sin $\delta^3_{1+}/f^{3,B}_{1+}$, for n=1,2,3). The constants λ_n are then determined by calculating from the dispersion relation (3.6) the values of Re $M^3_{1+;n}$ (n=1,2,3) at the resonance energy (W=1236 MeV), and requiring that these should be zero. (As input the values for $\frac{1}{\lambda_n}$ Im $M^3_{1+;n}$ are used, as obtained from the Ansatz (3.12), while E^3_{1+} and S^3_{1+} are set equal to zero for the moment.) In the second part of the first iteration step essentially the same procedure is repeated for the amplitude S^3_{1+}, using in the Ansatz the force

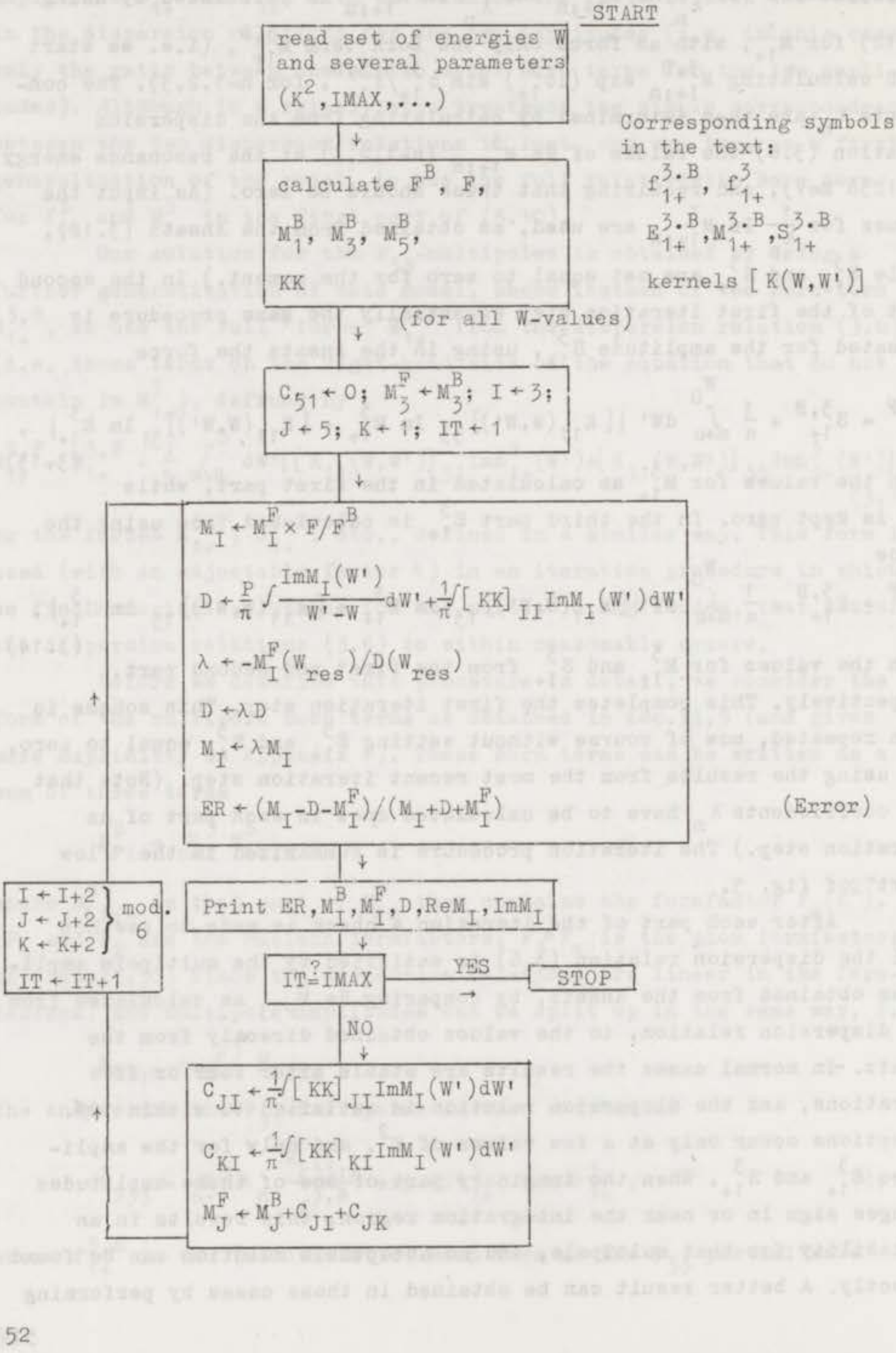
$$S^{3,F}_{1+} = S^{3,B}_{1+} + \frac{1}{\pi}\int_{m+\mu}^{W_0} dW' \{[K_{11}(W,W')]_{53} \text{ Im } M^3_{1+} + [K_{11}(W,W')]_{51} \text{ Im } E^3_{1+}\} , \quad (3.13)$$

with the values for M^3_{1+} as calculated in the first part, while E^3_{1+} is kept zero. In the third part E^3_{1+} is calculated, now using the force

$$E^{3,F}_{1+} = E^{3,B}_{1+} + \frac{1}{\pi}\int_{m+\mu}^{W_0} dW'\{[K_{11}(W,W')]_{13} \text{ Im } M^3_{1+} + [K_{11}(W,W')]_{15} \text{ Im } S^3_{1+}\} , \quad (3.14)$$

with the values for M^3_{1+} and S^3_{1+} from the first and second part, respectively. This completes the first iteration step. This scheme is then repeated, now of course without setting E^3_{1+} and S^3_{1+} equal to zero, but using the results from the most recent iteration step. (Note that the coefficients λ_n have to be calculated anew in each part of an iteration step.) The iteration procedure is summarized in the "flow chart" of fig. 5.

After each part of the iteration a check is made to see how well the dispersion relation (3.6) is satisfied by the multipole amplitudes obtained from the Ansatz, by comparing Re $\widetilde{M}_{P33}$ as calculated from the dispersion relation, to the values obtained directly from the Ansatz. In normal cases the results are stable after four or five iterations, and the dispersion relation is satisfied to within 10%. Exceptions occur only at a few values of K^2, and only for the amplitudes E^3_{1+} and S^3_{1+}. When the imaginary part of one of these amplitudes changes sign in or near the integration region, this results in an instability for that multipole, and no acceptable solution can be found directly. A better result can be obtained in those cases by performing fig. 5. Flow chart for the calculation of P_{33}-multipoles



the iterations with the combination (e.g.) $a_i E^3_{1+;i} + a_j E^3_{1+;j}$. Here $E^3_{1+;i}$ is the part of the amplitude that is instable against the iterations; $E^3_{1+;j}$ is one of the other parts, which is also calculated by itself, so that the value of $E^3_{1+;i}$ can be separated out afterwards. By varying the coefficients a_i and a_j we obtain different solutions, from which we select the one that gives the best fit to the dispersion relation (3.6).

The equations (3.7)..(3.9) are then used to calculate the real parts of the other multipole amplitudes; the imaginary parts of the S- and P-wave multipoles are found by using Watson's theorem in the form (3.5); imaginary parts for higher multipoles are set equal to zero. For P_{11} and D_{13} multipoles we use in the principal value integral of eqs. (3.7) and (3.8) a simple Breit-Wigner form.

Calculations have been performed for 11 different values of K^2 between .01 and 2.$(GeV/c)^2$, both for vector and axial vector amplitudes *).

III.4 CROSS-SECTION FORMULAE

In sec.I.1 a general expression (1.9) was given for the pion production cross-section. In the present section we will rewrite this in terms of πN_2-centre-of-mass amplitudes and of the multipole amplitudes. We thus have to express the square of the matrix element, i.e.

$$|T_{fi}|^2 = \varepsilon_\mu \, \varepsilon^*_\nu \; {}_{out}\langle \pi N_2 | J^L_\mu | N_1 \rangle_{in} \; {}_{out}\langle \pi N_2 | J^L_\mu | N_1 \rangle^*_{in} \quad , \qquad (3.15)$$

in these amplitudes, where ε_μ is given by eq.(1.8).

A. Electroproduction

Since the current J^{EM}_μ is conserved, i.e. $K_\mu \langle ..| J^{EM}_\mu | ..\rangle = 0$, we may replace the ε_μ in (3.15) for electroproduction by $b_\mu = \varepsilon_\mu - \frac{\vec{\varepsilon}.\vec{k}}{k} K_\mu$, which we also used in eqs.(1.26) for the matrices Λ_i (defining the amplitudes F_i). We sum over the final spins and average over the initial spin of the electrons (i.e. we consider only unpolarized

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*) for the calculation of the axial vector amplitudes essentially the same procedure was used, the main difference being that in this case also longitudinal amplitudes occur (cf. eq.(1.29)).

electrons) and define $\alpha_{\mu\nu} = \frac{1}{2}\sum_{\text{spins}} b_\mu b^*_\nu \zeta_\nu$, where $\zeta_i = 1$ for $i=1,2,3$ and $\zeta_4 = -1$. (No summation over ν in this case.) Apart from a common factor $e^4/(4m_\ell^2 K^4)$, the non-zero components of $\alpha_{\mu\nu}$ are given by

$$\begin{aligned} &\alpha_{11} \sim K^2 + \ell_1^2 && \alpha_{00} \sim K^2(4m_\ell^2 + \ell_1^2)/k^2 \\ &\alpha_{22} \sim K^2 && \alpha_{10} = \alpha_{01} \sim \ell_1[K^2(4m_\ell^2 + K^2 + \ell_1^2)]^{\frac{1}{2}}/k \end{aligned} \tag{3.16}$$

where we used $L = (\vec{\ell}, i\ell_0) = K_1 + K_2$, and $m_\ell = m_{\ell 1} = m_{\ell 2}$.

A useful parametrization of $\alpha_{\mu\nu}$ can be given in terms of the quantity ε [Ha63A], [Do69], defined by

$$\varepsilon^{-1} = 1 + \frac{2k^{L2}}{K^2} \tan^2 \tfrac{1}{2}\theta_\ell^L , \tag{3.17}$$

which is used frequently in the description of experimental results, (θ_ℓ^L is the electron scattering angle in the laboratory frame). When we neglect (where possible) the electron mass (i.e. the electrons are treated in the extreme relativistic limit, which is a very good approximation in these processes), we find

$$\ell_1^2 = K^2 \frac{2\varepsilon}{1-\varepsilon} ,$$

and obtain thus

$$\alpha_{\mu\nu} = \tfrac{1}{2} \sum_{\text{spins}} b_\mu b^*_\nu \zeta_\nu = \frac{e^4}{4m_\ell^2 K^4} \frac{2K^2}{1-\varepsilon} \rho_{\mu\nu} \tag{3.18^a}$$

with for the non-zero elements of $\rho_{\mu\nu}$

$$\begin{aligned} &\rho_{11} = \tfrac{1}{2}(1+\varepsilon) && \rho_{00} = \varepsilon K^2/k^2 \\ &\rho_{22} = \tfrac{1}{2}(1-\varepsilon) && \rho_{10} = \rho_{01} = [\tfrac{1}{2}\varepsilon(1+\varepsilon)K^2]^{\frac{1}{2}}/k . \end{aligned} \tag{3.18^b}$$

The space components of the matrix ρ are the same as for the polarization density matrix of a real photon with a partial linear transverse polarization, and so ε is often called the polarization parameter of the virtual photon. (Note that in our case scalar components are present as well, which are of course absent for real photons.)

Before substituting (3.18) in the expression for the matrix element squared (3.15), it is convenient to use eqs.(1.26)

$$\varepsilon_\mu \, {}_{\text{out}}\langle \pi N_2 | J_\mu^{EM} | N_1 \rangle_{\text{in}} = \chi_2^\dagger F \chi_1 = \chi_2^\dagger \sum_{i=1}^{6} F_i \Lambda_i \chi_1 .$$

From the definitions of the Λ_i it is clear that we can separate F into a transverse (T) and a scalar (S) part,

$$F_T = \sum_{i=1}^{4} F_i \Lambda_i ; \qquad F_S = \sum_{i=5}^{6} F_i \Lambda_i \quad . \tag{3.19}$$

Summing over final nucleon spins, we then have

$$\sum_{\substack{\text{final}\\ \text{spins}}} |T_{fi}|^2 = \chi_1^\dagger [F_T^\dagger F_T + (F_T^\dagger F_S + F_S^\dagger F_T) + F_S^\dagger F_S]\chi_1 \quad . \tag{3.20}$$

The first (purely transverse) term has the same form as in real photoproduction, where of course the other terms are absent. By explicit calculation we obtain

$$\begin{aligned}\sum_{\substack{\text{final}\\ \text{spins}}} \chi_1^\dagger F_T^\dagger F_T \chi_1 = \sum_{i,j=1}^{2} [& A\,\delta_{ij} + B\,\hat{q}_i\hat{q}_j + \mathrm{Im}C\,\vec{P}_N.(\hat{k}\times\hat{q})\hat{q}_i\hat{q}_j + \\ & + \mathrm{Im}D\,\hat{q}_i(\hat{k}\times\vec{P}_N)_j + \mathrm{Im}E\,(\hat{k}\times\hat{q})_i\,P_{Nj} + \\ & + \mathrm{Im}F\,\hat{q}_i(\hat{q}\times\vec{P}_N)_j + \mathrm{Im}G\,\vec{P}_N.\hat{k}\hat{q}_i(\hat{q}\times\hat{k})_j]\alpha_{ij}\end{aligned} \tag{3.21^a}$$

$$\begin{aligned}\sum_{\substack{\text{final}\\ \text{spins}}} \chi_1^\dagger (F_T^\dagger F_S + F_S^\dagger F_T)\chi_1 = \sum_{i=1}^{2} [& (\mathrm{Re}J + \vec{P}_N.(\hat{q}\times\hat{k})\mathrm{Im}K)\hat{q}_i + \\ & + \mathrm{Im}L(\vec{P}_N\times\hat{q})_i + \mathrm{Im}M(\vec{P}_N\times\hat{k})_i + \mathrm{Im}N(P_N.\hat{k})(\hat{q}\times\hat{k})_i]\alpha_{i0}\end{aligned} \tag{3.21^b}$$

$$\sum_{\substack{\text{final}\\ \text{spins}}} \chi_1^\dagger F_S^\dagger F_S \chi_1 = (H + \mathrm{Im}I\,\vec{P}_N.(\hat{q}\times\hat{k}))\alpha_{00} \quad . \tag{3.21^c}$$

In these equations $\alpha_{\mu\nu}$ can be expressed in terms of ε by means of eqs. (3.18). The initial nucleon polarization $\vec{P}_N$ is given by $\vec{P}_N = \chi_1^\dagger \vec{\sigma} \chi_1$, and the coefficients are defined as follows,

$$A = |F_1|^2 + |F_2|^2 - 2\cos\theta\,\mathrm{Re}(F_1^* F_2)$$

$$B = |F_3|^2 + |F_4|^2 + 2\mathrm{Re}(F_1^* F_4 + F_2^* F_3 + \cos\theta\,F_3^* F_4)$$

$$C = -2\,F_3^* F_4$$

$$D = -2[+F_1^* F_2 + F_1^* F_3 - F_2^* F_4 - \cos\theta\,F_2^* F_3]$$

$$E = -2\,F_1^* F_2 \qquad\qquad F = -2\,F_1^* F_4$$

$$G = -2\,F_2^* F_3 \qquad\qquad I = 2\,F_6^* F_5$$

$$H = |F_5|^2 + |F_6|^2 + 2\cos\theta\ \mathrm{Re}(F_5^* F_6)$$

$$J = F_5^*(F_1+F_4) + F_6^*(F_2+F_3) + \cos\theta(F_5^*F_3 + F_6^*F_4)$$

$$K = F_6^*F_4 - F_5^*F_3 \qquad\qquad L = F_5^*F_1$$

$$M = F_5^*F_2 - F_6^*F_1 + \cos\theta\ F_6^*F_2 \qquad\qquad N = F_6^*F_2 \tag{3.22}$$

For unpolarized nucleons we find from eqs.(3.18) through (3.21) with $\vec{P}_N=0$

$$\tfrac{1}{4}\sum_{\text{spins}}|T_{fi}|^2 = \tfrac{1}{2}\sum_{i,j=1}^{2}[A\,\delta_{ij} + B\,\hat{q}_i\hat{q}_j]\alpha_{ij} + \tfrac{1}{2}\sum_{i=1}^{2}\mathrm{Re}J\,\hat{q}_i\,\alpha_{i0} + \tfrac{1}{2}H\,\alpha_{00} =$$

$$= \frac{1}{4m_\ell^2}\frac{e^4}{K^2}\Big[A + \tfrac{1}{2}B\sin^2\theta + \tfrac{1}{2}\varepsilon B\sin^2\theta\cos 2\varphi + \varepsilon\frac{K^2}{k^2}H -$$

$$- 2\{\tfrac{1}{2}\varepsilon(1+\varepsilon)K^2/k^2\}^{\frac{1}{2}}\mathrm{Re}J\sin\theta\cos\varphi\Big] \tag{3.23}$$

The cross-section for electroproduction (unpolarized nucleons), which is found immediately from eq.(3.23), is given usually in terms of the so-called "cross-section for pion photoproduction with virtual photons", denoted as $\frac{d^2\sigma_\nu}{d\,\Omega_\pi}$, and defined by

$$\frac{d^5}{dk_{20}^L\,d\Omega_e^L\,d\Omega_\pi} = \frac{k_2^L}{k_1^L}\,\frac{k^L}{(2\pi)^3}\,\frac{e^2}{K^2}\,\frac{\Gamma}{1-\varepsilon}\,\frac{d^2\sigma_\nu}{d\,\Omega_\pi} \tag{3.24^a}$$

$$\frac{d^2\sigma_\nu}{d\,\Omega_\pi} = \frac{q}{k}\,\frac{m^2}{(4\pi W)^2}\,\frac{e^2}{\Gamma}\Big[A + \tfrac{1}{2}B\sin^2\theta + \tfrac{1}{2}\varepsilon B\sin^2\theta\cos 2\varphi +$$

$$+ \varepsilon K^2H/k^2 - 2\{\tfrac{1}{2}\varepsilon(1+\varepsilon)K^2/k^2\}^{\frac{1}{2}}\mathrm{Re}J\sin\theta\cos\varphi\Big]. \tag{3.24^b}$$

Most experimental results are given in terms of this quantity $\frac{d^2\sigma_\nu}{d\,\Omega_\pi}$, but the factor Γ is different with various authors. Either $\Gamma=1$ is taken (e.g. in refs. [Ak67], [Co67]) or $\Gamma=(W^2-m^2)/2Wk$ (e.g. in refs. [Mi69], [Ly67], [Si71]), which factor also reduces to 1 in the limit of real photoproduction, ($K^2=0$).

Via eq.(3.22) we thus have obtained expressions for the doubly differential cross-section (3.24^b) in terms of the centre-of-mass amplitudes F_i. In the calculations the substitution in the F_i of the multipole amplitudes as obtained in sec.III.3 is done numerically by computer, using eq.(1.31). The results of these calculations are given in the next section.

To obtain the cross-section for the situation where the direction of the final pion is not observed, evidently we have to integrate over $d\Omega_\pi$. The terms containing $\cos\varphi$ and $\cos 2\varphi$ then vanish, and the results can be expressed directly in terms of the multipole amplitudes, again using eq.(1.31). We find

$$\sigma_v = \int d\Omega_\pi \frac{d^2\sigma_v}{d\Omega_\pi} = \frac{q}{k} \frac{m^2}{(4\pi W)^2} \frac{e^2}{\Gamma} 2\pi \sum_{\ell=0}^{\infty} \{ \ell(\ell+1)^2 [|M_{\ell+}|^2 + |E_{(\ell+1)-}|^2]$$

$$+\ell^2(\ell+1)[|M_{\ell-}|^2 + |E_{(\ell-1)+}|^2] + 2\varepsilon \frac{K^2}{k^2}[(\ell+1)^3 |S_{(\ell+1)-}|^2 + \ell^3 |S_{(\ell-1)+}|^2]\} \quad (3.25)$$

Also for this case some results are given in sec.III.5.

Cross-section formulae, valid for electroproduction on polarized nucleons can be obtained in a similar way from the complete eqs. (3.21), with $\vec{P}_N \neq 0$. Since for this case no experimental information is available as yet, we will not discuss it further.

B. Neutrinoproduction

For neutrinoproduction we have to use in eq.(3.15) the current $J_\mu^W = J_\mu^V + J_\mu^A$ (Cf. sec.I.4), and we obtain

$$\sum_{\text{spins}} |T_{fi}|^2 = \sum_{\text{spins}} \varepsilon_\mu \varepsilon_\nu^* [{}_{out}\langle \pi N_2 | J_\mu^V | N_1 \rangle_{in} \, {}_{out}\langle \pi N_2 | J_\nu^V | N_1 \rangle_{in}^* +$$

$$+ {}_{out}\langle \pi N_2 | J_\mu^V | N_1 \rangle_{in} \, {}_{out}\langle \pi N_2 | J_\nu^A | N_1 \rangle_{in}^* +$$

$$+ {}_{out}\langle \pi N_2 | J_\mu^A | N_1 \rangle_{in} \, {}_{out}\langle \pi N_2 | J_\nu^V | N_1 \rangle_{in}^* +$$

$$+ {}_{out}\langle \pi N_2 | J_\mu^A | N_1 \rangle_{in} \, {}_{out}\langle \pi N_2 | J_\nu^A | N_1 \rangle_{in}^*] =$$

$$= \alpha_{\mu\nu} [\beta_{\mu\nu}^V + \beta_{\mu\nu}^{VA} + \beta_{\mu\nu}^A] , \quad (3.26)$$

where we have used

$$\alpha_{\mu\nu} = \sum_{\substack{\text{spins} \\ (\ell_2)}} \varepsilon_\mu \varepsilon_\nu^* \zeta_\nu = \frac{G^2}{2m_{\ell 1} m_{\ell 2}} (L_\mu L_\nu - K_\mu K_\nu - \tfrac{1}{2}(L^2 - K^2)\delta_{\mu\nu} \pm \varepsilon_{\mu\nu\rho\sigma} K_\rho L_\sigma), \quad (3.27^a)$$

and where $\beta_{\mu\nu}^V$, etc., denote the various products of hadron current matrix elements. The last term in (3.27^a) occurs because the initial lepton is completely polarized (due to the factor $(1+\gamma_5)$ in eq.(1.8^b)

for ε_μ). The + sign holds when neutrinos are used, the - sign for anti-neutrinos. Apart from the factor $G^2/2m_{\ell 1}m_{\ell 2}$ the non-zero matrix elements of $\alpha_{\mu\nu}$ are given by

$$\alpha_{11} \sim \ell_1^2 + K^2 + m_{\ell 2}^2 \qquad \alpha_{13} = \alpha_{31} \sim \ell_1 \ell_3$$

$$\alpha_{22} \sim K^2 + m_{\ell 2}^2 \qquad \alpha_{10} = \alpha_{01} \sim \ell_1 \ell_0$$

$$\alpha_{33} \sim \ell_3^2 - k_0^2 + m_{\ell 2}^2 \qquad \alpha_{23} = -\alpha_{32} \sim \mp i k_0 \ell_1$$

$$\alpha_{00} \sim \ell_0^2 - k^2 - m_{\ell 2}^2 \qquad \alpha_{20} = -\alpha_{02} \sim \mp i k \ell_1$$

$$\alpha_{12} = -\alpha_{21} \sim \pm i(k\ell_0 - k_0 \ell_3) \qquad \alpha_{30} = \alpha_{03} \sim \ell_3 \ell_0 - k k_0 \quad , \tag{3.27^b}$$

(upper or lower sign for ν or $\bar{\nu}$).

Since J_μ^V is a conserved current, but J_μ^A is not, only those factors ε_μ that are contracted with J_μ^V have to be replaced by b_μ (eq. (1.26)). When the different terms in (3.26) are calculated separately, we obtain for the first term the same form as given in (3.20) and (3.21) for electroproduction, with of course now a different definition for ε_μ in the quantities $\alpha_{\mu\nu}$, but with the coefficients still given by (3.22). For unpolarized nucleons this is

$$\tfrac{1}{2} \sum_{\substack{\text{nucl.}\\ \text{spins}}} \alpha_{\mu\nu}^V \beta_{\mu\nu}^V = \tfrac{1}{2} \sum_{i,j=1}^{2} [A\, \delta_{ij} + B\, \hat{q}_i \hat{q}_j] \alpha_{ij}^V + \tfrac{1}{2} \sum_{i=1}^{2} \mathrm{ReJ}\, \hat{q}_i\, \alpha_{i0}^V + $$
$$+ \tfrac{1}{2} H\, \alpha_{00}^V \ , \tag{3.28^a}$$

where $\alpha_{\mu\nu}^V = \sum_{\substack{\text{spins}\\ (\ell_2)}} b_\mu b_\nu^* \zeta_\nu$ can be obtained simply from eqs.(3.27). In the other terms of (3.26) it is easier to use $\alpha_{\mu\nu}$ as defined in (3.27), working out the products $b_\mu \varepsilon_\nu^*$, etc. We obtain, using the symmetric (s) and anti-symmetric (a) forms $\alpha_{\mu\nu}^{s,a} = \tfrac{1}{2}(\alpha_{\mu\nu} \pm \alpha_{\nu\mu})$,

$$\tfrac{1}{2} \sum_{\substack{\text{nucl.}\\ \text{spins}}} \alpha_{\mu\nu} \beta_{\mu\nu}^A = \tfrac{1}{2} \sum_{i,j=1}^{3} [T\, \delta_{ij} + 2\mathrm{Re}(U+V) \hat{k}_i \hat{q}_j + S\, \hat{q}_i \hat{q}_j + R\, \hat{k}_i \hat{k}_j] \alpha_{ij}^s -$$
$$- \sum_{i=1}^{3} [\mathrm{ReW}\, \hat{q}_i + \mathrm{ReX}\, \hat{k}_i] \alpha_{i0}^s + \tfrac{1}{2} Z \alpha_{00}^s +$$
$$+ \sum_{i,j=1}^{3} [i\, \mathrm{Im}(U-V) \hat{k}_i \hat{q}_j] \alpha_{ij}^a -$$
$$- \sum_{i=1}^{3} [i\, \mathrm{ImW}\, \hat{q}_i + i\, \mathrm{ImX}\, \hat{k}_i] \alpha_{i0} \ , \tag{3.28^b}$$

$$\frac{1}{2}\sum_{\substack{\text{nucl.}\\ \text{spins}}} \alpha_{\mu\nu}\beta^{VA}_{\mu\nu} = \sum_{i,j=1}^{3}\left[\mathrm{Im}(Fc-Cc)(\hat{k}\times\hat{q})_i\hat{k}_j + \mathrm{Im}(Ec-Dc)(\hat{k}\times\hat{q})_i\hat{q}_j\right]\alpha^{s}_{ij} +$$

$$+\ \frac{1}{2}\sum_{i,j=1}^{3}\left[i\,\varepsilon_{ijk}\,\hat{q}_k\ \mathrm{Re}(2Ac+Cc+\cos\theta(Dc+Ec)+Fc)\ +\right.$$

$$\left. +\ i\,\varepsilon_{ijk}\,\hat{k}_k\ \mathrm{Re}(2Bc-\cos\theta(Cc+Fc)-Dc-Ec)\right]\alpha^{a}_{ij} +$$

$$+\ \sum_{i=1}^{3}\left[(\hat{k}\times\hat{q})_i\ \mathrm{Im}(-Gc+Hc)\right]\alpha^{s}_{i0} + \sum_{i=1}^{3}\left[(\hat{k}\times\hat{q})_i\ i\ \mathrm{Re}(Gc+Hc)\right]\alpha^{a}_{i0}, \qquad (3.28^c)$$

with coefficients given by

$$R = 2\mathrm{Re}(Fa_1Fa_5^* + \cos\theta[Fa_5Fa_6^* - Fa_2Fa_5^*]) - |Fa_2|^2 + |Fa_5|^2 + |Fa_6|^2$$

$$S = 2\mathrm{Re}(Fa_1Fa_4^* + Fa_2Fa_3^* + \cos\theta\ Fa_3Fa_4^*) + |Fa_3|^2 + |Fa_4|^2$$

$$T = |Fa_1|^2 + |Fa_2|^2 - 2\cos\theta\ \mathrm{Re}(Fa_1Fa_2^*)$$

$$U = Fa_1Fa_2^* + Fa_1Fa_3^* - \cos\theta\ Fa_2Fa_3^*$$

$$V = Fa_1Fa_6^* + Fa_2Fa_5^* + Fa_3Fa_5^* + Fa_4Fa_6^* + \cos\theta(Fa_3Fa_6^* + Fa_4Fa_5^*)$$

$$W = Fa_1Fa_7^* + Fa_2Fa_8^* + Fa_3Fa_8^* + Fa_4Fa_7^* + \cos\ (Fa_3Fa_7^* + Fa_4Fa_8^*)$$

$$X = Fa_1Fa_8^* + Fa_5Fa_8^* + Fa_6Fa_7^* + \cos\theta(-Fa_2Fa_8^* + Fa_5Fa_7^* + Fa_6Fa_8^*)$$

$$Z = |Fa_7|^2 + |Fa_8|^2 + 2\cos\theta\ \mathrm{Re}(Fa_7Fa_8^*) \qquad (3.29^a)$$

$$Ac = Fa_1F_1^*$$

$$Bc = -Fa_1F_2^* - Fa_2F_1^* + \cos\theta\ Fa_2F_2^*$$

$$Cc = -Fa_1F_1^* - \cos\theta\ (Fa_1F_3^* + Fa_2F_4^*) + \frac{k_0}{k}(Fa_1F_6^* + Fa_2F_5^*)$$

$$Dc = Fa_1F_3^* + Fa_2F_4^*$$

$$Ec = Fa_3F_1^* + Fa_4F_2^*$$

$$Fc = Fa_5F_1^* + Fa_6F_2^*$$

$$Gc = Fa_1F_6^* + Fa_2F_5^*$$

$$Hc = -Fa_8F_1^* - Fa_7F_2^* \qquad (3.29^b)$$

The differential cross-section is given by

$$\frac{d^5\sigma}{dk_{20}\,d\Omega^L_\ell\,d\Omega_\pi} = \frac{k^L_2}{k^L_1}\,\frac{k^L}{(2\pi)^3}\,\frac{q}{k}\,\frac{m^2 m_{\ell 1} m_{\ell 2}}{(4\pi W)^2}\,\frac{1}{2}\sum_{\substack{\text{nucl.}\\ \text{spins}}}\left[\alpha^{V}_{\mu\nu}\beta^{V}_{\mu\nu} + \alpha_{\mu\nu}\beta^{VA}_{\mu\nu} + \alpha_{\mu\nu}\beta^{A}_{\mu\nu}\right] \qquad (3.30)$$

III.5 NUMERICAL CALCULATIONS AND RESULTS

Numerical values for the various parts of the multipole amplitudes (without the formfactors) were calculated on a computer by the method given in sec.III.3, at eleven different values of K^2 between $0.5\,\mu^2$ and $100\,\mu^2$ (μ = pion mars), i.e. between .01 and 2 $(\mathrm{GeV}/c)^2$, and at 43 energy points (31 for the axial vector multipoles), in the range between 7.722μ and 9.46μ (1080 to 1320 MeV). The results were stored on a magnetic tape and used as input for the computer program that performed the cross-section calculations. This program selects the appropriate multipoles, inserts the formfactors and the isospin coefficients and then applies the formulae from sec.III.4 to obtain the differential or total cross-sections.

