FLOW PHENOMENA OF PURE ^{4}HE AND ^{3}HE-^{4}HE MIXTURES IN THE HELIUM II REGION

F. A. STAAS

STELLINGEN

I

Tegen de wijze waarop Dekker de Boltzmann-vergelijking voor de electronen in een metaal afleidt, zijn bedenkingen aan te voeren.

Dekker, H. J., "Solid State Physics", p. 280.

II

De door Vinen, Careri c.s., Bhagat en Critchlow op allerlei manieren gevonden kritische verschijnselen kunnen verklaard worden met behulp van cohaerente turbulentie.

Vinen, W. F., Proc. Roy. Soc. A240, 1957, 128
Vinen, W. F., Proc. Roy. Soc. A243, 1957, 400.
Careri, G., Scaramuzzi, F. and mcCormick, W. D., Proc. VIIth Conf. on low temp. Phys. Toronto, Canada, 1960, p. 502.
Bhagat, S. M. and Critchlow, P. R., Cryogenics **2** (1961) 39.

III

De manier, waarop Donnelly en Simon het gedrag van de stroming tussen roterende cylinders in het boven-kritische gebied beschrijven kan eleganter geschieden door dimensieloze grootheden in te voeren.

Donnelly, R. J. and Simon, N. J., J. Fluid Mech **7** (1960) 401.

IV

Het vermogen warmte af te voeren d.m.v. Helium II in een buis kan aanmerkelijk vergroot worden door de buis te voorzien van een poreuze kern. Dit zou toegepast kunnen worden bij de koeling van een trefplaatje in het inwendige van een kernreactor.

V

De bewering van Atkins dat instabiliteiten niet zouden optreden, indien de buitenste cylinder van een roterende cylinderviscosimeter draait, is onjuist.

Atkins, K. R., "Liquid Helium", Cambridge Univ. Press. p. 191.
Taylor, G. I., Phil. Trans. A223, **289** (1923).
Taylor, G. I., Proc. Roy. Soc. A151, **494** (1935).

VI

Een nauwkeurige bepaling van de concentratie van ^{3}He- ^{4}He gasmengsels kan geschieden door de meting van de transmissie van een neutronenbundel door het mengsel.

VII

Bij de bestudering van stromingsverschijnselen in He II is het gewenst kanalen te kiezen met een zodanige doorsnede en oppervlaktegesteldheid dat het hydrodynamisch gedrag van gewone vloeistoffen hierin bekend is.

VIII

Bij de poging van Lovejoy en Shirley om de kernoriëntatie van Tb^{156} in neodymium ethylsulfaat te verklaren zijn te grote vereenvoudigingen gemaakt.

Lovejoy, C. A. and Shirley, D. A., Proc. VIIth Conf. on low temp. Phys. Toronto, Canada 1960, p. 164.

IX

Het is onjuist om de effectieve massa van het ^{3}He atoom in verdunde ^{3}He–^{4}He mengsels, waarbij weinig ^{3}He opgelost is in het superfluide ^{4}He, te berekenen met behulp van de structuurfactor van vloeibaar ^{4}He.

X

Voor de studie van het excitatiespectrum van He II zou men met vrucht gebruik kunnen maken van de Rayleigh-verstrooiing van gamma-stralen, waarbij de frequentieverschuiving wordt gemeten met behulp van het Mössbauereffect.

XI

In de tweede ronde van het V.H.M.O. verdient het aanbeveling de elektriciteitsleer en de leer van het magnetisme te baseren op de wetten van Maxwell.

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FLOW PHENOMENA OF PURE ^{4}HE AND ^{3}HE-^{4}HE MIXTURES IN THE HELIUM II REGION

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKSUNIVERSITEIT TE LEIDEN OP GEZAG VAN DE RECTOR MAGNIFICUS DR. G. SEVENSTER, HOOGLERAAR IN DE FACULTEIT DER GODGELEERDHEID, TEGEN DE BEDENKINGEN VAN DE FACULTEIT DER WIS- EN NATUURKUNDE TE VERDEDIGEN OP DONDERDAG 16 NOVEMBER 1961 TE 16 UUR

DOOR

FRANS ADRIANUS STAAS

GEBOREN TE KLATEN (IND.) IN 1927

Promotor: PROF. DR. K. W. TACONIS

AAN ANS

Teneinde te voldoen aan het verlangen van de Faculteit der Wis- en Natuurkunde volgt hieronder een beknopt overzicht van mijn studie.

Na het beëindigen van mijn hbs-b opleiding in 1946 liet ik mij inschrijven aan de Technische Hogeschool te Delft voor de studie van vliegtuigbouwkundig ingenieur. In 1949 legde ik daar het propaedeutisch examen af.

In 1950 liet ik mij aan de Rijksuniversiteit te Leiden inschrijven voor de studie in de Wis- en Natuurkunde. Het candidaatsexamen letter a legde ik af in 1953; het doctoraal examen experimentele natuurkunde met bijvak mechanica in 1956. Inmiddels was ik sedert 1953 werkzaam in het Kamerlingh Onnes Laboratorium als medewerker in de Heliumwerkgroep onder leiding van Prof. Dr. K. W. Taconis. Tot 1956 assisteerde ik Dr. D. H. N. Wansink bij metingen over ^{3}He–^{4}He mengsels in nauwe spleten. Mijn eerste zelfstandig onderzoek dateert van 1955–1956, in welke periode de metingen verricht werden over kritische snelheid en wederkerige wrijving in ^{4}He, waarbij ik geholpen werd door Dr. R. de Bruyn Ouboter. In 1957 werden enige oriënterende metingen uitgevoerd die niet tot direct resultaat hebben geleid. Hierna volgden in 1958 de onderzoekingen betreffende de ^{3}He–^{4}He mengsels, waarbij ik werd geassisteerd door de heer K. Fokkens. Deze metingen werden in 1959 afgesloten. Bij de daarop volgende metingen aan zuiver ^{4}He werd ik aanvankelijk bijgestaan door de heer K. Fokkens en later door de heer W. M. van Alphen. Na mijn huwelijk in 1958 vond ik een toegewijde medewerkster in mijn echtgenote. Sinds 1956 assisteerde ik op het practicum voor prae-candidaten en vanaf 1960 vervulde ik de functie van hoofdassistent van het natuurkundig practicum voor de studenten in de chemie. Met ingang van 1 januari 1960 werd ik benoemd tot wetenschappelijk ambtenaar.

CONTENTS

INTRODUCTION

Liquid helium II shows an anomalous behaviour of the heat conductivity and the flow properties. This thesis presents the results of a number of investigations in this field which are discussed on the basis of the two-fluid model.

The research originated from an attempt to determine the viscosity of the normal fluid of ^{3}He–^{4}He mixtures [1]), with an apparatus described in the third part of the second chapter. The results obtained were not understood as the viscosities appeared to be velocity dependent. This indicated turbulent flow although the calculated Reynolds numbers containing the density of the normal fluid were so small that only laminar flow could be expected.

Therefore a second apparatus was constructed which is described in the first chapter. With this method it was possible to create a flow of the liquid in such a way that the normal fluid containing the ^{3}He atoms moves in the capillary with almost the same velocity as the superfluid. The fluid behaves as an ordinary fluid and hence we expected in this special case that the corresponding Reynolds number might contain the total density. Even if the Reynolds number was calculated using the total density, this Reynolds number was sufficiently small that laminar flow was assured.

The viscosities of the ^{3}He–^{4}He mixtures obtained from this experiment were found to be in disagreement with those determined by the oscillating disk method [2]). Extrapolation to zero concentration (pure ^{4}He) showed satisfactory agreement at temperatures below 1.6°K with the measurements of the viscosity of the normal fluid performed with the rotating cylinder method, however at 1.9°K deviations were found .The agreement below 1.6°K can be considered as an experimental proof of the statement that ^{3}He moves with the same velocity as the normal fluid. Furthermore the dependence of the obtained viscosities on concentration and temperature was elucidated by Zharkov's theory [3]).

Because of the deviations between the two methods near 1.9°K mentioned above we decided to measure the normal fluid viscosity of pure ^{4}He by the capillary method. For several reasons it was evident that in order to obtain accurate results the measurements should be done with wide capillaries, but due to the appearance of mutual friction this seemed to be a difficult task.

We succeeded in constructing an apparatus which made it possible to measure the viscosity of the normal fluid using wide capillaries. As in the experiment with mixtures a normal fluid flow was produced in the same

direction and with the same velocity as the superfluid. The measured viscosities were in disagreement with the results obtained with the rotating cylinder above 1.8°K, but confirmed the results of the extrapolated ^3He–^4He viscosity measurements.

It was possible to study turbulent flow with the same apparatus. These measurements are discussed in the second chapter. If a Reynolds number containing the total density and the normal fluid velocity was used, then in the high velocity range it appeared that the relation between the pressure gradient and the normal fluid velocity was given by the classical rule of Blasius [4]). It was interesting to investigate whether deviations of this rule appeared in the case that superfluid and normal fluid move with different velocities. A striking result was that the superfluid velocity did not affect the above- mentioned relation. The relation holds even in the case of ordinary heat conduction experiments where superfluid and normal fluid flow in opposite directions in such a way that there is no transport of mass.

In addition experiments on gravitational flow, performed by other investigators [5])[6]) are discussed in connection with turbulence. Here the difficulty arises that only a weighted average of superfluid velocity and normal fluid velocity can be measured. The measurements can still be interpreted assuming that mutual friction is not appreciably different passing from laminar to turbulent flow. In connection with this can be said that the early heat conductivity measurements of Keesom and Saris [7]) within experimental accuracy show no difference of mutual friction between laminar and turbulent flow.

The measurements with ^3He–^4He mixtures mentioned at the beginning of this introduction could be satisfactorily explained assuming turbulence if the Reynolds number does not contain the previously expected normal fluid density but the total density instead.

It should be pointed out that the critical velocity of the onset of mutual friction has nothing to do with the critical velocity of the onset of turbulence which can be calculated from the above- mentioned Reynolds number. Some early experiments on mutual friction and the critical velocity related to the onset of mutual friction are discussed in the last chapter. The measured critical velocity in a 260 μ capillary shows a maximum at 1.5°K. The mutual friction can be described by a third power dependence as was supposed by Gorter and Mellink.

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CHAPTER I

VISCOSITY OF LIQUID ^{3}He–^{4}He MIXTURES IN THE HELIUM II REGION ABOVE 1°K

Summary

The viscosity of liquid ^{3}He–^{4}He mixtures has been derived from isothermal flow through a capillary 75.6 μ in diameter. The apparatus consists of two identical glass vessels, connected to each other by the glass capillary. The positions of both levels were measured as a function of time. It appeared that the logarithm of the level difference always decreased linearly with time, which means that Poiseuille's law is obeyed. The analysis of the flow has been given. The values of the viscosity extrapolated to zero concentration (pure ^{4}He) are in agreement with the measurements with the rotating cylinder by Heikkila and Hollis Hallett. There is also satisfactory agreement in the low concentration region with the theory given by Zharkov.

1. *Introduction.* Several investigators have measured the viscosity of the normal fluid η_n of pure ^{4}He below the λ-point. One can distinguish between several methods, the oldest method being the observation of the damping of an oscillating disk used by Keesom and MacWood [1]) and later on followed by Andronikashvilli [2]); De Troyer, Van Itterbeek and Van den Berg [3]); Hollis Hallett [4]) and Dash and Taylor [6])[7]). From these observations one can calculate the product $\rho_n\eta_n$, ρ_n and η_n being respectively the density and the viscosity of the normal fluid. The value of the viscosity of the normal fluid is obtained from an independent measurement of the normal fluid density.

Independent of the knowledge of the normal fluid density Heikkila and Hollis Hallett [8]) were able to measure the normal fluid viscosities by using the rotating cylinder viscometer. The results of the torsion pendulum method and the rotating cylinder viscometer are still in quantitative disagreement. Zinov'eva [9]) has derived values of the viscosity of the normal fluid from the attenuation of second sound in a resonant cavity which are in better agreement with those values found by Heikkila and Hollis Hallett.

Capillary methods have been used in heat conduction experiments. Since in pure ^{4}He below the λ-point heat is transferred only by the normal fluid, the heat current is related to the normal fluid velocity v_n. The normal

fluid viscosity may be calculated from laminar flow if the pressure gradient corresponding with the velocity of the normal fluid is known. The normal fluid flow is accompanied by a superfluid counterflow in such a way that the resultant flow is zero. Now the pressure gradient can only be easily calculated from the temperature gradient if a critical velocity, above which mutual friction comes into play is not exceeded. The critical velocity depends on the capillary diameter and is lower for a wider capillary. This makes measurements with narrow capillaries necessary unless one is able to measure very small temperature gradients. On the other hand the choice of the capillary or slit is limited by mean free path effects. The mean free path of phonons increases with decreasing temperatures and is in the case of pure ^{4}He at lower temperatures comparable with the slit width. Heat conduction methods were employed by Broese van Groenou, Poll, Delsing and Gorter [10]) and recently by Brewer and Edwards [11]). From their heat conduction method with wide capillaries of 52.0 μ and 107.6 μ diameter Brewer and Edwards derived values for the viscosity which are in good agreement with those of the rotating cylinder viscometer.

Until now viscosity measurements on ^{3}He–^{4}He dilute mixtures have been reported by Pellam [12]), Wansink and Taconis [13]), Dash and Taylor [14]). Wansink and Taconis were able to deduce the viscosity from flow experiments through a narrow slit of 0.3 μ, while Pellam and Dash and Taylor used the torsion pendulum method. The latter measured the viscosity of three samples with liquid concentrations of about 3%, 5% and 10% ^{3}He.