We have performed several tests to investigate the influence of the various approximations (made in sec.III.2) on our results. Varying the value of the cut-off energy W_0 was found to have little effect, although it is of course difficult to predict the effect of a complete integration without cut-off, since there is not much information about the multipoles at those high energies. Also in that case the question of convergence becomes more important, and related to this we might need several subtraction constants.

Including the imaginary parts of non-P_{33}-multipoles in the integrals in (2.33) was found to be unnecessary, except possibly for the S-wave multipoles, which can have some effect on the P-waves (other than P_{33}). Numerical evaluation showed however, that even here we can expect the effect to be less than 5%, so we have not included these terms in our calculations (as indicated in (3.6)..(3.9)), since other factors will cause greater uncertainties.

For electroproduction we performed cross-section calculations in order to make a comparison with three sets of experiments, done at Stanford [Ly67], Harvard (CEA) [Mi69] and Daresbury [He71], [Si71]. For the nucleon formfactors we use results from ref.[Go67], where a fit to experimental data for elastic electron-proton scattering is given in terms of the Sachs formfactors [Sa62], that are defined by

$$G_E = F_1 - \frac{K^2}{2m} F_2$$
$$G_M = F_1 + 2mF_2; \tag{3.31}$$

(we have omitted here the upper indices p, n, or S, V; cf. sec.II.5). These formfactors can be described by the "scaling law" parametrization [Go67]

$$G_{EM}(K^2) = G_M^p(K^2)/\mu_p = G_E^p(K^2) = (1+K^2/0.71(\text{GeV}/c)^2)^{-2} \,, \quad (3.32)$$

(μ_p is the total magnetic moment of the proton). For $G_M^n(K^2)/\mu_n$ we assume the same form, and we take $G_E^n(K^2)=0$ (see e.g. [Ru69]). This leads to

$$F_1^{S,V} = F_1^p \pm F_1^n = G_{EM}(K^2)\{ 1+K^2(\mu_p' \pm \mu_n')/(K^2+4m^2) \} \,,$$
$$F_2^{S,V} = F_2^p \pm F_2^n = G_{EM}(K^2)\{ 2m(\mu_p' \pm \mu_n')/(K^2+4m^2)\} \,, \quad (3.33)$$

(upper or lower sign for isoscalar (S) or isovector (V) formfactors, respectively). For the pion formfactor we use a form $(1+K^2/m'^2)^{-1}$, with a mass m'=560MeV ([Mi69]; cf. also [Ak67]), i.e.

$$F_\pi(K^2) = (1 + K^2/0.31(\text{GeV}/c)^2)^{-1} \,. \quad (3.34)$$

(For comparison, in the case of π^+ production a few calculations have been done as well, using an other form for F_π (see figs.9 and 10); for π^0 production the cross-section calculations are rather insensitive to the precise form of F_π).

The Stanford group [Ly67] gives only the total cross-section σ_v, which via eq.(3.25) can be calculated directly from the multipole amplitudes. A selection of their results is shown in fig.6, where our calculations are compared with the experimental data. The two other groups have performed coincidence experiments, in which the two charged particles in the final state are both measured. Thus they obtained angular distributions for the cross-sections; the results are expressed in terms of $\frac{d\sigma_v}{d\Omega_\pi}$, eq.(3.24). To make a comparison with these experiments, the computer program first uses eq.(1.31) to calculate the centre-of-mass amplitudes F_i, then expresses the coefficients A, B, H and J in terms of the F_i (eq.(3.22)), and uses eq.(3.24) to obtain $\frac{d\sigma_v}{d\Omega_\pi}$. The Harvard group [Mi69] measured both π^0 and π^+ production. Some of their results for π^0 production together with our calculations are given in fig.7, while fig.9 gives some experimental data and calculations for π^+ production. (Figs.8 and 10 give separate terms in the cross-section.) The Daresbury group [He71], [Si71] give extensive results on

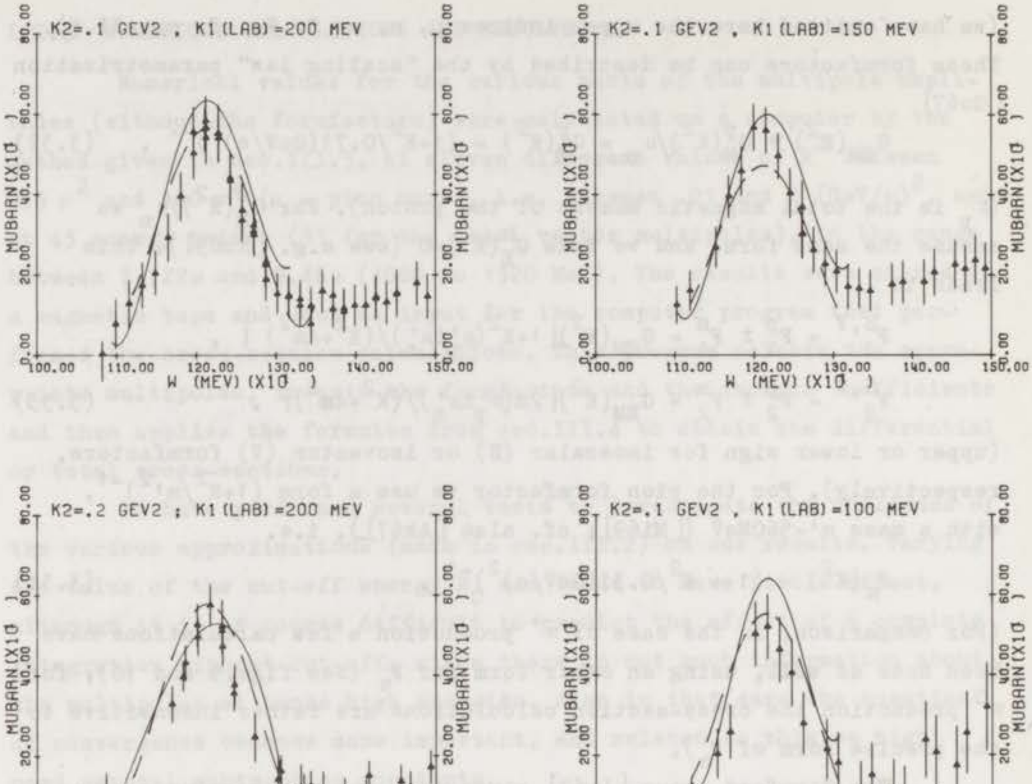


Fig. 6^a. Total cross-section for $ep \rightarrow ep\pi^0$ and $en\pi^+$
Experimental points: Lynch et al. [Ly67]; solid line: our calculation; dashed line: Adler [Ad68].

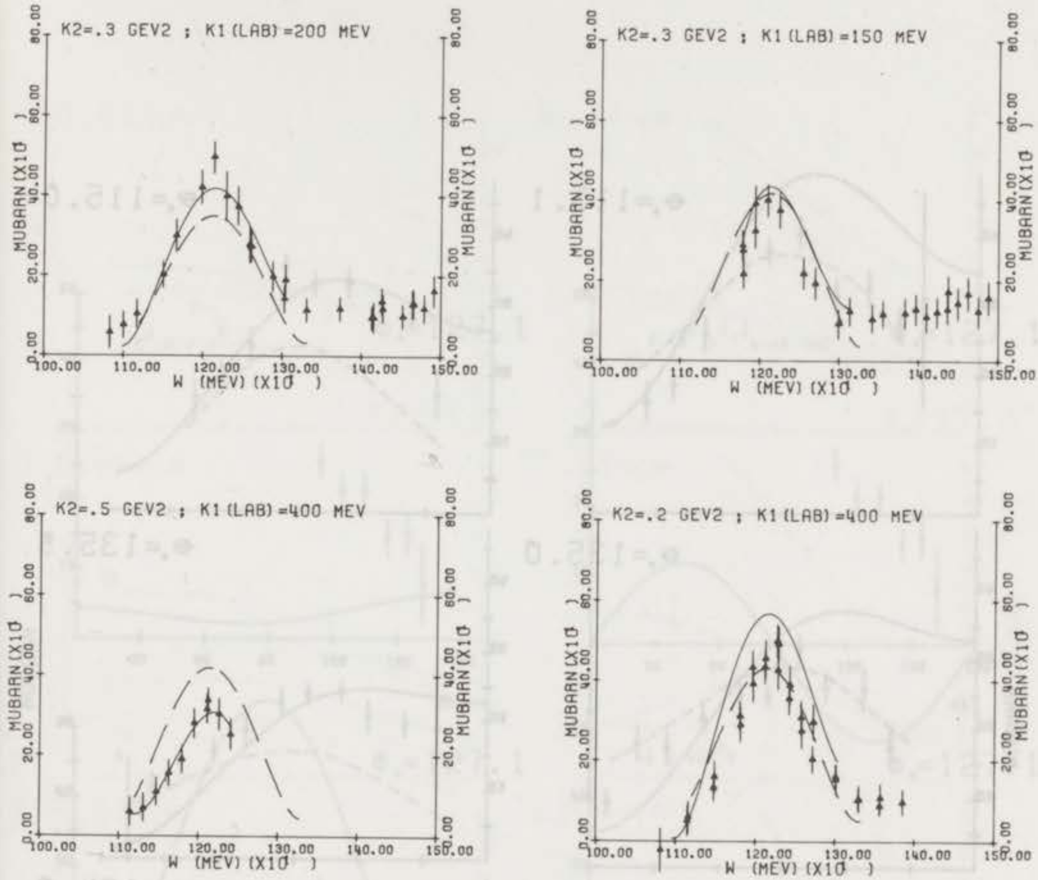
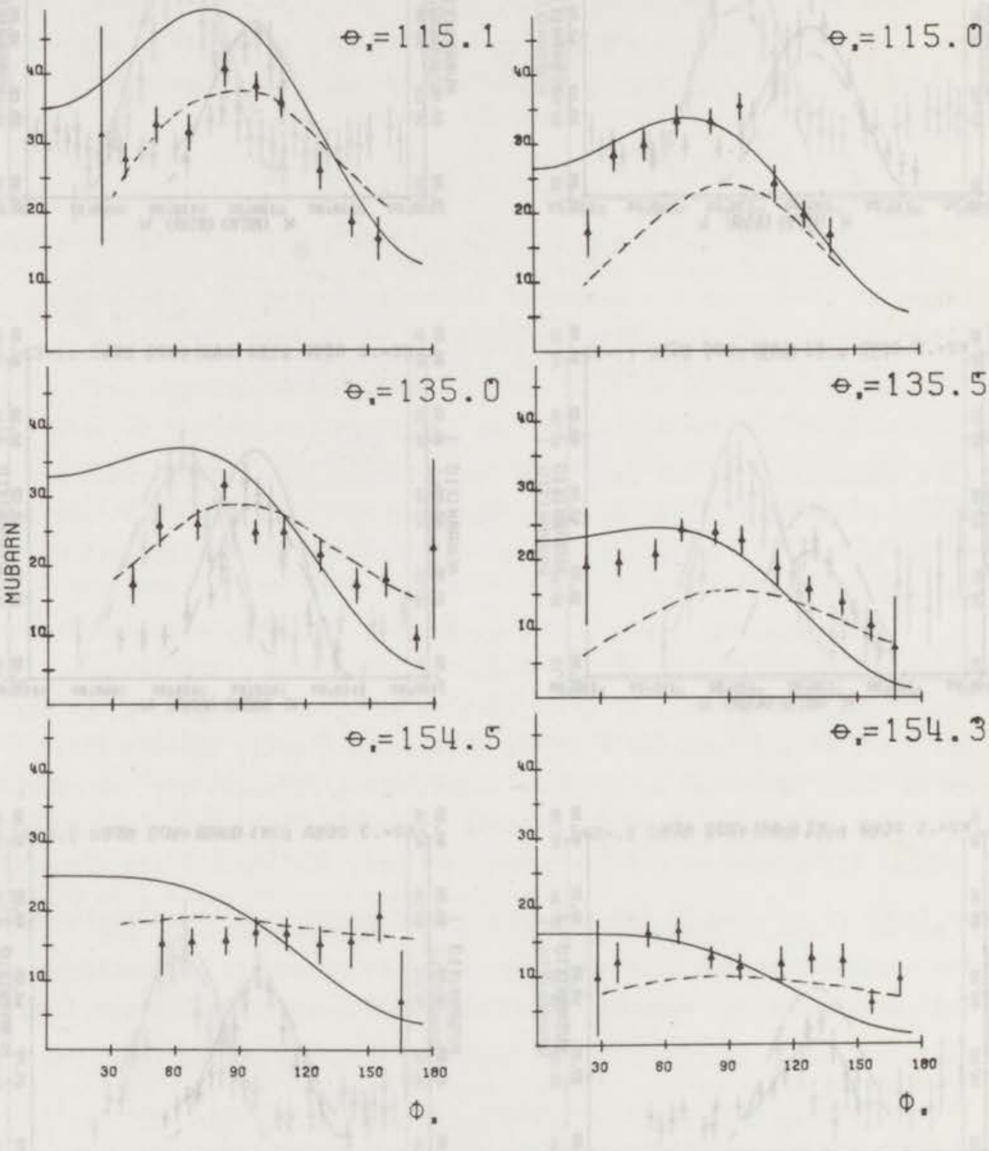


Fig. 6^b. Total cross-section for $ep \rightarrow ep\pi^0$ and $en\pi^+$

Experimental points: Lynch et al. [Ly67] ; solid line: our calculation; dashed line: Adler [Ad68].



$K^2 = 0.25\ GeV^2$
$W = 1228\ MeV$

$K^2 = 0.25\ GeV^2$
$W = 1259\ MeV$

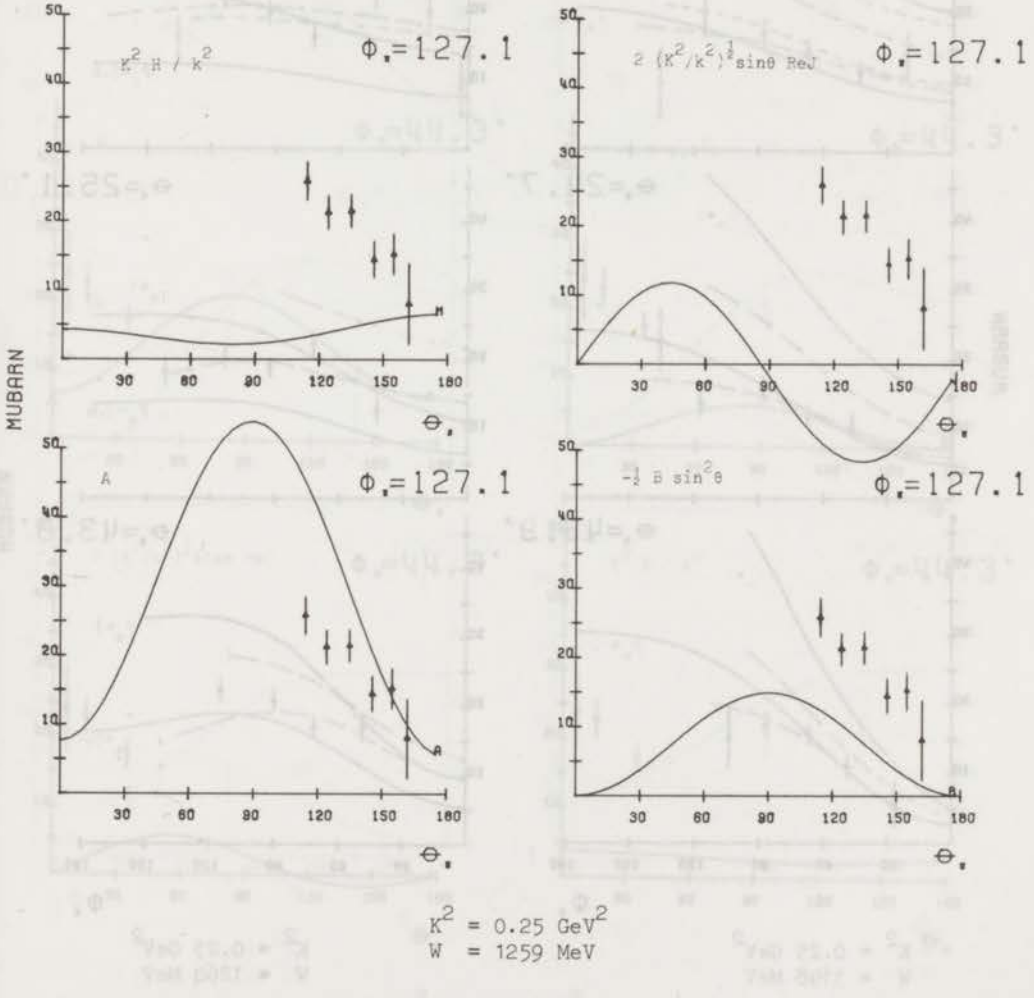
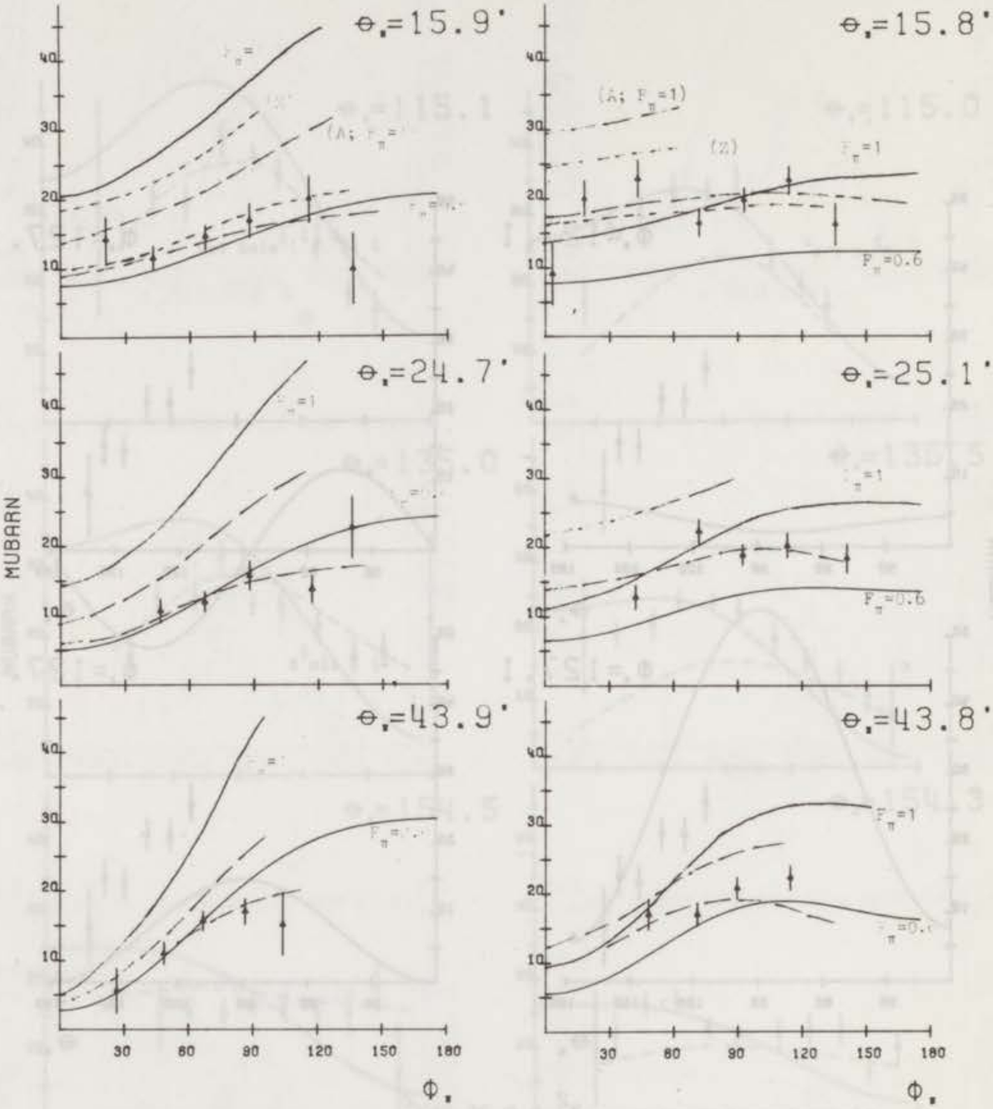


Fig. 8. Separate terms in the cross-section for $ep \rightarrow ep\pi^o$.(cf.(3.24[b]))

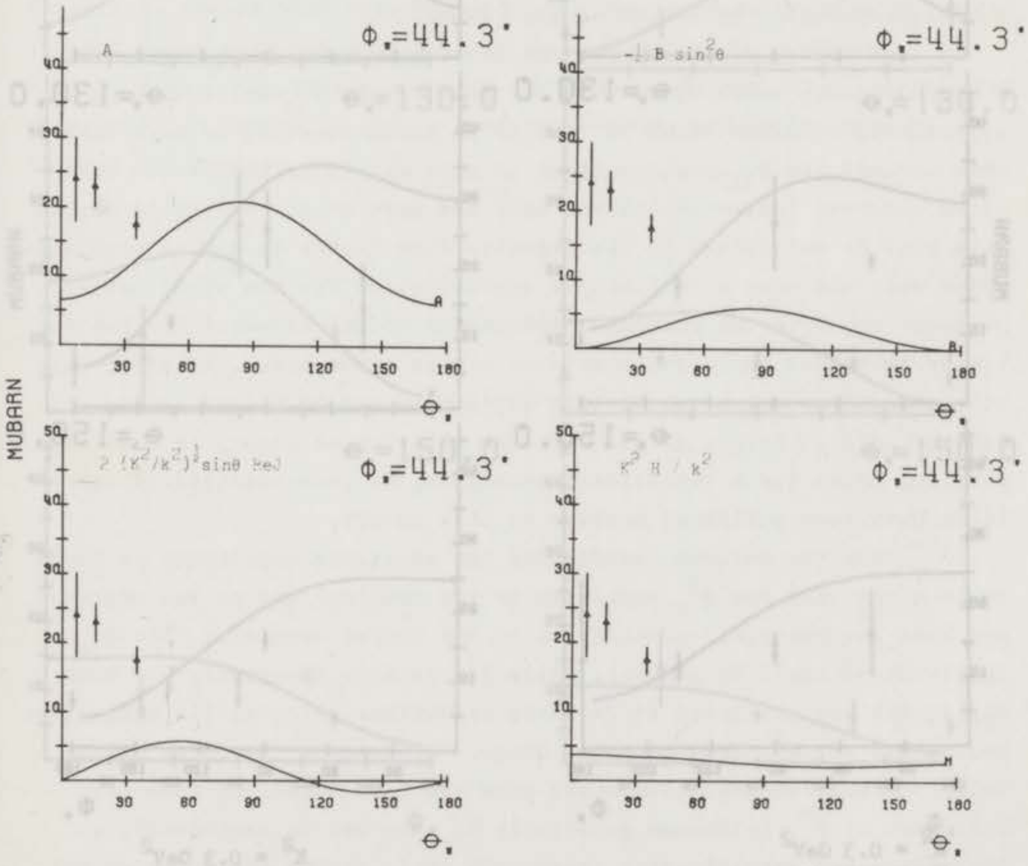
Fig. 7.(page 64). Angular distributions for $ep \rightarrow ep\pi^o$
Experimental points: Mistretta et al. [Mi69]; solid line: our calculation; dashed line: Adler(A) [Ad68], Zagury [Za66], [Za67],cf [Mi69].



K^2 = 0.25 GeV^2
W = 1198 MeV

K^2 = 0.25 GeV^2
W = 1260 MeV

Fig. 9. Angular distributions for $ep \rightarrow en\pi^+$
Experimental points: Mistretta et al. [Mi69]; solid line: our calculation; dashed line: Adler [Ad68], Zagury [Za66], [Za67], cf [Mi69].



K^2 = 0.25 GeV^2
W = 1260 MeV

Fig. 10. Separate terms in the cross-section for $ep \to en\pi^+$.(cf.(3.24^b))

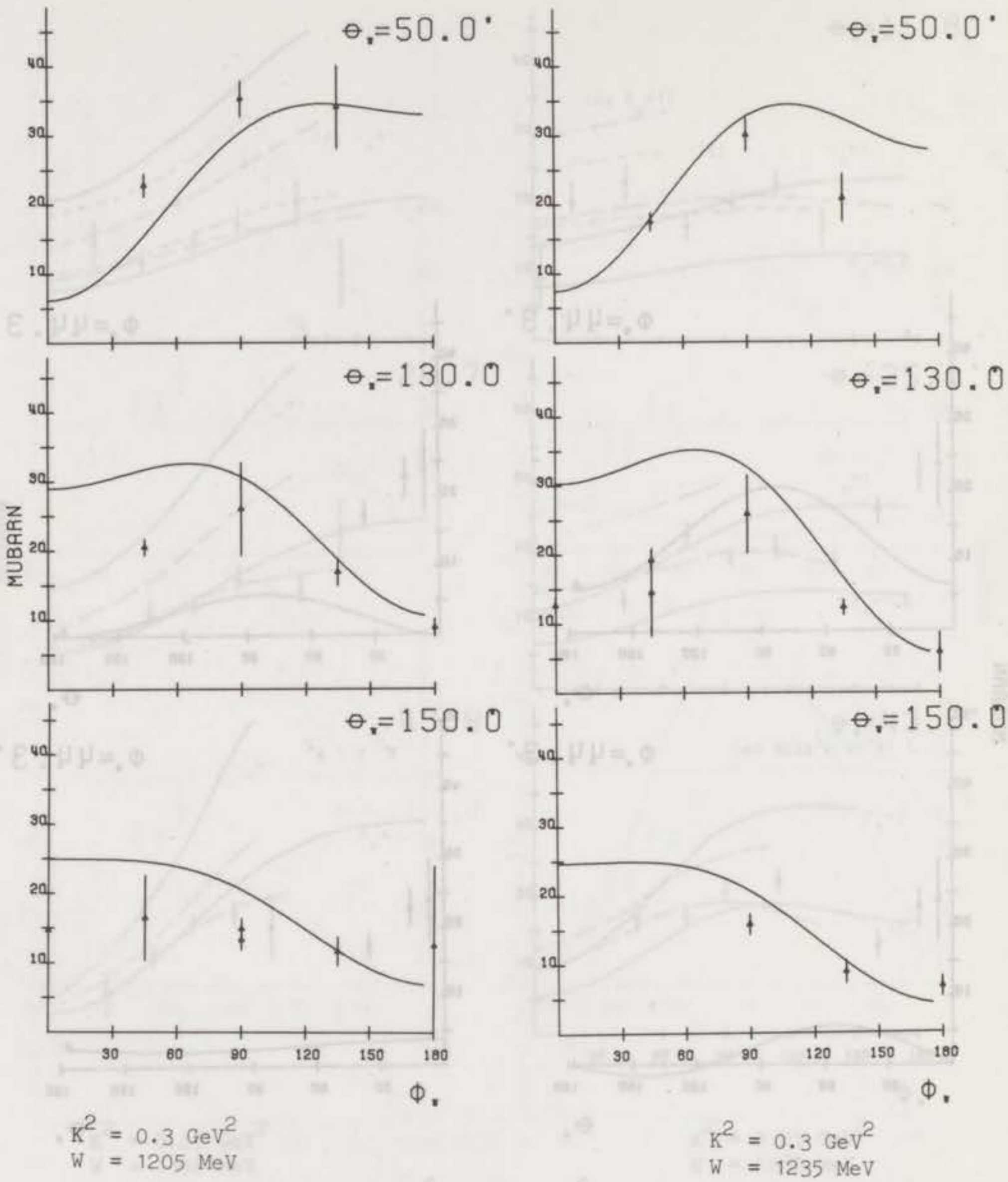


Fig. 11. Angular distributions for $ep \rightarrow ep\pi^{\circ}$

Experimental points: Siddle, Hellings et al. [Si71], [He71]; solid line: our calculation.

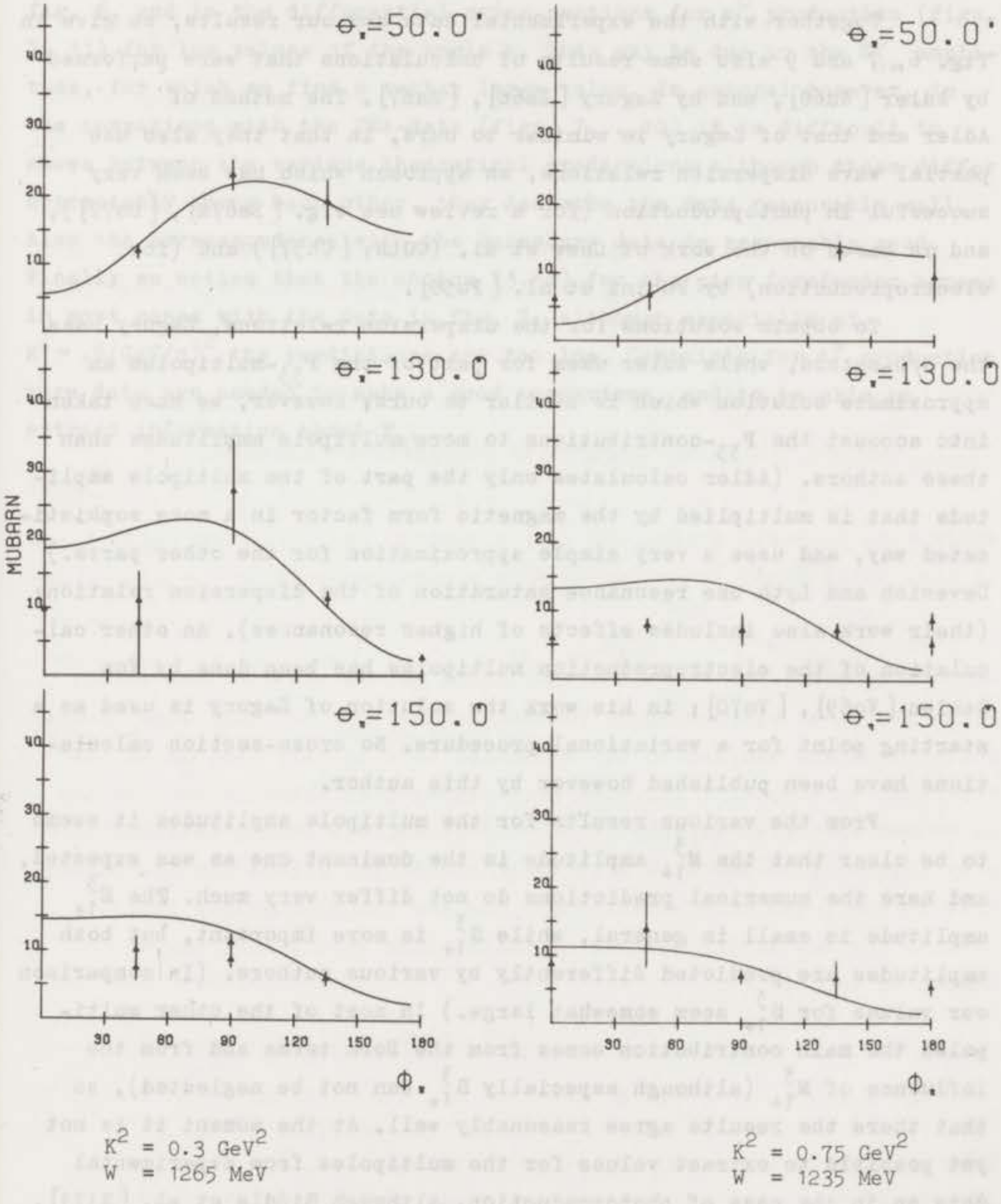


Fig. 12. Angular distributions for $ep \rightarrow ep\pi^o$

Experimental points: Siddle , Hellings et al. [Si71], [He71]; solid line: our calculation.