We have determined the viscosity of the normal fluid straightforward using isothermal flow in a capillary 75.6 μ in diameter, making use of the relation between the mean velocity and the pressure gradient according to Poiseuille's law.

In liquid ^{3}He–^{4}He mixtures below the λ-point a kind of osmotic pressure exists between two vessels containing mixtures of different concentrations separated by a narrow slit [15])[16]).

In the case where one connects two vessels containing mixtures of different compositions by a capillary, superfluid will flow to the high concentration side until the osmotic pressure is balanced by a level difference. In our case using a capillary 76 μ in diameter, this difference adjusts itself within a few seconds. A concentration difference of 10^{-4} corresponds to an osmotic pressure of about 2 cm helium column at a temperature near 1°K.

Subsequent to the establishment of the level difference one can expect a flow of the normal fluid under influence of the hydrostatic pressure head, according to Poiseuille's law, while the superfluid flow obeys the condition of permanent equilibrium between the hydrostatic pressure head and the osmotic pressure difference. Therefore in general at not too low concentrations where the concentration difference is relatively small superfluid flows with almost the same velocity as do the normal and ^{3}He atoms. As we shall

show later on diffusion is negligible and we can calculate the viscosity easily from Poiseuille's law.

2. *The apparatus.* The apparatus consists of two identical thin-walled glass vessels V_1 and V_2 connected to each other by a glass capillary. A filter of fine nylon fabric N is used for protecting the capillary against the entrance of impurities. The vessels V_1 and V_2 are connected to the manometer system by means of two identical glass capillaries C, which are

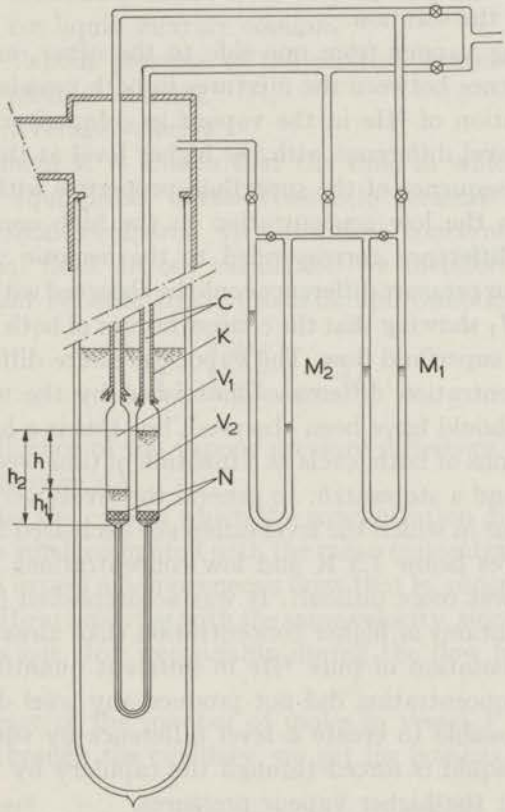

Fig. 1. The apparatus.

partially wrapped with cotton cord K to prevent the increase of heat leaks with the falling of the surrounding helium bath. During the experiment the helium bath temperature was kept constant within 10^{-5} degrees monitored by a carbon resistance thermometer.

It was necessary to purify the ^3He–^4He mixtures before using them. We purified the gaseous mixtures by leading them through a spiral capillary at a temperature of about 4°K in a separate cryostat. Measurements of the vapour pressure of the liquid mixtures were carried out with the aid

of oil manometers M_1 and M_2 filled with octoil S. We deduced the concentration of the solution from the known temperatures of the helium bath and vapour pressure differences with respect to the bath pressure [17][18][19]. Both parts of the apparatus could be connected to a mercury filled Toepler pump system.

3. *The method.* The measurements of the viscosity were preceded by the creation of an initial level difference of the required extent (about 2 cm). This could be done in several ways depending on the temperature and the concentration of the solution.

By transferring vapour from one side to the other, one creates a concentration difference between the mixtures in both vessels due to the much higher concentration of ^{3}He in the vapour in comparison with the liquid. We observed a level difference with the higher level at the high concentration side. In consequence of the superfluid properties within a few seconds ^{4}He flowed from the low concentration to the high concentration vessel until the level difference corresponded to the osmotic pressure. At that moment no vapour pressure difference could be observed with any precision on oil manometer M_1 showing that the concentrations of both levels were nearly equalised by the superfluid flow. The vapour pressure difference due to the very small concentration difference diminished by the weight of a small vapour column should have been observed, but this is a hardly measurable effect. The positions of both levels as a function of time were measured using a cathetometer and a stopwatch. In general the levels were followed during about half an hour in which the level difference decreased by a factor of ten.

At temperatures below 1.3°K and low concentrations the creation of a level difference was more difficult. It was accomplished by condensing on one side only solutions of higher concentration than already present in the apparatus. Condensation of pure ^{4}He in different quantities or a mixture of much lower concentration did not produce any level difference.

It was also possible to create a level difference by suddenly admitting gas so that the liquid is forced through the capillary by the overpressure. This was done at the higher vapour pressures.

4. *The calculation of the flow.* We shall use the following symbols:

X_l, X_v:	Molar concentration of the liquid or vapour phase, being the ratio of the number of ^{3}He atoms to the total number of atoms.
ρ_l, ρ_v:	Molar density of the liquid and the vapour phase.
ρ_{3l}, ρ_{3v}:	Molar density of ^{3}He in the liquid and in the vapour phase.
M:	Molar weight of the mixture.
r, l:	Radius and total length of the capillary.

R, L: Radius and length of the vessels V_1 and V_2.

$A = \frac{r^4 g}{4lR^2}$: A constant of the apparatus.

g: Acceleration due to the gravity.

h_1, h_2: Level heights in vessels V_1 and V_2.

$h = h_2 - h_1$: Level difference.

$\Delta X = X_{1_2} - X_{1_1}$: The small concentration difference between the vessels.

t: Time in seconds.

p_0: Osmotic pressure with respect to pure ^{4}He expressed in cm liquid mixture column.

p_v: Vapour pressure of the mixture expressed in cm liquid mixture column.

$\bar{v}$: Average velocity.

From this experiment it is known that the time in which the superfluid flow adjusts the equilibrium between osmotic pressure and hydrostatic pressure is very small compared with the time concerning the Poiseuille flow of the normal fluid. In our calculation we therefore assume instantaneous equilibrium between the hydrostatic and osmotic pressure:

$$h = \left(\frac{\partial p_0}{\partial X}\right)_T \Delta X, \tag{1}$$

neglecting the influence of the vapour pressure difference due to the small concentration difference ΔX.

First we consider the case in which the concentration difference between the vessels, ΔX, is small compared with the mean concentration of the liquid $\bar{X}$. In this case we expect a homogeneous flow, that is, superfluid and normal fluid containing ^{3}He are moving with the same velocity, since the composition of the liquid does not alter appreciably during the flow from vessel V_2 to vessel V_1.

Since the increase of the number of moles in vessel V_1 is equal to the amount passing through the capillary, we get for laminar flow:

$$\frac{\pi r^4}{8\eta l} M \rho_1^2 g h = \frac{\mathrm{d}}{\mathrm{d}t}(\pi R^2 \rho_1 h_1) = -\frac{\pi R^2 \rho_1}{2}\dot{h}. \tag{2}$$

Neglecting two vapour corrections mentioned below we obtain the differential equation:

$$\frac{\dot{h}}{h} = -AM\frac{\rho_1}{\eta} \qquad \text{with} \quad A = \frac{r^4 g}{4lR^2}, \tag{3}$$

which expresses the linear relation between velocity ($\propto \dot{h}$) and pressure head ($\propto h$). Conversely a straight line dependence of log h on the time obtained as information from the experiment attests to the laminar flow

of the liquid, and the viscosity can be calculated easily from the slope of the straight line.

The vapour corrections are of two kinds:

a. The correction dealing with the replacement of vapour by liquid on one side (condensation) and the replacement of liquid by vapour on the other side (evaporation). This can be considered as a correction of the measured value of the flow $\dot{h}$ this being too large by a factor $(1 - \rho_{3v}/\rho_{3l})^{-1}$. This will be derived more accurately later on.

b. The correction arising from a difference in vapour pressure corresponding to a concentration difference at both sides. This makes the driving force no longer equal to the level difference h but increases it by a factor

$$\left(1 - \frac{\partial p_v}{\partial X} \Big/ \frac{\partial p_0}{\partial X}\right)^{-1}.$$

After taking these vapour corrections into account equation (3) is transformed into:

$$\frac{\dot{h}}{h} = -AM\frac{\rho_l}{\eta}\left(\frac{1}{1 - \frac{\partial p_v}{\partial X} \Big/ \frac{\partial p_0}{\partial X}}\right)\left(\frac{1}{1 - \rho_{3v}/\rho_3}\right). \tag{4}$$

In the case where the concentration difference is comparable with the mean concentration $\bar{X}$, the dilution of the highest concentration becomes important, since now the decrease of h with time and the corresponding decrease of the concentration difference gives rise to a non-negligible difference in flow velocity for superfluid and normal fluid containing ^{3}He. This is so because the concentration in the vessel having the higher level must decrease and the concentration in the other one must increase. This means that a measurement of $\dot{h}$ no longer correctly accounts for the velocity of the normal fluid and the ^{3}He atoms.

Moreover, we have taken into consideration the variation of the concentration along the capillary according to the equilibrium between the driving force and the osmotic pressure:

$$\text{grad}\, p = (\partial p_0/\partial X)_T \,\text{grad}\, X. \tag{5}$$

According to the relation for laminar flow in which ^{3}He participates:

$$\bar{v}_3 = -\frac{r^2}{8\eta}\rho_l g M \,\text{grad}\, p, \tag{6}$$

in which the pressure p is expressed in cm He column.

After substitution of (5) this becomes:

$$\bar{v}_3 = -\frac{r^2}{8\eta}\rho_l\, g M \left(\frac{\partial p_0}{\partial X}\right)_T \text{grad}\, X. \tag{7}$$

As an approximation we assume a stationary flow. The constant number of moles of ^{3}He per second passing through an arbitrary cross-section of the capillary is then:

$$\pi r^2 \rho_1 X \bar{v}_3 = \frac{\pi r^4}{16\eta} \rho_1^2 g M \left(\frac{\partial p_0}{\partial X}\right)_T \operatorname{grad} X^2 = \frac{\pi r^4}{16\eta l} \rho_1 g M \left(\frac{\partial p_0}{\partial X}\right)_T (X_2^2 - X_1^2). \quad (8)$$

The total number of moles of ^{3}He already present in vessel I is

$$\pi R^2 \{h_1 X_{l_1} \rho_l + (L - h_1) X_{v_1} \rho_{v_1}\}.$$

The ^{3}He balance gives:

$$\pi R^2 \frac{d}{dt} \{h_1 X_{l_1} \rho_{l_1} + (L - h_1) X_{v_1} \rho_{v_1}\} = \frac{\pi r^4}{16\eta l} \rho_1^2 g M \left(\frac{\partial p_0}{\partial X}\right)_T (X_2^2 - X_1^2). \quad (9)$$

For simplicity we introduce h_{tot} and $\bar{X}$:

$$h_{tot} = h_1 + h_2 \quad (10)$$

(a measure of the total quantity of liquid which remains constant within experimental accuracy).

$$\bar{X} = \frac{h_1 X_{l_1} + h_2 X_{l_2}}{h_{tot}}, \quad (11)$$

the mean molar concentration of the liquid.

Equilibrium between osmotic pressure and the driving force corrected for the vapour pressure difference gives:

$$\left(\frac{\partial p_0}{\partial X}\right)_T (X_{l_2} - X_{l_1}) = h + \left(\frac{\partial p_v}{\partial X}\right)_T (X_{l_2} - X_{l_1}). \quad (12)$$

With (10), (11) and (12) we are able to express X_{l_1} and X_{l_2} and h_1 in terms of the constants h_{tot}, $\bar{X}$ and the level difference h.

$$X_{l_2} = \bar{X} + \frac{(h_{tot} - h)h}{2h_{tot}\left(\frac{\partial p_0}{\partial X} - \frac{\partial p_v}{\partial X}\right)}, \quad (13)$$

$$X_{l_1} = \bar{X} - \frac{(h_{tot} + h)h}{2h_{tot}\left(\frac{\partial p_0}{\partial X} - \frac{\partial p_v}{\partial X}\right)}, \quad (14)$$

$$h_1 = \frac{h_{tot} - h}{2} \quad (15)$$

and after substitution of (13), (14) and (15) in (9) we arrive at the equation:

$$(1 + \varepsilon + \delta(h)) \frac{\dot{h}}{h} = - AM \frac{\rho_l}{\eta} \left(\frac{1}{1 - \frac{\partial p_v}{\partial X} \Big/ \frac{\partial p_0}{\partial X}}\right) \left(\frac{1}{1 - \rho_{3v}/\rho_{3l}}\right) \quad (16)$$

in which:

$$A = \frac{r^4 g}{4lR^2},$$

$$\varepsilon = \frac{h_{\text{tot}} + \mu}{2\left(\frac{\partial p_0}{\partial X} - \frac{\partial p_v}{\partial X}\right)\bar{X}} \quad \text{with} \quad \mu = \frac{2L}{\rho_3/\rho_{3v} - 1},$$

$$\delta(h) = \frac{(2-\varepsilon)h^2 - 2\mu h}{2h_{\text{tot}}\left(\frac{\partial p_0}{\partial X} - \frac{\partial p_v}{\partial X}\right)\bar{X} - h^2},$$

assuming ρ_l and ρ_{3l}/ρ_{3v} constant.