π^0 production, of which a selection is presented together with our calculations in figs. 11 and 12.

Together with the experimental data and our results, we give in fig. 6, 7 and 9 also some results of calculations that were performed by Adler [Ad68], and by Zagury [Za66], [Za67]. The method of Adler and that of Zagury is similar to ours, in that they also use partial wave dispersion relations, an approach which has been very succesful in photoproduction (for a review see e.g. [Be67A], [Do72]), and is based on the work of Chew et al. (CGLN; [Ch57]) and (for electroproduction) by Fubini et al. [Fu58].

To obtain solutions for the dispersion relations, Zagury uses the N/D-method, while Adler uses for part of the P_{33}-multipoles an approximate solution which is similar to ours; however, we have taken into account the P_{33}-contributions to more multipole amplitudes than these authors. (Adler calculates only the part of the multipole amplitude that is multiplied by the magnetic form factor in a more sophisticated way, and uses a very simple approximation for the other parts.) Devenish and Lyth use resonance saturation of the dispersion relation; (their work also includes effects of higher resonances). An other calculation of the electroproduction multipoles has been done by Von Gehlen [Vo69], [Vo70]; in his work the solution of Zagury is used as a starting point for a variational procedure. No cross-section calculations have been published however by this author.

From the various results for the multipole amplitudes it seems to be clear that the M^3_{1+} amplitude is the dominant one as was expected, and here the numerical predictions do not differ very much. The E^3_{1+} amplitude is small in general, while S^3_{1+} is more important, but both amplitudes are predicted differently by various authors. (In comparison our values for S^3_{1+} seem somewhat large.) In most of the other multipoles the main contribution comes from the Born terms and from the influence of M^3_{1+} (although especially S^3_{1+} can not be neglected), so that there the results agree reasonably well. At the moment it is not yet possible to extract values for the multipoles from experimental data as in the case of photoproduction, although Siddle et al. [Si71], [He71] give some results, based on their data combined with a few extra assumptions.

From the cross-section values, as presented in figs. 6,..,12, we see that there are a few points where our results are too high compared with experiments, especially in the total cross-sections in fig. 6, and in the differential cross-sections for π^0 production (figs. 7, 11) for low values of the angle φ. This may be due to the S^{3}_{1+} amplitude, for which we find a rather large value. In general however, in the comparison with the CEA-data (figs. 7,..,10) it is difficult to chose between the various theoretical predictions; although these differ appreciably among each other, they describe the data reasonably well. Also the correspondence with the Daresbury data is reasonably good. Finally we notice that the choice (3.34) for the pion formfactor agrees in most cases with the data in fig. 7, although especially at $K^2 \approx .6(\mathrm{GeV}/c)^2$ the predictions are too low. Certainly for π^+ production more data are needed to make a good comparison, and to be able to extract information about F_π.

C H A P T E R   I V

G E N E R A L   F O R M A L I S M   F O R   C O M P T O N   S C A T T E R I N G   O N   N U C L E O N S

IV.1 S-MATRIX AND CROSS-SECTION

In the following chapters we consider the process

$$\gamma p \rightarrow \gamma p \, , \tag{4.1}$$

where γ is a (real) photon and p is a proton. The formalism that we use is also suitable to describe the more general process

$$\gamma N \rightarrow \gamma N \, ,$$

where N is a nucleon (proton or neutron), but we restrict the actual calculations to scattering on protons, since only for that case experimental data are available. The situation here is very similar to that described in Chapter I for pion production; i.e. although the process is caused by the electromagnetic interaction, the strong interaction is very important as well. We will again retain all orders for the strong interaction , and only lowest order for the electromagnetic interaction. For the matrix element we obtain then in the same way as indicated in sec.I.1,

$$\langle \gamma_2 N_2 | S | \gamma_1 N_1 \rangle = \langle \gamma_2 N_2 | \gamma_1 N_1 \rangle + i(2\pi)^{-2} \delta^{(4)}(P_1+K_1-P_2-K_2)\left(\frac{m^2}{4p_{10}p_{20}k_{10}k_{20}}\right)^{\frac{1}{2}} T_{fi} \tag{4.2}$$

where we denote the 4-momenta of the nucleons by P_i and those of the photons by K_i (i=1,2 for initial and final particles). The conventions are the same as before. Further we have

$$T_{fi} = \varepsilon^*_{2\nu} \; {}_{out}\langle N_2 | J_\mu J_\nu | N_1 \rangle_{in} \; \varepsilon_{1\mu} \, , \tag{4.3}$$

where $\varepsilon_{i\mu}$ are the polarization vectors of the photons (see Appendix A). The cross-section is given by

$$d^6\sigma = (2\pi)^{-2}\delta^{(4)}(P_1+K_1-P_2-K_2)m^2(4P_1.K_1)^{-1}|T_{fi}|^2 \frac{d\vec{p}_2}{p_{20}} \frac{d\vec{k}_2}{k_{20}} \quad (4.4^a)$$

In the centre-of-mass system we obtain for the differential cross-section per solid angle

$$\frac{d\sigma}{d\Omega} = \Sigma\left(\frac{m}{4\pi W}\right)^2 |T_{fi}|^2 , \quad (4.4^b)$$

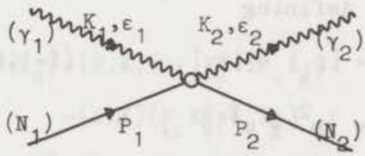
where W is the total centre-of-mass energy, and Σ denotes a sum over final spins and polarizations and an average over initial ones.

IV.2 KINEMATICS

The momenta in the Compton scattering process are labelled as indicated in fig. 13.

fig. 13

The labels K_i, P_i represent the momenta; ε_i the polarization vectors $(K_i.\varepsilon_i=0)$



Energy-momentum conservation implies

$$P_1 + K_1 = P_2 + K_2 . \quad (4.5)$$

Instead of using the momentum variables P_i and K_i, it is more convenient to work with the variables P, K and Q, defined by

$$P = \tfrac{1}{2}(P_1 + P_2)$$

$$K = \tfrac{1}{2}(K_1 + K_2) \quad (4.6^a)$$

$$Q = K_1 - K_2 = P_2 - P_1 ,$$

or with orthogonal set P', K, Q, N, where P' and N are defined by *)

$$P'_\mu = P_\mu - \frac{1}{K^2}(P.K)K_\mu \quad (4.6^b)$$

$$N_\mu = -i\,\varepsilon_{\mu\nu\rho\sigma} P'_\nu K_\rho Q_\sigma .$$

*) Our choice of sign in N_μ agrees with refs. [He62], [Ra66]; the opposite sign is taken by [Ba68], [He61], [Pr58], [Ko68].

We also define the Lorentz-invariant variables

$$s = -(K_1 + P_1)^2$$
$$t = -(K_1 - K_2)^2 \tag{4.7}$$
$$u = -(K_1 - P_2)^2 \ ,$$

satisfying the relation

$$s + t + u = 2m^2 \ . \tag{4.8}$$

We then find $P'^2N^2 = \frac{1}{4}(su-m^4)^2$, and we define $(P'^2N^2)^{\frac{1}{2}} = \frac{1}{2}(su-m^4)$. The photon polarization vectors satisfy the relations

$$\varepsilon_i . K_i = 0 \qquad (i=1,2) \tag{4.9}$$

due to the transverse polarization of the photons (see Appendix A).

We will work in the centre-of-mass system, with a coordinate frame for which the x_3-axis lies along the direction of $\vec{p}_1$, and the x_1x_3-plane is the scattering plane. Denoting the polar angles of $\vec{p}_2$ by (θ,φ), and defining

$$p = |p_i| = |k_i| \qquad (i=1,2)$$
$$E = (p^2 + m^2)^{\frac{1}{2}} \ ,$$

we have in this coordinate frame (since $\varphi=0$)

$$K_1 = (-\vec{p}_1, ip) = (0,0,-p,ip)$$
$$K_2 = (-\vec{p}_2, ip) = (-p \sin\theta,\ 0, -p \cos\theta, ip)$$
$$P_1 = (\vec{p}_1, iE) = (0,0,p,iE) \tag{4.10}$$
$$P_2 = (\vec{p}_2, iE) = (p \sin\theta, 0, p \cos\theta, iE) \ .$$

The total centre-of-mass energy is W=p+E.

IV.3 SPACE-TIME STRUCTURE OF THE SCATTERING-MATRIX ELEMENT

In this section we expand the matrix element T_{fi} for Compton scattering, as defined in eq.(4.3), in terms of various sets of amplitudes similar to those introduced in sec.I.4 for pion production. Two different sets of invariant amplitudes, depending only on the invariants s, t and u, will be defined in parts A and B of this section, while in part C we introduce helicity amplitudes.

If only parity conservation (and no C- or T-invariance) is assumed, the number of amplitudes in each set will be eight; if C-invariance is imposed as well, this is reduced to six. We will treat the general case where C-invariance is not assumed a priori, so that we will be able to investigate the consequences of C- or T-violation for the results.

A. I n v a r i a n t a m p l i t u d e s (A_i)

We define a set of eight amplitudes A_i by writing the matrix element T_{fi} as

$$T_{fi} = \varepsilon^*_{2\nu}\ \bar{u}(\vec{p}_2)\left[\sum_{i=1}^{8} L^i_{\mu\nu}\ A_i\right]u(\vec{p}_1)\varepsilon_{1\mu} \tag{4.11}$$

where the matrices $L^i_{\mu\nu}$ are given by

$$L^1_{\mu\nu} = K^2\,\delta_{\mu\nu} - 2K_\mu K_\nu$$

$$L^2_{\mu\nu} = \tfrac{1}{2}K^2[\gamma_\mu(i\gamma.K)\gamma_\nu - \gamma_\nu(i\gamma.K)\gamma_\mu] + (P.K)(i\gamma_\mu K_\nu + i\gamma_\nu K_\mu) - (i\gamma.K)(P_\mu K_\nu + P_\nu K_\mu)$$

$$L^3_{\mu\nu} = \tfrac{1}{2}K^2[\gamma_\mu,\gamma_\nu] + \tfrac{1}{2}K_\mu[(i\gamma.K), i\gamma_\nu] + \tfrac{1}{2}[i\gamma_\mu,(i\gamma.K)]K_\nu - m(i\gamma_\mu K_\nu + i\gamma_\nu K_\mu) + m\,\delta_{\mu\nu}(i\gamma.K) - (P.K)\delta_{\mu\nu} + (P_\mu K_\nu + P_\nu K_\mu)$$

$$L^4_{\mu\nu} = -K^2(i\gamma_\mu P_\nu + i\gamma_\nu P_\mu) + (P.K)(i\gamma_\mu K_\nu + i\gamma_\nu K_\mu) + (i\gamma.K)(P_\mu K_\nu + P_\nu K_\mu) - (P.K)\delta_{\mu\nu}(i\gamma.K) - mK^2\delta_{\mu\nu} + 2mK_\mu K_\nu$$

$$L^5_{\mu\nu} = -K^2 P_\mu P_\nu + (P.K)(P_\mu K_\nu + P_\nu K_\mu) + \tfrac{1}{2}(P^2K^2 - (P.K)^2)\delta_{\mu\nu} - P^2 K_\mu K_\nu$$

$$L^6_{\mu\nu} = -P_\mu(i\gamma.K)P_\nu + \tfrac{1}{2}(P.K)(i\gamma_\mu P_\nu + i\gamma_\nu P_\mu) + \tfrac{1}{4}(P.K)[\gamma_\mu(i\gamma.K)\gamma_\nu - \gamma_\nu(i\gamma.K)\gamma_\mu] - \tfrac{1}{4}mK^2[\gamma_\mu,\gamma_\nu] + \tfrac{1}{2}m(P.K)\delta_{\mu\nu} + \tfrac{1}{2}P^2(i\gamma.K)\delta_{\mu\nu} - (i\gamma.K)K_\mu K_\nu + \tfrac{1}{2}m^2(i\gamma_\mu K_\nu + i\gamma_\nu K_\mu) - \tfrac{1}{2}m(P_\mu K_\nu + P_\nu K_\mu) - \tfrac{1}{4}m\{K_\mu[(i\gamma.K), i\gamma_\nu] + K_\nu[i\gamma_\mu,(i\gamma.K)]\}$$

$$L^7_{\mu\nu} = \tfrac{1}{2}K^2\{P_\mu[i\gamma_\nu,(i\gamma.K)] + P_\nu[i\gamma_\mu,(i\gamma.K)]\} - \tfrac{1}{2}(P.K)\{K_\mu[i\gamma_\nu,(i\gamma.K)] + K_\nu[i\gamma_\mu,(i\gamma.K)]\} + m\{P_\mu(i\gamma.K)K_\nu - K_\mu(i\gamma.K)P_\nu\} - (P.K)(P_\mu K_\nu - P_\nu K_\mu)$$

$$L^8_{\mu\nu} = -K^2(i\gamma_\mu P_\nu - i\gamma_\nu P_\mu) + (P.K)(i\gamma_\mu K_\nu - i\gamma_\nu K_\mu) - (i\gamma.K)(P_\mu K_\nu - P_\nu K_\mu) \tag{4.12}$$

When C- (or T-) invariance holds, only the first six amplitudes will be non-zero. These are the same as the six amplitudes introduced by Bardeen and Tung [Ba68]. In the case that these invariances may be violated we will need the complete set of eight amplitudes.

The amplitudes A_i are free of kinematical singularities, and furthermore they have the important property that there are no constraint equations that have to be satisfied, since the matrices $L^i_{\mu\nu}$ are explicitly gauge-invariant and regular. The procedure that leads to this special set of matrices is given in ref. [Ba68] and can be briefly summarized as follows. We start from an expansion for T_{fi}, in which all possible matrices constructed from P, K and γ are used. (We do not need Q, since this can be easily eliminated by using the Dirac equation and the polarization conditions, $\varepsilon_i . K_i = 0$ (i=1,2).) Denoting these matrices by $\ell^i_{\mu\nu}$ and the corresponding amplitudes by a_i, we have instead of eq. (4.11),

$$T_{fi} = \varepsilon^*_{2\nu}\, \bar{u}(\vec{p}_2)[\sum_{i=1}^{10} \ell^i_{\mu\nu}\, a_i]\, u(\vec{p}_1)\varepsilon_{1\mu} \qquad (4.13)$$

where the matrices are given by

$$\begin{aligned}
\ell^1_{\mu\nu} &= \delta_{\mu\nu} & \ell^6_{\mu\nu} &= \tfrac{1}{2}P_\mu[i\gamma_\nu,(i\gamma.K)] - \tfrac{1}{2}P_\nu[i\gamma_\mu,(i\gamma.K)] \\
\ell^2_{\mu\nu} &= \delta_{\mu\nu}(i\gamma.K) & \ell^7_{\mu\nu} &= \tfrac{1}{2}[\gamma_\mu,\gamma_\nu] \\
\ell^3_{\mu\nu} &= P_\mu P_\nu & \ell^8_{\mu\nu} &= \tfrac{1}{2}[\gamma_\mu(i\gamma.K)\gamma_\nu - \gamma_\nu(i\gamma.K)\gamma_\mu] \\
\ell^4_{\mu\nu} &= P_\mu P_\nu(i\gamma.K) & \ell^9_{\mu\nu} &= i\gamma_\mu P_\nu - i\gamma_\nu P_\mu \\
\ell^5_{\mu\nu} &= i\gamma_\mu P_\nu + i\gamma_\nu P_\mu & \ell^{10}_{\mu\nu} &= \tfrac{1}{2}P_\mu[i\gamma_\nu,(i\gamma.K)] + \tfrac{1}{2}P_\nu[i\gamma_\mu,(i\gamma.K)]
\end{aligned} \qquad (4.14)$$

It can be verified [He61] that the amplitudes a_i, defined by (4.13), are free of kinematical singularities. They are however not independent, since we now have ten amplitudes instead of the eight needed, and even when two of these are eliminated, gauge invariance imposes further constraints. The next step is then to introduce gauge-invariant matrices $\ell^{i'}_{\mu\nu}$,

$$\ell^{i'}_{\mu\nu} = I_{\mu\rho}\, \ell^i_{\rho\sigma}\, I_{\sigma\nu}\ , \qquad (4.15)$$

where $I_{\mu\nu}$ is the operator that projects out the gauge-invariant part of the $\ell^i_{\mu\nu}$. It is given by

$$I_{\mu\nu} = \delta_{\mu\nu} - K_{2\mu}K_{1\nu}/2K^2\ . \qquad (4.16)$$

The matrices $\ell^{i'}_{\mu\nu}$ contain kinematical singularities, due to the K^2-denominator in (4.16), which would lead to zeroes in the corresponding amplitudes a_i'. In the last step these singularities are eliminated by taking appropriate linear combinations of the $\ell^{i'}_{\mu\nu}$, while at this stage we also reduce the number of invariants to eight, using relations between the invariants, obtained in ref. [Sc68]. This finally leads us to the invariants $L^i_{\mu\nu}$ given in (4.12).

B. I n v a r i a n t   a m p l i t u d e s (B_i)

An other useful set of amplitudes, B_i, is defined by

$$T_{fi} = \varepsilon^*_{2\nu}\, \bar{u}(\vec{p}_2)[\sum_{i=1}^{8} M^i_{\mu\nu}\, B_i]u(\vec{p}_1)\varepsilon_{1\mu} \tag{4.17}$$

and

$$M^1_{\mu\nu} = P'_\mu P'_\nu/P'^2$$

$$M^2_{\mu\nu} = N_\mu N_\nu/N^2$$

$$M^3_{\mu\nu} = (P'_\mu N_\nu - P'_\nu N_\mu)i\gamma_5/(P'^2N^2)^{\frac{1}{2}}$$

$$M^4_{\mu\nu} = (i\gamma.K)P'_\mu P'_\nu/P'^2$$

$$M^5_{\mu\nu} = (i\gamma.K)N_\mu N_\nu/N^2$$

$$M^6_{\mu\nu} = (P'_\mu N_\nu + P'_\nu N_\mu)i\gamma_5(i\gamma.K)/(P'^2N^2)^{\frac{1}{2}}$$

$$M^7_{\mu\nu} = (P'_\mu N_\nu + P'_\nu N_\mu)i\gamma_5/(P'^2N^2)^{\frac{1}{2}}$$

$$M^8_{\mu\nu} = (P'_\mu N_\nu - P'_\nu N_\mu)i\gamma_5(i\gamma.K)/(P'^2N^2)^{\frac{1}{2}} . \tag{4.18}$$

In the case of C- (or T-)invariance again the last two amplitudes, B_7 and B_8, will be zero. The first six amplitudes are the same as those introduced by Hearn and Leader [He62], (cf. also [Pr58]). Since some of the gauge-invariant matrices $M^i_{\mu\nu}$ are singular in the points t=0 and $su=m^4$, the following extra conditions have to be imposed on the amplitudes B_i to ensure that T_{fi} is regular, (cf. part C of this section)

at t=0:

$$m(B_1 + B_2) - \tfrac{1}{2}(s - m^2)(B_4 + B_5) = 0 ,$$

$$\frac{1}{2m}(s - m^2)(B_4 + B_5) + 2B_3 = 0 , \qquad B_8 = 0 ; \tag{4.19}$$

at $su=m^4$:

$$\tfrac{1}{2}(s - m^2)(B_4 - B_5) + (s + m^2)B_6 = 0 ,$$

$$\frac{1}{2m}(B_1 - B_2) + B_6 = 0 , \qquad B_7 = 0. \tag{4.20}$$

Due to the simple forms of the matrices $M^i_{\mu\nu}$, as compared to the $L^i_{\mu\nu}$, the connection with helicity amplitudes (cf. part C of this section) and Born terms (cf. Sec.V.4) is less complicated for the amplitudes B_i than for the A_i. Thus we will use the B_i as an intermediate set between the A_i and other forms. A simple method to obtain the relations between these two sets of amplitudes is given in Appendix G. These relations are summarized as

$$\tilde{A} = [Z]\tilde{B} \tag{4.21}$$

(using the same vector notation as in sec.I.4.D), with [Z] given by eq.(G.6).

C. Helicity amplitudes

We define helicity amplitudes $f_{\lambda_2\mu_2;\lambda_1\mu_1}(\theta,\varphi,p)$ for Compton scattering by taking the matrix element of the T-operator between states with definite photon and nucleon helicities, writing in the centre-of-mass frame [Ja59]

$$\langle \vec{k}_2,\lambda_2;\vec{p}_2,\mu_2|T|\vec{k}_1,\lambda_1;\vec{p}_1,\mu_1\rangle =$$

$$= \bar{u}^{(r_2)}(\vec{p}_2)\varepsilon^{*(\lambda_2)}_{2\nu}(\vec{k}_2)T_{\mu\nu}\varepsilon^{(\lambda_1)}_{1\mu}(\vec{k}_1)u^{(r_1)}(\vec{p}_1)=\frac{4\pi W}{m}f_{\lambda_2\mu_2;\lambda_1\mu_1}(\theta,\varphi,p) \tag{4.22}$$

where $\vec{p}_i$ is the momentum of the initial (i=1) or final nucleon (i=2), and $\vec{k}_i = -\vec{p}_i$ the momentum of the initial or final photon (cf. sec.IV.2); μ_i is the nucleon helicity and λ_i that of the photon ($\mu_i=\pm\frac{1}{2}$;$\lambda_i=\pm1$; we will also use $r_i=2\mu_i=\pm1$); θ and φ are the polar angles of the vector $\vec{p}_2$. For the polarization vectors we have

$$\varepsilon_1^{(\lambda_1)}(-\vec{p}_1) = \varepsilon_1^{(-\lambda_1)}(\vec{p}_1)$$

and (4.23)

$$\varepsilon_2^{(\lambda_2)}(\vec{p}_2) = \sum_{\lambda'=\pm1} \exp(-i(\lambda'-\lambda_2)\varphi)d^1_{\lambda'\lambda_2}(\theta)\varepsilon_2^{(\lambda')}(\vec{p}_1) ,$$

where the $d^{j}_{\lambda\lambda'}(\theta)$ are defined by Jacob and Wick [Ja59]. Denoting the spinor for a nucleon at rest with helicity $\mu_1 = \frac{1}{2}r_1$ in the direction $\hat{p}_1$ by

$$u^{(r_1)}_{\hat{p}_1} = \begin{pmatrix} \chi(r_1) \\ 0 \end{pmatrix}, \text{ with } \chi(+1) = \binom{1}{0} \text{ and } \chi(-1) = \binom{0}{1},$$

we have for the spinors the relation

$$u^{(r_2)}_{\hat{p}_2} = \sum_{s=\pm 1} \exp(\tfrac{1}{2}i(r_2 - s)\varphi)\, d^{\frac{1}{2}}_{\frac{1}{2}s,\frac{1}{2}r_2}(\theta)\, u^{(s)}_{\hat{p}_1} . \tag{4.24}$$

The spinors for a moving particle are then obtained from

$$u^{(r_i)}(\vec{p}_i) = \frac{m - i\gamma . P_i}{(2m(E+m))^{\frac{1}{2}}}\, u^{(r_i)}_{\hat{p}_i} . \tag{4.25}$$

Using these formulae and the explicit forms for the polarization vectors from Appendix A we obtain, substituting in eq.(4.22) for $T_{\mu\nu}$ the expansion in terms of the amplitudes B_i,

$$8\pi W f_{\lambda_2\mu_2;\lambda_1\mu_1}(\theta,\varphi,p) = 2m\bar{u}^{(r_2)}(\vec{p}_2)\varepsilon^{*(\lambda_2)}_{2\nu}(\vec{k}_2)T_{\mu\nu}\varepsilon^{(\lambda_1)}_{1\mu}(\vec{k}_1)u^{(r_1)}(\vec{p}_1) =$$

$$= m\bar{u}^{(r_2)}(\vec{p}_2)\{[\lambda_2(\cos^2\tfrac{1}{2}\theta - e^{2i\lambda_2\varphi}\sin^2\tfrac{1}{2}\theta)\hat{e}_1 + i(\cos^2\tfrac{1}{2}\theta + e^{2i\lambda_2\varphi}\sin^2\tfrac{1}{2}\theta)\hat{e}_2 -$$

$$- \lambda_2 \sin\theta e^{i\lambda_2\varphi}\hat{e}_3]_\nu [\textstyle\sum_i M^i_{\mu\nu} B_i][\lambda_1\hat{e}_1 - i\hat{e}_2]_\mu\} u^{(r_1)}(\vec{p}_1) =$$

$$= \{\delta_{r_1 r_2}\cos\tfrac{1}{2}\theta[m(B_2 - \lambda_1\lambda_2 B_1) - Wp(B_5 - \lambda_1\lambda_2 B_4) + r_1(\lambda_2 - \lambda_1)WpB_8 -$$

$$- r_1(\lambda_1 + \lambda_2)WpB_6] + \delta_{r_1,-r_2}\sin\tfrac{1}{2}\theta[-r_1 E(B_2 - \lambda_1\lambda_2 B_1) +$$

$$+ r_1 mp(B_5 - \lambda_1\lambda_2 B_4) + (\lambda_1 - \lambda_2)pB_3 + (\lambda_1 + \lambda_2)pB_7]\}\exp(i(\lambda_2 - \lambda_1 + \tfrac{1}{2}(r_1 - r_2))\varphi). \tag{4.26}$$

(This calculation is a generalization to eight amplitudes of the method given by Rasche and Woolcock [Ra66], cf. also [Ra65].) Since we have chosen our coordinate system in such a way that $\varphi = 0$, the exponential factor in eq.(4.26) equals one, and will be omitted from now on. We introduce a shorter notation for the helicity amplitudes by defining

$$\Phi_1 = f_{1,\frac{1}{2};1,\frac{1}{2}} \qquad \Phi_4 = f_{1,-\frac{1}{2};1,\frac{1}{2}}$$
$$\Phi_2 = f_{-1,-\frac{1}{2};1,\frac{1}{2}} \qquad \Phi_5 = f_{1,-\frac{1}{2};1,-\frac{1}{2}}$$
$$\Phi_3 = f_{-1,\frac{1}{2};1,\frac{1}{2}} \qquad \Phi_6 = f_{-1,\frac{1}{2};1,-\frac{1}{2}}$$

$$\Phi_7 = f_{-1,-\frac{1}{2};1,-\frac{1}{2}} \qquad \Phi_8 = f_{1,\frac{1}{2};1,-\frac{1}{2}} \tag{4.27}$$

The relations between the Φ_i and the B_i can be read off immediately from eq.(4.26), and using the formulae from Appendix G (eq.(G.6)) we obtain the relations with the amplitudes A_i:

$$\Phi_1 = -\frac{1}{32\pi W}\cos\tfrac{1}{2}\theta[2((s-m^2)^2+tm^2)A_4-m(su-m^4)A_5-((s-m^2)^2-tm^2)A_6]$$

$$\Phi_2 = -\frac{1}{32\pi W^2}\sin\tfrac{1}{2}\theta[t(s+m^2)A_1+2((s-m^2)^2-(su-m^4))A_3-mt(s-m^2)A_2]$$

$$\Phi_3 = \frac{1}{32\pi W}\cos\tfrac{1}{2}\theta[2mt(A_1+A_3) - t(s-m^2)A_8]$$

$$\Phi_4 = -\frac{1}{32\pi W^2}\sin\tfrac{1}{2}\theta(su-m^4)[2mA_4+\tfrac{1}{2}(s+m^2)A_5+mA_6+(s-m^2)A_7]$$

$$\Phi_5 = \frac{1}{32\pi W}\cos\tfrac{1}{2}\theta(su-m^4)[2A_4 + mA_5 + A_6]$$

$$\Phi_6 = \frac{1}{32\pi W^2}\sin\tfrac{1}{2}\theta[t(s+m^2)A_1+mt(s-m^2)A_2+2m^2tA_3]$$

$$\Phi_7 = \frac{1}{32\pi W}\cos\tfrac{1}{2}\theta[2mt(A_1+A_3) + t(s-m^2)A_8]$$

$$\Phi_8 = \frac{1}{32\pi W^2}\sin\tfrac{1}{2}\theta(su-m^4)[2mA_4+\tfrac{1}{2}(s+m^2)A_5+mA_6-(s-m^2)A_7] \tag{4.28}$$

As we have remarked before, the amplitudes A_7 and A_8 are zero in the absence of C- or T-violation. We see from (4.28) that we obtain in that case $\Phi_3=\Phi_7$ and $\Phi_4=\Phi_8$, leaving six amplitudes.
The inverse relations are

$$A_1 = \frac{8\pi W}{mt\cos\frac{1}{2}\theta}[1+\tfrac{1}{2}t(s+m^2)/(s-m^2)^2](\Phi_3+\Phi_7)+\frac{8\pi W^2}{\sin\frac{1}{2}\theta}(\Phi_2-\Phi_6)/(s-m^2)^2$$

$$A_2 = \frac{1}{m^2(s-m^2)^2}\Big\{-\frac{4\pi W}{\cos\frac{1}{2}\theta}(s+m^2)[1+2(s-m^2)/t](\Phi_3+\Phi_7)+\frac{8\pi mW^2}{\sin\frac{1}{2}\theta}[-\Phi_2 + +(1+4(s-m^2)/t)\Phi_6]\Big\}$$

$$A_3 = \frac{1}{m(s-m^2)^2}\Big\{-\frac{4\pi W}{\cos\frac{1}{2}\theta}(s+m^2)(\Phi_3+\Phi_7)-\frac{8\pi mW^2}{\sin\frac{1}{2}\theta}(\Phi_2-\Phi_6)\Big\}$$

$$A_4 = \frac{1}{(s-m^2)^2}\Big\{-\frac{8\pi W}{\cos\frac{1}{2}\theta}\Phi_1+\frac{8\pi W}{\cos\frac{1}{2}\theta}\frac{[(s-m^2)(s+3m^2)+m^2t]}{(su-m^4)}\Phi_5 + + \frac{8\pi mW^2}{\sin\frac{1}{2}\theta}\frac{[t+2(s-m^2)]}{(su-m^4)}(\Phi_4-\Phi_8)\Big\}$$

$$A_5 = \frac{1}{(su-m^4)(s-m^2)} \left\{ -\frac{64\pi mW}{\cos\frac{1}{2}\theta} \Phi_5 - \frac{32\pi W^2}{\sin\frac{1}{2}\theta} (\Phi_4 - \Phi_8) \right\}$$

$$A_6 = \frac{1}{(s-m^2)^2} \left\{ \frac{16\pi W}{\cos\frac{1}{2}\theta} \Phi_1 - \frac{16\pi W}{\cos\frac{1}{2}\theta} \frac{[m^2t-(s-m^2)^2]}{(su-m^4)} \Phi_5 - \frac{16\pi mW^2}{\sin\frac{1}{2}\theta} \frac{t}{(su-m^4)} (\Phi_4 - \Phi_8) \right\}$$

$$A_7 = -\frac{16\pi W^2}{\sin\frac{1}{2}\theta} \frac{1}{(su-m^4)(s-m^2)} (\Phi_4 + \Phi_8)$$

$$A_8 = -\frac{16\pi W}{\cos\frac{1}{2}\theta} \frac{1}{t(s-m^2)} (\Phi_3 - \Phi_7) . \tag{4.29}$$

Due to angular momentum conservation, in the forward direction ($\theta=0$) all helicity amplitudes except Φ_1 and Φ_5 must vanish, and in the backward direction ($\theta=\pi$) only Φ_2 and Φ_6 should be non-zero. Moreover, from the partial wave expansion for the helicity amplitudes [Ja59],

$$\Phi_i = \langle \lambda_2 \mu_2 | T | \lambda_1 \mu_1 \rangle = \frac{1}{2p} \sum_J (2J+1) \Phi^J_{\nu_1 \nu_2} d^J_{\nu_1 \nu_2}(\theta) \quad , \tag{4.30}$$

with $\nu_i = \mu_i - \lambda_i$) and using the explicit forms for the $d^J_{\nu_1\nu_2}(\theta)$, it can be shown (see e.g. [Co68], Ch. IV) that the amplitudes Φ_i should contain factors

$$(\sin\tfrac{1}{2}\theta)^{|\nu_1 - \nu_2|} (\cos\tfrac{1}{2}\theta)^{|\nu_1 + \nu_2|} \quad ,$$

(which automatically ensure that the restrictions for $\theta=0$ and π are satisfied). Comparing with eqs.(4.28) and using the relations

$$t = -\sin^2\tfrac{1}{2}\theta \, (s-m^2)^2/s$$

and

$$su - m^4 = -st - (s-m^2)^2 = -\cos^2\tfrac{1}{2}\theta (s-m^2)^2 ,$$

we can immediately verify that the expressions for Φ_i in terms of the amplitudes A_j explicitly contain the correct factors. For the corresponding expressions in terms of the amplitudes B_j this is not the case, and this results in the restrictions (4.19) and (4.20) for these amplitudes.