The factor $(1 + \varepsilon + \delta(h))$ in the left hand side of (16) deals with the fact that the highest concentration is diluted during the flow. However, the subsequent delivery of ^{3}He from the vapour phase and the effect of a concentration gradient along the capillary also affects this factor.

From our experiments we found a linear dependence of log h on the time. This indicates that $(1 + \varepsilon + \delta(h))$ does not depend strongly on the level difference h. We regarded $\delta(h)$ as a correction of $(1 + \varepsilon)$.

To get an idea of the magnitude of these corrections we assume: $h_{\text{tot}} = 6$ cm, $L = 8$ cm, $h = 1$ cm and the temperature is 1°K. $\delta(h)$ is given in the following table:

Calculated from (16):		
X	$1 + \varepsilon$	$\delta(h)$
10^{-4}	2.511	-6.39×10^{-3}
5×10^{-4}	1.302	$+8.3 \times 10^{-3}$
10^{-3}	1.151	$+4.8 \times 10^{-3}$
5×10^{-3}	1.036	$+1.07 \times 10^{-3}$
10^{-2}	1.015	$+5.5 \times 10^{-4}$
5×10^{-2}	1.003	$+1.2 \times 10^{-4}$

This shows, that $\delta(h)$ is of no significance since $\delta(h)$ is small compared with $1 + \varepsilon$.

$\varepsilon = 0$ means that we are dealing with a homogeneous flow. In other words: the superfluid is moving with the same velocity as normal fluid and ^{3}He atoms. The amount flowing through the capillary per unit area and per second being ρv_n. In the case where $\varepsilon \neq 0$ the amount flowing through the capillary per unit area and per second equals $\rho_n v_n + \rho_s v_s$ with $\rho_n + \rho_s = \rho$.

We can easily conclude that $(\rho_n v_n + \rho_s v_s)(1 + \varepsilon) = \rho v_n$ yielding:

$$v_s - v_n = \frac{\frac{1}{1+\varepsilon} - 1}{1 - x} v_n$$

with $x = \rho_n/\rho$, expressing $v_s < v_n$ as we expected.

In our experiment the velocity of the normal fluid is of the order of 1 cm/s.

Corresponding with this value $(v_s - v_n)$ is calculated and given in the following table:

	X	$(v_s - v_n)$ cm/s		X	$(v_s - v_n)$ cm/s
$T = 1°K$	10^{-4}	-0.60	$T = 2.1°K$	10^{-4}	-1.75
	10^{-3}	-0.14		10^{-3}	-0.37
	10^{-2}	-0.01		10^{-2}	-0.00

Since we considered the quantity of ^{3}He transferred by the laminar flow, we also have to regard the transfer caused by diffusion under influence of the concentration gradient. Beenakker and Taconis measured concentration diffusion coefficients [20]). According to these measurements the diffusion coefficient is of the order of 10^{-1} at $T = 1.1°K$. Our minimum values of the concentration gradient in the capillary are of the order of 10^{-5}. The number of moles ^{3}He transported by diffusion per second and per unit area is equal to ρD grad X(of the order of 10^{-7}). The ratio of transfer by diffusion and transfer by the laminar flow is:

$$\frac{D \text{ grad } X}{\bar{v}_3 X}.$$

Using (6) this turns out to be

$$8\eta D/Xr^2\rho_1 gM\left(\frac{\partial p_0}{\partial X}\right)$$

(of order 10^{-3} in the case where $X = 10^{-4}$), hence $D \text{ grad } X/\bar{v}_3 X \ll 1$, meaning that diffusion is negligible.

5. *Calibration of the apparatus.* Calibration of the apparatus was performed in two different ways. Since the constant A of the apparatus contains the radius of the vessel R and the capillary dimensions, a gas flow experiment made it necessary to make independent measurements of both. The laminar flow constant r^4/l, depending on cross-section and length of the capillary was determined by a helium gas flow experiment at room temperature and atmospheric pressure, using a small initial pressure head of about 15 cm oil. One end of the capillary was therefore connected to a thermostated and calibrated volume filled with helium gas at 15 cm oil pressure difference with respect to atmospheric pressure while the other end was maintained at constant atmospheric pressure. The decrease of the pressure difference was measured at the differential oil manometer M_1.

However, a flow experiment with mercury at constant temperature supplied us with the complete constant A in one measurement according to (3).

The constants A derived from both methods were equal to within 1%:

$$A = 1.347 \times 10^{-7}\ \mathrm{cm^2\,s^{-2}},$$

$$\frac{\pi r^4}{l} = 4.93 \times 10^{-11}\ \mathrm{cm^3},$$

the radius of vessels $R = 0.1690$ cm and the capillary length l is about 13.0 cm.

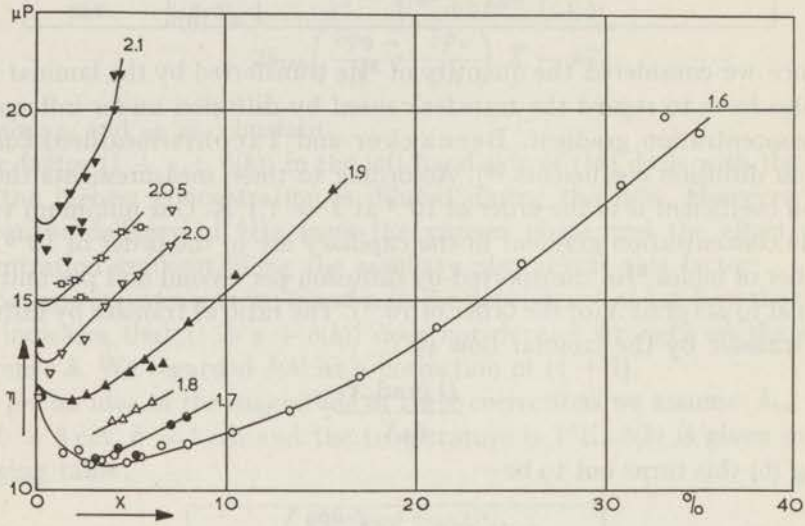

Fig. 2. The viscosity as a function of the concentration at temperatures above 1.6°K.

○ $T = 1.6$°K ▽ $T = 2.0$°K
● $T = 1.7$°K –□– $T = 2.045$°K
△ $T = 1.8$°K ▼ $T = 2.1$°K
▲ $T = 1.9$°K

TABLE I

$T = 1.1$°K					
X %	η' μP	$\dfrac{1}{1 - \dfrac{\partial p_v}{\partial X} \Big/ \dfrac{\partial p_0}{\partial X}}$	$\dfrac{1}{1 - \rho_{3v}/\rho_{31}}$	$1 + \varepsilon$	η μP
2.61	13.73	1.019	1.023	1.003	14.28
4.03	13.09	1.018	1.023	1.003	13.59
6.36	13.21	1.018	1.023	1.002	13.75
7.69	13.21	1.018	1.023	1.002	13.75
0.855	14.84	1.019	1.023	1.008	15.35
0.23	17.56	1.018	1.023	1.031	17.74
7.99	13.26	1.018	1.023	1.001	13.79
30.35	14.00	1.007	1.016	1.001	14.29
34.99	14.28	1.006	1.013	1.001	14.52
0.018	26.30	1.018	1.023	1.243	22.18
0.085	18.60	1.018	1.023	1.051	18.41
0.013	27.20	1.018	1.023	1.331	21.28
0.167	18.71	1.018	1.023	1.026	19.00

6. *Results and discussions of the results.* The results of the measurements are listed in tables I and II and are given in figures 2, 3 and 4 in which we

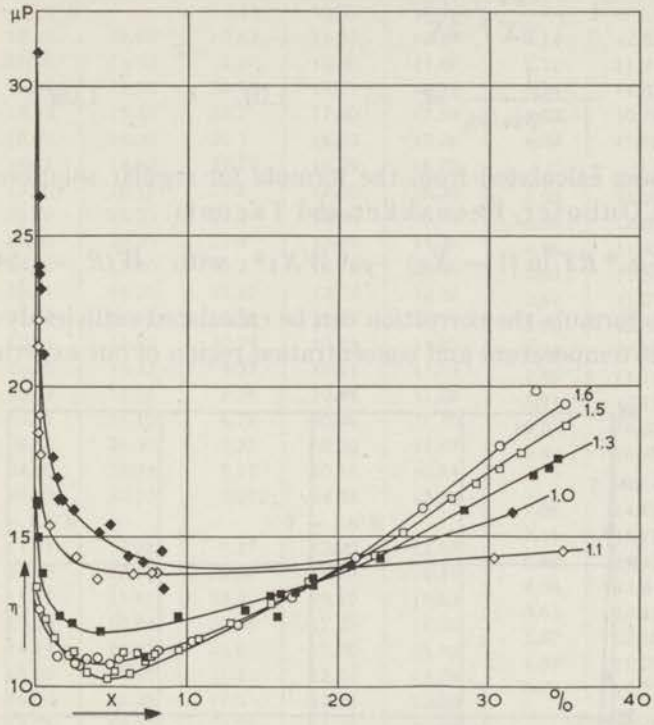

Fig. 3. The viscosity as a function of the concentration at temperatures below 1.6°K

◆ $T = 1.0°K$. □ $T = 1.5°K$
◇ $T = 1.1°K$ ○ $T = 1.6°K$
■ $T = 1.3°K$

drew the curves for constant temperatures. η' is calculated according to equation (3). The corrected value η is calculated from equation (16). The factors

$$\frac{1}{1 - \frac{\partial p_v}{\partial X} \Big/ \frac{\partial p_0}{\partial X}}$$

and

$$\frac{1}{1 - \rho_{3v}/\rho_{3l}}$$

depend slightly on temperature and even much less on the concentration. In order to give an idea of the magnitude of the corrections we give the following table (see also table I):

	$T = 1.0°K$	$T = 2.1°K$
$\dfrac{1}{1 - \dfrac{\partial p_v}{\partial X} \Big/ \dfrac{\partial p_0}{\partial X}} =$	1.01	1.05
$\dfrac{1}{1 - \rho_{3v}/\rho_{3l}} =$	1.01	1.08

$(\partial p_0/\partial X)_T$ was calculated from the formula for regular solutions given by De Bruyn Ouboter, Beenakker and Taconis:

$$p_0 = -\rho_4{}^0\, RT \ln (1 - X_L) - \rho_4{}^0\, W X_L{}^2 \quad \text{with} \quad W/R = 1.54°K.$$ [21]

With this formula the correction can be calculated sufficiently accurately in the whole temperature and concentration region of our experiment.

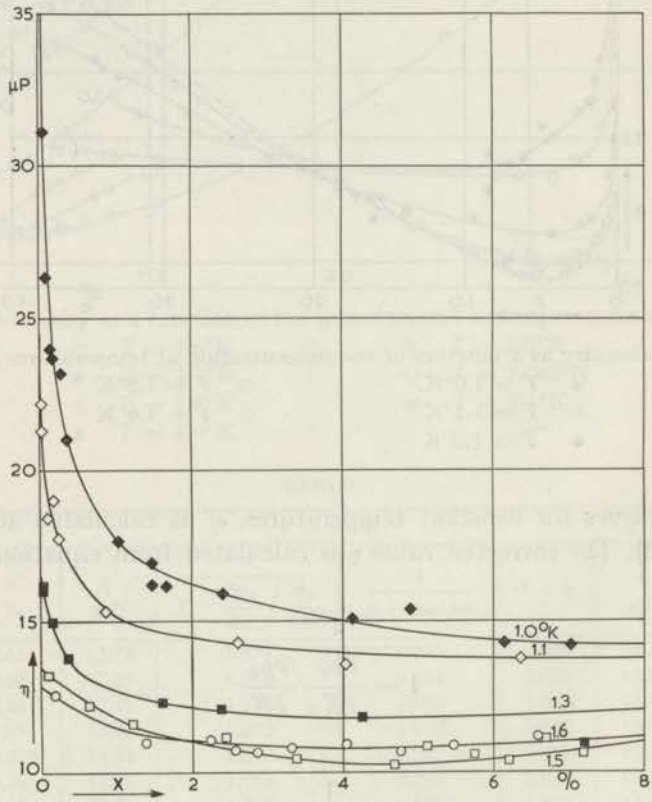

Fig. 4. The viscosity as a function of the concentration at temperatures below 1.6°K. (concentrations below 8%)

◆ $T = 1.0°K$ □ $T = 1.5°K$
◇ $T = 1.1°K$ ○ $T = 1.6°K$
■ $T = 1.3°K$

TABLE II

X %	η' μP	η μP
$T = 1.0°K$		
1.025	17.18	17.61
1.475	16.45	16.92
2.41	15.41	15.90
4.12	14.63	15.10
6.14	13.85	14.30
8.39	14.11	14.53
1.635	15.70	16.15
0.352	20.78	21.00
0.275	23.00	23.20
4.89	14.91	15.40
1.472	15.71	16.20
7.03	13.70	14.19
8.49	12.84	13.22
22.6	14.20	14.43
31.6	15.59	15.81
0.051	33.21	31.10
0.084	26.82	26.30
0.134	24.00	23.98
0.178	23.50	23.70
$T = 1.3°K$		
1.599	11.69	12.31
2.39	11.43	12.08
4.25	11.19	11.81
7.21	10.35	10.94
0.145	14.14	14.97
0.029	16.85	16.18
0.038	16.60	16.05
0.35	13.21	13.80
15.55	12.30	12.95
17.2	12.60	13.14
18.25	12.80	13.38
22.8	13.75	14.29
33.0	16.59	17.08
9.94	11.65	12.28
13.81	11.92	12.53
16.2	12.49	13.07
21.0	13.50	14.03
28.4	15.40	15.91
34.0	16.80	17.26
34.6	17.10	17.57
18.3	13.00	13.59
$T = 1.5°K$		
7.67	10.20	10.95
7.19	9.90	10.62
6.20	9.60	10.36
4.68	9.50	10.22
3.37	10.00	10.43

X %	η' μP	η μP
2.44	10.38	11.15
0.62	11.45	12.21
1.2	10.80	11.60
35.15	18.11	18.73
32.1	17.20	17.88
30.7	16.60	17.26
29.15	16.09	16.77
27.6	15.60	16.30
24.3	14.46	15.13
21.5	13.69	14.36
18.2	12.79	13.50
15.52	12.02	12.75
12.81	11.40	12.14
10.74	10.84	11.57
9.37	10.61	11.38
7.76	10.34	11.08
6.76	10.36	11.10
5.75	10.30	11.07
5.11	10.18	10.84
0.072	14.26	13.30
$T = 1.6°K$		
0.17	12.20	12.57
35.08	18.70	19.41
33.2	19.10	19.83
30.85	17.25	18.00
25.6	15.20	15.95
21.1	13.51	14.29
17.0	12.20	13.00
13.35	11.28	12.09
10.35	10.70	11.51
7.94	10.34	11.19
6.59	10.31	11.18
5.46	10.05	10.89
4.76	9.85	10.68
4.04	10.09	10.92
3.31	9.95	10.78
2.86	9.83	10.63
2.57	9.91	10.73
2.24	10.21	11.06
1.375	10.14	10.97
$T = 1.7°K$		
7.96	10.97	11.90
7.08	10.75	11.70
5.28	10.01	10.91
4.30	10.15	11.08
3.08	10.19	10.77
0.14	12.02	12.45
0.09	12.41	12.43

X %	η' μP	η μP
$T = 1.8°K$		
0.14	12.05	12.57
6.11	11.21	12.31
5.24	11.03	12.12
4.57	10.79	11.86
4.03	10.73	11.81
$T = 1.9°K$		
8.03	13.09	14.36
6.09	12.00	13.21
4.96	11.85	13.09
3.63	11.50	12.70
2.61	11.37	12.52
6.75	12.08	13.30
0.14	12.89	13.87
1.89	11.18	12.34
5.71	12.19	13.41
10.52	14.30	15.64
15.6	16.46	17.88
$T = 2.0°K$		
7.06	14.89	16.41
7.11	15.70	17.35
5.40	14.10	15.63
4.58	13.84	15.37
3.61	13.31	14.79
2.57	12.58	13.92
1.37	12.25	13.58
0.74	11.79	13.04
$T = 2.045°K$		
5.52	15.20	16.88
4.54	15.10	16.77
3.44	14.50	16.12
2.49	13.50	15.02
2.06	13.89	15.47
1.35	13.81	15.39
$T = 2.1°K$		
4.45	18.81	20.93
2.55	16.92	18.19
1.81	15.89	17.76
0.53	14.47	16.25
0.12	15.10	16.58
1.61	14.24	15.96
4.19	18.75	20.86
4.10	17.91	19.98
3.08	16.71	18.62
2.36	15.25	17.01
1.84	15.08	16.82
1.25	15.58	17.41

ρ_{3v} was calculated from $\rho_{3v} = \dfrac{p - p_4{}^0(1 - X_1)}{p_3{}^0} \rho_{3v}{}^0$

with the value of $\rho_{3v}{}^0$ measured by Kerr [22]).

The spread of the data amounts to less than 1% in most of the cases with the exception of some data showing a greater deviation from the curves. In the neighbourhood of the λ-point the deviation is of the order of 6%.

From figures 2, 3 and 4 we derived figure 5: "The behaviour of the viscosity as a function of the temperature". Dash and Taylor [14]) found a minimum in the viscosity for pure ^{4}He: 14.75 μP at a temperature $T = 1.8°$K and

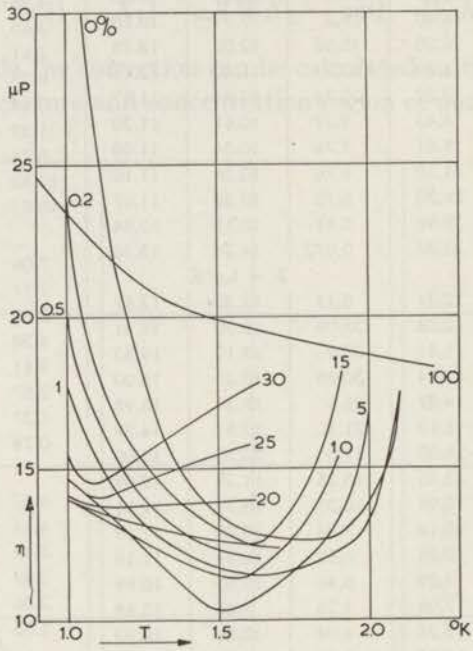

Fig. 5. The viscosity as a function of the temperature. Concentrations: 0, 0.2, 0.5, 1, 5, 10, 15, 20, 25, 30, 100%.

for a concentration of about 10% ^{3}He: 14.45μP at $T = 1.66°$K, whereas our values are 12.65μP extrapolated at $T = 1.8°$K and for 10% ^{3}He: 11.40μP at $T = 1.55°$K. Our minimum shifts to lower temperatures for increasing concentration. This shift is about 3×10^{-2} degree per percent ^{3}He while going from 0 to 5% ^{3}He the viscosity minimum decreases from 12.6μP to 10.3 μP and above $X = 5\%$ increases again by about 0.14 μP per percent ^{3}He.

A very striking result of this study is found in extending the measurements to zero concentration. If one dilutes the mixture and extrapolates the determined viscosity to $X = 0$ it appears very essential to apply the ε correction in order to arrive at the value found by Heikkila and Hollis

Hallett. The ε's however grow to values so large, that they can no longer be considered as corrections.

From the measurement of the level heights and the mean concentration one is able to calculate with (10), (11) and (12) the total quantity of ^{3}He present in one of the vessels at any instant during the flow. The time derivative of this quantity is proportional to the flow velocity of ^{3}He in the capillary and was found to be proportional to the level difference within experimental accuracy and we could conclude the validity of $\bar{v}_3 = -C \operatorname{grad} p$.

In the case of pure ^{4}He the flow of the normal fluid through a capillary with circular cross section according to the two fluid model is given by

$$\bar{v}_n = -\frac{r^2}{8\eta_n} \operatorname{grad} p.$$

From the viscosity measurements by Heikkila and Hollis Hallett we are able to calculate values of $\bar{v}_n$ as a function of grad p. These are in good

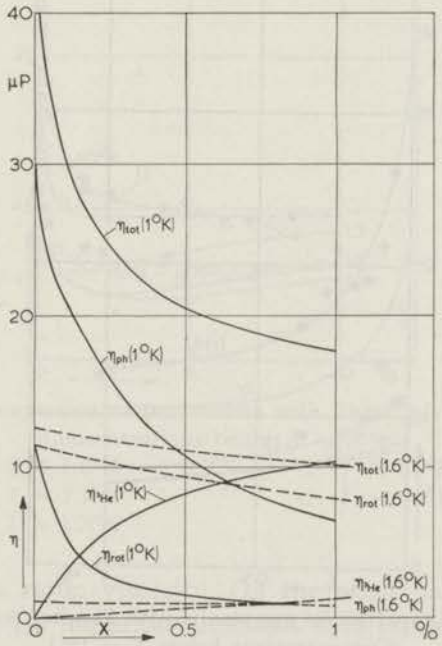

Fig. 6. The viscosity calculated from Zharkov's theory.
——— $T = 1.0°K$
----- $T = 1.6°K$

agreement with our values of $\bar{v}_3$ extrapolated to zero concentration. All of this allows the conclusion that the normal fluid part of the ^{4}He and the ^{3}He together behave as one viscous fluid. The application of the two fluid model is therefore strongly confirmed.

7. *Theoretical consideration.* The behaviour of the viscosity for pure ^{4}He at temperatures below 1.6°K is described by the theory of Landau and Khalatnikov [25][26].

Zharkov [27]) has extended the theory of Khalatnikov for the case of ^{3}He impurities.

According to this theory one can split up the viscosity into three parts:
1) the roton viscosity: η_r
2) the phonon viscosity: η_{ph}
3) the impurity viscosity: η_3,

the measured viscosity being the sum of these parts:

$$\eta = \eta_r + \eta_{ph} + \eta_3.$$

This is a consequence of the basic relation of proportionality of transfer of momentum and drift velocity gradient in case phonon gas, roton gas and

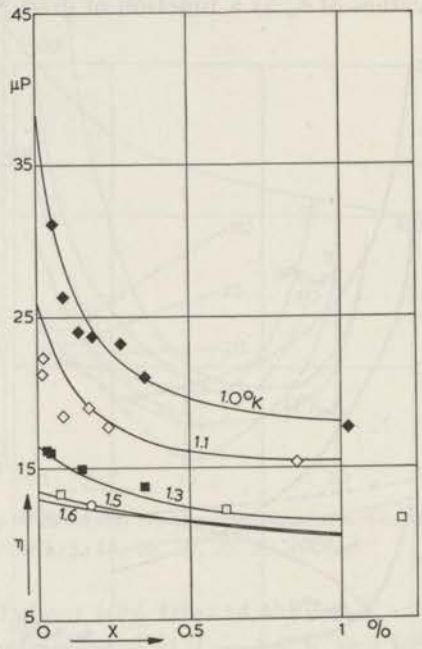

Fig. 7. Comparison of our results with Zharkov's formula.
(Concentrations below $X = 1\%$)
◆ $T = 1{,}0$°K □ $T = 1.5$°K
◇ $T = 1.1$°K ○ $T = 1.6$°K
■ $T = 1.3$°K

impurity gas obey this relation and travel with the same drift velocity at all points in the liquid. The rotons and impurity excitations can be regarded as heavy particles and thus phonons are obstructed in their motion by rotons and impurities. Transfer of momentum by phonons is therefore strongly

affected by the density of rotons and impurities. The decrease of the viscosity with increasing temperature in the case of pure ^{4}He below $T = 1.6°$K is due to the rapid increase of the roton density.

We calculated the three partial viscosities according to the theory of Zharkov. They are shown in fig. 6 in the case of $T = 1.0°$K (full curve) and $T = 1.6°$K (dotted curve).

At 1°K the phonon viscosity decreases strongly with increasing concentration due to the obstruction of the phonons by ^{3}He impurities. The same

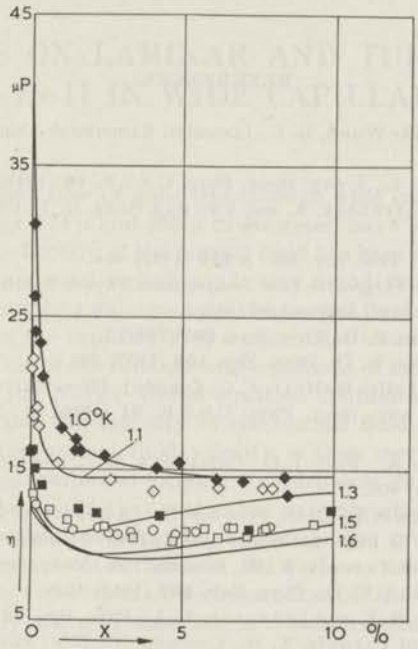

Fig. 8. Comparison of our results with Zharkov's formula.
(Concentrations below $X = 10\%$)

◆ $T = 1.0°$K □ $T = 1.5°$K
◇ $T = 1.1°$K ○ $T = 1.6°$K
■ $T = 1.3°$K

thing occurs to the roton viscosity. On the other hand the transfer of momentum by the ^{3}He impurities increases with increasing concentration and one can expect an increase of the total viscosity for the higher concentrations in case the decrease of η_{ph} and η_r is smaller than the increase of η_3 (see fig. 3). At higher temperatures ($T = 1.6°$K) the phonon viscosity does not change appreciably with variable ^{3}He concentration. This is due to the very high roton density and transfer of momentum by phonons is mainly limited by rotons. At this temperatures we see that the transfer of momentum by ^{3}He excitations is also less than at 1°K because of the density of the rotons.

Though the theory given by Zharkov is only valid for concentrations smaller than 10^{-4} we compared it with our measurements. The constants δ and a in Zharkov's formula depending on phonon-impurity and impurity-impurity interaction respectively are found to be $\delta = 1.86$ and $a = 1.50$. The results of the calculation are compared with experimental data in fig. 7, low concentrations and in fig. 8, high concentrations. The spread of the data in fig. 7 is mainly due to the insensitive determination of the concentration.

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CHAPTER II

EXPERIMENTS ON LAMINAR AND TURBULENT FLOW OF He II IN WIDE CAPILLARIES

Summary

Laminar and turbulent flow of pure ^{4}He through wide capillaries of circular cross section, respectively 82μ, 174 μ and 255 μ in diameter, have been studied in the He II region above 1°K. The viscosity of the normal fluid has been derived from the laminar flow behaviour using a special technique. It was found that the Reynolds number $Re = d\rho v_n/\eta_n$, containing the total density ρ, the normal fluid velocity v_n ,the viscosity η_n and the diameter of the capillary d, is a parameter characterising the flow in the whole velocity range. One of the fundamental equations of motion in the high velocity range appeared to be the Navier Stokes equation containing the total density, the normal fluid velocity and the viscosity of the normal fluid. The dependence of the pressure gradient on the normal fluid velocity is then represented by an equation analogous to the classical empirical formula for turbulent flow of ordinary fluids as given by Blasius. It was found that within the experimental accuracy direction and magnitude of the superfluid velocity did not affect the pressure gradient. In connection with the phenomena of turbulence, the various results on gravitational flow are discussed. Finally an experiment on turbulent flow of ^{3}He–^{4}He mixtures in the He II region, using a capillary 156 μ in diameter, is communicated.