Via the unitarity relation (cf. sec.V.3) Compton scattering is connected with pion photoproduction, and in the calculations we will need helicity amplitudes for this latter process. We denote these amplitudes by Ψ_i, and write

$$\Psi_i = \langle \vec{q},0;\vec{p}_n,\mu | T | \vec{k}_1,\lambda_1;\vec{p}_1,\mu_1 \rangle = \tfrac{1}{2}(pq)^{-\frac{1}{2}} \sum_J (2J+1)\Psi^J_{\nu_1\mu}(\theta) d^J_{\nu,\mu}(\theta) \tag{4.31^a}$$

or in a shorter notation

$$\Psi_i = \langle \mu | T | \lambda_1,\mu_1 \rangle,$$

with

$$\Psi_1 = \langle \tfrac{1}{2} | T | 1,\tfrac{1}{2} \rangle \qquad \Psi_3 = \langle \tfrac{1}{2} | T | -\tfrac{1}{2},1 \rangle$$

$$\Psi_2 = \langle -\tfrac{1}{2} | T | 1,\tfrac{1}{2} \rangle \qquad \Psi_4 = \langle -\tfrac{1}{2} | T | -\tfrac{1}{2},1 \rangle . \tag{4.31^b}$$

The helicity of the final nucleon in the photoproduction process is denoted with μ; the helicities of the initial photon and nucleon are again λ_1 and μ_1, and $\nu_1=\mu_1-\lambda_1$. The kinematics for this process is given in sec.I.2 (cf. fig. 2; there is a minor change in notation: the initial photon here has momentum K_1, and is real ($K_1^2=0$); the final nucleon momentum is P_n, ($\vec{p}_n=-\vec{q}$).)

IV.4 LOW-ENERGY BEHAVIOUR OF THE INVARIANT AMPLITUDES

It is well-known ([Lo54], [Ge54]) that to first order in the photon energy the Compton scattering amplitude depends only on the mass, charge, and magnetic moment of the scatterer (i.e. the proton). This result can be derived, using only gauge invariance and relativistic invariance [Lo54]. Denoting this first-order approximation at low energies (in the limit $p\to 0$, θ fixed) for the scattering amplitude by T^{ℓ}_{fi}, we have in the centre-of-mass system

$$\begin{aligned} T^{\ell}_{fi} = \chi_2^{\dagger}\{ &-\frac{e^2}{m}\vec{\varepsilon}_1.\vec{\varepsilon}^*_2 - 2i\mu_p^2 p\vec{\sigma}.[(\hat{p}_2\times\vec{\varepsilon}^*_2)\times(\hat{p}_1\times\vec{\varepsilon}_1)] - \\ &- \frac{1}{m}ie\mu_p p[\vec{\sigma}.\hat{p}_1(\hat{p}_1\times\vec{\varepsilon}_1).\vec{\varepsilon}^*_2+\vec{\sigma}.(\hat{p}_1\times\vec{\varepsilon}_1)(\hat{p}_1.\vec{\varepsilon}^*_2)-\vec{\sigma}.\hat{p}_2(\hat{p}_2\times\vec{\varepsilon}^*_2).\vec{\varepsilon}_1 - \\ &- \vec{\sigma}.(\hat{p}_2\times\vec{\varepsilon}^*_2)(\hat{p}_2.\vec{\varepsilon}_1)] + \frac{1}{m}ie\mu'_p p\vec{\sigma}.(\vec{\varepsilon}^*_2\times\vec{\varepsilon}_1)]\chi_1 \end{aligned} \tag{4.32}$$

where e is the charge and μ_p the magnetic moment of the proton; μ'_p is the anomalous part of the magnetic moment; χ_i are the Pauli-spinors of the nucleon. By a straightforward calculation one can derive from this expression for T^{ℓ}_{fi} the low-energy limits for the amplitudes A_i or B_i. Denoting these limits by A^{ℓ}_i and B^{ℓ}_i we obtain (using now the proton formfactors $F^p_1(0)=1$ and $F^p_2(0)=\mu'_p/2m$, or briefly F_1 and F_2; cf. sec.V.4),

$$A_1^\ell = e^2F_1^2/mp^2$$
$$A_2^\ell = e^2F_1(F_1 + 2mF_2)/m^2p^2$$
$$A_3^\ell = 2e^2F_2(F_1 + mF_2)/mp$$
$$A_4^\ell = e^2F_1(F_1 + 2mF_2)/m^2p^2$$
$$A_5^\ell = -4e^2F_1F_2/m^2p^2$$
$$A_6^\ell = -4e^2F_2^2/mp$$
$$A_7^\ell = A_8^\ell = 0 \tag{4.33^a}$$

$$B_1^\ell = -(1 - \cos\theta)e^2F_1^2/m$$
$$B_2^\ell = 4e^2F_2(F_1 + mF_2)$$
$$B_3^\ell = -(1-\cos\theta)e^2F_1(F_1+2mF_2)/2m - 2e^2F_2(F_1+mF_2)$$
$$B_4^\ell = -e^2F_1^2/mp$$
$$B_5^\ell = e^2(F_1+2mF_2)^2/mp$$
$$B_6^\ell = (1-\cos\theta)e^2F_1(F_1+2mF_2)/2m^2 + 2e^2F^2$$
$$B_7^\ell = B_8^\ell = 0 \tag{4.33^b}$$

When we write down dispersion relations for the invariant amplitudes (which will be done in the following chapters), we have to ascertain that the amplitudes will correctly satisfy these low-energy limits (cf. sec.VI.1).

IV.5 OUTLINE OF THE CALCULATION

In the following two chapters we present the formalism for the calculation of Compton-scattering cross-sections, and we compare the numerical values obtained in this way with experimental results. The first step in this procedure is described in Chapter V, where we make use of the analytic properties of the invariant amplitudes, to derive a set of integral equations (dispersion relations) for these amplitudes. This derivation is similar to the one given in Chapter II for pion production, but in the case of Compton scattering it is not necessary to transform the equations to integral equations for multipole amplitudes.

The method of solution for the dispersion relations is described in Chapter VI. Via an expansion in terms of helicity amplitudes, it is possible (in the energy region under consideration, i.e. near the first resonance) to express the integrands in terms of experimental data from pion photoproduction. The equations are then solved numerically, and amplitudes and cross-sections are calculated.

CHAPTER V

ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDES FOR COMPTON SCATTERING

V.1 ANALYTICITY

The analytic properties of the invariant amplitudes for Compton scattering are quite similar to those for pion production. Therefore we will mention in this chapter only those aspects that are typical for Compton scattering, and refer to Chapter II for a more detailed treatment.

We consider the three related processes

$$
\begin{aligned}
&\gamma_1 N_1 \rightarrow \gamma_2 N_2 && \text{(s-channel)}\\
&N_1 \bar{N}_2 \rightarrow \gamma_1 \gamma_2 && \text{(t-channel)}\\
&\gamma_2 N_1 \rightarrow \gamma_1 N_2 && \text{(u-channel)} .
\end{aligned}
\tag{5.1}
$$

The analyticity postulate for the invariant amplitudes for these processes is the same as stated in sec.II.2, i.e. we need only one set of amplitudes for a description of these three processes, and the amplitudes are meromorphic functions of s, t and u. Since we consider electromagnetic interactions only in lowest order, the singularities which are determined by the possible intermediate states in reactions (5.1) are the same as for pion production, as can be easily verified. Thus, also the positions of poles and cuts are the same in both cases (fig. 3). For the Compton-scattering amplitudes A_j, with $j=1,\ldots 6$, we can then derive again a dispersion relation as in eq.(2.5).

$$\mathrm{Re}\ A_j(s,t) = \frac{R_s^j}{s-m^2} + \frac{R_t^j}{t-\mu^2} + \frac{R_u^j}{u-m^2} + \frac{R_{su}^j}{(s-m^2)(u-m^2)} +$$

$$+ \frac{P}{\pi}\int_a^\infty \frac{ds'}{s'-s}\ \mathrm{Im}[A_j(s',t) + A_j(u',t)]. \quad (j=1,..6) \tag{5.2}$$

The residues are of course different from those in eq.(2.5); they are calculated in sec.V.4. (Notice the appearance of a double pole in (5.2), which was absent in (2.5).) For the amplitudes A_7 and A_8 however, we find a different equation, since the amplitudes are not real analytic, which is due to the fact that these are C-violating amplitudes. It can be shown that iA_j (j=7,8) is real analytic, so that for real s we obtain

$$\mathrm{disc}\,[A_j] = + 2\,\mathrm{Re}\,A_j \qquad (j=7,8)\ ,$$

where $\mathrm{disc}[A_j]$ is the discontinuity of the amplitude across the branch cut (cf. sec.II.2). This leads to a dispersion relation

$$\mathrm{Im}\ A_j(s,t) = -\frac{P}{\pi}\int_a^\infty \frac{ds'}{s'-s}\ \mathrm{Re}[A_j(s',t)+A_j(u',t)] \quad (j=7,8), \tag{5.3}$$

where no pole terms appear, since the Born terms do not contain a C-violating part (cf. sec.V.4).

V.2 CROSSING SYMMETRY

Since for Compton scattering the s- and u-channel reactions are identical (5.1), the crossing symmetry relations can be derived quite easily. In fact, we have in the s-channel

$$\langle N_2(P_2),\gamma_2(K_2)|T|N_1(P_1),\gamma_1(K_1)\rangle =$$

$$= \varepsilon^*_{2\nu}(\vec{k}_2)\bar{u}(\vec{p}_2)\sum_i A_i(s,t,u)L^i_{\mu\nu}(P,K)u(\vec{p}_1)\varepsilon_{1\mu}(\vec{k}_1)\ , \tag{5.4}$$

and in the u-channel

$$\langle N_2(P_2),\gamma_1(-K_1)|T|N_1(P_1),\gamma_2(-K_2)\rangle =$$

$$= \varepsilon^*_{1\mu}(-\vec{k}_1)\bar{u}(\vec{p}_2)\sum_i A_i(s,t,u)L^i_{\mu\nu}(P,K)u(\vec{p}_1)\varepsilon_{2\nu}(-\vec{k}_2)\ . \tag{5.5}$$

Of course, the matrix elements can not depend on the way in which the photons are numbered, so that we may rewrite (5.5) as

$$\langle N_2(P_2),\gamma_2(-K_1)|T|N_1(P_1),\gamma_1(-K_2)\rangle =$$

$$= \varepsilon^*_{2\mu}(-\vec{k}_1)\bar{u}(\vec{p}_2)\sum_i A_i(s,t,u)L^i_{\mu\nu}(P,K)u(\vec{p}_1)\varepsilon_{1\nu}(-\vec{k}_2)\ . \tag{5.6}$$

Making in (5.6) the substitution $\mu \rightleftarrows \nu$ and $K_1 \rightleftarrows -K_2$, we find

$$\langle N_2(P_2),\gamma_2(K_2)|T|N_1(P_1),\gamma_1(K_1)\rangle =$$

$$= \varepsilon^*_{2\nu}(\vec{k}_2)\bar{u}(\vec{p}_2) \sum_i A_i(u,t,s)L^i_{\mu\nu}(P,-K)u(\vec{p}_1)\varepsilon_{1\mu}(\vec{k}_1) \ . \tag{5.7}$$

Comparing (5.4) and (5.7), and using the explicit forms for the $L^i_{\mu\nu}$ (eq.(4.12)), we obtain

$$A_i(u,t,s) = \eta^A_i \ A_i(s,t,u) \tag{5.8^a}$$

with

$$\{\eta^A_i\} = (+1,+1,-1,+1,+1,-1,-1,-1). \tag{5.8^b}$$

Similarly, we obtain by replacing $\sum_i A_i \ L^i_{\mu\nu}$ in eqs.(5.4)ff. by $\sum_i B_i \ M^i_{\mu\nu}$

$$B_i(u,t,s) = \eta^B_i \ B_i(s,t,u) \tag{5.9^a}$$

with

$$\{\eta^B_i\} = (+1,+1,+1,-1,-1,+1,-1,-1). \tag{5.9^b}$$

The same results should be obtained by using the charge conjugation properties of the T-matrix element. For the C-conserving part we have

$$\langle N_2(P_2),\gamma_2(K_2)|T^c|N_1(P_1),\gamma_1(K_1)\rangle = \langle \overline{N}_2(P_2),\gamma_2(K_2)|T^c|\overline{N}_1(P_1),\gamma_1(K_1)\rangle \ ; \tag{5.10}$$

a similar relation with a minus sign holds for the C-non-conserving part (T^{nc}). Comparing the right-hand side of (5.10) with the inverse u-channel reaction matrix element for T^c (denoting the corresponding amplitudes by A^c_i, and the crossing factors by η^c_i),

$$\langle \overline{N}_1(-P_1),\gamma_2(K_2)|T^c|\overline{N}_2(-P_2),\gamma_1(K_1)\rangle =$$

$$= -\varepsilon^*_{2\nu}(\vec{k}_2)\bar{v}(-\vec{p}_2)\left[\sum_{i=1}^{8} A^c_i(s,t,u)L^i_{\mu\nu}(P,K)\right]v(-\vec{p}_1)\varepsilon_{1\mu}(\vec{k}_1) \ , \tag{5.11}$$

and proceeding as in sec.II.3 (eq.(2.10)) we find

$$\eta^A_i = \eta^c_i \qquad (i=1,\ldots,6)$$

$$\eta^A_i = -\eta^c_i \qquad (i=7,8) \ .$$

Using T^{nc} and η^{nc}_i, we obtain the opposite sign. For compatibility with eqs.(5.8) clearly we must have $A^c_i=0$ for $i=7,8$, and $A^{nc}_i=0$ for $i=1,\ldots,6$, thus verifying that indeed A_7 and A_8 are the two C-non-conserving amplitudes. The same results can be obtained for the amplitudes B_7 and B_8.

V.3 UNITARITY

As in sec.II.4, it can be shown that from the unitarity relation for the T-operator, i.e. $(T-T^\dagger) = iT^\dagger T$, follows

$$\varepsilon^*_{2\nu}\bar{u}(\vec{p}_2)\sum_{i=1}^{8}(A_i(s+i\varepsilon,t)-A_i(s-i,\varepsilon t))L^i_{\mu\nu}u(p_1)\varepsilon_{1\mu} =$$

$$= \varepsilon^*_{2\nu}\bar{u}(p_2)\sum_{i=1}^{8}\mathrm{disc}[A_i(s,t)]_{s,u}L^i_{\mu\nu}u(\vec{p}_1)\varepsilon_{1\mu} = i\sum_n\langle f|T^\dagger|n\rangle\langle n|T|i\rangle . \qquad (5.12)$$

The intermediate states n, that give non-zero values for the right-hand side of this equation, are of the same type as for pion production so that the same poles and branch cuts occur as in Ch.II. The discontinuities are given by [0162]

$$\mathrm{disc}\,[A_j(s,t)] = 2i\,\mathrm{Im}\,A_j(s,t) \quad \text{for } j=1,\ldots 6$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5.13)

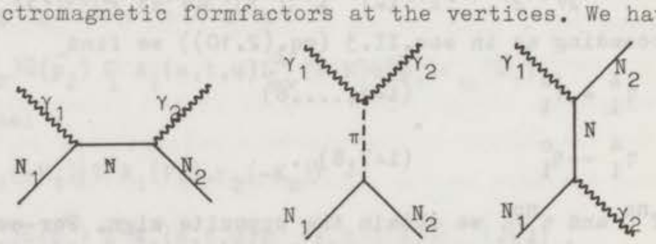
$$\mathrm{disc}\,[A_j(s,t)] = 2\,\mathrm{Re}\,A_j(s,t) \quad \text{for } j=7,8 ,$$

due to the fact that the first six amplitudes are real analytic, while A_7 and A_8 are imaginary analytic.

V.4 POLE TERMS

The pole terms in the dispersion relations for the amplitudes A_i (or B_i) are found by making an expansion of the renormalized Born terms, using the matrices $L^i_{\mu\nu}$ (or $M^i_{\mu\nu}$). These Born terms correspond to the diagrams in fig. 14, calculated with normal Feynman rules, but including the electromagnetic formfactors at the vertices. We have *)

fig.14



s-channel

$$e^2\,\bar{u}(\vec{p}_2)\varepsilon^*_{2\nu}[F_1\gamma_\nu - F_2\sigma_{\nu\alpha}K_{2\alpha}]\,\frac{i\gamma\cdot(K_1+P_1)-m}{(K_1+P_1)^2+m^2}\,[F_1\gamma_\mu + F_2\sigma_{\mu\beta}K_{1\beta}]\varepsilon_{1\mu}u(\vec{p}_1) \qquad (5.14)$$

---

*) a factor e^2 appears in eqs.(5.14), (5.15), due to our normalization of the formfactors (cf. sec.II.5).

u-channel

$$e^2\ \bar{u}(\vec{p}_2)\varepsilon_{1\mu}[F_1\gamma_\mu+F_2\sigma_{\mu\alpha}K_{1\alpha}]\ \frac{i\gamma.(P_2-K_1)-m}{(P_2-K_1)^2+m^2}[F_1\gamma_\nu-F_2\sigma_{\nu\beta}K_{2\beta}]\varepsilon_{2\nu}u(\vec{p}_1) \qquad (5.15)$$

t-channel

$$-\frac{gG}{\mu}\ \bar{u}(\vec{p}_2)\gamma_5\ u(\vec{p}_1)\ \frac{1}{(P_1-P_2)^2+\mu^2}\ \varepsilon_{\mu\nu\rho\sigma}\ \varepsilon_{1\mu}\ \varepsilon_{2\nu}\ K_{1\rho}\ K_{2\sigma} \qquad (5.16)$$

The formfactors F_i are the nucleon formfactors $F_i^p(0)$ or $F_i^n(0)$, introduced in sec.II.5 (p for protons; n for neutrons); g is again the pion-nucleon coupling constant, and G is the $\pi^0\rightarrow 2\gamma$ decay constant

$$G = G(\mu^2) = -8(\pi/\tau\mu)^{\frac{1}{2}}$$

where τ is the $(\pi^0\rightarrow 2\gamma)$-lifetime. (The sign of G is taken from ref. [La62]; cf. also [He62], [Ko68].) In the rest of this work we will consider only the case that the nucleon is a proton, and we omit the index p on the formfactors F_i^p from now on.

The expansion of the Born terms is done most easily in terms of the invariants $M^i_{\mu\nu}$, using the method described in Appendix G. Via eq. (4.21) we then obtain the pole termsfor the amplitudes A_i. We find in the s-channel:

$$e^2\ \bar{u}(\vec{p}_2)\varepsilon^*_{2\nu}\ [F_1\gamma_\nu-F_2\sigma_{\nu\alpha}K_{2\alpha}]\ \frac{i\gamma.(K_1+P_1)-m}{(K_1+P_1)+m^2}[F_1\gamma_\mu+F_2\sigma_{\mu\beta}K_{1\beta}]\varepsilon_{1\mu}\ u(\vec{p}_1) =$$

$$= e^2\ \bar{u}(\vec{p}_2)\varepsilon^*_{2\nu}\{\frac{F_1^2}{m^2-s}[\gamma_\nu(i\gamma.K)\gamma_\mu+P_{2\nu}i\gamma_\mu+P_{1\mu}i\gamma_\nu] + \frac{F_1F_2}{m^2-s}[2(m^2-s)\gamma_\nu\gamma_\mu +$$

$$+ 2i\gamma_\nu(i\gamma.K)P_{1\mu}+2P_{2\nu}(i\gamma.K)i\gamma_\mu+4m\gamma_\nu(i\gamma.K)\gamma_\mu+2m(P_{2\nu}i\gamma_\mu+P_{1\mu}i\gamma_\nu)-2P_{2\nu}P_{1\mu}] +$$

$$+ \frac{F_2^2}{m^2-s}[(m^2-s)(-\gamma_\nu(i\gamma.K)\gamma_\mu+P_{2\nu}i\gamma_\mu+P_{1\mu}i\gamma_\nu+2m\gamma_\nu\gamma_\mu) - 4P_{2\nu}(i\gamma.K)P_{1\mu} +$$

$$+ 4m(i\gamma_\nu(i\gamma.K)P_{1\mu}+P_{2\nu}(i\gamma.K)i\gamma_\mu)+4m^2\gamma_\nu(i\gamma.K)\gamma_\mu]\}\varepsilon_{1\mu}\ u(\vec{p}_1) =$$

$$= e^2\ \bar{u}(\vec{p}_2)\varepsilon^*_{2\nu}\{\frac{F_1^2}{s-m^2}[2mM^1_{\mu\nu}+mM^3_{\mu\nu}-(1+\frac{4}{t}(s-m^2))M^4_{\mu\nu}-M^5_{\mu\nu}+M^6_{\mu\nu}] +$$

$$+ \frac{F_1F_2}{s-m^2}[\frac{2}{t}(u-s)(s-m^2)M^1_{\mu\nu}+2(s-m^2)M^2_{\mu\nu}+(2m^2+\frac{2}{t}(s-m^2)^2)M^3_{\mu\nu}\ -\frac{8m}{t}(s-m^2)M^4_{\mu\nu}\ +$$

$$+ 4mM^5_{\mu\nu}-2mM^6_{\mu\nu}]+\frac{F_2^2}{s-m^2}[sm(s-m^2)M^2_{\mu\nu}-m(s-m^2)M^3_{\mu\nu}-(s-m^2)M^4_{\mu\nu}$$

$$(4m^2+(s-m^2))M^5_{\mu\nu}+(s-m^2)M^6_{\mu\nu}]\}\varepsilon_{1\mu}\ u(\vec{p}_1) \qquad (5.19)$$

From eq.(5.19) the contributions from the s-channel Born term to the amplitudes B_i can be read off immediately. Using (4.21) we find for the contributions (a_i^s) to the amplitudes A_i:

$$a_1^s = \frac{e^2}{s-m^2}\left[\frac{1}{mt^2}\left(4(s-m^2)^2 + 2t(s+m^2)\right)F_1^2 - 2F_1F_2 - 2mF_2^2\right]$$

$$a_2^s = \frac{e^2}{s-m^2}\left[\frac{1}{m^2t^2}\left(4(s-m^2)(u-m^2)+2t(s+m^2)\right)F_1^2+\frac{2}{mt}(4m^2+t)F_1F_2+2F_2^2\right]$$

$$a_3^s = \frac{e^2}{s-m^2}\left[-\frac{2}{mt}(s-m^2)F_1^2 + \left(2 - \frac{4}{t}(s-m^2)\right)F_1F_2 + 2mF_2^2\right]$$

$$a_4^s = \frac{e^2}{s-m^2}\left[\frac{4}{t}F_1^2 + \frac{8m}{t}F_1F_2 + 2F_2^2\right]$$

$$a_5^s = \frac{e^2}{s-m^2}\left[-\frac{16}{t}F_1F_2\right]$$

$$a_6^s = \frac{e^2}{s-m^2}\left[-4F_2^2\right]$$

$$a_7^s = 0$$

$$a_8^s = 0 \tag{5.20}$$

The pole residues for the s-channel are then found by taking the limit

$$R_i^s = \lim_{s\to m^2} (s-m^2)a_i^s \,. \tag{5.21}$$

The u-channel residues can be found in the same way, but it is much simpler to use the crossing properties from sec.V.2, which give $R_i^u = \eta_i R_i^s$. The t-channel pole can be found to contribute only to A_2 (or B_3). Summarizing these results, we obtain for the complete pole terms for the amplitudes A_i:

$$A_1^B = \left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\left[-2e^2F_2(F_1+mF_2)\right] + \frac{1}{s-m^2}\,\frac{1}{u-m^2}\left[-4e^2mF_1^2\right]$$

$$A_2^B = \left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\left[\frac{2e^2}{m}F_2(F_1+mF_2)\right] + \frac{1}{s-m^2}\,\frac{1}{u-m^2}\left[-4e^2F_1(F_1+2mF_2)\right] + \frac{2gG}{\mu m}\,\frac{1}{t-\mu^2}$$

$$A_3^B = \left(\frac{1}{s-m^2} - \frac{1}{u-m^2}\right)\left[2e^2F_2(F_1+mF_2)\right]$$

$$A_4^B = \left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\left[2e^2F_2^2\right] + \frac{1}{s-m^2}\,\frac{1}{u-m^2}\left[-4e^2F_1(F_1+2mF_2)\right]$$

$$A_5^B = \frac{1}{s-m^2}\,\frac{1}{u-m^2}\left[16e^2F_1F_2\right]$$

$$A_6^B = \left(\frac{1}{s-m^2} - \frac{1}{u-m^2}\right)\left[-4e^2F_2^2\right]$$

$$A_7^B = A_8^B = 0 \tag{5.22}$$

CHAPTER VI

SOLUTION OF THE DISPERSION RELATIONS AND CROSS-SECTION CALCULATIONS FOR COMPTON SCATTERING

VI.1 DISPERION RELATIONS AND LOW-ENERGY LIMITS

Using the crossing properties (5.8) of the amplitudes A_i, we rewrite the dispersion relations (5.2) and (5.3) as

$$\mathrm{Re}\, A_j(s,t) = A_j^B(s,t) + \frac{P}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \left(\frac{1}{s'-s} + \frac{\eta_j}{s'-u}\right) \mathrm{Im}\, A_j(s',t) \qquad (j=1,..6) \qquad (6.1^a)$$

$$\mathrm{Im}\, A_j(s,t) = -\frac{P}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \left(\frac{1}{s'-s} + \frac{\eta_j}{s'-u}\right) \mathrm{Re}\, A_j(s',t) \;, \qquad (j=7,8) \qquad (6.1^b)$$

or in a shorter notation

$$\mathrm{Re}\, A_j = A_j^B + A_j^C \qquad (j=1,..6) \qquad (6.2^a)$$

$$\mathrm{Im}\, A_j = A_j^C \quad , \qquad (j=7,8) \qquad (6.2^b)$$

where A_j^B are the pole terms (5.22), and we have denoted the integral terms by A_j^C (continuum contributions). In the derivation of these equations we assumed that no subtractions were necessary. For this to be true, it is at least necessary that the amplitudes appearing in eqs. (6.1) should have the correct low-energy behaviour. This is in fact the case, since the terms A_j^C can not contain dynamical singularities for $p \to 0$ (i.e. $s \to m^2$) [Ba68], and thus will become constants, while it can be verified easily that the pole terms A_j^B have the same low-energy limit as given in eqs.(4.33^a). It might be that subtractions are necessary which do not spoil the low-energy limit; but we will assume that this is not the case.

We note however, that for the amplitudes B_j the low-energy limit of the pole terms does not agree with eqs.(4.33[b]). Thus, even assuming $\lim_{p\to 0} B_j^C=0$, we still would have to introduce subtractions in the dispersion relations for the amplitudes B_i [He62], [Ko68]. This, and the related fact [Ba68] that for these amplitudes B_i we have to take into account the constraint equations (4.19), (4.20), leads us to write down the dispersion relations (6.1) for the A_i, and to use the B_i only as a set of intermediate amplitudes to simplify several calculations.

VI.2 CONNECTION WITH PION PHOTOPRODUCTION

With the assumption that the unitarity relation (5.12) holds even for unphysical values of s, the integrands in eqs.(6.1) can be found in principle from the results of sec.V.3. We will restrict our calculations to the energy region from threshold (940 MeV) to about 1350 MeV in the centre-of-mass system (i.e. photon laboratory energies below 500 MeV). In this region the process is dominated by the first pion-nucleon resonance (1236 MeV). The dispersion integrals are cut off at an energy $W_0 \approx 1800$ MeV ($s_0 \approx 3.2$ GeV^2), which will be a reasonable approximation for low s. Further, we simplify the right-hand side of eq.(5.12) by taking as intermediate states $|n\rangle$ only states, containing one pion and one nucleon, thus reducing the expression to

$$i \sum_{ch} \langle \gamma N|T^{\dagger}|\pi N\rangle\langle\pi N|T|\gamma N\rangle \ ,$$

i.e. essentially a product of two matrix elements for pion photoproduction, where the Σ denotes a sum over the two possible charge states. Below the two-pion threshold this is correct, and for higher energies we can expect it to be a good approximation, since inspection of photoproduction and pion-nucleon scattering data shows, that in the region considered, the matrix elements involving two (or more) pions will be small (< 5%) compared to those for single pion photoproduction, (cf. the discussion in sec.III.2). To express Im $A_j(s',t)$ and Re $A_j(s',t)$ in terms of photoproduction amplitudes, both sides of eq.(5.12) are expanded in terms of partial wave helicity amplitudes $\Phi^J_{\nu_1\nu_2}$ and $\Psi^J_{\nu_1\nu_2}$, which were introduced in eqs.(4.30) and (4.31). Due to orthogonality of the $d^J_{\nu_1\nu_2}$-functions, a very simple relation results. If T-invariance is assumed we obtain (for details about isospin see sec.VI.3),

$$\mathrm{Im}\,\Phi^J_{\nu_1\nu_2} = \tfrac{1}{2} \sum_{\nu=\pm\frac{1}{2}} \Psi^J_{\nu_1\nu}\,\Psi^J_{\nu_2\nu} \tag{6.3}$$

where ν_1 and ν_2 are the total helicities for the photon-nucleon states, and ν for the pion-nucleon state (cf. sec.IV.3.C). In the more general case (no C- or T-invariance), the scattering operator can be separated into a C-conserving part (T^c) and a C-non-conserving part (T^{nc}). It was shown in sec.V.2 that the matrix elements of T^c can be expanded in terms of the amplitudes $A_1 \ldots A_6$, and T^{nc} in terms of A_7 and A_8. From eqs.(4.28) it is then clear that the helicity amplitudes Φ_i can be written as a sum of a C-conserving amplitude (φ_i) and a C-non-conserving amplitude (φ'_i). A similar decomposition will be possible for the pion photoproduction amplitudes Ψ_i, and for the corresponding partial wave amplitudes,

$$\Phi^J_{\nu_1\nu_2} = \varphi^J_{\nu_1\nu_2} + \varphi'^J_{\nu_1\nu_2}$$

$$\Psi^J_{\nu_1\nu_2} = \psi^J_{\nu_1\nu_2} + \psi'^J_{\nu_1\nu_2} \,. \tag{6.4}$$

We then obtain from the unitarity relation a generalization of eq.(6.3),

$$\mathrm{Im}\,\varphi^J_{\nu_1\nu_2} = \tfrac{1}{2} \sum_{\nu=\pm\frac{1}{2}} \left(\psi^J_{\nu_1\nu}\,\psi^{J*}_{\nu_2\nu} + \psi'^J_{\nu_1\nu}\,\psi'^{J*}_{\nu_2\nu}\right) \tag{6.5^a}$$

$$\mathrm{Re}\,\varphi'^J_{\nu_1\nu_2} = \tfrac{1}{2}i \sum_{\nu=\pm\frac{1}{2}} \left(\psi^J_{\nu_1\nu}\,\psi'^{J*}_{\nu_2\nu} + \psi'^J_{\nu_1\nu}\,\psi^{J*}_{\nu_2\nu}\right) . \tag{6.5^b}$$

Via eq.(4.30),

$$\Phi_i = \frac{1}{2p} \sum_J (2J+1)\Phi^J_{\nu_1\nu_2}\, d^J_{\nu_1\nu_2}(\theta) \,,$$

we obtain from eqs(6.5) expressions for $\mathrm{Im}\,\varphi_i$ and $\mathrm{Re}\,\varphi'_i$. Using eqs. (4.29) we can thus express $\mathrm{Im}\,A_j$ (for j=1,...6) and $\mathrm{Re}\,A_j$ (j=7,8) in terms of partial wave helicity amplitudes for photoproduction. Via the dispersion relations (6.1) we then obtain $\mathrm{Re}\,A_j$ (j=1,..6) and $\mathrm{Im}\,A_j$ (j=7,8).

We noted already that s' may reach unphysical values in the integrands, and the same is true for θ. It can be verified however, that the factors $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ that occur in eqs.(4.29) cancel against the factors in the functions $d^J_{\nu_1\nu_2}(\theta)$, so that this does not give further complications for the calculation.

VI.3 AMPLITUDES FOR PION PHOTOPRODUCTION

Results for pion photoproduction are summarized often in terms of multipole amplitudes, obtained from an analysis of experimental data. These amplitudes are defined in essentially the same way as in sec.I.4.C for electroproduction. There is only a difference in normalization (an extra factor $\frac{4\pi W}{m}$ has to be inserted on the right-hand side of eq. (1.28^a) for photoproduction), and no scalar or longitudinal multipoles appear, since now the photons are real. In terms of these multipole amplitudes the partial wave helicity amplitudes are given by

$$\Psi^J_{-\frac{1}{2}\,\frac{1}{2}} = (\tfrac{1}{2}pq)^{\frac{1}{2}}[\ell(M_{\ell+} - E_{(\ell+1)-}) + (\ell+2)(E_{\ell+} + M_{(\ell+1)-})]$$

$$\Psi^J_{-\frac{1}{2}-\frac{1}{2}} = (\tfrac{1}{2}pq)^{\frac{1}{2}}[(\ell+2)(E_{\ell+} - M_{(\ell+1)-}) + \ell(M_{\ell+} + E_{(\ell+1)-})]$$

$$\Psi^J_{-\frac{3}{2}\,\frac{1}{2}} = (\tfrac{1}{2}pq)^{\frac{1}{2}}[\ell(\ell+2)]^{\frac{1}{2}}[-E_{\ell+} + M_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}]$$

$$\Psi^J_{-\frac{3}{2}-\frac{1}{2}} = (\tfrac{1}{2}pq)^{\frac{1}{2}}[\ell(\ell+2)]^{\frac{1}{2}}[-E_{\ell+} + M_{\ell+} + E_{(\ell+1)-} + M_{(\ell+1)-}] \quad ,(6.6)$$

where $J=\ell+\frac{1}{2}$. The isospin decomposition of these amplitudes (Ψ, M and E) follows directly from eq.(1.15), if we omit both e^-, and add a γ in the initial state. For a given initial state (γn) two distinct pion-nucleon states can occur. Clearly the amplitudes $\Psi^J_{\nu_1\nu_2}$ as used in (6.3), or (6.5), should represent a sum over these two states. Introducing amplitudes with distinct isospin indices, $\Psi^{nJ}_{\nu_1\nu_2}$, with n=0,1,3 (or 0,+,-), (cf.sec.I.3), and dropping the indices J, ν_1 and ν_2 for the moment, we find

$$\Psi\Psi^* = \chi_I[(\Psi^+\Psi^{+*} + 2\Psi^-\Psi^{-*} + 3\Psi^0\Psi^{0*})I + (\Psi^+\Psi^{0*} + \Psi^0\Psi^{+*} + 2\Psi^-\Psi^{0*} + 2\Psi^0\Psi^{-*})\tau_3]\chi_I \quad (6.7)$$

where we have introduced isospinors χ_I for the nucleon; I and τ_3 are the familiar 2x2 matrices.

The present experimental situation is such, that for the photoproduction reaction with initial protons far more data are available than for photoproduction from neutrons. As can be seen from (1.15), from the proton data alone, only the amplitudes $E^3_{\ell\pm}$ and $M^3_{\ell\pm}$, and the combinations $(M^0_{\ell\pm}+\frac{1}{3}M^1_{\ell\pm})$ and $(E^0_{\ell\pm}+\frac{1}{3}E^1_{\ell\pm})$ can be extracted. Thus it is useful to rewrite (6.7) in the following form (valid only for protons)

$$\Psi\Psi^* = 3(\Psi^0+\tfrac{1}{3}\Psi^1)(\Psi^0+\tfrac{1}{3}\Psi^1)^* + \tfrac{2}{3}\Psi^3\Psi^{3*} \; . \tag{6.8}$$

Similar relations hold for the various terms in eqs.(6.5).

To introduce T-violation in our calculations, obviously we should use T-violating photoproduction amplitudes. No direct multipole analyses have been performed however, that include the possibility of T-violation, but we will follow the arguments given by Berends and Weaver [Be71b] to obtain a phenomenological form for these amplitudes, containing one parameter. We write the multipole amplitudes as $M^n_{\ell\pm}=m^n_{\ell\pm}+m^{n'}_{\ell\pm}$, (etc.), where again $m^{n'}_{\ell\pm}$ denotes the T-violating part, and n is the isospin index. Using the unitarity relation, it can be shown [Be71b] that the phases of the amplitudes m and m' differ by 90^o, i.e.

$$m = \pm|m|\,e^{i\delta} \tag{6.9^a}$$

$$m' = \pm i|m'|\,e^{i\delta} \;, \tag{6.9^b}$$

where δ is the corresponding pion-nucleon phaseshift (cf. sec.III.1). Defining x by

$$x = \pm|m'|/|m| \;, \tag{6.10}$$

where the sign is the product of the signs that are chosen in eqs.(6.9^a) and (6.9^b), we can write

$$M^n_{\ell\pm} = |m^n_{\ell\pm}|\,(1+x^2)^{\frac{1}{2}}\exp[\,i(\delta^n_{\ell\pm}+\tan^{-1}x)] \; . \tag{6.11}$$

We have assumed here that the absolute value of the amplitude $M^n_{\ell\pm}$ remains the same, with or without T-violation. Information about the parameter x has to be obtained from a comparison between photoproduction and the inverse reaction $(\pi N\rightarrow\gamma N)$. Some results are given in ref. [Be71b].

VI.4 NUMERICAL CALCULATIONS AND RESULTS

Expressing the centre-of-mass cross-section for Compton scattering (4.4^b) (with unpolarized particles) in terms of the helicity amplitudes Φ_i, we obtain the simple expression

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{2}\sum_{i=1}^{8}|\Phi_i|^2 \tag{6.12}$$

As we have shown in the previous sections, we can obtain numerical solutions for the amplitudes A_j by using experimental data from pion photoproduction. In these numerical calculations we used as our main source of input a set of photoproduction multipole amplitudes (with $J \leq \frac{7}{2}$) from an analysis by Berends and Weaver [Be71a], covering the centre-of-mass energy range between 1100 and 1315 MeV (i.e. photon lab. energies between 180 and 450 MeV). For the energies in the dispersion integral that lie above this region (up to 1770 MeV in the centre-of-mass, i.e. 1200 MeV photon lab.energy) we used results from a multipole analysis by Walker [Wa69], with $J \leq \frac{5}{2}$. (The effects of higher multipoles are estimated to be small (<5%) in this energy region.) In some cases (see below) a T-violating part was introduced in the multipole amplitudes via eq.(6.11).

By means of a computer we applied the method given in the previous sections, to calculate from these multipole amplitudes the Compton scattering amplitudes and cross-sections. The steps in this calculation can be summarized as follows. From eqs.(6.6) and (6.8) the photoproduction helicity amplitudes are obtained (with or without a T-violating part). The imaginary parts of the Compton scattering helicity amplitudes φ_i, and the real parts of φ'_i are then found from eqs.(6.5) and (4.30). Since it is clear that the amplitudes A_j (with j=1,...6) depend only on φ_i, we can calculate via (4.29) $\mathrm{Im}A_j$ for these six amplitudes. Similarly, A_7 and A_8 depend only on φ'_i so that their real parts can be found. Via the dispersion relations (6.1) we obtain the corresponding real or imaginary parts. The cross-section is then calculated via eqs.(4.28) and (6.12). In figs. 15 through 18 the results of these calculations are given, and compared with experimental data and other theoretical work. In fig. 15 our "normal" calculation (no T-violation; sign of t-channel pole as given in eq.(5.22)) is compared to the "unitary limit", i.e. a calculation with all $\mathrm{Re}\,\Phi_i$ set to zero. From eq.(6.12) it is clear that this must give a lower limit for the cross-section, but we see in fig.15 that some of the experimental points at energies near the pion-nucleon resonance (especially at angles $\theta \approx 90^\circ$) lie very close to, or even below this limit. A similar picture emerges from a recent calculation by Pfeil et al. [Pf73], who used a Bonn multipole analysis ([No71],[Pf72]) as input. The theoretical assumptions involved here, are essentially only

C-, P-, and T-invariance, analyticity and unitarity of the S-matrix. If one assumes these to be correct, this means that the experimental data for Compton scattering are incompatible with the photoproduction multipole analysis. The data for Compton scattering are not very abundant however, while for photoproduction there are some uncertainties, since several new experiments have been performed since these multipole analyses were made.

Although the conflict between the data may be resolved by new experiments, it is worthwhile to investigate the possibility of obtaining agreement by relaxing the theoretical assumptions. The introduction of T-violation *) can in principle affect the unitary limit and the values of the cross-section. It can be verified that eq.(6.12) may be written as

$$\frac{d\sigma}{d\Omega} = \Sigma_{i=1}^{8} \left(|\varphi_i|^2 + |\varphi_i'|^2\right) , \tag{6.13}$$

i.e. without cross-terms between T-conserving and T-violating parts of the helicity amplitudes, but from eqs.(6.5) and (6.6) we find that the amplitudes φ_i will depend not only on the absolute value of the multipole amplitudes, but also on their real and imaginary parts. These parts may change if T-violation is introduced in the way, described in sec.VI.3, so that in principle it is possible that the absolute value of φ_i becomes smaller.

In fig.16 we give the results of a calculation in which a T-violating effect is introduced in the photoproduction amplitudes, by assuming for the dominant multipole amplitude (M^3_{1+}) a T-violating phase $\tan^{-1}x = -20^{\circ}$,(cf. eq.(6.11)); the other multipoles are taken without a T-violating part. This form for the T-violating amplitudes in pion photoproduction was taken from an analysis by Berends and Weaver [Be71b], who could account in this way for a small discrepancy between experimental data on photoproduction and its inverse reaction. For comparison, we give in fig. 16 also the results of the calculation without T-violation, and some experimental points. We see that for energies below the resonance ($W < 1236$ MeV) the cross-section is in fact reduced somewhat by the introduction of T-violation in this way, but

*) P-invariance seems to hold quite well in e.m. and strong interactions; there is however a possibility that T- and C-invariance may be violated by e.m. interactions.

near the resonance the effect is very small, so that the discrepancy remains.

Fig. 17 gives our normal calculation compared to one with an opposite sign for the residue of the t-channel pole. For energies below the resonance the normal calculation results in a somewhat larger value for the cross-section than the one with the opposite sign, while near and above the resonance there is a small difference in the other direction. Although for a scattering angle $\theta = 135^{\circ}$ the situation is not very clear, we see that at $\theta = 90^{\circ}$ the experiments favour the "normal" sign of the residue, as given in (5.22). This result is in accordance with earlier analyses ([La62], [He62], [Ko68]).

In fig. 18 finally, we compare our results to those of the Bonn group (W.Pfeil et al. [Pf73]) and Köberle [Ko68]. Both of these calculations use the set of amplitudes B_i (in our notation), introduced by Hearn and Leader [He62], and based on work by Prange [Pr58]. In ref. [Pf73] the constraint equations between these amplitudes (eqs.(4.19), (4.20)) are taken fully into account, while this is not the case in ref. [Ko68]. We do not need these constraints, since we use the amplitudes A_i, where the conditions leading to the constraints are automatically satisfied. The t-channel continuum contribution, which involves the pion-pion interaction, has only been included by Köberle. Since for these contributions only model calculations can be used, and Köberle finds a small effect, we have omitted them, although they can be fitted into our formalism without much trouble. (They contribute an extra term to the fixed-t dispersion relations (6.1).) The input from photoproduction is different for the three calculations. Köberle used theoretical predictions for pion photoproduction, the Bonn group used a multipole analysis ([No71], [Pf72]), and we used a different multipole analysis [Be71a]. (For the high-energy data Pfeil et al. used [Mo73], and we took [Wa69], but this different choice hardly influences the results.) A special feature of the calculations by Pfeil et al. [Pf73] is, that they also performed a simultaneous partial wave fit to the experimental data for both Compton scattering and photoproduction. The photoproduction amplitudes agree more or less with normal analyses, and they obtain phenomenological amplitudes for Compton scattering. The results for the cross-section, calculated with these amplitudes is also shown in fig.18; they are clearly smaller that the dispersion calculations.

Figs.15,..,18: Cross-sections for Compton scattering on protons. In these figures a solid line represents a fit to our calculations. The experimental data are denoted by triangles (with error bars); they are quoted from ref.[Pf73]. The meaning of the other lines is explained below. The energy (EG) is the laboratory energy of the initial photon.

Fig.15 (page 100,101). The "unitary limit" is given by the crosses and the dashed line, respectively. The solid line is the normal calculation (no T-violation).

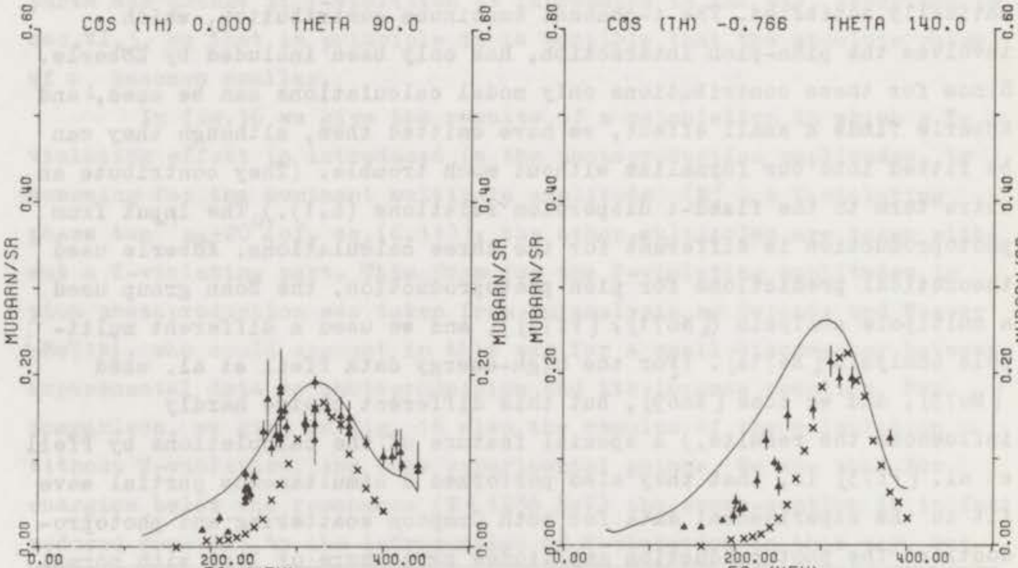
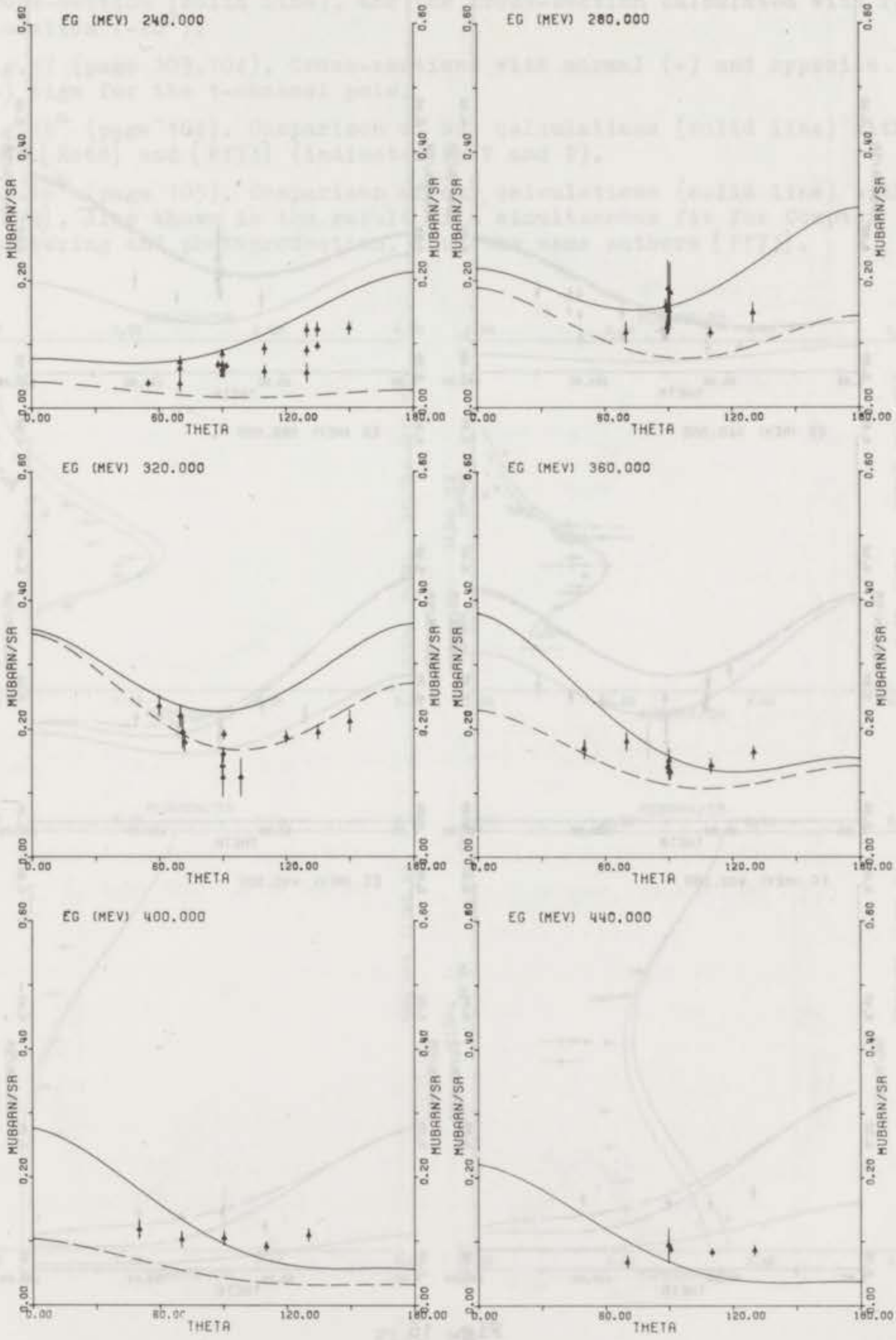


Fig. 15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EG (MEV) 280.000
EG (MEV) 320.000
EG (MEV) 360.000
EG (MEV) 400.000
EG (MEV) 440.000
MUBARN/SR
THETA
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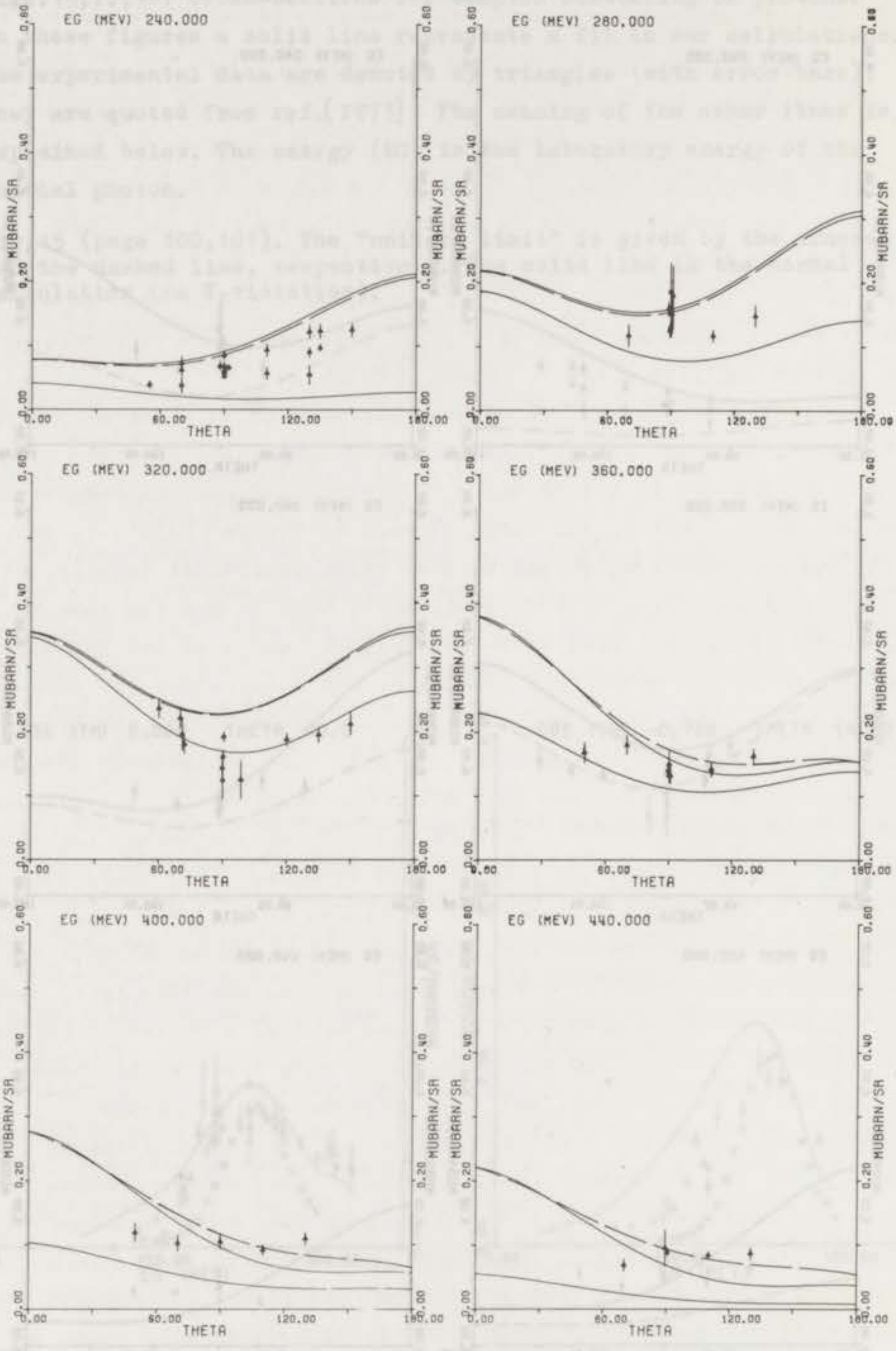
0.20
0.40
0.60
60.00
120.00
180.00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EG (MEV) 360.000
EG (MEV) 400.000
EG (MEV) 440.000
MUBARN/SR
THETA
0.00
0.20
0.40
0.60
60.00
120.00
180.00

Fig. 16

Fig.16 (page 102). The "unitary limit" with a T-violating phase of -20° is given by the lower line. The other two lines give the normal cross-section (solid line), and the cross-section calculated with T-violation (-20°).

Fig.17 (page 103,104). Cross-sections with normal (+) and opposite (-) sign for the t-channel pole.

Fig.18[a] (page 104). Comparison of our calculations (solid line) with refs.[Ko68] and [Pf73] (indicated by K and P).

Fig.18[b] (page 105). Comparison of our calculations (solid line) with [Pf73]. Also shown is the result of a simultaneous fit for Compton scattering and photoproduction, from the same authors [Pf73].

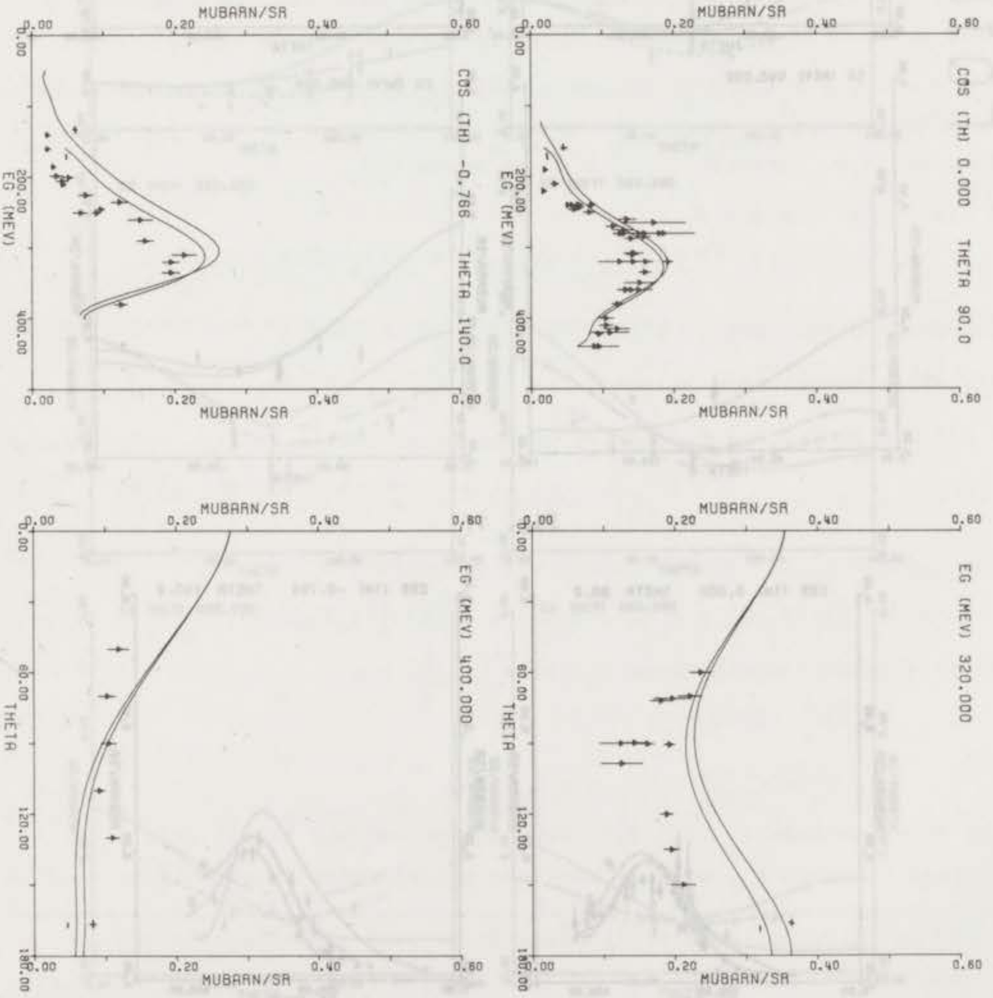


Fig. 17

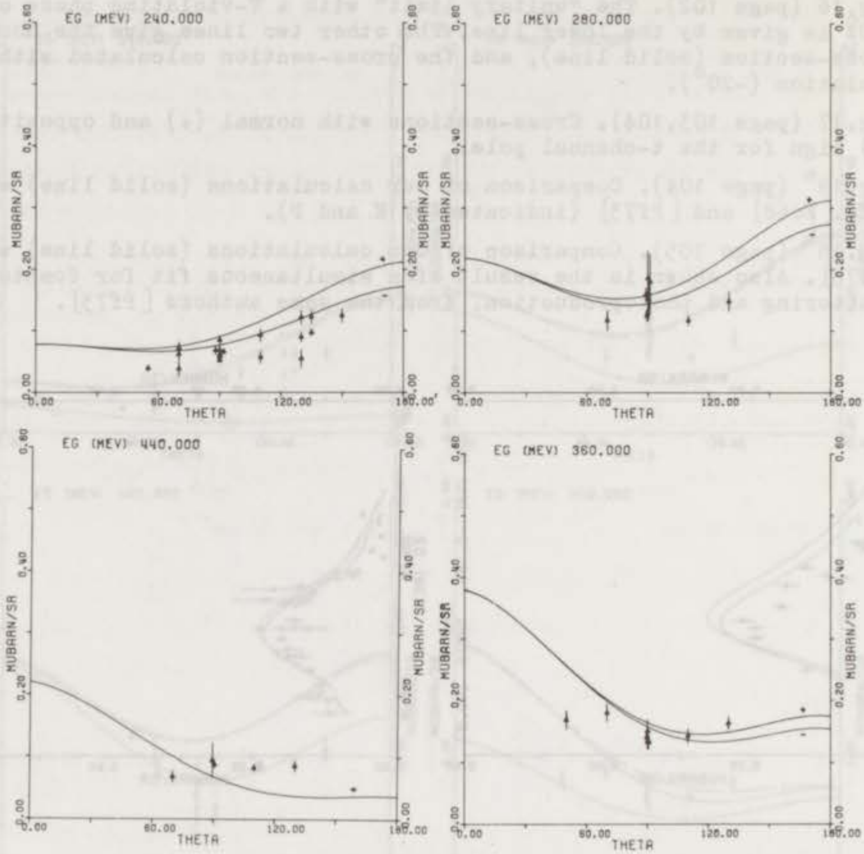


Fig. 17

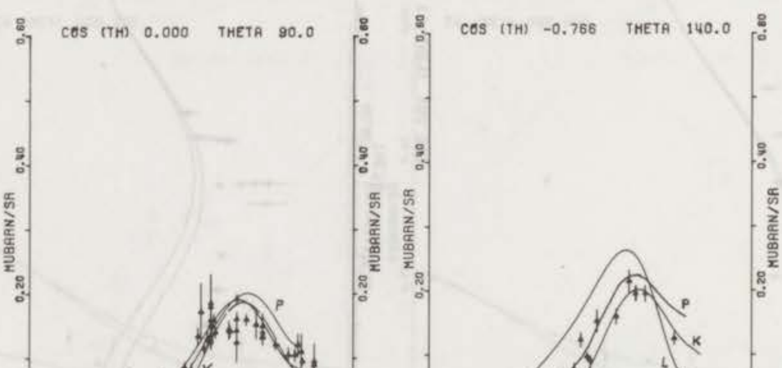
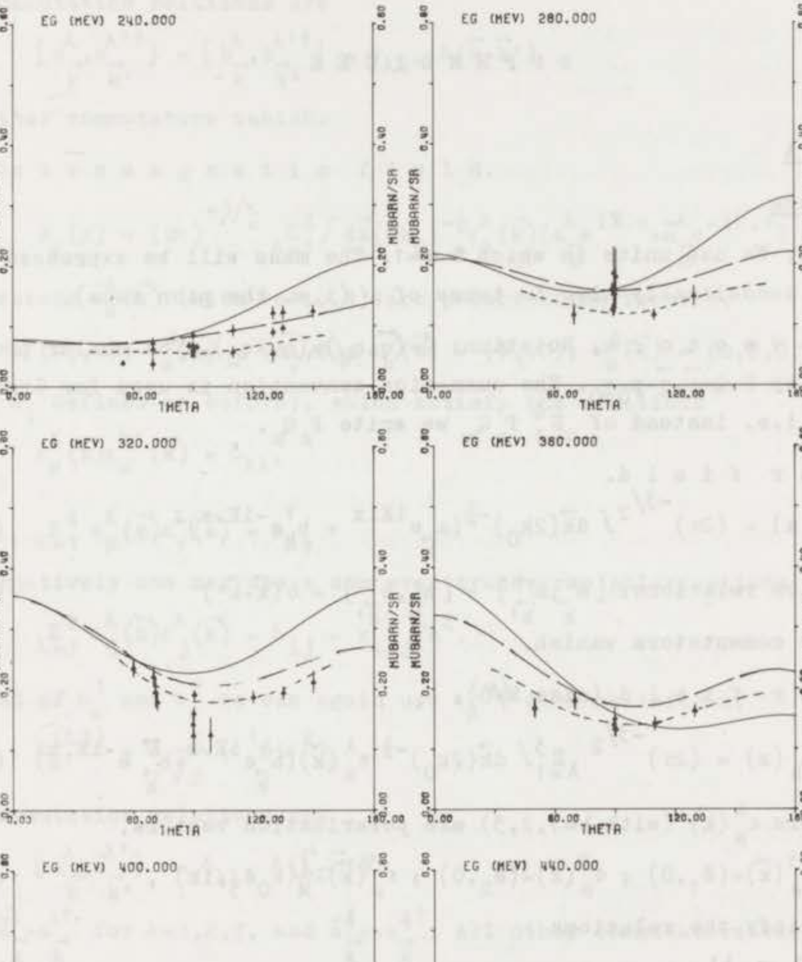


Fig. 18[a]



EG (MEV) 240.000
EG (MEV) 280.000
EG (MEV) 320.000
EG (MEV) 360.000
EG (MEV) 400.000
EG (MEV) 440.000
MUBARN/SR
THETA
0.00
0.20
0.40
0.60
60.00
120.00
180.00

Fig. 18^{b}

APPENDICES

Appendix A

Conventions

U n i t s. We use units in which $\hbar=c=1$. The mass will be expressed in MeV, and occasionally also in terms of μ (i.e. the pion mass).