Introduction. According to the microscopic theory of He II the collective thermal motion in the fluid can be considered as a system of excitations which behave like quasi particles moving in the volume occupied by the fluid [1]). A flow of the normal fluid is then to be considered as a flow of the excitation gas in the background fluid which behaves like a superfluid. The normal fluid exchanges momentum with a wall and shows a viscous behaviour, while the superfluid lacks viscosity and is therefore to be considered as an ideal fluid. The motion of the normal fluid involves a certain effective mass indicated by ρ_n. The effective mass of the superfluid is:

$$\rho_s = \rho - \rho_n.$$

The ratio ρ_n/ρ_s is a function of the temperature; ρ_n vanishes at absolute zero and ρ_s vanishes at the λ-temperature.

The macroscopic flow properties of He II are described in terms of the two usually simultaneous motions of the normal fluid and the superfluid

with the respective velocities v_n and v_s. The mass flow density is defined by:

$$\rho v = \rho_n v_n + \rho_s v_s. \tag{1}$$

Introducing the relative velocity $v_r = v_n - v_s$ we can rewrite this equation:

$$\rho v = \rho v_n - \rho_s v_r. \tag{2}$$

As the entropy is determined by the statistical distribution of the elementary excitations entropy can only be transferred by the normal fluid and hence the entropy flow density is given by $\rho S v_n$. Consequently the heat flow density is given by:

$$\varphi = \rho S T v_n. \tag{3}$$

The heat conduction of He II in a capillary or slit is described by a flow of the normal fluid from the high temperature side to the low temperature side without net mass transfer; therefore at the same time a counterflow of the superfluid takes place in such a way that equation (1) is satisfied with v averaged over the cross section equal to zero. Due to the viscous behaviour of the normal fluid a heat flow in He II is accompanied by a pressure gradient.

Simultaneous measurements of the pressure gradient and the temperature gradient associated with pure heat conduction by He II in a capillary or slit have been carried out by several investigators [2) 3) 4) 5)]. In general, below a certain critical value of the heat flux, these experiments have shown a linear dependence of the temperature gradient on the heat flux. In this region within experimental error London's equation [6)]

$$\text{grad}\, p = \rho S \,\text{grad}\, T \tag{4}$$

is confirmed. Above the critical heat input, an approximately cubic dependence of the temperature gradient on the heat flux φ is found [7)] and the validity of London's equation breaks down. Gorter and Mellink [8)] have attributed the cubic dependence of the temperature gradient on the heat flux to the appearance of a mutual friction between the normal fluid and the superfluid at relative velocities exceeding the critical velocity. As the heat flux is related linearly with the relative velocity, according to equation (2) and (3)

$$\varphi = \rho_s S T v_r$$

from the cubic relation between the temperature gradient and the heat flux φ, the cubic dependence of the mutual friction on the relative velocity was concluded. Below the critical relative velocity no mutual friction is observed [9)].

In general the aforementioned experiments have also shown a linear dependence of the pressure gradient on the heat flux in a wide range independent of the presence of mutual friction [10)]. This experimental fact

is known as the rule of Allen and Reekie. Since the heat flux is proportional to the normal fluid velocity v_n, it is this rule which justifies the description of the normal fluid flow with a coefficient of viscosity η_n, analogous to Poiseuille's law.

However, deviations from the rule of Allen and Reekie in the high velocity range have also been found by various investigators [10][11]), and a reasonable explanation was lacking up to now. The most important aim of our research was to clear up these departures. The very striking result is that the deviations from Allen and Reekie's rule are analogous to the deviations from Poiseuille's law due to turbulence in an ordinary fluid and hence that the flow behaviour of the fluid is quite similar to the behaviour of an ordinary fluid also in the case of turbulence.

This chapter consists of three parts. The first deals with the determination of the normal fluid viscosity of ^{4}He from laminar flow in wide capillaries and in the second part the turbulent flow behaviour of ^{4}He at higher velocities in the same capillaries is discussed. In the third part is described an early experiment on turbulent flow of liquid ^{3}He–^{4}He mixtures through a capillary.

I. LAMINAR FLOW

1. *Viscosity of the normal fluid.* The relation between the velocity and the pressure gradient, for a capillary with circular cross-section of radius r is given, according to Poiseuille's law by:

$$\bar{v}_n = -\frac{r^2}{8\eta_n} \operatorname{grad} p \tag{5}$$

where $\bar{v}_n$ is the mean value of the normal fluid velocity over a cross-section of the capillary.

The relation between the heat current

$$W = \pi r^2 \varphi$$

and the pressure gradient can be derived by using (3) and (5):

$$W = -\frac{\pi r^4}{8\eta_n} \rho S T \operatorname{grad} p. \tag{6}$$

If we are dealing with a relative velocity smaller than the critical velocity and thus no mutual friction is present, then equation (4) holds and equation (6) can be transformed into:

$$W = -\frac{\pi r^4}{8\eta_n} \rho^2 S^2 T \operatorname{grad} T. \tag{7}$$

Equation (6) and (7) are commonly used to determine the viscosity coefficient η_n of the normal fluid from a heat conduction experiment. However the use of equation (7) has in practice a great disadvantage. The value of the critical velocity decreases with increasing diameter of the capillary or slit. Because of the very small critical velocity in wide capillaries, the subcritical region is limited to very small temperature differences and therefore in choosing the proper capillary the upper limit of the diameter is determined by the sensitiveness of the temperature measurement. This implies that only capillaries or slits of small diameter can be used.

This in turn gives rise to the following complications:

i) it is difficult to make a homogeneous narrow slit or capillary unless one uses special techniques. The practicability of a calibration to obtain the slit's geometry is usually very difficult.

ii) since the mean free path of the phonons increases with decreasing temperature it is possible that the mean free path becomes comparable with the channel's size and the phonon contribution to the effective viscosity is decreased by the mean free path effect.

iii) the heat conductivity of the material of which the capillary or slit has been made becomes comparable with the heat conductivity of the helium. According to equation (7) the gradient of the temperature is not constant along the channel, but depends on the temperature as the reverse of S^2T. So if the temperature gradients in the wall and in the helium are different one can expect an exchange of heat between the helium and the boundary material. Actually, heat is transferred from the helium to the wall over the entire length of the channel.

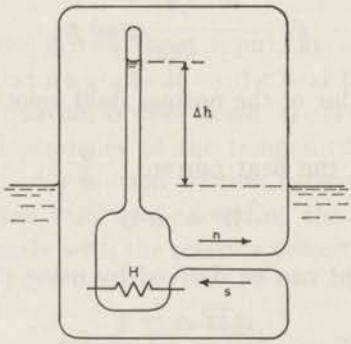

Fig. 1. Schematic diagram of a flow experiment in He II (heat conduction method).

Another possibility to measure the viscosity is to derive it straightforward from equation (6). The experimental arrangement is as follows (see fig. 1): A heater H is put in a thermally insulated vessel which is filled with liquid He II. The vessel is connected with the bath by means of a calibrated capillary and is supplied with a manometer tube in which a level can be

observed. If the heater is switched on, a level difference h is shown corresponding with a pressure difference given by equation (6). According to Allen and Reekie's rule the pressure difference increases linearly with the heat input. However, as soon as the critical heat input is exceeded the temperature difference between the helium in the vessel and the helium of the bath increases very fast with increasing heat input due to the mutual friction between the normal fluid and the superfluid. The vapour pressure difference corresponding with the temperature difference of both levels becomes very soon much greater than the pressure difference which corresponds with the laminar flow of the normal fluid. In this case the level difference is reversed and the resulting pressure difference has to be obtained by subtracting the pressure difference due to the liquid helium column from the vapour pressure difference. This subtraction of two pressure differences of about the same order of magnitude leads of course to inaccurate results.

All difficulties appearing in the measurements mentioned above are connected with the phenomenon of mutual friction between the normal fluid and the superfluid. In order to be able to study the flow accurately we eliminated the effect of mutual friction by using a special technique described hereafter.

2. *Normal fluid flow in wide capillaries without effective mutual friction.* Let us now study the case in which we connect the vessels not only through the capillary but parallel to it also through a superleak S to the bath (see fig. 2). From other experiments we know that the pressure differences and the temperature differences across a superleak are always related to each other by the equation of London [3][4][12][13].

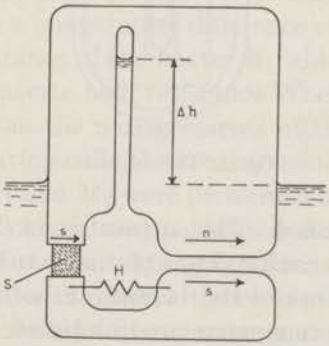

Fig. 2. Schematic diagram of a flow experiment in He II. (heat conduction without mutual friction).

When the heater is switched on the normal fluid flows through the capillary to the helium bath. Because of the very narrow channels in the superleak the normal fluid flow in the superleak is negligibly small. The critical

velocity in the superleak is very high (of the order of 50 cm/s) and if the capacity of the superleak is large enough, the superleak is able to supply the required superfluid flow to avoid the appearance of the extra temperature difference which would arise from mutual friction by a counterflow in the capillary. Indeed, a counterflow of superfluid will no longer appear, and the superfluid in the capillary will now move with the normal fluid in the same direction such that no effective mutual friction occurs. All the difficulties arising from mutual friction mentioned in the preceding section have been avoided and it is even possible to study turbulent flow by using this method.

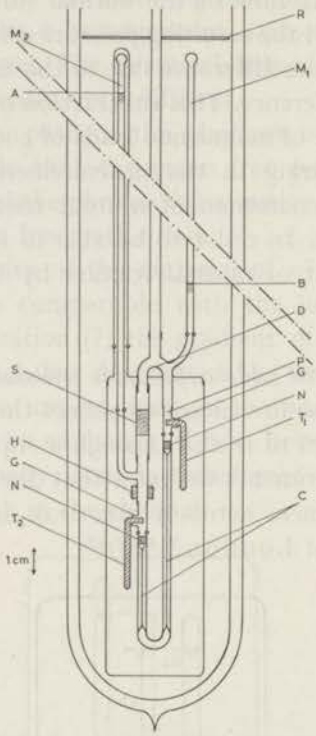

Fig. 3. The apparatus.

3. *Apparatus and method.* The apparatus is illustrated in fig. 3. The capillary C was sealed at each end to a platinum tube P. In order to measure accurately the temperature of the helium at both ends of the capillary it is necessary that the thermometers are in intimate contact with the liquid; moreover the body of the thermometer should be rather free from the wall, which may have a slightly different temperature due to the small heat flow through the wall of the platinum tube. Therefore into each of the platinum tubes thin walled germansilver side tubes G were soldered, which served as holders for the "De Vroomen-type" carbon thermometers T_1 and T_2. The copper bodies of the thermometers were soldered only at the end of the

side tube, thus thermally "isolating" the thermometers from the platinum wall. The superleak consisted of a thin walled stainless steel tube, 5 millimeters in diameter, containing jewellers rouge powder highly compressed by hammering. The capillary was connected with the superleak by means of a copper piece H around which a heating coil was wound. A manometer tube M_2 was mounted between the heater and the superleak. Filters of fine nylon fabric N protected the capillary against the entrance of impurities. The whole was enclosed in a vacuum jacket. The other end of the capillary was connected with the upper end of the superleak by means of a copper tube on which the manometer tube M_1 was mounted. As the heat transferred through the capillary should flow off to the helium bath, the upper part of the wide copper tube was immersed in the bath to ensure a good thermal contact.

Before the helium gas was condensed in the apparatus it was purified by leading it successively through a tube filled with charcoal at liquid nitrogen temperature and through a spiral capillary in a separate cryostat with helium at the normal boiling point.

After the evacuation of the vacuum jacket the thermometers T_1 and T_2 were calibrated against the vapour pressure of the helium bath in steps of 0.05 degree. In order to check whether during the calibration the temperatures of the helium inside the apparatus and of the bath were the same, a differential oil manometer was placed on top of the cryostat. No measurable difference in vapour pressure was observed. The measurements of the resistances of the carbon thermometers were performed with the aid of a high resistance d.c. Wheatstone bridge using a Philips electronic d.c. millivoltmeter with a sensitivity of 10^{-6} volt per scale division as a zero instrument. Using a thermometer power of 3.5×10^{-7} watt, one scale division corresponded to a temperature difference of about 3×10^{-5} degrees at $T = 1.7°K$. The resistance of the heater H, which was about 1030 Ohm, was measured with a separate high resistance Wheatstone bridge. The heat input was calculated from the measurements of the electrical current and the resistance of the heating coil. The measurements of the level heights in the manometer tubes M_1 and M_2 were performed by cathetometer readings.

The capillary C was chosen and calibrated carefully by the following procedure: From a great number of long capillaries pieces were cut off with perpendicular ends in order to observe the cross-section with a microscope. If the cross section was found to be circular and of a suitable diameter a drop of mercury was put in the capillary. The length of the mercury thread was determined by means of a measuring microscope. A piece of the capillary was chosen in which the length varied less than 1%; this was cut out and blown in the form shown in fig. 3. In order to obtain an accurate calibration of the capillary, each end of it was connected to one of the two identical cylindrical glass vessels of known diameter $2R$, and a flow experiment

was done with mercury. The vessels were partially filled with mercury whereupon the level difference h of the mercury in the vessels was measured as a function of the time t. From the laminar flow of the mercury we obtained the constant r^4/l which could be calculated according to the formula:

$$\ln h = -\frac{r^4 \rho g}{4 l \eta R^2} t + \text{constant}$$

in which r and l are the radius and the length of the capillary and ρ and η the density and viscosity of mercury. The length of the capillary was measured independently. The results of the calibrations are given in the following table:

	r^4/l	l	d
capillary I:	2.297×10^{-11} cm³	12.29 cm	82.0×10^{-4} cm
capillary II:	4.436×10^{-10} cm³	12.76 cm	173.5×10^{-4} cm
capillary III:	2.213×10^{-9} cm³	11.97 cm	255.2×10^{-4} cm

4. *Measurements and results on laminar flow.* In a stationary state measurements of the level difference h and the temperature difference ΔT across the capillary were performed simultaneously as a function of the heat input, while the temperature T_1 was kept constant by regulating the bath temperature.

As could be expected on account of the presence of the superleak, the measured pressure differences and the temperature differences across the capillary were always found to be related within experimental error by London's equation:

$$\Delta p = \int_{T_1}^{T_2} \rho S \, \mathrm{d}T \simeq \rho S \Delta T. \tag{8}$$

A typical example of the linear dependence of Δp on ΔT is illustrated in fig. 4. A correction had to be applied to the hydrostatic pressure difference $\rho g h$ due to the vapour pressure difference corresponding to the temperature difference. If we denote the vapour pressure by p_v, the corrected pressure difference is:

$$\Delta p = \rho g h + \frac{\mathrm{d}p_v}{\mathrm{d}T} \Delta T.$$

After eliminating ΔT with (8) one obtains:

$$\Delta p = \rho g h \left(1 + \frac{\mathrm{d}p_v/\mathrm{d}T}{\rho S - \mathrm{d}p_v/\mathrm{d}T} \right). \tag{9}$$

A typical example of the linear relation between h and W at low heat inputs is shown in fig. 4. From the slope of the straight line we calculated the

viscosity according to eqs. (6) and (9):

$$\eta_n = \frac{\pi r^4}{8l} \rho^2 S T g \left(1 + \frac{dp_v/dT}{\rho S - dp_v/dT} \right) \frac{h}{W} .$$

The viscosities have been measured with capillaries of 255 μ, 173.5 μ and 82 μ in diameter. The results are shown in fig. 5 and are given in table I. Above 1.7°K the viscosities thus obtained do not agree with the viscosities measured by the rotating cylinder method [14]), indicated in fig. 5 by a

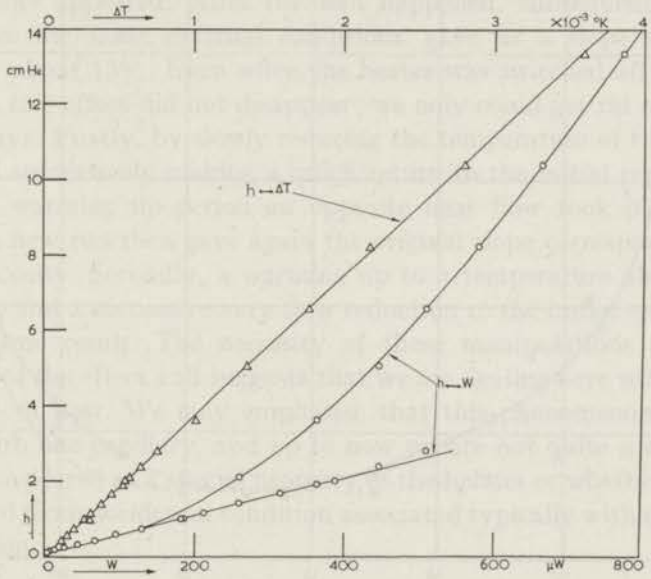

Fig. 4. The total heat input W and the temperature difference ΔT as a function of the level difference h at a temperature of 1.703°K, using a capillary 173.5 μ in diameter.

TABLE I

d = 173.5 μ		d = 82.0 μ		d = 255.2 μ	
T(°K)	η(μP)	T(°K)	η(μP)	T(°K)	η(μP)
1.019	34.00	1.394	14.61	1.012	31.94
1.083	19.97	1.405	14.51	1.189	17.19
1.136	19.15	1.500	12.59	1.405	14.50
1.190	17.87	1.506	13.31	1.804	12.83
1.280	16.14	1.598	12.90	1.900	14.10
1.404	14.59	1.602	12.89	2.008	16.13
1.496	13.32	1.605	12.89		
1.598	13.00	1.903	14.15		
1.703	12.60	1.999	15.81		
1.703	12.73	2.003	15.95		
1.803	13.34	2.100	19.41		
1.899	14.10	2.109	19.78		
1.999	15.85	2.135	21.32		
2.104	19.57				
2.142	21.98				

dotted curve, but on the other hand, they support the measurements by the oscillating disk method in the vicinity of the λ-point [15]). Below 1.7°K a good agreement was found with the rotating cylinder method. The viscosities derived by Zinov'eva from the attenuation of second sound in a resonant cavity are also shown in the figure [16]).

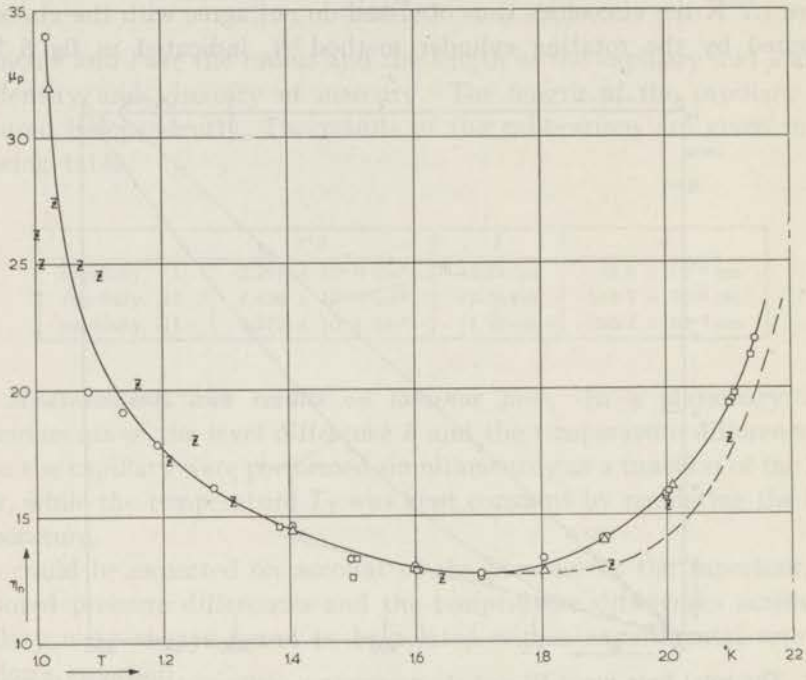

Fig. 5. The viscosity of the normal fluid as a function of the temperature.

□: $d = 82\ \mu$ Z: Zinov'eva.
○: $d = 173.5\ \mu$ ----: Heikkila andHollis Hallet.
△: $d = 255\ \mu$

The viscosity data are less accurate near 1.1°K for various reasons:

i) in this region the accuracy of the thermometer calibration is small due to the low vapour pressure against which the calibration takes place, and since the coëfficient in eq. (6) is strongly temperature dependent, a small error in the absolute temperature gives rise to a great error in the calculated viscosity.

ii) the viscosity increases strongly with decreasing temperature.

iii) in spite of the precautions made to reduce radiation effects, there was always a small constant heat leak due to absorption of radiation in manometer tube M_2.

As at low temperatures the normal fluid density is small, the normal fluid velocity is already high due to radiation only and therefore the region for

laminar flow becomes relatively small. Clearly this limitation is more pronounced in the capillary of the smallest diameter.

Sometimes at temperatures near 1.6°K a higher viscosity was found with the capillary of 82 μ in diameter. It differed from the real viscosity by about 15%. This has to be attributed to a kind of turbulence. In the graph of the level difference h against the total heat input W, we initially found a linear dependence between h and W with a slope corresponding to the real viscosity, but after a certain rapid increase of the heat input a sudden jump of the level difference appeared. After this had happened, subsequent measurements under the same external conditions gave us a slope which was enlarged by about 15%. Even when the heater was switched off, for about 45 minutes, this effect did not disappear; we only could get rid of it in two different ways: Firstly, by slowly reducing the temperature of the bath for a while and successively making a quick return to the initial temperature. During the warming up period an opposite heat flow took place in the capillary. A new run then gave again the original slope corresponding with the real viscosity. Secondly, a warming up to a temperature above the λ-temperature and a successive very slow reduction to the initial temperature gave the same result. The necessity of these manipulations shows the persistancy of the effect and suggests that we are dealing here with a rather stable kind of flow. We may emphasize that this phenomenon was only observed with one capillary, and up to now we are not quite sure whether it is to be considered as a special property of the helium or whether it should be attributed to an accidental condition associated typically with the capillary in question.

II. TURBULENT FLOW

1. *Introduction.* Deviations from the rule of Allen and Reekie were observed in case of a flow at higher velocities (see fig. 4). In order to understand these deviations, we consider the flow of an ordinary incompressible viscous fluid which is governed by the Navier Stokes equation of motion:

$$\rho \left\{ \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \mathrm{grad})\, \boldsymbol{v} \right\} = - \mathrm{grad}\, \mathrm{p} + \eta \nabla^2 \boldsymbol{v}.$$

Assuming stationary laminar flow we can write $\partial \boldsymbol{v}/\partial t = 0$ and the equation becomes:

$$\rho (\boldsymbol{v} \cdot \mathrm{grad})\, \boldsymbol{v} = - \mathrm{grad}\, p + \eta \nabla^2 \boldsymbol{v}. \tag{10}$$

An exact general solution of this equation is only possible in the case where the acceleration term on the left hand side is zero or vanishingly small. The equation then expresses equilibrium between the pressure gradient as the driving force and the viscous drag force. Together with the condition

of no slip at the wall, this leads to Poiseuille's equation for laminar flow.

In the case where the acceleration term has to be taken into consideration some more information concerning the flow can be obtained by applying the "principle of dynamical similarity" *i.e.* by considering two geometrical similar flow patterns. (For example the flow patterns in two different but *geometrical similar* channels with characteristic lengths l_1 and l_2, respectively).

Let us assume that the velocity and the pressure gradient in an arbitrary point of the velocity field of the first system equals v_1 and $\text{grad}_1 p$. Let η_1 and ρ_1 be the viscosity and the density of the fluid present in this system. In the corresponding point of the second system (with characteristic length l_2) these values are: v_2, $\text{grad}_2 p$, η_2 and ρ_2. Because of the similarity, the two sets of the values mentioned above correspond with identical solutions of equation (10). After insertion of these values in the Navier Stokes equation we obtain two conditions which should be simultaneously satisfied to yield identical solutions. These conditions may *e.g.* be written:

$$\frac{\rho_1 v_1 l_1}{\eta_1} = \frac{\rho_2 v_2 l_2}{\eta_2} \quad \text{and} \quad \frac{\rho_1 l_1^3 \,\text{grad}_1 p}{\eta_1^2} = \frac{\rho_2 l_2^3 \,\text{grad}_2 p}{\eta_2^2} \tag{11}$$

(or in any other form obtained from arbitrary combinations of the two conditions mentioned above).

Obviously the pattern of the flow governed by the equation (10) is always characterized by the geometry of the system (for instance the cross-section of the capillary) and *e.g.* the numbers:

$$Re_v = \frac{2r\rho}{\eta}\bar{v} \quad \text{and} \quad Re_p = \frac{\rho r^3}{4\eta^2}\,\text{grad}\, p. \tag{12}$$

Apart from a constant these numbers are the same as the numbers in (11). The introduction of $\bar{v}$, the mean velocity over a cross section, also affects the numbers (11) only by a constant factor, because the ratio of the local velocity to the mean velocity in a point of one system is equal to that in the corresponding point of the similar system.

Hence the relation between the pressure gradient and the velocity which characterizes the flow through channels with similar cross-sections is given by an equation of the form:

$$Re_p = f(Re_v).$$

In the case of a laminar flow through a channel with circular cross-section this relation becomes:

$$Re_p = Re_v \tag{13}$$

which is equivalent to Poiseuille's law.

A complete study of the transition from laminar flow to turbulent flow has been made by Reynolds using glass tubes with flared entrances.

Reynolds found a critical velocity below which there was always a laminar flow. The value of the critical velocity at which the eddies in originally turbulent flow die out corresponds to the Reynolds number $Re = 2r\rho\bar{v}/\eta = 2320$ in the case of a tube with circular cross-section.