F o u r - v e c t o r s. Notation: $Q=(\vec{q},q_4)=(\vec{q},iq_0)$. The scalar product is given by $P.Q=\vec{p}.\vec{q}-p_0q_0$. The summation convention is used for Greek indices, i.e. instead of $\sum_{\mu=1}^{4} P_\mu Q_\mu$ we write $P_\mu Q_\mu$.

S c a l a r f i e l d.

$$\varphi(x) = (2\pi)^{-3/2}\int d\vec{k}(2k_0)^{-\frac{1}{2}}(a_{\vec{k}}e^{iK.x} + b^{\dagger}_{k}e^{-iK.x}) \tag{A.1}$$

Commutation relations: $[a_{\vec{k}},a^{\dagger}_{\vec{k}'}] = [b_{\vec{k}},b^{\dagger}_{\vec{k}'}] = \delta(\vec{k}-\vec{k}')$ (A.2)

All other commutators vanish.

V e c t o r f i e l d (mass $M\neq 0$).

$$A_\mu(x) = (2\pi)^{-3/2}\sum_{\lambda=1}^{3}\int d\vec{k}(2k_0)^{-\frac{1}{2}}\,\varepsilon^{\lambda}_{\mu}(\vec{k})(a^{\lambda}_{\vec{k}}e^{iK.x}+b^{\lambda\dagger}_{\vec{k}}\,e^{-iK.x}) \tag{A.3}$$

The vectors $\varepsilon^{\lambda}_{\mu}(\vec{k})$ (with $\lambda=1,2,3$) are polarization vectors,

$$\varepsilon^{1}_{\mu}(\vec{k})=(\hat{e}_1,0)\ ;\ \varepsilon^{2}_{\mu}(\vec{k})=(\hat{e}_2,0)\ ;\ \varepsilon^{3}_{\mu}(\vec{k})=\tfrac{1}{M}(k_0\hat{e}_3,ik)\ , \tag{A.4}$$

which satisfy the relations

$$\varepsilon^{\lambda}_{\mu}(\vec{k})\varepsilon^{\lambda'}_{\mu}(k) = \delta_{\lambda\lambda'} \tag{A.5}$$

and

$$\sum_{\lambda=1}^{3}\varepsilon^{\lambda}_{\mu}(\vec{k})\varepsilon^{\lambda}_{\nu}(\vec{k}) = \delta_{\mu\nu} - K_\mu K_\nu/K^2\ . \tag{A.6}$$

The three-vectors $\hat{e}_i$ ($i=1,2,3$) form an orthonormal basis, with $\hat{e}_3=\hat{k}$.*) Instead of these $\varepsilon^{\lambda}_{\mu}$, we will also use polarization vectors $\varepsilon^{(s)}_{\mu}$,

---

*) In defining helicity amplitudes, it is more convenient to use a coordinate frame with $\hat{e}_3=-\hat{k}$, etc. In eqs.(A.4) and (A.10) the $\hat{e}_i$ are then replaced by $-\hat{e}_i$, and $\varepsilon^{4}_{\mu} = (0,0,0,-1)$.

corresponding to states with helicity-eigenvalue s,

$$\varepsilon_\mu^{(\pm 1)} = \mp \frac{1}{\sqrt{2}} (\varepsilon^1 \pm i\varepsilon^2)_\mu \text{ , and } \varepsilon_\mu^{(0)} = \varepsilon_\mu^3 \text{ .} \tag{A.7}$$

The commutation relations are

$$[a^\lambda_{\vec{k}}, a^{\lambda'\dagger}_{\vec{k}'}] = [b^\lambda_{\vec{k}}, b^{\lambda'\dagger}_{\vec{k}'}] = \delta_{\lambda\lambda'}\, \delta(\vec{k}-\vec{k}') \text{ .} \tag{A.8}$$

All other commutators vanish.

E l e c t r o m a g n e t i c   f i e l d.

$$A_\mu(x) = (2\pi)^{-3/2} \sum_{\lambda=1}^{4} \int d\vec{k}(2k_0)^{-\frac{1}{2}} \varepsilon_\mu^\lambda(\vec{k})(a^\lambda_{\vec{k}} e^{iK.x} + \bar{a}^\lambda_{\vec{k}} e^{-iK.x}) \tag{A.9}$$

The vectors $\varepsilon_\mu^\lambda(\vec{k})$ (with $\lambda=1,..4$) are polarization vectors,

$$\varepsilon_\mu^1(\vec{k}) = (\hat{e}_1,0);\ \varepsilon_\mu^2(\vec{k}) = (\hat{e}_2,0);\ \varepsilon_\mu^3(\vec{k}) = (\hat{e}_3,0);\ \varepsilon_\mu^4(\vec{k}) = (0,0,0,1) \tag{A.10}$$

(with $\hat{e}_i$ defined as before), which satisfy the relations

$$\varepsilon_\mu^\lambda(\vec{k})\varepsilon_\mu^{\lambda'}(\vec{k}) = \delta_{\lambda\lambda'} \tag{A.11}$$

and

$$\sum_{\lambda=1}^{4} \varepsilon_\mu^\lambda(\vec{k})\varepsilon_\nu^\lambda(\vec{k}) = \delta_{\mu\nu} \tag{A.12}$$

(Alternatively one may use a sum over transverse polarizations only,

$$\sum_{\lambda=1}^{2} \varepsilon_i^\lambda(\vec{k})\varepsilon_j^\lambda(\vec{k}) = \delta_{ij} - k_i k_j/k^2.) \tag{A.13}$$

Instead of ε_μ^1 and ε_μ^2 we can again use $\varepsilon_\mu^{(s)}$ with helicity s,

$$\varepsilon_\mu^{(\pm 1)} = \mp \frac{1}{\sqrt{2}} (\varepsilon^1 \pm i\varepsilon^2)_\mu \text{ .} \tag{A.14}$$

The commutation relations are

$$[a^\lambda_{\vec{k}}, \bar{a}^{\lambda'}_{\vec{k}'}] = \delta_{\lambda\lambda'}\, \delta(\vec{k}-\vec{k}') \text{ ,} \tag{A.15}$$

with $\bar{a}^\lambda_{\vec{k}} = a^{\lambda\dagger}_{\vec{k}}$ for $\lambda=1,2,3$, and $\bar{a}^4_{\vec{k}} = -a^{4\dagger}_{\vec{k}}$. All other commutators vanish.

D i r a c   f i e l d.

$$\psi(x) = (2\pi)^{-3/2} \sum_{j=1}^{2} \int d\vec{k}(m/k_0)^{\frac{1}{2}}(a_{\vec{k}j}\, e^{iK.x} u_j(\vec{k}) + b^\dagger_{\vec{k}j}\, e^{-iK.x} v_j(\vec{k})) \text{ .} \tag{A.16}$$

We have anti-commutation relations

$$\{a_{\vec{k}j}, a^\dagger_{\vec{k}'j'}\} = \{b_{\vec{k}j}, b^\dagger_{\vec{k}'j'}\} = \delta_{jj'}\, \delta(\vec{k}-\vec{k}') \text{ .} \tag{A.17}$$

The Dirac equation

$$(\gamma_\mu \partial_\mu + m)\psi(x) = 0 \tag{A.18}$$

implies

$$(i\gamma.K + m)u(\vec{k}) = 0$$
$$(i\gamma.K - m)v(\vec{k}) = 0$$
$$\bar{u}(\vec{k})(i\gamma.K + m) = 0$$
$$\bar{v}(\vec{k})(i\gamma.K - m) = 0 \quad , \qquad (A.19)$$

where $\bar{u}=u^\dagger\gamma_4$ and $\bar{v}=v^\dagger\gamma_4$.

The Dirac matrices are defined by

$$\gamma_\mu^\dagger = \gamma_\mu$$
$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$$
$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \ . \qquad (A.20)$$

We use the representation

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & \end{pmatrix} ; \ \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \ \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} ; \qquad (A.21)$$

where all matrices are 2x2 matrices, and j=1,2,3. The Pauli matrices σ_j are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \qquad (A.22)$$

they satisfy the relation $\sigma_j\sigma_k=\delta_{jk}+i\varepsilon_{jk\ell}\sigma_\ell$. We define the 4x4 matrices $\sigma_{\mu\nu}$ as

$$\sigma_{\mu\nu} = \tfrac{1}{2}i[\gamma_\mu,\gamma_\nu] \ . \qquad (A.23)$$

The spinors $u_j(\vec{k})$ and $v_j(\vec{k})$, with j=1,2, are

$$u_j(\vec{k}) = \left(\frac{m+k_0}{2m}\right)^{\frac{1}{2}} \begin{pmatrix} \chi_j \\ \dfrac{\vec{\sigma}.\vec{k}}{m+k_0} \quad \chi_j \end{pmatrix} \qquad (A.24^a)$$

$$v_j(\vec{k}) = -(-1)^j \left(\frac{m+k_0}{2m}\right)^{\frac{1}{2}} \begin{pmatrix} \dfrac{\vec{\sigma}.\vec{k}}{m+k_0} \quad \chi_j \\ \chi_j \end{pmatrix} , \qquad (A.24^b)$$

with $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The normalization is

$$u_i^\dagger u_j = v_i^\dagger v_j = \delta_{ij}\ k_0/m$$
$$\bar{u}_i u_j = \bar{v}_i v_j = \delta_{ij} \ , \qquad (A.25)$$

and the sum over polarizations

$$\sum_{j=1,2} u_{j\beta}(\vec{k})\bar{u}_{j\alpha}(\vec{k}) = \frac{1}{2m}(m-i\gamma.K)_{\beta\alpha}$$

$$\sum_{j=1,2} v_{j\beta}(\vec{k})\bar{v}_{j\alpha}(\vec{k}) = \frac{1}{2m}(-m-i\gamma.K)_{\beta\alpha} \tag{A.26}$$

D i s c r e t e t r a n s f o r m a t i o n s: P, C, and T.

The action of the transformations P, C and T on the Dirac field is given by

$$P\, a_{\vec{k}j}\, P^{-1} = a_{-\vec{k}j} \quad ; \quad P\, b_{\vec{k}j}\, P^{-1} = -b_{-\vec{k}j}$$

$$C\, a_{\vec{k}j}\, C^{-1} = b_{\vec{k}j'}$$

$$T\, a_{\vec{k}j}\, T^{-1} = (-1)^{j'}\, a^{\dagger}_{-\vec{k}j'} \quad ; \quad T\, b^{\dagger}_{\vec{k}j}\, T^{-1} = (-1)^{j}\, b_{-\vec{k}j'} \quad , \tag{A.27}$$

with $j \neq j'$. In connection with these transformations it is convenient to use the following properties of some special products of Dirac matrices (denoted here with the same symbol as the corresponding transformation),

$$P = \gamma_4 \quad : \gamma_4\, u_j(\vec{k}) = u_j(-\vec{k})$$

$$\gamma_4\, v_j(\vec{k}) = -v_j(-\vec{k}) \tag{A.28}$$

$$C = \gamma_2\gamma_4 \quad : u_j(\vec{k}) = C\, \tilde{\bar{v}}_{j'}(\vec{k})$$

$$v_j(\vec{k}) = C\, \tilde{\bar{u}}_{j'}(\vec{k}) \tag{A.29}$$

$$C\gamma_\mu C^{-1} = -\tilde{\gamma}_\mu \quad ; \quad C\gamma_5 C^{-1} = \tilde{\gamma}_5 \tag{A.30}$$

$$T = -\gamma_1\gamma_3\gamma_4 : T\, \tilde{\bar{u}}_j(\vec{k}) = (-1)^j\, u_{j'}(-\vec{k})$$

$$T\, \tilde{\bar{v}}_j(\vec{k}) = (-1)^{j'} v_{j'}(-\vec{k}) \quad , \tag{A.31}$$

again with $j \neq j'$.

For the scalar field eqs.(A.27) hold without the minus sign in front of the operators $a_{\vec{k}}$ and $b_{\vec{k}}$, and without the indices j or j'.

For the vector field we have

$$P\, a^{\lambda}_{\vec{k}}\, P^{-1} = -a^{\lambda}_{-\vec{k}} \quad ; \quad P\, b^{\lambda}_{\vec{k}}\, P^{-1} = -b^{\lambda}_{\vec{k}}$$

$$C\, a^{\lambda}_{\vec{k}}\, C^{-1} = -b^{\lambda}_{\vec{k}}$$

$$T\, a^{\lambda}_{\vec{k}}\, T^{-1} = -a^{\lambda\dagger}_{-\vec{k}} \quad ; \quad T\, b^{\lambda}_{\vec{k}}\, T^{-1} = -b^{\lambda\dagger}_{-\vec{k}} \quad , \tag{A.32}$$

and for the electromagnetic field eqs.(A.32) hold if b is replaced by a, and if we use $\bar{a}^{\lambda}_{\vec{k}}$ instead of $a^{\lambda\dagger}_{\vec{k}}$.

I s o s p i n. The proton and neutron fields (ψ_p and ψ_n) are combined to a nucleon field $\Psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$, so that we have to use the isospinors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for proton and neutron, respectively.

The pion fields $(\pi^{\pm 0})$ can be combined to give a vector $\vec{\varphi}$ in isospin space,

$$\varphi_1 = \frac{1}{\sqrt{2}}(\varphi^\dagger + \varphi)$$

$$\varphi_2 = \frac{i}{\sqrt{2}}(\varphi^\dagger - \varphi)$$

$$\varphi_3 = (\pi^0) \quad ,$$

with $\quad \varphi = (\pi^+)^\dagger = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$

$$\varphi = (\pi^-)^\dagger = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2) .$$

This leads for $\pi^\pm$ states to an isovector

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad , \tag{A.33^a}$$

and for π^0 states to

$$V = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{A.33^b}$$

Appendix B

Multipole amplitudes

In the πN_2-centre-of-mass system we write the matrix element for the process $XN_1 \to \pi N_2$ (cf. fig.2) between states with definite momenta and spins as

$$T_{fi} = \langle \vec{q}s' | T | \vec{k}s\lambda \rangle = \langle \hat{q}s' | T' | \hat{k}s\lambda \rangle , \tag{B.1}$$

labelling the states by their momenta and spins, and omitting the particle labels. In the second step we have absorbed the energy dependence in the operator T'. The variable λ indicates the state of the incoming X-particle, which we take as an eigenstate of angular momentum and parity, (λ=E, M, L or S; cf. table B.I).

Following ref. [Pe57] we make a decomposition in terms of angular momentum eigenstates of both initial and final states, and write (using variables, defined in sec.I.2),

$$T_{fi} = \sum_{\{L\}} \langle \hat{q}s' | \ell m_\ell s' \rangle \langle \ell m_\ell s' | T' | LM_L s\lambda \rangle \langle LM_L s\lambda | \hat{k}s\lambda \rangle =$$

$$\sum_{\{L\}} Y_{\ell m_\ell}(\hat{q}) \langle \ell m_\ell s' | J\ell M_J \rangle \langle J\ell M_J | T' | JLM_J\lambda \rangle \langle JLM_J\lambda | LM_L s\lambda \rangle R_\lambda Y^*_{LM_L}(\hat{k}). \tag{B.2}$$

We have used here the relations

$$\langle \hat{q}s' | \ell m_\ell s' \rangle = \langle \hat{q} | \ell m_\ell \rangle = Y_{\ell m_\ell}(\hat{q})$$

$$\langle LM_L s\lambda | \hat{k}s\lambda \rangle = \langle LM_L\lambda | \hat{k}\lambda \rangle = R_\lambda Y^*_{LM_L}(\hat{k}) , \tag{B.3}$$

with $Y_{\ell m}$ the usual spherical harmonic functions. The brackets $\{L\}$ denote the set of indices $\{J, M_J, L, M_L, \ell, m_\ell\}$, and R_λ are the operators (see e.g. [Go64]),

$$R_E = -i \frac{\vec{\varepsilon}.(\hat{k}\times\vec{\ell}_k)}{\sqrt{L(L+1)}} \qquad R_L = \vec{\varepsilon}.\hat{k}$$

$$R_M = - \frac{\vec{\varepsilon}.\vec{\ell}_k}{\sqrt{L(L+1)}} \qquad R_S = -\varepsilon_0 \tag{B.4}$$

which give, when acting on $Y^*_{LM_L}(\hat{k})$, the correct behaviour for a particle with respectively an electric, magnetic, longitudinal or scalar polarization character, (R_E and R_M are the same as for real photons).

In the summation over $\{L\}$ in (B.2), non-zero terms appear only for $J=L\pm\frac{1}{2}$, and $\ell=J\pm\frac{1}{2}$. Furthermore, $\langle J\ell M_J|T'|JLM_J\lambda\rangle$ will vanish for all combinations of ℓ, J and L, other than those in table B.I, due to the parity character of the incoming X-particle. Thus we can write

$$T_{fi} = \sum_{JL}\langle J\ell'|T'|JL\lambda\rangle \sum_{\ell m_\ell M_L} Y_{\ell m_\ell}(\hat{q})\langle \ell m_\ell s'|P^{\lambda}_{JL\ell'}|LM_L s\rangle R_\lambda Y^*_{LM_L}(\hat{k}) \tag{B.5}$$

where $P^{\lambda}_{LJ\ell'}$ are projection operators that select the appropriate value (ℓ') of ℓ for a given value of J and L, depending on the type of polarization λ. These operators $P^{\lambda}_{JL\ell}(\hat{q})$ are given by [Go64]

$$P_{\ell+\frac{1}{2},\ell,\ell} = \frac{1}{2\ell+1}\,(\ell+1+\vec{\sigma}.\vec{\ell}_q) \qquad P_{\ell-\frac{1}{2},\ell,\ell} = \frac{1}{2\ell+1}\,(\ell-\vec{\sigma}.\vec{\ell}_q) \tag{B.6^a}$$

$$P_{\ell+\frac{1}{2},\ell+1,\ell} = \frac{1}{2\ell+3}\,\vec{\sigma}.\hat{q}(\ell+1-\vec{\sigma}.\vec{\ell}_q) \qquad P_{\ell-\frac{1}{2},\ell-1,\ell} = \frac{1}{2\ell-1}\,\vec{\sigma}.\hat{q}(\ell+\vec{\sigma}.\vec{\ell}_q) \tag{B.6^b}$$

As can be seen from table B.I, for the vector amplitudes we have to use (B.6^a) for λ=E,L,S, and (B.6^b) for λ=M. For the axial vector amplitudes the opposite choice should be made. Using now (cf. [Er53])

$$\langle \ell m_\ell s'|P^{\lambda}_{JLs'}|LM_L s\lambda\rangle = \chi_2^\dagger[\int d\Omega_r Y^*_{\ell m_\ell}(\hat{r})P^{\lambda}_{JL\ell'}(\hat{r})Y_{LM_L}(\hat{r})]\chi_1 ,$$

$$\sum_{M_L} Y_{LM_L}(\hat{r})\, Y^*_{LM_L}(\hat{k}) = \frac{2L+1}{4\pi}\, P_L(\hat{r}.\hat{k})$$

and
$$\sum_{\ell m_\ell} Y_{\ell m_\ell}(\hat{q})\, Y^*_{\ell m_\ell}(\hat{r}) = \delta(\hat{q}-\hat{r}) , \tag{B.7}$$

we find

$$T_{fi} = \sum_{JL}\frac{1}{4\pi}\langle J\ell'|T'|JL\lambda\rangle\chi_2^\dagger[(2L+1)P_{JL\ell'}(\hat{q})R_\lambda(\hat{k})P_L(\hat{q}.\hat{k})]\chi_1, \tag{B.8}$$

where $P_L(\hat{q}.\hat{k})$ are Legendre polynomials.

For the vector amplitudes, current conservation can be imposed in the same way as in sec.I.4.B, by substituting b_μ for ε_μ in R_λ (see e.g. (1.28)). This will eliminate the terms with λ=L in eq.(B.8). since R_L becomes $R_L=\vec{b}.\hat{k}=0$.

We finally introduce multipole amplitudes by defining

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell+1,E\rangle = 4\pi i\, E_{\ell+}\sqrt{(\ell+1)(\ell+2)} \qquad (\ell\geq 0)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell-1,E\rangle = -4\pi i\, E_{\ell-}\sqrt{\ell(\ell-1)} \qquad (\ell\geq 2)$$

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell,M\rangle = 4\pi i\, M_{\ell+}\sqrt{\ell(\ell+1)} \qquad (\ell\geq 1)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell,M\rangle = 4\pi i\, M_{\ell-}\sqrt{\ell(\ell+1)} \qquad (\ell\geq 1)$$

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell+1,S\rangle = 4\pi i\, S_{\ell+}(\ell+1) \qquad (\ell\geqq 0)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell-1,S\rangle = 4\pi i\,\ell\, S_{\ell-} \qquad (\ell\geqq 1) \qquad (B.9)$$

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell,E\rangle = -4\pi i\, Ea_{\ell+}\sqrt{\ell(\ell+1)} \qquad (\ell\geqq 1)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell,E\rangle = 4\pi i\, Ea_{\ell-}\sqrt{\ell(\ell+1)} \qquad (\ell\geqq 1)$$

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell+1,M\rangle = 4\pi i\, Ma_{\ell+}\sqrt{(\ell+1)(\ell+2)} \qquad (\ell\geqq 0)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell-1,M\rangle = 4\pi i\, Ma_{\ell-}\sqrt{\ell(\ell-1)} \qquad (\ell\geqq 2)$$

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell,L\rangle = 4\pi i(\ell+1)La_{\ell+} \qquad (\ell\geqq 0)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell,L\rangle = 4\pi i\,\ell\, La_{\ell-} \qquad (\ell\geqq 1)$$

$$\langle \ell+\tfrac{1}{2},\ell|T'|\ell+\tfrac{1}{2},\ell,S\rangle = 4\pi i(\ell+1)\, Sa_{\ell+} \qquad (\ell\geqq 0)$$

$$\langle \ell-\tfrac{1}{2},\ell|T'|\ell-\tfrac{1}{2},\ell,S\rangle = 4\pi i\,\ell\, Sa_{\ell-}, \qquad (\ell\geqq 1) \qquad (B.10)$$

where the index $\ell\pm$ indicates that $J=\ell\pm\frac{1}{2}$.

With the notations

$$M_{\ell i} = (E_{\ell+}, E_{\ell-}, M_{\ell+}, M_{\ell-}, S_{\ell+}, S_{\ell-}) \qquad (i=1,\ldots 6)$$

and

$$Ma_{\ell i} = (Ea_{\ell+}, Ea_{\ell-}, Ma_{\ell+}, Ma_{\ell-}, La_{\ell+}, La_{\ell-}, Sa_{\ell+}, Sa_{\ell-}) \qquad (i=1,\ldots 8)$$

and the definitions

$$Z_{\ell 1} = \vec{\sigma}.\hat{q}(\ell+1-\vec{\sigma}.\vec{\ell}_q)\vec{b}.(\hat{k}\times\vec{\ell}_k)P_{\ell+1}(\hat{k}.\hat{q})$$

$$Z_{\ell 2} = -\vec{\sigma}.\hat{q}(\ell+\vec{\sigma}.\vec{\ell}_q)\vec{b}.(\hat{k}\times\vec{\ell}_k)P_{\ell-1}(\hat{k}.\hat{q})$$

$$Z_{\ell 3} = -i(\ell+1+\vec{\sigma}.\vec{\ell}_q)\vec{b}.\vec{\ell}_k\, P_{\ell}(\hat{k}.\hat{q})$$

$$Z_{\ell 4} = -i(\ell-\vec{\sigma}.\vec{\ell}_q)\vec{b}.\vec{\ell}_k\, P_{\ell}(\hat{k}.\hat{q})$$

$$Z_{\ell 5} = -i(\ell+1)\vec{\sigma}.\hat{q}(\ell+1-\vec{\sigma}.\vec{\ell}_q)b_0\, P_{\ell+1}(\hat{k}.\hat{q})$$

$$Z_{\ell 6} = -i\ell\,\vec{\sigma}.\hat{q}(\ell+\vec{\sigma}.\vec{\ell}_q)b_0\, P_{\ell-1}(\hat{k}.\hat{q}) \qquad (B.11)$$

and

$$Za_{\ell 1} = -(\ell+1+\vec{\sigma}.\vec{\ell}_q)\vec{\varepsilon}.(\hat{k}\times\vec{\ell}_k)P_\ell(\hat{k}.\hat{q})$$
$$Za_{\ell 2} = (\ell-\vec{\sigma}.\vec{\ell}_q)\vec{\varepsilon}.(\hat{k}\times\vec{\ell}_k)P_\ell(\hat{k}.\hat{q})$$
$$Za_{\ell 3} = -i\,\vec{\sigma}.\hat{q}(\ell+1-\vec{\sigma}.\vec{\ell}_q)\vec{\varepsilon}.\vec{\ell}_k\ P_{\ell+1}(\hat{k}.\hat{q})$$
$$Za_{\ell 4} = -i\,\vec{\sigma}.\hat{q}(\ell+\vec{\sigma}.\vec{\ell}_q)\vec{\varepsilon}.\vec{\ell}_k\ P_{\ell-1}(\hat{k}.\hat{q})$$
$$Za_{\ell 5} = i(\ell+1)(\ell+1+\vec{\sigma}.\vec{\ell}_q)\vec{\varepsilon}.\hat{k}\ P_\ell(\hat{k}.\hat{q})$$
$$Za_{\ell 6} = i\,\ell(\ell-\vec{\sigma}.\vec{\ell}_q)\vec{\varepsilon}.\hat{k}\ P_\ell(\hat{k}.\hat{q})$$
$$Za_{\ell 7} = -i(\ell+1)(\ell+1+\vec{\sigma}.\vec{\ell}_q)\varepsilon_0\ P_\ell(\hat{k}.\hat{q})$$
$$Za_{\ell 8} = -i\,\ell(\ell-\vec{\sigma}.\vec{\ell}_q)\varepsilon_0\ P_\ell(\hat{k}.\hat{q}) \qquad (B.12)$$

we can then write

$$T_{fi} = \chi_2^\dagger \sum_{\ell=0}^{\infty}\left[\sum_{i=1}^{6} M_{\ell i}\, Z_{\ell i} + \sum_{i=1}^{8} Ma_{\ell i}\, Za_{\ell i}\right]\chi_1 \qquad (B.13)$$

for neutrinoproduction; for electroproduction we have of course only the terms with $M_{\ell i}$.

Table B.I

Multipole transition		J	ℓ	Vector multipole	Vector parity	Axial vector multipole	Axial vector parity
electric	$2^{\ell+1}$	$\ell+\frac{1}{2}$	L-1	$E_{\ell+}$	$(-1)^L$	$Ea_{(\ell+1)-}$	$-(-1)^L$
	$2^{\ell-1}$	$\ell-\frac{1}{2}$	L+1	$E_{\ell-}$		$Ea_{(\ell-1)+}$	
magnetic	2^{ℓ}	$\ell+\frac{1}{2}$	L	$M_{\ell+}$	$-(-1)^L$	$Ma_{(\ell+1)-}$	$(-1)^L$
	2^{ℓ}	$\ell-\frac{1}{2}$	L	$M_{\ell-}$		$Ma_{(\ell-1)+}$	
longitudinal	$2^{\ell+1}$	$\ell+\frac{1}{2}$	L-1	$L_{\ell+}$	$(-1)^L$	$La_{(\ell+1)-}$	$-(-1)^L$
	$2^{\ell-1}$	$\ell-\frac{1}{2}$	L+1	$L_{\ell-}$		$La_{(\ell-1)+}$	
scalar	$2^{\ell+1}$	$\ell+\frac{1}{2}$	L-1	$S_{\ell+}$	$(-1)^L$	$Sa_{(\ell+1)-}$	$-(-1)^L$
	$2^{\ell-1}$	$\ell-\frac{1}{2}$	L+1	$S_{\ell-}$		$Sa_{(\ell-1)+}$	

L: angular momentum of the lepton pair (i.e.: of the virtual "K-particle"). J: total angular momentum. ℓ: orbital momentum of the final pion.

Appendix C

Transformation matrices between sets of amplitudes (Pion production)

Explicit expressions are given for the matrices, introduced in sec. I.4.D

$$[B(s,t)] = \left[\begin{array}{c|c|c|c|c|c}
1 & 0 & -\frac{K.Q}{W-m} & W-m+\frac{K.Q}{W-m} & 0 & \frac{K^2}{W-m} \\
\hline
-1 & 0 & -\frac{K.Q}{W+m} & W+m+\frac{K.Q}{W+m} & 0 & \frac{K^2}{W+m} \\
\hline
0 & \frac{1}{W+m}\left[W^2-m^2+\frac{1}{2}K^2\right] & 1 & -1 & -\frac{K^2}{W+m} & 0 \\
\hline
0 & -\frac{1}{W-m}\left[W^2-m^2+\frac{1}{2}K^2\right] & 1 & -1 & \frac{K^2}{W-m} & 0 \\
\hline
-(E_1-m) & \begin{array}{l}\frac{1}{2k_0}\left[k^2(3K.Q+2k_0W)\right. \\ \left.-k.q(K^2+2(W^2-m^2))\right]\end{array} & K.Q+q_0(W-m) & \begin{array}{l}-K.Q-q_0(W-m) \\ +(E_1-m)(W+m)\end{array} & q_0K^2-k_0K.Q & -(E_1-m)(W+m) \\
\hline
(E_1+m) & \begin{array}{l}-\frac{1}{2k_0}\left[k^2(3K.Q+2k_0W)\right. \\ \left.-k.q(K^2+2(W^2-m^2))\right]\end{array} & K.Q+q_0(W+m) & \begin{array}{l}-K.Q-q_0(W+m) \\ +(E_1+m)(W-m)\end{array} & -q_0K^2+k_0K.Q & -(E_1+m)(\text{"}-m)
\end{array}\right] \qquad (C.1)$$

$$\beta \equiv W^2 - m^2 + \tfrac{1}{2}K^2$$

$$\delta \equiv \tfrac{3}{2}K.Q + k_0 W$$

$$[B^{-1}(s,t)] =$$

$$
\begin{array}{c}
\frac{1}{4W^2k_0} \\[3em]
\frac{K^2}{2k_0W^2(t-\mu^2)} \\[3em]
\frac{1}{4W^2k_0} \\[3em]
\frac{1}{4W^2k_0} \\[3em]
\frac{\beta}{2W^2k_0(t-\mu^2)} \\[3em]
\frac{W^2-m^2}{4W^2k_0}
\end{array}
\left[
\begin{array}{c|c|c|c|c|c}
\begin{array}{l}(W-m)[\,(W+m)^2 \\ -\,K^2\frac{E_1+m}{E_1-m}\,]\end{array} & \begin{array}{c}-\frac{k_0(W-m)}{E_1+m}\;\cdot \\ (W^2-m^2+K^2)\end{array} & 2m(W+m)(K.Q-\frac{K^2}{k^2}\vec{k}.\vec{q}) & 2m(W-m)(K.Q-\frac{K^2}{k^2}k.q) & 2m\frac{k_0K^2}{k^2} & 2m\frac{k_0K^2}{k^2} \\
\hline
-k_0\,\frac{W-m}{E_1-m} & k_0\,\frac{W+m}{E_1+m} & (W^2-m^2)(\frac{K.Q}{K^2}-\frac{\vec{k}.\vec{q}}{k^2}) & (W^2-m^2)(-\frac{K.Q}{K^2}+\frac{\vec{k}.\vec{q}}{k^2}) & -(W+m)\frac{k_0}{k^2} & (W-m)\frac{k_0}{k^2} \\
\hline
k_0\,\frac{(W-m)^2}{E_1-m} & k_0\,\frac{(W+m)^2}{E_1+m} & \begin{array}{c}(W+m)\;\cdot \\ (K.Q-\frac{K^2}{k^2}\vec{k}.\vec{q}+2k_0W)\end{array} & \begin{array}{c}(W-m)\;\cdot \\ (K.Q-\frac{K^2}{k^2}\vec{k}.\vec{q}+2k_0W)\end{array} & \frac{k_0K^2}{k^2} & \frac{k_0K^2}{k^2} \\
\hline
k_0\,\frac{(W-m)^2}{E_1-m} & k_0\,\frac{(W+m)^2}{E_1+m} & (W+m)(K.Q-\frac{K^2}{k^2}\vec{k}.\vec{q}) & (W-m)(K.Q-\frac{K^2}{k^2}\vec{k}.\vec{q}) & \frac{k_0K^2}{k^2} & \frac{k_0K^2}{k^2} \\
\hline
-k_0\,\frac{W-m}{E_1-m} & k_0\,\frac{W+m}{E_1+m} & (W^2-m^2)(\frac{\delta}{\beta}-\frac{\vec{k}.\vec{q}}{k^2}) & (W^2-m^2)(-\frac{\delta}{\beta}+\frac{\vec{k}.\vec{q}}{k^2}) & -(W+m)\frac{k_0}{k^2} & (W-m)\frac{k_0}{k^2} \\
\hline
\frac{k_0}{E_1-m} & \frac{k_0}{E_1+m} & -\frac{K.Q}{W-m}+(W+m)\frac{\vec{k}.\vec{q}}{k^2} & -\frac{K.Q}{W+m}+(W-m)\frac{\vec{k}.\vec{q}}{k^2} & -\,\frac{k_0}{k^2} & -\,\frac{k_0}{k^2}
\end{array}
\right]
\qquad (C.2)
$$

$$[C(s)] = \frac{2m}{(W-m)[(E_1+m)(E_2+m)]^{\frac{1}{2}}} \begin{bmatrix} 1 & & & & & \\ & \frac{(E_1+m)(E_2+m)(W-m)}{qk(W+m)} & & & 0 & \\ & & \frac{(E_1+m)(W-m)}{qk(W+m)} & & & \\ & & & \frac{1}{E_2-m} & & \\ & 0 & & & \frac{(E_2+m)(W-m)}{q} & \\ & & & & & \frac{(E_1+m)(W-m)}{k} \end{bmatrix} \qquad (C.3)$$

$[C(s)^{-1}]$ is the inverse of $[C(s)]$, and can be obtained directly from (C.3) . (C.4)

$$
[D_\ell(x)] =
\begin{array}{c}
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell} \\[2ex]
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell} \\[2ex]
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell}
\end{array}
\left[\begin{array}{c|c|c|c|c|c}
P_\ell(x) & -P_{\ell+1}(x) & \frac{\ell}{2\ell+1}(P_{\ell-1}(x)-P_{\ell+1}(x)) & \frac{\ell+1}{2\ell+3}(P_\ell(x)-P_{\ell+2}(x)) & 0 & 0 \\[1ex]
\hline
P_\ell(x) & -P_{\ell-1}(x) & \frac{\ell+1}{2\ell+1}(P_{\ell+1}(x)-P_{\ell-1}(x)) & \frac{\ell}{2\ell-1}(P_\ell(x)-P_{\ell-2}(x)) & 0 & 0 \\[1ex]
\hline
P_\ell(x) & -P_{\ell+1}(x) & \frac{1}{2\ell+1}(P_{\ell+1}(x)-P_{\ell-1}(x)) & 0 & 0 & 0 \\[1ex]
\hline
-P_\ell(x) & P_{\ell-1}(x) & \frac{1}{2\ell+1}(P_{\ell-1}(x)-P_{\ell+1}(x)) & 0 & 0 & 0 \\[1ex]
\hline
0 & 0 & 0 & 0 & P_{\ell+1}(x) & P_\ell(x) \\[1ex]
\hline
0 & 0 & 0 & 0 & P_{\ell-1}(x) & P_\ell(x)
\end{array}\right]
\qquad (C.5)
$$

$[G_\ell(x)] =$

$P'_{\ell+1}(x)$	$P'_{\ell-1}(x)$	$\ell P'_{\ell+1}(x)$	$(\ell+1)P'_{\ell-1}(x)$	0	0
0	0	$(\ell+1)P'_\ell(x)$	$\ell P'_\ell(x)$	0	0
$P''_{\ell+1}(x)$	$P''_{\ell-1}(x)$	$-P''_{\ell+1}(x)$	$P''_{\ell-1}(x)$	0	0
$-P''_\ell(x)$	$-P''_\ell(x)$	$P''_\ell(x)$	$-P''_\ell(x)$	0	0
0	0	0	0	$-(\ell+1)P'_\ell(x)$	$\ell P'_\ell(x)$
0	0	0	0	$(\ell+1)P'_{\ell+1}(x)$	$-\ell P'_{\ell-1}(x)$

(C.6)

$[Ba(s,t)] =$

$W+m$	0	0	$-m$	0	0	0	0
$-(W-m)$	0	0	$-m$	0	0	0	0
1	1	-1	0	$W+m$	$-(W+m)$	0	0
-1	-1	1	0	$W-m$	$-(W-m)$	0	0
0	1	0	0	$W+m$	0	-1	$-(W+m)$
$-\frac{W-m}{E_1+m}$	-1	0	$-\frac{m}{E_1+m}$	$W-m$	0	1	$-(W-m)$
$-(E_2-m)$	$-(E_1+E_2)$	$-q_0$	$-m$	$(W-m)(E_1+E_2)$	$q_0(W-m)$	$-k_0$	$k_0(W-m)$
E_2+m	E_1+E_2	q_0	$-m$	$(W+m)(E_1+E_2)$	$q_0(W+m)$	k_0	$k_0(W+m)$

(C.7)

$$[Ba^{-1}(s,t)] = \frac{1}{4W^2}\left[\begin{array}{cccccccc}
2W & -2W & 0 & 0 & 0 & 0 & 0 & 0 \\
-(W-m) & \frac{(W+m)^2}{E_1+m} & q_0(W-m) & -q_0(W+m) & k_0(W-m) & -k_0(W+m) & -(W+m) & W-m \\
W+m & -\frac{K^2}{E_1+m} & -(E_2+W)(W-m) & (E_2+W)(W+m) & k_0(W-m) & -k_0(W+m) & -(W+m) & W-m \\
-\frac{2W}{m}(W-m) & -\frac{2W}{m}(W+m) & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -\frac{W+m}{E_1+m} & q_0 & q_0 & k_0 & k_0 & 1 & 1 \\
-1 & -\frac{W+m}{E_1+m} & -(E_2+W) & -(E_2+W) & k_0 & k_0 & 1 & 1 \\
-(W-m) & -\frac{W^2-m^2}{E_1+m} & q_0(W-m) & -q_0(W+m) & -(E_1+W)(W-m) & (E_1+W)(W+m) & -(W+m) & W-m \\
-1 & \frac{W-m}{E_1+m} & q_0 & q_0 & -(E_1+W) & -(E_1+W) & 1 & 1
\end{array}\right] \tag{C.8}$$

$$[\,Ca(s)\,] = 2m \begin{bmatrix} \frac{1}{qO_2} & & & & & & & \\ & \frac{O_2}{K} & & & & & 0 & \\ & & \frac{O_1}{q^2k} & & & & & \\ & & & \frac{1}{O_1q} & & & & \\ & & & & \frac{O_1}{qk^2} & & & \\ & 0 & & & & \frac{1}{O_1k} & & \\ & & & & & & \frac{1}{O_1} & \\ & & & & & & & \frac{O_1}{qk} \end{bmatrix} \qquad (C.9)$$

$$O_1 = \sqrt{(E_1+m)(E_2+m)}$$

$$O_2 = \sqrt{\frac{(E_1+m)}{(E_2+m)}}$$

$[\,Ca^{-1}(s)\,]$ is the inverse of $[\,Ca(s)\,]$ (C.10)

$$[Da_\ell(x)] =$$

$$
\begin{array}{c}
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell} \\[2ex]
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell} \\[2ex]
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell} \\[2ex]
\frac{1}{2(\ell+1)} \\[2ex]
\frac{1}{2\ell}
\end{array}
\left[
\begin{array}{cccc|cc|cc}
P_{\ell+1}(x) & -P_\ell(x) & -(1-x^2)\frac{P'_{\ell+1}(x)}{\ell+1} & -(1-x^2)\frac{P'_\ell(x)}{\ell} & & & & \\[1ex]
P_{\ell-1}(x) & -P_\ell(x) & (1-x^2)\frac{P'_{\ell-1}(x)}{\ell} & (1-x^2)\frac{P'_\ell(x)}{\ell+1} & & 0 & & \\[1ex]
-P_{\ell+1}(x) & P_\ell(x) & -(1-x^2)\frac{P'_{\ell+1}(x)}{(\ell+1)(\ell+2)} & 0 & & & & \\[1ex]
P_{\ell-1}(x) & -P_\ell(x) & -(1-x^2)\frac{P'_{\ell-1}(x)}{\ell(\ell-1)} & 0 & & & & \\[1ex]
\hline
P_{\ell+1}(x) & 0 & xP_{\ell+1}(x) & xP_\ell(x) & P_{\ell+1}(x) & P_\ell(x) & & \\[1ex]
P_{\ell-1}(x) & 0 & xP_{\ell-1}(x) & xP_\ell(x) & P_{\ell-1}(x) & P_\ell(x) & & \\[1ex]
\hline
 & & & & & & P_\ell(x) & P_{\ell+1}(x) \\[1ex]
 & & 0 & & & & P_\ell(x) & P_{\ell-1}(x)
\end{array}
\right]
\qquad (C.11)
$$

$$
[\,Ga_{\ell}(x)] =
\left[\begin{array}{cccc|cc|cc}
P'_{\ell}(x) & P'_{\ell}(x) & (\ell+2)P'_{\ell}(x) & (\ell-1)P'_{\ell}(x) & & & & \\
0 & 0 & (\ell+1)P'_{\ell+1}(x) & \ell P'_{\ell-1}(x) & & & & \\
P''_{\ell}(x) & P''_{\ell}(x) & P''_{\ell}(x) & -P''_{\ell}(x) & & 0 & & \\
-P''_{\ell+1}(x) & -P''_{\ell-1}(x) & -P''_{\ell+1}(x) & P''_{\ell-1}(x) & & & & \\
\hline
-P'_{\ell}(x)-xP''_{\ell}(x) & -P'_{\ell}(x)-xP''_{\ell}(x) & -P''_{\ell+1}(x) & P''_{\ell-1}(x) & -(\ell+1)P'_{\ell}(x) & \ell P'_{\ell}(x) & & \\
xP''_{\ell+1}(x) & xP''_{\ell-1}(x) & xP''_{\ell+1}(x) & -xP''_{\ell-1}(x) & (\ell+1)P'_{\ell+1}(x) & -\ell P'_{\ell-1}(x) & & \\
\hline
 & & & 0 & & & (\ell+1)P'_{\ell+1}(x) & -\ell P'_{\ell-1}(x) \\
 & & & & & & -(\ell+1)P'_{\ell}(x) & \ell P'_{\ell}(x)
\end{array}\right]
\qquad (C.12)
$$

Appendix D

Watson's theorem

To derive the theorem, we make use of helicity amplitudes for the process $XN_1 \to \pi N_2$ (cf. fig.2), which are defined by

$$f_{\mu_2,\lambda\mu_1}(\theta,\varphi) = \langle\mu_2|T|\lambda\mu_1\rangle \; , \tag{D.1}$$

where μ_1,μ_2 are the helicities of the initial and final nucleons, and λ the helicity of the X-particle, in a centre-of-mass coordinate frame with the x_1x_3-plane as the production plane and the x_3-axis along $\vec{p}_1=-\vec{k}$. The angular momentum decomposition [Ja59] is given by

$$f_{\mu_2,\lambda\mu_1} = \tfrac{1}{2}\sum_J (2J+1)\exp[-i(\lambda-\mu_1+\mu_2)\varphi]\langle\mu_2|T^J|\lambda\mu_1\rangle d^J_{\mu_1-\lambda,\mu_2}(\theta). \tag{D.2}$$

The partial wave helicity amplitudes $\langle\mu_2|T^J|\lambda\mu_1\rangle$ can be expressed in terms of the multipole amplitudes with the same values of J, and vice versa. This connection can be found by expressing first $\langle\mu_2|T^J|\lambda\mu_1\rangle$ in terms of the centre-of-mass amplitudes F_i, defined in sec.I.4.B, and then using the relations between F_i and the multipole amplitudes from sec.I.4.C. We further need helicity amplitudes for pion-nucleon scattering, given by

$$g_{\mu_2\mu_1}(\theta_2\varphi_2,\theta_1\varphi_1) = \frac{1}{2\pi}\sum_J\sum_M (J+\tfrac{1}{2})\langle\mu_2|T^J|\mu_1\rangle D^J_{M\mu'}(\varphi_2,\theta_2,-\varphi_2)D^J_{M\mu_1}(\varphi_1,\theta_1,-\varphi_1) \tag{D.3}$$

where the angles θ_i,φ_i specify the direction of the initial and final nucleon. If we make the approximation of retaining only πN-states for the states $|n\rangle$ in the unitarity equation (3.1), then we obtain

$$\mathrm{Im}\, f_{\mu_2,\lambda\mu_1}(\theta,\varphi) = \tfrac{1}{2}\sum_{\mu'}\int d\Omega'\, g^*_{\mu'\mu_1}(\theta_2\varphi_2,\theta_1\varphi_1)f_{\mu',\lambda\mu_1}(\theta_2,\varphi_2) \; , \tag{D.4}$$

if we take for $|i\rangle$ and $|f\rangle$ states with helicities (λ,μ_1) and (μ_2). Using the orthonormality relations for the $D^J_{M\mu}$-functions,

$$\int d\Omega'\; D^{J*}_{mn}(\varphi,\theta,-\varphi)D^{J'}_{m'n'}(\varphi,\theta,-\varphi) = \frac{4\pi}{2J+1}\,\delta_{JJ'}\delta_{mm'}\delta_{nn'} \; , \tag{D.5}$$

we find from eqs.(D.2)..(D.4)

$$\mathrm{Im}\,\langle\mu_2|T^J|\lambda\mu_1\rangle = \tfrac{1}{2}\sum_{\mu'}\langle\mu'|T^J|\mu_2\rangle^{\dagger}\langle\mu'|T^J|\lambda\mu_1\rangle. \tag{D.6}$$

When this is expressed again in terms of the multipole amplitudes, we have

$$\text{Im } M_{\ell i} = \tfrac{1}{2} M_{\ell i}[\langle \tfrac{1}{2}| T^J | \tfrac{1}{2}\rangle \pm \langle \tfrac{1}{2}| T^J | -\tfrac{1}{2}\rangle]^* \quad , \qquad \text{(D.7)}$$

(+ or - for i odd or even)

where we have used the relation

$$\langle \tfrac{1}{2}| T^J | \pm\tfrac{1}{2}\rangle = -\langle -\tfrac{1}{2}| T^J | \mp\tfrac{1}{2}\rangle . \qquad \text{(D.8)}$$

Since for the πN-scattering partial wave amplitudes we have

$$f_{\ell\pm} \sim [\langle \tfrac{1}{2}| T^J | \tfrac{1}{2}\rangle \pm \langle \tfrac{1}{2}| T^J | -\tfrac{1}{2}\rangle]$$

and

$$f_{\ell\pm} = \frac{1}{q} e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm} \qquad \text{(D.9)}$$

(cf. sec.III.1), we obtain finally

$$M_{\ell i'} = |M_{\ell i'}| \exp[i'(\delta_{\ell\pm} + in\pi)] , \qquad \text{(D.10)}$$

(with n an integer, and again with + or - for i' odd or even).

For completeness we give here also the relations between helicity amplitudes and multipole amplitudes. Defining

$$T_J(1) = \langle \tfrac{1}{2}| T^J | -1\ \tfrac{1}{2} \rangle$$

$$T_J(2) = \langle \tfrac{1}{2}| T^J | 1\ -\tfrac{1}{2} \rangle$$

$$T_J(3) = \langle \tfrac{1}{2}| T^J | -1\ -\tfrac{1}{2} \rangle$$

$$T_J(4) = \langle \tfrac{1}{2}| T^J | 1\ \tfrac{1}{2} \rangle$$

$$T_J(5) = \langle \tfrac{1}{2}| T^J | 0\ \tfrac{1}{2} \rangle$$

$$T_J(6) = \langle \tfrac{1}{2}| T^J | 0\ -\tfrac{1}{2} \rangle \quad , \qquad \text{(D.11)}$$

we can write

$$E_{\ell+} = -\frac{i\sqrt{2}}{4(\ell+1)} [\sqrt{\frac{\ell}{\ell+2}}(-T_{\ell+\frac{1}{2}}(1)+T_{\ell+\frac{1}{2}}(2))+T_{\ell+\frac{1}{2}}(3)-T_{\ell+\frac{1}{2}}(4)]$$

$$E_{(\ell+1)-} = -\frac{i\sqrt{2}}{4(\ell+1)} [\sqrt{\frac{\ell+2}{\ell}}\ (T_{\ell+\frac{1}{2}}(1)+T_{\ell+\frac{1}{2}}(2))+T_{\ell+\frac{1}{2}}(3)+T_{\ell+\frac{1}{2}}(4)]$$

$$M_{\ell+} = -\frac{i\sqrt{2}}{4(\ell+1)} [\sqrt{\frac{\ell+2}{\ell}}\ (T_{\ell+\frac{1}{2}}(1)-T_{\ell+\frac{1}{2}}(2))+T_{\ell+\frac{1}{2}}(3)-T_{\ell+\frac{1}{2}}(4)]$$

$$M_{(\ell+1)-} = -\frac{i\sqrt{2}}{4(\ell+1)} [\sqrt{\frac{\ell}{\ell+2}}\ (T_{\ell+\frac{1}{2}}(1)+T_{\ell+\frac{1}{2}}(2))-(T_{\ell+\frac{1}{2}}(3)+T_{\ell+\frac{1}{2}}(4))]$$

$$S_{\ell+} = \frac{i}{2(\ell+1)} \frac{k}{k_0} [\, T_{\ell+\frac{1}{2}}(5) + T_{\ell+\frac{1}{2}}(6)]$$

$$S_{(\ell+1)-} = \frac{i}{2(\ell+1)} \frac{k}{k_0} [\, T_{\ell+\frac{1}{2}}(5) - T_{\ell+\frac{1}{2}}(6)] \qquad \text{(D.12)}$$

and the inverse

$$i\sqrt{2}\,T_{\ell+\frac{1}{2}}(1) = \sqrt{\ell(\ell+2)}\,[E_{\ell+} - M_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}]$$

$$i\sqrt{2}\,T_{\ell+\frac{1}{2}}(2) = \sqrt{\ell(\ell+2)}\,[M_{\ell+} - E_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}]$$

$$i\sqrt{2}\,T_{\ell+\frac{1}{2}}(3) = [(\ell+2)(M_{(\ell+1)} - E_{\ell+}) - \ell(M_{\ell+} + E_{(\ell+1)-})]$$

$$i\sqrt{2}\,T_{\ell+\frac{1}{2}}(4) = [\ell(M_{\ell+} - E_{(\ell+1)-}) + (\ell+2)(M_{(\ell+1)-} + E_{\ell+})]$$

$$i\,T_{\ell+\frac{1}{2}}(5) = \frac{k_0}{k}(\ell+1)[S_{\ell+} + S_{(\ell+1)-}]$$

$$i\,T_{\ell+\frac{1}{2}}(6) = \frac{k_0}{k}(\ell+1)[S_{\ell+} - S_{(\ell+1)-}]. \tag{D.13}$$

For the axial vector amplitudes we have to replace the last two equations of (D.12) by

$$La_{\ell+} = \frac{i}{2(\ell+1)}[Ta_{\ell+\frac{1}{2}}(5) + Ta_{\ell+\frac{1}{2}}(6)]$$

$$La_{(\ell+1)-} = \frac{i}{2(\ell+1)}[Ta_{\ell+\frac{1}{2}}(5) - Ta_{\ell+\frac{1}{2}}(6)]$$

$$Sa_{\ell+} = \frac{i}{2(\ell+1)}\frac{k}{k_0}[Ta_{\ell+\frac{1}{2}}(7) + Ta_{\ell+\frac{1}{2}}(8)]$$

$$Sa_{(\ell+1)-} = \frac{i}{2(\ell+1)}\frac{k}{k_0}[Ta_{\ell+\frac{1}{2}}(7) - Ta_{\ell+\frac{1}{2}}(8)] \tag{D.14}$$

and the last two equations of (D.13) by the corresponding inverse relations. The other equations remain the same (except for the added letters a).

Notice that the partial wave helicity amplitudes $T_J(i)$ differ slightly from the photoproduction amplitudes $\Psi^J_{\nu_1\nu_2}$ that were used in Ch.IV and VI. Apart from a factor $4\pi W/m$, due to the different normalization for photoproduction, we have the correspondence

$$T_J(1) \sim i\,\Psi^J_{-\frac{3}{2}-\frac{1}{2}}$$

$$T_J(2) \sim -i\,\Psi^J_{-\frac{3}{2}\frac{1}{2}}$$

$$T_J(3) \sim i\,\Psi^J_{-\frac{1}{2}-\frac{1}{2}}$$

$$T_J(4) \sim -i\,\Psi^J_{-\frac{1}{2}\frac{1}{2}} \tag{D.15}$$

The amplitudes $T_J(5)$ and $T_J(6)$ do not occur in photoproduction.

Appendix E

A different set of invariant amplitudes

In sec.I.4 we introduced two sets of invariant amplitudes, denoted by $\{A_i\}$ and $\{B_i\}$, respectively. The set $\{B_i\}$ is free of kinematical singularities [Ba61], but vector current conservation imposes the two restrictions (1.21) on these amplitudes, so that they are not independent. Two of them can be eliminated, which is done in eqs.(1.22) by introducing the set $\{A_i\}$, [De61]. This elimination causes the appearance of kinematical singularities in the amplitudes A_2 and A_5; these have a pole at $t=\mu^2$ (outside the physical region). In the scattering matrix elements T_{fi} the amplitudes A_i always occur in the combination $\sum_i A_i M_i$ (eq.(1.22a)), where these singularities must be canceled. In fact, we have

$$A_2M_2 + A_5M_5 = i\gamma_5(P.\varepsilon)(t-\mu^2)A_2 + i\gamma_5(-2P.K\, A_2 - K^2)(Q.\varepsilon) =$$

$$= i\gamma_5(P.\varepsilon)(t-\mu^2)A_2 + 2i\gamma_5 B_3(Q.\varepsilon), \qquad \text{(E.1)}$$

where in the first term the singular factor $\frac{1}{t-\mu^2}$ in A_2 is canceled explicitly, while the second term can be expressed in terms of the amplitude B_3, which is free of kinematical singularities.

As has been argued by Berends [Be70] (cf. also [Ad68]), in numerical work the cancellation in the second term of (E.1) can cause problems, which are avoided by using instead of the amplitudes A_i, a new set $\{A'_i\}$, where

$$A'_i = A_i \qquad (i\neq 5)$$

and

$$A'_5 = B_3 = -P.K\, A_2 - \tfrac{1}{2}K^2 A_5 = \frac{1}{Q.K}\left[\tfrac{1}{2}K^2B_1 + P.K\, B_2 + K^2B_4\right]. \qquad \text{(E.2)}$$

For these amplitudes the dispersion relations (2.31) still hold, only with residues $\widetilde{\Gamma}'$ and $\widetilde{\Gamma}'_t$ which differ from eqs.(2.30), and are given by

$$\Gamma'_i = \Gamma_i \qquad (i\neq 5)$$

$$\Gamma'^{\pm,0}_5 = -\tfrac{1}{4}gF_1^{V,S}$$

$$\Gamma'_{t5} = -\delta_{j5}gF_\pi . \qquad \text{(E.3)}$$

It can be verified, that this new set of dispersion relations is the same as the old one, except for an additional subtraction constant in the relation for A_5, since we can write

$$\mathrm{Re}\; B_3(s,t) = -\left(\frac{1}{s-m^2} - \frac{\xi}{u-m^2}\right)\tfrac{1}{4}gF_1 - \tfrac{1}{2}(1-\xi)\,\frac{g(F_\pi - F_1)}{t-\mu^2} +$$

$$+\frac{P}{\pi}\int ds'\left\{\frac{1}{s'-s} - \frac{\xi}{s'-u}\right\}\mathrm{Im}\left[\tfrac{1}{4}(s'-u')A_2(s',t) - \tfrac{1}{2}K^2A_5(s',t)\right] \qquad (E.4^a)$$

and

$$(P.K)\mathrm{Re}A_2(s,t) = -\left(\frac{1}{s-m^2} + \frac{\xi}{u-m^2}\right)\frac{g(u-s)}{4(t-\mu^2)}F_1 +$$

$$+\frac{P}{\pi}\int ds'\left\{\frac{1}{s'-s} + \frac{\xi}{s'-u}\right\}\mathrm{Im}\left[-\tfrac{1}{4}(s-u)A_2(s',t)\right]. \qquad (E.4^b)$$

Combining these two equations, we find

$$-\tfrac{1}{2}K^2\mathrm{Re}A_5(s,t) = -\tfrac{1}{2}K^2\left(\frac{1}{s-m^2} - \frac{\xi}{u-m^2}\right)\Gamma_5 - \tfrac{1}{2}K^2\,\frac{(1-\xi)}{2}\,\frac{\Gamma_{t5}}{t-\mu^2} -$$

$$-\tfrac{1}{2}K^2\,\frac{P}{\pi}\int ds'\left\{\frac{1}{s'-s} - \frac{\xi}{s'-u}\right\}\mathrm{Im}A_5(s',t) + \frac{1}{2\pi}(1-\xi)\int ds'\,\mathrm{Im}A_2(s',t) \qquad (E.5)$$

i.e. a dispersion relation for A_5 with an extra subtraction constant

$$-\frac{(1-\xi)}{\pi K^2}\int_{(m+\mu)^2}^{\infty} ds'\;\mathrm{Im}A_2(s',t)\;. \qquad (E.6)$$

If we use the amplitudes $A_i^!$, corresponding changes must be made in the matrices $[B]$ and $[B^{-1}]$ (see Appendix C). These changes can be summarized as

$$B^!_{i5} = -\frac{2}{K^2}B_{i5}$$

$$B^!_{i2} = B_{i2} - \frac{2P.K}{K^2}B_{i5}$$

$$B^!_{ij} = B_{ij} \qquad (j\neq 2 \text{ or } 5) \qquad (E.7)$$

and

$$B^{!-1}_{i5} = \frac{K^2(t-\mu^2)}{4\beta}B_{i5}$$

$$B^{!-1}_{ij} = B^{-1}_{ij} \qquad (j\neq 5)\;, \qquad (E.8)$$

with $\beta = W^2 - m^2 + \tfrac{1}{2}K^2$. We have performed a series of calculations with these new amplitudes $A_i^!$, and found the difference with the original calculation to be less than a few percent, i.e. within the errors (which are $\lesssim 10\%$) the results are the same.

Appendix F

Multipole Born terms and integral kernels

B o r n t e r m s.