The relation between Re_v and Re_p for a turbulent flow may be obtained from the empirical equation given by Blasius [17]) and is then represented by *):

$$Re_p = 4.94 \times 10^{-3}\, Re_v^{1.75}. \tag{14}$$

2. *The principle of dynamical similarity applied on He II.* As proposed by Gorter and Mellink [8]), the equations governing the flow of the superfluid respectively the normal fluid may be written:

$$\rho_s \frac{d\boldsymbol{v}_s}{dt} = -\frac{\rho_s}{\rho} \operatorname{grad} p + \rho_s S \operatorname{grad} T - A\rho_s\rho_n |\boldsymbol{v}_s - \boldsymbol{v}_n|^2 (\boldsymbol{v}_s - \boldsymbol{v}_n) \tag{15}$$

and

$$\rho_n \frac{d\boldsymbol{v}_n}{dt} = -\frac{\rho_n}{\rho} \operatorname{grad} p - \rho_s S \operatorname{grad} T + \eta_n \nabla^2 \boldsymbol{v}_n + A\rho_s\rho_n |\boldsymbol{v}_s - \boldsymbol{v}_n|^2 (\boldsymbol{v}_s - \boldsymbol{v}_n). \tag{16}$$

Addition of these two equations gives:

$$\rho_s \frac{d\boldsymbol{v}_s}{dt} + \rho_n \frac{d\boldsymbol{v}_n}{dt} = -\operatorname{grad} p + \eta_n \nabla^2 \boldsymbol{v}_n. \tag{17}$$

In the case that the relative velocity $\boldsymbol{v}_r = \boldsymbol{v}_n - \boldsymbol{v}_s$ is small compared with the velocities $\boldsymbol{v}_n$ and $\boldsymbol{v}_s$ this equation transforms into:

$$\rho \frac{d\boldsymbol{v}_n}{dt} = -\operatorname{grad} p + \eta_n \nabla^2 \boldsymbol{v}_n \tag{18}$$

which is exactly the equation of Navier Stokes with $\boldsymbol{v}$ and η replaced by $\boldsymbol{v}_n$ and η_n and we may expect that the dependence of the pressure gradient on the normal fluid velocity for the flow through a capillary with circular cross-section is given by:

$$Re_p = Re_{v_n} \tag{19}$$

and

$$Re_p = 4.94 \times 10^{-3}\, Re_{v_n}^{1.75} \tag{20}$$

analogous to (13) and (14), in which $\bar{v}$ and η are replaced by $\bar{v}_n$ and η_n.

We may still emphasize that in this analysis we deliberately excluded the possibility of interaction of the superfluid with the wall.

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*) The dimensionless quantities Re_p and Re_v defined by (12) are different from the dimensionless quantities originally used by Blasius. We especially introduced the quantities Re_p and Re_v, because the first contains the pressure gradient only and the second the velocity only, so that each of them represents only one direct measured quantity.

3. *Measurements and results on turbulent flow.* We found that the pressure differences and temperature differences across the capillary obeyed the equation of London even in the high velocity range. The capacity of the superleak was large enough in the whole temperature range except at temperatures very near the λ-point ($T > 2.15°K$), when dealing with high velocities. Therefore we may assume that the mean relative velocity $\bar{v}_n - \bar{v}_s$ in the capillary does not exceed the critical velocity and as the critical velocity in these wide capillaries is small, eq. (18) should be valid.

From the measurement of the level difference (see fig. 3) we calculated:

$$Re_p = \frac{\rho r^3}{4\eta_n^2} \operatorname{grad} p.$$

With eq. (9) we obtained:

$$Re_p = \frac{\rho^2 r^3 g}{4\eta_n^2 l}\left(1 + \frac{\mathrm{d}p_v/\mathrm{d}T}{\rho S - \mathrm{d}p_v/\mathrm{d}T}\right) h.$$

It was also possible to calculate Re_p from the measurements of the temperature using London's equation:

$$Re_p = \frac{\rho^2 r^3}{4\eta_n^2 l}\int_{T_1}^{T_2} S\,\mathrm{d}T.$$

From the total heat input we calculated:

$$Re_{v_n} = \frac{2r\rho\bar{v}_n}{\eta_n}.$$

With eq. (3) we obtained:

$$Re_{v_n} = \frac{2W}{\pi r \eta_n S T}. \tag{21}$$

It should be pointed out that the calculated number Re_p is an average over the length of the capillary, since we replaced grad p by $\Delta p/l$. The correct value of Re_v, corresponding with the thus obtained Re_p, should be found by averaging $Re_v^{1.75}$ over the length of the capillary. In our experiment, however, only such small temperature differences occurred, that it was justified to use the data corresponding with the high temperature side to calculate Re_{v_n}. This may be seen as follows:

We can write:

$$\operatorname{grad} p \propto \frac{\eta_n^2}{r^3\rho}(Re_{v_n})^{1.75}$$

and with eqs. (4) and (21):

$$\operatorname{grad} T \propto \frac{\eta_n^2}{r^3\rho^2 S}\left(\frac{2W}{\pi r \eta_n S T}\right)^{1.75} \propto T^{-17}.$$

This shows that the temperature gradient is smallest at the high temperature side, thus the average value of $Re_{v_n}^{1.75}$ is very near to the value of $Re_{v_n}^{1.75}$ at the high temperature side.

The results are shown in the figures 6 and 7. Equations (19) and (20) corresponding with respectively laminar flow and turbulent flow are represented by the solid lines. For ease in distinguishing the several temperature regions we have shifted the scale of Re_p in the logarithmic plot. As one can see, the agreement with the empirical formula of Blasius for ordinary liquids is striking. However, contrary to ordinary fluids, where

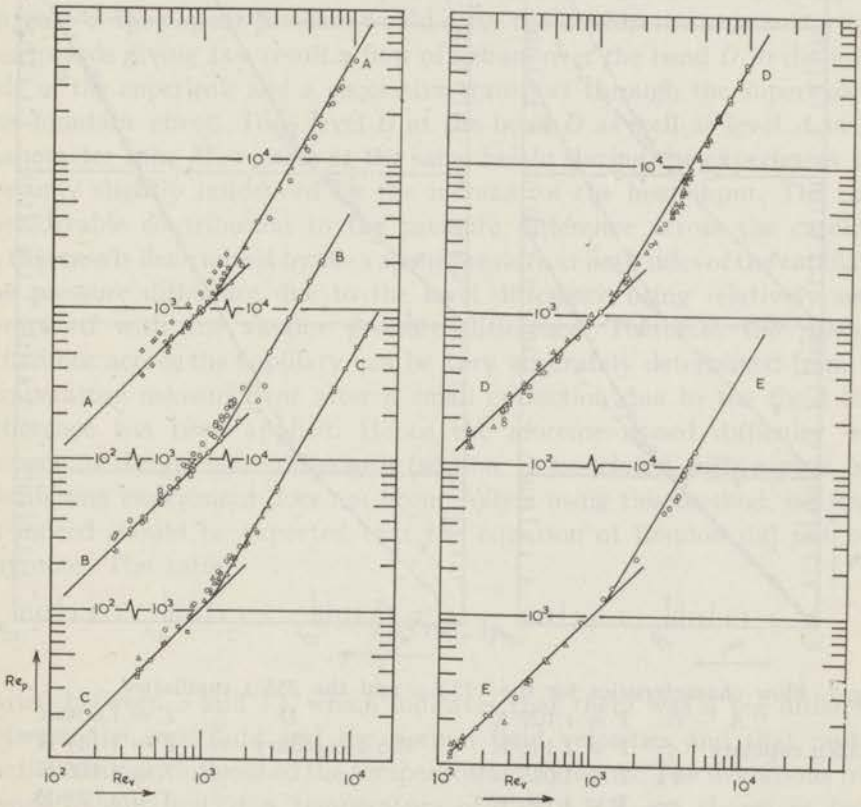

Fig. 6. Flow characteristic for the 82 μ capillary.

A	○:	$T = 1.292°K$	D	○:	$T = 2.001°K$
	△:	$T = 1.394°K$		△:	$T = 2.100°K$
	◇:	$T = 1.503°K$			
	□:	$T = 1.405°K$	E	○:	$T = 2.109°K$
				△:	$T = 2.135°K$
B	○:	$T = 1.602°K$			
C	○:	$T = 1.703°K$			
	△:	$T = 1.802°K$			
	□:	$T = 1.903°K$			

the value of the lowest critical Reynolds number equals 2300, here all the diagrams show that this number is much smaller, $Re_v = 1200$, and is, within experimental error, the intersection point of the two branches; whereas with increasing velocity it is possible to establish laminar flow up to Reynolds numbers as high as 5000.

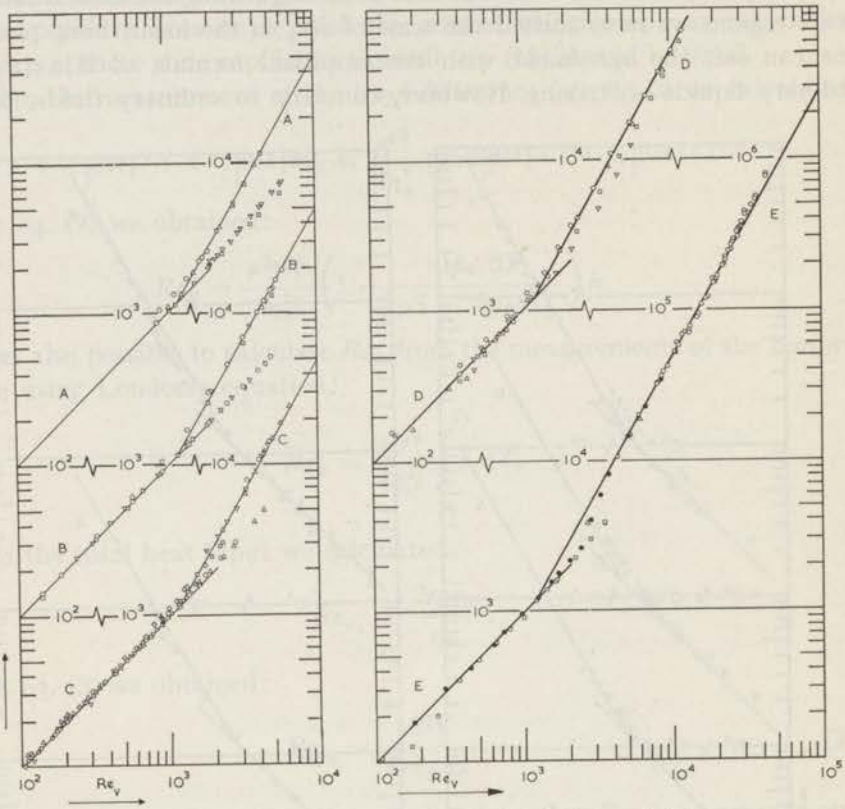

Fig. 7. Flow characteristics for the 173,5 μ and the 255 μ capillaries.

A	○:	$T = 1.109°K$	D	○:	$T = 1.604°K$
173.5 μ capillary	△:	$T = 1.136°K$	255 μ capillary	□:	$T = 1.189°K$
	▽:	$T = 1.083°K$		△:	$T = 1.405°K$
	□:	$T = 1.190°K$		▽:	$T = 1.022°K$
B	○:	$T = 1.496°K$	E	○:	"overflow method"
173.5 μ capillary	□:	$T = 1.280°K$	255 μ capillary	●:	$T = 2.008°K$
	▽:	$T = 1.404°K$		□:	$T = 1.807°K$
				△:	$T = 1.900°K$
C	□:	$T = 1.999°K$			
173.5 μ capillary	△:	$T = 1.703°K$			
	▽:	$T = 1.803°K$			
	◇:	$T = 2.104°K$			
	○:	$T = 1.598°K$			
	▷:	$T = 1.899°K$			

4. *The "overflow method"*. In order to investigate whether indeed the relative velocity should be equal to and not surpasses the critical velocity for the validity of eq. (20), we also performed measurements with large relative velocities. For this purpose the same apparatus was used; the only difference in technique being that the apparatus is initially filled with helium up to the bend D of the copper capillary (see fig. 3). After supplying heat with the heater H, the space above the superleak is emptied into the manometer tube M_2 due to the fountain effect across the superleak. A further increase of the heat input gives rise to an increase of the temperature and an allied increase of the vapour pressure in manometer tube M_2. An increase of the vapour pressure would drive the level in the manometer tube downwards giving as a result a flow of helium over the bend D to the upper side of the superleak and a successive transport through the superleak by the fountain effect. Thus level B at the bend D as well as level A in the manometer tube M_2 remain at the same height during the experiment and are only slightly influenced by the increase of the heat input. The most considerable contribution to the pressure difference across the capillary in this case is determined by the vapour pressure at both sides of the capillary; the pressure difference due to the level difference being relatively small compared with the vapour pressure difference. Therefore the pressure difference across the capillary can be very accurately determined from the temperature measurement after a small correction due to the fixed level difference has been applied. Hence the aforementioned difficulty with respect to the pressure difference (section 1) associated with a pure heat conduction experiment does not occur. When using this method, we found as indeed should be expected that the equation of London did not hold anymore. The ratio

$$\int_{T_1}^{T_2} \rho S \, \mathrm{d}T / \Delta p$$

varied between 3 and 12, which indicates that there was a big difference between the superfluid and the normal fluid velocities and that mutual friction strongly influenced the temperature differences. The deviations from London's equation at a temperature of $T = 1.7°\mathrm{K}$ are shown in fig. 8, where Δp and ΔT are plotted logarithmically. The slightly curved full drawn line A represents London's equation $\Delta p = \int \rho S \, \mathrm{d}T$, whereas curve B represents the measurements with the "overflow method" at the same temperature. The shift of the curve corresponds with a factor of 10 in the pressure difference.

Nevertheless we found that the results still satisfied eq. (20). They are shown in fig. 7, indicated by "overflow method" .We may emphasize that Reynolds numbers Re_v up to values as high as 35000 have been reached by this method, while the velocities of the superfluid and the normal fluid

were very different. These results therefore prove that there is no restrictive condition with respect to the relative velocity for eq. (20) to be valid.

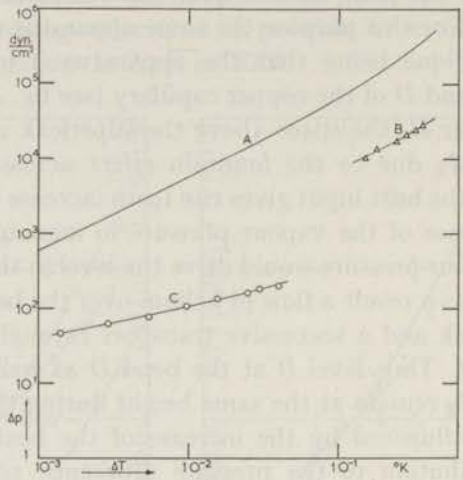

Fig. 8. Comparison of the measured pressure differences obtained from the respective methods at a temperature $T = 1.7°K$, using the 255 μ capillary.