The multipole Born terms $\widetilde{M}_\ell^B$ (cf. eqs.(2.32),(2.33)) have the following form for the amplitudes with isospin index 0,+ (ξ=+1) or - (ξ=-1); (we omit the index S or V on the formfactors F_i for convenience).

$$E_{\ell+}^B = \frac{\frac{1}{2}gC(W-m)}{4m(\ell+1)}\Bigg(\frac{2\delta_{\ell 0}}{W^2-m^2}\,[F_1-(W-m)F_2 + \xi(W+m)F_2] - \xi[F_1+2mF_2]T_{\ell+} - $$

$$- 2(1-\xi)F_\pi\,[\frac{\ell R_\ell^\pi}{(E_1+m)(W-m)} - \frac{q(\ell+1)R_{\ell+1}^\pi}{k(E_2+m)(W-m)}] -$$

$$- 2\xi[F_1 + (W+m)F_2]\,\frac{\ell R_\ell^N}{(E_1+m)(W-m)} + 2\xi[F_1-(W-m)F_2]\,\frac{q(\ell+1)R_{\ell+1}^N}{k(E_2+m)(W-m)}\Bigg) \tag{F.1}$$

$$E_{\ell-}^B = \frac{\frac{1}{2}gC(W-m)}{4m\ell}\Bigg(-\xi[F_1+2m_1F_2]T_{\ell-}+2(1-\xi)F_\pi[\frac{(\ell+1)R_\ell^\pi}{(E_1+m)(W-m)} - \frac{q\ell R_{\ell-1}^\pi}{k(E_2+m)(W-m)}] +$$

$$+ 2\xi[F_1+(W+m)F_2]\frac{(\ell+1)R_\ell^N}{(E_1+m)(W-m)} - 2\xi[F_1-(W-m)F_2]\,\frac{q\ell R_{\ell-1}^N}{k(E_2+m)(W-m)}\Bigg) \tag{F.2}$$

$$M_{\ell+}^B = \frac{\frac{1}{2}gC(W-m)}{4m(\ell+1)}\Bigg(-\xi[F_1+2mF_2]T_{\ell+} + 2(1-\xi)F_\pi\,\frac{R_\ell^\pi}{(E_1+m)(W-m)} +$$

$$+ 2\xi[F_1 + (W+m)F_2]\frac{R_\ell^N}{(E_1+m)(W-m)}\Bigg) \tag{F.3}$$

$$M_{\ell-}^B = \frac{\frac{1}{2}gC(W-m)}{4m\ell}\Bigg(\frac{2qk\delta_{\ell 1}}{C^2(W-m)^2}[-F_1-(W+m)F_2+\xi(W-m)F_2] + \xi[F_1+2mF_2]T_{\ell-} -$$

$$-2(1-\xi)F_\pi\frac{R_\ell^\pi}{(E_1+m)(W-m)} - 2\xi[F_1+(W+m)F_2]\,\frac{R_\ell^N}{(E_1+m)(W-m)} \tag{F.4}$$

$$S^B_{\ell+} = \frac{\frac{1}{2}gC(W-m)}{4m(\ell+1)}\Bigg(-\frac{2k\delta_{\ell 0}}{(E_1+m)(W^2-m^2)}\Big[F_1 + (E_1+m)F_2 + \xi(W+m)F_2 +$$

$$+ (1-\xi)\frac{k_0}{K^2}(W+m)(F_1-F_\pi)\Big] + (1-\xi)\frac{2q_0-k_0}{W-m}F_\pi\Big[\frac{Q_\ell(\bar{q}_0)}{q(E_1+m)} - \frac{Q_{\ell+1}(\bar{q}_0)}{k(E_2+m)}\Big] +$$

$$+ \frac{\xi(-1)^{\ell+1}}{W-m}\Big[(2q_0-W)F_1 + m(2q_0-k_0)F_2\Big]\Big[\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)} + \frac{Q_{\ell+1}(\bar{E}_2)}{k(E_2+m)}\Big] +$$

$$+ \frac{\xi(-1)^{\ell+1}}{W-m}\Big[mF_1 - (k_0W+K^2-\mu^2)F_2\Big]\Big[\frac{Q_\ell(\bar{E}_2)}{q(E_2+m)} - \frac{Q_{\ell+1}(\bar{E}_2)}{k(E_2+m)}\Big]\Bigg) \qquad (F.5)$$

$$S^B_{\ell-} = \frac{\frac{1}{2}gC(W-m)}{4m\ell}\Bigg(\frac{2q\delta_{\ell 1}}{(E_2+m)(W-m)^2}\Big[F_1 - (E_1-m)F_2' - \xi(W-m)F_2 +$$

$$+ (1-\xi)\frac{k_0}{K^2}(W-m)(F_1-F_\pi)\Big] + (1-\xi)\frac{2q_0-k_0}{W-m}F_\pi\Big[\frac{Q_\ell(\bar{q}_0)}{q(E_1+m)} - \frac{Q_{\ell-1}(\bar{q}_0)}{k(E_2+m)}\Big] +$$

$$+ \frac{\xi(-1)^{\ell+1}}{W-m}\Big[(2q_0-W)F_1 + m(2q_0-k_0)F_2\Big]\Big[\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)} + \frac{Q_{\ell-1}(\bar{E}_2)}{k(E_2+m)}\Big] +$$

$$+ \frac{\xi(-1)^{\ell+1}}{W-m}\Big[mF_1 - (k_0W+K^2-\mu^2)F_2\Big]\Big[\frac{Q_\ell(\bar{E}_2)}{q(E_1+m)} - \frac{Q_{\ell-1}(\bar{E}_2)}{k(E_2+m)}\Big]\Bigg) \qquad (F.6)$$

$$Ea^B_{\ell+} = \frac{gC(W-m)}{4m(\ell+1)}\Bigg(-\xi G_A X_{\ell+} + 2\xi G_A\frac{(\ell+2)qR^N_{\ell+1}}{C^2(W-m)} - 2\xi G_A\frac{(\ell+1)R^N_\ell}{k(W-m)}\Bigg) \qquad (F.7)$$

$$Ea^B_{\ell-} = \frac{gC(W-m)}{4m\ell}\Bigg(-\frac{2\delta_{\ell 1}qG_A}{(W-m)^2(E_2+m)} - \xi G_A X_{\ell-} - 2\xi G_A\frac{(\ell-1)qR^N_{\ell-1}}{C^2(W-m)} + 2\xi G_A\frac{\ell R^N_\ell}{k(W-m)}\Bigg) \qquad (F.8)$$

$$Ma^B_{\ell+} = \frac{gC(W-m)}{4m(\ell+1)}\Bigg(\frac{2\delta_{\ell 0}kG_A}{(E_1+m)(W^2-m^2)} + \xi G_A X_{\ell+} - 2\xi G_A\frac{qR^N_{\ell+1}}{C^2(W-m)}\Bigg) \qquad (F.9)$$

$$Ma^B_{\ell-} = \frac{gC(W-m)}{4m\ell}\Bigg(-\xi G_A X_{\ell-} + 2\xi G_A\frac{qR^N_{\ell-1}}{C^2(W-m)}\Bigg) \qquad (F.10)$$

$$\mathrm{La}^{B}_{\ell+} = \frac{gC(W-m)}{4m(\ell+1)}\Bigg(\frac{2\delta_{\ell 0}}{k(W^2-m^2)}\left[(E_1-m)G_A + \xi(W+m)G_A + k^2H_A\right] +$$

$$+ \frac{\xi(-1)^\ell}{k(W-m)}\left[(2m^2 - k_0(W-2q_0))G_A - mk^2H_A\right]\left[\frac{Q_{\ell+1}(\overline{E}_2)}{C^2} + \frac{Q_\ell(\overline{E}_2)}{kq}\right] +$$

$$+ \frac{\xi(-1)^\ell}{k(W-m)}\left[m(2W-k_0)G_A - Wk^2H_A\right]\left[\frac{Q_{\ell+1}(\overline{E}_2)}{C^2} - \frac{Q_\ell(\overline{E}_2)}{kq}\right]\Bigg) \tag{F.11}$$

$$\mathrm{La}^{B}_{\ell-} = \frac{gC(W-m)}{4m\ell}\Bigg(-\frac{2q\delta_{\ell 1}}{C^2(W-m)^2}\left[(E_1+m)G_A + \xi(W-m)G_A - k^2H_A\right] +$$

$$+ \frac{\xi(-1)^\ell}{k(W-m)}\left[(2m^2 - k_0(W-2q_0))G_A - mk^2H_A\right]\left[\frac{Q_{\ell-1}(\overline{E}_2)}{C^2} + \frac{Q_\ell(\overline{E}_2)}{kq}\right] +$$

$$+ \frac{(-1)}{k(W-m)}\left[m(2W-k_0)G_A - Wk^2H_A\right]\left[\frac{Q_{\ell-1}(\overline{E}_2)}{C^2} - \frac{Q_\ell(\overline{E}_2)}{kq}\right]\Bigg) \tag{F.12}$$

$$\mathrm{Sa}^{B}_{\ell+} = \frac{gC(W-m)}{4m(\ell+1)}\Bigg(\frac{2\delta_{\ell 0}}{W^2-m^2}\left[G_A - k_0H_A\right] +$$

$$+ \frac{\xi(-1)^\ell}{W-m}\left[(W-2q_0)G_A + mk_0H_A\right]\left[\frac{Q_\ell(\overline{E}_2)}{kq} + \frac{Q_{\ell+1}(\overline{E}_2)}{C^2}\right] +$$

$$+ \frac{\xi(-1)^\ell}{W-m}\left[mG_A + Wk_0H_A\right]\left[\frac{Q_\ell(\overline{E}_2)}{kq} - \frac{Q_{\ell+1}(\overline{E}_2)}{C^2}\right]\Bigg) \tag{F.13}$$

$$\mathrm{Sa}^{B}_{\ell-} = \frac{gC(W-m)}{4m\ell}\Bigg(-\frac{2qk\delta_{\ell 1}}{C^2(W-m)^2}\left[G_A + k_0H_A\right] +$$

$$+ \frac{\xi(-1)^\ell}{W-m}\left[(W-2q_0)G_A + mk_0H_A\right]\left[\frac{Q_{\ell-1}(\overline{E}_2)}{C^2} + \frac{Q_\ell(\overline{E}_2)}{kq}\right] +$$

$$+ \frac{\xi(-1)^\ell}{W-m}\left[mG_A + Wk_0H_A\right]\left[\frac{Q_{\ell-1}(\overline{E}_2)}{C^2} - \frac{Q_\ell(\overline{E}_2)}{kq}\right]\Bigg) \tag{F.14}$$

In these formulas we have used the following definitions

$$C = \left[(E_1+m)(E_2+m)\right]^{\frac{1}{2}}$$

$$\overline{E}_2 = (2k_0E_2 + K^2)/2kq$$

$$\overline{q}_0 = (2k_0q_0 + K^2)/2kq$$

$$Q_\ell(y) = \tfrac{1}{2}\int_{-1}^{1} dx\, P_\ell(x)/(y-x) \tag{F.15}$$

(P_ℓ and Q_ℓ are Legendre functions of the first and second kind.)

$$R_\ell^N = \frac{(-1)^\ell}{2\ell+1}\,[\,Q_{\ell+1}(\bar{E}_2) - Q_{\ell-1}(\bar{E}_2)]$$

$$R_\ell^\pi = \frac{1}{2\ell+1}\,[\,Q_{\ell+1}(\bar{q}_0) - Q_{\ell-1}(\bar{q}_0)]$$

$$T_{\ell\pm} = (-1)^\ell[\frac{1}{qk}\,Q_\ell(\bar{E}_2) - \frac{W+m}{C^2(W-m)}\,Q_{\ell\pm 1}(\bar{E}_2)]$$

$$X_{\ell\pm} = (-1)^\ell[\frac{1}{q(E_1+m)}\,Q_\ell(\bar{E}_2) - \frac{W+m}{k(E_2+m)(W-m)}\,Q_{\ell\pm 1}(\bar{E}_2)]\ . \qquad (F.16)$$

The Born terms for multipole amplitudes with isospin index 1 or 3 (isospin $\frac{1}{2}$ or 3/2) can be obtained from eqs.(F.1)..(F.14) by using eq. (1.17).

I n t e g r a l k e r n e l s.

The integral kernels in eq.(2.33) that connect $\mathrm{Im}\widetilde{M}_{\ell'}^{n'}(W')$ to $\mathrm{Re}\widetilde{M}_\ell^n(W)$ are defined as

$$[\,K_{\ell\ell'}^{nn'}(W,W')] = -\frac{1}{W'-W}[I]\delta_{\ell\ell'}\delta_{nn'} + 2W'\int_{-1}^{1} dx[\,DCB_\ell]\{\frac{c_{nn'}}{s'-s} + [\xi]\frac{d_{nn'}}{s'-u}\}[\,BCG_{\ell'}] \qquad (F.17)$$

where [I] is a unit matrix (6x6 for vector, 8x8 for axial vector amplitudes), and we have explicitly indicated the isospin indices (n,n'). Further we used the abbreviations

$$[\,DCB_\ell] = [\,D_\ell(x)][\,C^{-1}(s)][\,B(s,t)] \qquad (F.18^a)$$

$$[\,BCG_{\ell'}] = [\,B^{-1}(s',t)][\,C(s')][\,G_{\ell'}(x')]\ , \qquad (F.18^b)$$

and defined

$$x' = (kqx + k_0'q_0' - k_0q_0)/k'q'$$

and

$$s' = W^2\ .$$

If isospin indices +,-,0 are used, the factors c and d are given by

$$c_{nn'} = d_{nn'} = \quad_{nn'}\ . \qquad (F.19)$$

With indices 1 or 3 (i.e. isospin $\frac{1}{2}$ or 3/2 states) we find from (F.19) and (1.17)

$$\begin{array}{ll} c_{11} = 1 & d_{11} = -1/3 \\ c_{33} = 1 & d_{33} = 1/3 \\ c_{13} = 0 & d_{13} = 4/3 \\ c_{31} = 0 & d_{31} = 2/3 \end{array}$$

if we define $\xi=1$ for these cases (i.e. $[\xi]=\delta_{ij}\eta_j$; cf. sec.II.3).

The matrices $[DCB_\ell]$ and $[BCG_{\ell'}]$ can simply be calculated by using the formulae of Appendix C. In our calculations this was done explicitly, after which the kernels were obtained by further evaluating eq.(F.17) numerically on a computer.

T h r e s h o l d   f a c t o r s.

As has been noticed in sec.III.2, for the numerical calculations we extract threshold factors (powers of q and k) from the multipole amplitudes. According to this, the formulae in this Appendix should be slightly modified: eqs.(F.1)..(F.14) must be divided by the appropriate factors of q and k; the same holds for (F.18^a), while (F.18^b) has to be multiplied by factors q' and k'.

Appendix G

Transformation matrices between sets of amplitudes (Compton scattering)

To obtain the connection between the different sets of invariant amplitudes that were introduced in sec.IV.3, it is useful to introduce the following set of momentum tensors X^i :

$$\begin{aligned}
X^1_{\mu\nu} &= P'_\mu P'_\nu/P'^2 \\
X^2_{\mu\nu} &= N_\mu N_\nu/N^2 \\
X^3_{\mu\nu} &= (2P'^2N^2)^{-\frac{1}{2}}(P'_\mu N_\nu - P'_\nu N_\mu) \\
X^4_{\mu\nu} &= (2P'^2N^2)^{-\frac{1}{2}}(P'_\mu N_\nu + P'_\nu N_\mu)
\end{aligned} \tag{G.1}$$

Since P.N=0, we have clearly $X^i_{\mu\nu}X^j_{\mu\nu}=\delta_{ij}$, and it can be shown (cf.[He62], [Ba68]) that if we write the matrix element T_{fi} as

$$T_{fi} = \langle\gamma_2 N_2|T|\gamma_1 N_1\rangle = \varepsilon^*_{2\nu}(\vec{k}_2)\bar{u}(\vec{p}_2)\ T_{\mu\nu}\ u(\vec{p}_1)\varepsilon_{1\mu}(\vec{k}_1) =$$

$$= \varepsilon^*_{2\nu}(\vec{k}_2)\bar{u}(\vec{p}_2)[\textstyle\sum_{i=1}^4 X^i_{\mu\nu}\ Y_i]u(\vec{p}_1)\varepsilon_{1\mu}(\vec{k}_1)\quad , \tag{G.2}$$

the forms Y_i will contain, apart from scalar coefficients, only the matrices γ_5 and $(\gamma.K)$. Further we have

$$Y_j = X^j_{\mu\nu}\ T_{\mu\nu} \tag{G.3}$$

due to the orthonormality of the $X^i_{\mu\nu}$, and the fact that the set $\{X^i_{\mu\nu}\}$ is complete, in a sense that is evident from eq.(G.2). Using the results from table G.I, we can express the forms Y_j both in terms of the amplitudes A_i and in terms of B_i, e.g. (omitting the spinors and polarization vectors)

$$\begin{aligned}
Y_1 = X^1_{\mu\nu}\ T_{\mu\nu} &= B_1 + (i\gamma.K)B_4 = \\
&= [K^2A_1 - (P.K)A_3 + mK^2A_4 - \tfrac{1}{2}K^2P'^2A_5 - \tfrac{1}{2}m(P.K)A_6] + \\
&\quad + (i\gamma.K)[mA_3 + (P.K)A_4 - \tfrac{1}{2}P^2A_6]\ ,
\end{aligned}$$

and similar relations for j=2,3,4. From these expressions we can find immediately the matrix $[Z^{-1}]$, defined by the inverse of eq.(4.21),

$$\tilde{B} = [Z^{-1}]\tilde{A}\ . \tag{G.4}$$

$[Z^{-1}] =$

$$\left[\begin{array}{cccccc|cc}
K^2 & 0 & -(P.K) & mK^2 & -\frac{1}{2}K^2P'^2 & -\frac{1}{2}m(P.K) & & \\
K^2 & 0 & -(P.K) & -mK^2 & \frac{1}{2}K^2P'^2 & \frac{1}{2}m(P.K) & & \\
0 & mK^2 & (P.K) & 0 & 0 & 0 & & \\
0 & 0 & m & (P.K) & 0 & -\frac{1}{2}P^2 & & \\
0 & 0 & m & -(P.K) & 0 & \frac{1}{2}P^2 & & \\
0 & 0 & 0 & -K^2 & 0 & \frac{1}{2}(P.K) & & \\
\hline
 & & & & & & -K^2P'^2 & 0 \\
 & & & & & & 0 & K^2
\end{array}\right] \tag{G.5}$$

The matrix $[Z]$ is given by

$[Z] =$

$$\begin{array}{c}
\frac{1}{2K^2} \\[4pt]
\frac{1}{mK^2} \\[4pt]
\frac{1}{2m} \\[4pt]
\frac{-1}{K^2P'^2} \\[4pt]
\frac{1}{K^2P'^2} \\[4pt]
\frac{1}{K^2P'^2} \\[4pt]
\frac{-1}{K^2P'^2} \\[4pt]
\frac{1}{K^2}
\end{array}
\left[\begin{array}{cccccc|cc}
1 & 1 & 0 & \frac{(P.K)}{m} & \frac{(P.K)}{m} & 0 & & \\
0 & 0 & 1 & -\frac{(P.K)}{2m} & -\frac{(P.K)}{2m} & 0 & & \\
0 & 0 & 0 & 1 & 1 & 0 & & \\
0 & 0 & 0 & \frac{1}{2}(P.K) & -\frac{1}{2}(P.K) & P^2 & & \\
-1 & 1 & 0 & 0 & 0 & -2m & & \\
0 & 0 & 0 & -K^2 & K^2 & -2(P.K) & & \\
\hline
 & & & & & & 1 & 0 \\
 & & & & & & 0 & 1
\end{array}\right] \tag{G.6}$$

We can use the same method to express the Born terms (5.14).. (5.16) in terms of the invariants $M^i_{\mu\nu}$ (eq.(5.19)). We then need the results from table G.II, where the various terms in the expressions for the Born terms have been denoted by $R_{\mu\nu}$.

Table G.I[a]

i	$X^1_{\mu\nu} L^i_{\mu\nu}$	$X^2_{\mu\nu} L^i_{\mu\nu}$	$\frac{1}{\sqrt{2}} X^3_{\mu\nu} L^i_{\mu\nu}$	$\frac{1}{\sqrt{2}} X^4_{\mu\nu} L^i_{\mu\nu}$
1	K^2	K^2	0	0
2	0	0	$mK^2 i\gamma_5$	0
3	$-(P.K)+m(i\gamma.K)$	$-(P.K)+m(i\gamma.K)$	$(P.K)i\gamma_5$	0
4	$mK^2+(P.K)(i\gamma.K)$	$-mK^2-(P.K)(i\gamma.K)$	0	$-K^2 i\gamma_5(i\gamma.K)$
5	$-\frac{1}{2}K^2P'^2$	$\frac{1}{2}K^2P'^2$	0	0
6	$-\frac{1}{2}m(P.K)-\frac{1}{2}P^2(i\gamma.K)$	$\frac{1}{2}m(P.K)+\frac{1}{2}P^2(i\gamma.K)$	0	$\frac{1}{2}(P.K)i\gamma_5(i\gamma.K)$
7	0	0	0	$-K^2P'^2 i\gamma_5$
8	0	0	$K^2 i\gamma_5(i\gamma.K)$	0

Table G.I[b]

i	$X^1_{\mu\nu} M^i_{\mu\nu}$	$X^2_{\mu\nu} M^i_{\mu\nu}$	$\frac{1}{\sqrt{2}} X^3_{\mu\nu} M^i_{\mu\nu}$	$\frac{1}{\sqrt{2}} X^4_{\mu\nu} M^i_{\mu\nu}$
1	1	0	0	0
2	0	1	0	0
3	0	0	$i\gamma_5$	0
4	$(i\gamma.K)$	0	0	0
5	0	$(i\gamma.K)$	0	0
6	0	0	0	$i\gamma_5(i\gamma.K)$
7	0	0	0	$i\gamma_5$
8	0	0	$i\gamma_5(i\gamma.K)$	0

Table G.II

$R_{\mu\nu}$	$X^1_{\mu\nu}\ R_{\mu\nu}$	$X^2_{\mu\nu}\ R_{\mu\nu}$	$\frac{1}{\sqrt{2}}\ X^3_{\mu\nu}\ R_{\mu\nu}$	$\frac{1}{\sqrt{2}}\ X^4_{\mu\nu}\ R_{\mu\nu}$
$\gamma_\nu\gamma_\mu$	1	1	$-i\gamma_5(P.K)/K^2$	0
$\gamma_\nu(i\gamma.K)\gamma_\mu$	$-(i\gamma.K)$	$-(i\gamma.K)$	$-mi\gamma_5$	0
$\gamma_\mu(i\gamma.K)\gamma_\nu$	$-(i\gamma.K)$	$-(i\gamma.K)$	$+mi\gamma_5$	0
$i\gamma_\nu(i\gamma.K)P_{j\mu}$	0	0	$-\frac{1}{2}P'^2 i\gamma_5$	$-\frac{1}{2}P'^2 i\gamma_5$
$P_{j\nu}(i\gamma.K)i\gamma_\mu$	0	0	$-\frac{1}{2}P'^2 i\gamma_5$	$+\frac{1}{2}P'^2 i\gamma_5$
$i\gamma_\mu P_{j\nu}$	$-m-(P.K)(i\gamma.K)/K^2$	0	$-\frac{1}{2}i\gamma_5(i\gamma.K)$	$+\frac{1}{2}i\gamma_5(i\gamma.K)$
$i\gamma_\nu P_{j\mu}$	$-m-(P.K)(i\gamma.K)/K^2$	0	$+\frac{1}{2}i\gamma_5(i\gamma.K)$	$+\frac{1}{2}i\gamma_5(i\gamma.K)$
$P_{2\nu}P_{1\mu}$	P'^2	0	0	0
$P_{2\nu}(i\gamma.K)P_{1\mu}$	$(i\gamma.K)P'^2$	0	0	0
$\gamma_5\varepsilon_{\mu\nu\rho\sigma}Q_\rho K_\sigma$	0	0	$2K^2 i\gamma_5$	0

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S A M E N V A T T I N G

In dit proefschrift komen twee verschillende soorten processen aan de orde:

1) electro- en neutrinoproductie van π-mesonen, dat wil zeggen: inelastische verstrooiing van electronen of neutrino's aan nucleonen, waarbij een pion geproduceerd wordt:

$$e\,N \to e\,N\,\pi$$

$$\nu_e N \to e\,N\,\pi$$

$$\nu_\mu N \to \mu\,N\,\pi\ .$$

De eerste reactie wordt veroorzaakt door de electromagnetische wisselwerking; de twee andere door de zwakke wisselwerking.

2) Compton verstrooiing aan nucleonen, ofwel elastische verstrooiing van fotonen aan nucleonen:

$$\gamma\,N \to \gamma\,N\ ,$$

welk proces weer van electromagnetische aard is.

Hoewel deze processen slechts kunnen verlopen door middel van de electromagnetische of zwakke wisselwerking, geldt in beide gevallen dat men rekening moet houden met een belangrijke invloed van de sterke wisselwerking, aangezien er hadronen aanwezig zijn (in casu: pionen en/of nucleonen). De invloed van de electromagnetische en zwakke wisselwerking kan goed beschreven worden door middel van storingstheorie in laagste orde; voor de sterke wisselwerking is dit echter niet mogelijk, zodat deze in zijn geheel in rekening gebracht dient te worden. Aangezien de theoretische kennis omtrent de sterke wisselwerking niet voldoende is om de invloed ervan op bevredigende wijze te kunnen berekenen, kunnen geen numerieke waarden voor de botsingsdoorsneden van de beschouwde processen verkregen worden, als men alleen uitgaat van theoretische beginselen. Daarom is het nodig een andere methode te volgen voor onze berekeningen. Dit is mogelijk door aan te nemen dat de verstrooiingsamplitudes voor de beschouwde processen bepaalde analytische eigenschappen bezitten, als functies van geschikt gekozen kinematische variabelen. Deze aanname leidt tot dispersierelaties voor de amplitudes, welke samen met de eis dat de verstrooiingsmatrix unitair dient te zijn, de basis vormen voor de theorie der dispersierelaties. In het kader van deze theorie kan de invloed van de sterke wisselwerking op de verstrooiingsamplitudes voor de beschouwde processen in verband gebracht worden met andere processen, waarbij deze wisselwerking een rol speelt. Op deze manier kan men experimentele gegevens omtrent deze andere processen gebruiken om het gebrek aan theoretische kennis aangaande de sterke interacties te compenseren. In ons geval houdt dit in dat we gebruik maken van experimentele gegevens over pion-nucleon verstrooiing bij de berekening van electro- en neutrinoproductie, en van experimentele gegevens over pionfotoproductie bij de berekening van Compton verstrooiing. De zo verkregen resultaten kunnen vergeleken worden met experimentele gegevens over deze processen (i.e. over electro- en neutrinoproductie, respectievelijk Compton verstrooiing), voorzover deze al bekend zijn, of zij kunnen dienen als theoretische voorspellingen van toekomstige experimenten. Bezien vanuit theoretisch standpunt kan anderzijds overeenstemming tussen theoretische resultaten en experimentele gegevens beschouwd worden als ondersteuning van de veronderstellingen waarop de theorie der dispersierelaties is gebaseerd. In dit proefschrift zijn de eerste drie hoofdstukken gewijd aan electro- en neutrinoproductie, de laatste drie aan Compton verstrooiing. De behandeling van deze twee soorten processen verloopt grotendeels parallel.

In Hoofdstuk I worden formele uitdrukkingen voor de botsingsdoorsneden voor electro- en neutrinoproductie gegeven, alsmede de kinematica voor deze processen. Verschillende stelsels van amplitudes worden beschreven, waarin de matrixelementen van de verstrooiingsoperator kunnen worden ontbonden. (De belangrijkste hiervan zijn de Lorentz-invariante amplitudes en de multipoolamplitudes.) In Hoofdstuk II worden de dispersierelaties verkregen, uitgaande van de veronderstelde analyticiteitseigenschappen van de invariante amplitudes, waarna een transformatie volgt tot een stelsel gekoppelde integraalvergelijkingen voor de multipoolamplitudes. In Hoofdstuk III gebruiken we experimentele gegevens over pion-nucleon verstrooiing, met behulp waarvan de fases van de multipoolamplitudes voor electro- en neutrinoproductie gevonden kunnen worden, binnen een bepaald energiegebied. De multipool-dispersierelaties zijn dan bij benadering oplosbaar, zodat we numerieke waarden voor de multipoolamplitudes kunnen vinden. Deze amplitudes dienen vervolgens vermenigvuldigd te worden met de geschikte pion- of nucleon-vormfactoren, die weer uit experimentele gegevens afgeleid zijn. Voor electroproductie zijn de benodigde nucleon-vormfactoren bekend, terwijl de pion-vormfactor verkregen zou moeten worden uit een vergelijking van experiment en theorie voor electroproductie. Zoals in dit hoofdstuk blijkt, zijn de resultaten nog te onzeker om deze vormfactor geheel vast te leggen. Voor enkele verschillende keuzes van deze vormfactor zijn hier botsingsdoorsneden voor electroproductie berekend. Voor een volledige berekening van neutrinoproductie zijn dezelfde vormfactoren nodig als bij electroproductie, vermeerderd met de twee axiale nucleon-vormfactoren, waarvan er een slecht bekend is. Aangezien er bovendien weinig experimentele informatie over dit proces beschikbaar is, zijn hier alleen de amplitudes berekend, en niet de botsingsdoorsneden. Voor deze laatste worden alleen formele uitdrukkingen gegeven.

De drie hoofdstukken over Compton verstrooiing volgen ongeveer hetzelfde schema als de eerste drie. Hoofdstuk IV geeft formele uitdrukkingen voor de botsingsdoorsnede, behandelt de kinematica en introduceert diverse stelsels van amplitudes. In Hoofdstuk V worden de analyticiteitseigenschappen van de invariante amplitudes behandeld, waarbij vaak terugverwezen wordt naar Hoofdstuk II. In Hoofdstuk VI wordt aangetoond dat in het energiegebied van de eerste resonantie de dispersierelaties voor Compton verstrooiing opgelost kunnen worden met behulp van amplitudes voor pion fotoproductie, welke goed bekend zijn uit analyses van experimentele gegevens. Amplitudes en botsingsdoorsneden voor Compton verstrooiing kunnen dan berekend worden en vergeleken met experimentele waarden. Verder kan uit de fotoproductie-amplitudes een ondergrens berekend worden voor de botsingsdoorsnede voor Compton verstrooiing, waarbij alleen gebruik gemaakt hoeft te worden van de unitariteitsrelatie en de gebruikelijke invariantie aannamen voor de S-matrix (C, P, T). Het blijkt nu dat deze ondergrens geschonden wordt door enkele experimentele punten. Om na te gaan of deze discrepantie opgeheven kan worden door de eis van T-invariantie te laten vallen, is het formalisme dienovereenkomstig gegeneraliseerd, en hebben we enkele berekeningen gedaan waarbij op eenvoudige manier een schending van deze invariantie werd ingevoerd in de fotoproductie amplitudes. De resultaten tonen aan dat hierdoor de genoemde discrepantie iets kleiner kan worden, maar vermoedelijk niet geheel zal verdwijnen.

Tenslotte zijn enkele appendices bijgevoegd, waarin de gebruikte conventies zijn samengevat, alsmede enkele details van de berekeningen.

S T U D I E O V E R Z I C H T

Na een schoolopleiding aan de Rijks H.B.S. te Middelburg begon ik in 1960 met mijn studie aan de Rijksuniversiteit te Leiden. In mei 1964 legde ik het candidaatsexamen af, met hoofdvakken natuurkunde en wiskunde, en bijvak sterrekunde. Hierna koos ik de studierichting theoretische natuurkunde, waarin het doctoraalexamen werd afgelegd in december 1967. In 1965 was ik gedurende een half jaar werkzaam op het Kamerlingh Onnes Laboratorium, in de werkgroep Metalen. Sinds juni 1967 ben ik verbonden aan het Instituut-Lorentz voor Theoretische Natuurkunde, (in dienst van de Stichting voor Fundamenteel Onderzoek der Materie), waar ik onder leiding van Prof. Dr. J.A.M. Cox en Dr. F.A. Berends heb gewerkt aan enkele onderzoekingen op het gebied van de hoge-energie fysica in verband met dispersierelaties. Tevens heb ik enige malen werkcolleges in de quantummechanica verzorgd. In 1968 en 1969 heb ik tweemaal een werkbezoek van enkele maanden gebracht aan Prof. Dr. A. Donnachie van de Universiteit van Glasgow. Deze reizen werden mogelijk gemaakt door een financiele bijdrage van de Stichting F.O.M.