A. without effective mutual friction.
B. "overflow method".
C. "pure heat conduction" method.

5. *The "pure heat conduction method"*. Subsequently we have completed our research on turbulence by considering the case of no net mass transfer. For this purpose we removed the superleak S from the apparatus and closed the ends of the tubes between which the superleak previously had been located. As we mentioned in the introductory part of this paper, such a heat conduction experiment in a tube filled with Helium II is described by a normal fluid flow, which transfers the heat, accompanied with a counterflow of superfluid in such a way that there is no net mass transfer. The counterflow of the superfluid gives rise to mutual friction resulting in the break-down of London's relation. According to Keesom and Saris [7]), the temperature gradient and the heat flux are now related by the empirical equation:

$$\varphi = \text{constant} \times T^5 \, (\text{grad}\, T)^{1/3}$$

from which one can see that the temperature gradient increases much faster with increasing heat input than in the case without mutual friction.

We measured the temperatures T_2 and the level differences h as a function of the heat input keeping the temperature T_1 constant. The effective pressure difference is given by:

$$\Delta p_{\text{eff}} = \Delta p_{\text{v}} + \rho g h$$

where Δp_v denotes the vapour pressure difference. The observed level difference h appeared to be in general negative. With our apparatus it was possible to make suitable measurements in a restricted temperature interval only because:

i) Δp_{eff} should be measurable with enough accuracy (see section 1).

ii) the critical Reynolds number should be exceeded appreciably, for we were especially interested in the turbulent region. These conditions imply that the heat conductivity must be as high as possible and at the same time the entropy, which enters in the Reynolds number (21), as low as possible.

The results of the measurements performed with the capillary of 255 μ in diameter at a temperature of 1.703°K are given in table II.

TABLE II

W μW	T_2 °K	h mm He	$10^{-3} Re_v$	$10^{-3} Re_p$
686	1.7438	−122.06	3.34	7.71
620	1.7335	− 86.42	3.15	7.12
590	1.7288	− 71.70	3.06	6.21
515	1.7195	− 42.72	2.76	5.22
440	1.7135	− 23.80	2.43	4.07
381	1.7093	− 12.15	2.15	2.98
322	1.7068	− 5.09	1.83	2.43
268	1.7053	− 1.28	1.53	1.82
175	1.7040	+ 1.61	1.01	0.90
74.6	1.7039	+ 0.80	0.43	0.45

For completeness the relation between Δp_{eff} and the measured temperature differences in this heat conduction experiment is also represented in fig. 8

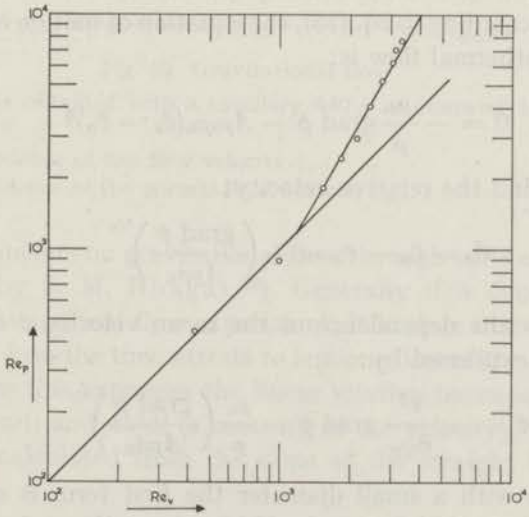

Fig. 9. Flow characteristic for the "pure heat conduction" method. $T = 1.7$°K, 255 μ capillary.

curve C. From this graph one may compare the magnitudes of mutual friction appearing in the various experiments. Also in this case eq. (20) is found to be satisfied (see fig. 9), and we may finally conclude that eq. (20) is always valid independent of the magnitude of the relative velocity.

The result gives a good support for the original statement encountered in the equations of Gorter and Mellink, that mutual friction directly influences only the temperature differences but not the pressure differences and moreover that within the limits of our experimental superfluid velocities there is no observable interaction with the wall since this would directly affect the pressure difference.

6. *Turbulence in gravitational flow.* Experiments on isothermal gravitational flow have been performed by Allen and Misener [18] (1939), Atkins [19] (1951) and recently by Bhagat [20] (1960).

The early paper of Allen and Misener does not contain sufficient information to enable us to make rigorous recalculations and therefore we will not consider these results here, except to remark that Jones [21] indicated that the results of Allen and Misener for the flow in wide capillaries could qualitatively be interpreted by assuming that turbulent flow occurred.

Atkins [19] studied the dependence of the mean flow velocity on the pressure gradient. According to eq. (2) the observed mean velocity is:

$$\bar{v} = \bar{v}_{\mathrm{n}} - \frac{\rho_{\mathrm{s}}}{\rho}\,\bar{v}_{\mathrm{r}}$$

in which $\bar{v}_{\mathrm{n}}$ can be calculated from eq. (5) and $\bar{v}_{\mathrm{r}}$ can be estimated in the following way: According to eq. (15), the equation of motion of the superfluid in the case of isothermal flow is:

$$0 = -\frac{\rho_{\mathrm{s}}}{\rho}\,\mathrm{grad}\,p - A\rho_{\mathrm{s}}\rho_{\mathrm{n}}(\bar{v}_{\mathrm{s}} - \bar{v}_{\mathrm{n}})^3$$

from which we find the relative velocity:

$$\bar{v}_{\mathrm{r}} = \bar{v}_{\mathrm{n}} - \bar{v}_{\mathrm{s}} = +\left(\frac{\mathrm{grad}\,p}{A\rho\rho_{\mathrm{n}}}\right)^{1/3}. \tag{22}$$

For laminar flow the dependence of the mean velocity $\bar{v}$ on the pressure gradient is then expressed by:

$$\bar{v} = -\frac{r^2}{8\eta_{\mathrm{n}}}\,\mathrm{grad}\,p - \frac{\rho_{\mathrm{s}}}{\rho}\left(\frac{\mathrm{grad}\,p}{A\rho\rho_{\mathrm{n}}}\right)^{1/3}. \tag{23}$$

For capillaries with a small diameter the first term is small compared with the second one. Atkins noticed that the values calculated from the first term exceeded the observed velocity for the wide capillary of 440 μ

in diameter. Estimating the range of Reynolds numbers for this capillary we found that the flow should have been turbulent. Therefore we calculated the first term of eq. (23) using eqs. (12) and (20):

$$\bar{v}_n = -\frac{\eta_n}{2r\rho}\left(\frac{1}{4.94 \times 10^{-3}} \cdot \frac{r^3\rho \operatorname{grad} p}{4\eta_n^2}\right)^{4/7}.$$

The dependence of $\bar{v}_n$ on the pressure gradient is shown in fig. 10 curve C and the dependence of the mean velocity $\bar{v}$ on the pressure gradient, calculated from:

$$\bar{v} = -\frac{\eta_n}{2r\rho}\left(\frac{1}{4.94 \times 10^{-3}} \cdot \frac{r^3\rho \operatorname{grad} p}{4\eta_n^2}\right)^{4/7} - \frac{\rho_s}{\rho}\left(\frac{\operatorname{grad} p}{A\rho\rho_n}\right)^{1/3}$$

is shown in fig. 10 curve B. We used the value $A = 40$ cm s g^{-1}. It seems that there is still no reason here to introduce a friction term $F_s(v_s)$ in the equation of motion as was suggested by Atkins to explain these results.

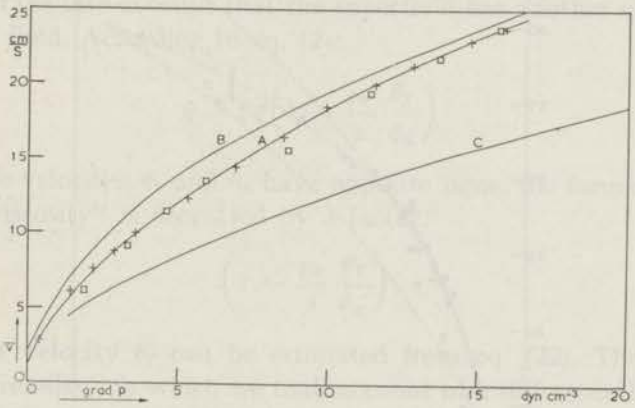

Fig. 10. Gravitational flow.

A. result of Atkins obtained with a capillary 440 μ in diameter at a temperature of 1.22°K.

B. calculated dependence of the flow velocity $\bar{v}$.

C. calculated dependence of the normal fluid velocity $\bar{v}_n$.

Recently experiments on gravitational flow through wide capillaries have been performed by S. M. Bhagat [20]). Generally if h denotes the level difference in a gravitational flow experiment, it is clear that a straight line dependence of log h on the time attests to laminar flow of the liquid through the capillary, since this expresses the linear relation between h (a measure of the pressure head) and dh/dt (a measure of the velocity). The kinematic viscosity can be calculated from the slope of the straight line according to the relation:

$$\frac{d}{dt}\ln h = \frac{1}{h}\frac{dh}{dt} = B\frac{\rho}{\eta} \tag{24}$$

where B is a constant of the apparatus. As eq. (24) is not satisfied in the case of Helium II, Bhagat introduced an "effective viscosity", η_{eff} from the slope of a graph of log $(h + c)$ versus time, where the constant c was for each run adapted in such a way, that $\log(h + c)$ showed a linear dependence on the time in the considered range of level differences and analogous to eq. (24) the "effective viscosity" was then calculated from:

$$\frac{1}{h+c}\frac{\mathrm{d}h}{\mathrm{d}t} = B\frac{\rho}{\eta_{\text{eff}}}. \tag{25}$$

As we shall show now, Bhagat's "effective viscosity" can be calculated from turbulent flow. In the paper of Bhagat the "effective viscosity" was given as a function of the initial level difference h_i (fig. 11 curve A).

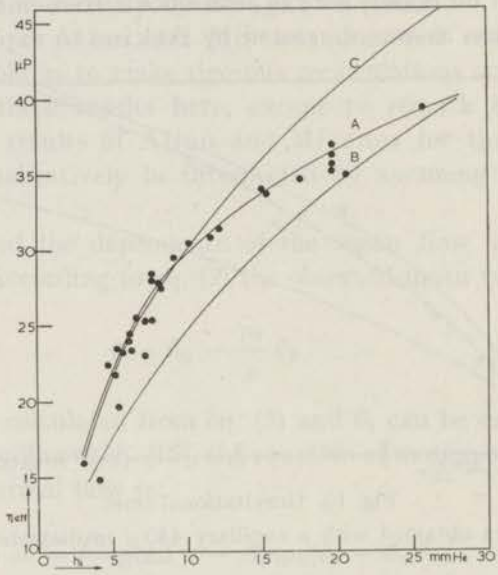

Fig. 11. A. Bhagat's "effective viscosity" as a function of the initial level difference.
B. calculated "effective viscosity" with $v_n \neq v_s$
C. calculated "effective viscosity" with $v_n = v_s$
●: Bhagat's experimental data.

If we assume that the constant c is small compared with the initial level difference we may approximate eq. (25) by eq. (24) for h equal to h_i, but this comes to the same as an equation analogous to Poiseuille's formula:

$$\eta_{\text{eff}} = \frac{r^2}{8\bar{v}}\,\text{grad}_i\, p \tag{26}$$

in which $\text{grad}_i p$ is the initial pressure gradient. If we assume in a first approximation, that there is no difference between superfluid and normal

fluid velocity, $\bar{v}$ equals $\bar{v}_n$ and according to eq. (12) we can write:

$$\bar{v} = \bar{v}_n = \frac{\eta_n Re_v}{2r\rho} \quad \text{and} \quad \text{grad}\, p = \frac{4\eta_n^2 Re_p}{\rho r^3}.$$

Inserting these expressions in eq. (26) we obtain:

$$\eta_{eff} = \eta_n \frac{Re_p}{Re_v}.$$

As the flow is turbulent we assume eq. (20) to be valid and we find:

$$\eta_{eff} = 4.80 \times 10^{-2} \eta_n Re_p^{3/7} \tag{27}$$

and we are able to calculate the "effective viscosity" as a function of the initial level difference h_i (see fig. 11 curve C).

However, in the calculations performed above, we assumed that the superfluid and the normal fluid move together with the same velocity and we did not take into account that the superfluid has another velocity than the normal fluid. According to eq. (2):

$$\bar{v} = \bar{v}_n \left(1 - \frac{\rho_s}{\rho} \frac{\bar{v}_r}{\bar{v}_n}\right)$$

in which the velocities $\bar{v}_r$ and $\bar{v}_n$ have opposite signs, the former calculated "effective viscosity" is decreased by a factor:

$$\left(1 - \frac{\rho_s}{\rho} \frac{\bar{v}_r}{\bar{v}_n}\right).$$

The relative velocity $\bar{v}_r$ can be estimated from eq. (22). The calculated "effective viscosity" in which we took account of a difference between the velocities of the normal fluid and the superfluid are also shown in fig. 11 (curve B). We used the value $A = 70$ cm s g^{-1} [22]). It should be pointed out that according to eq. (25) the "effective viscosity" is enlarged by the introduction of the constant c. Thus our calculated "effective viscosity" must be smaller than the value given by Bhagat. The curve B confirms this requirement. As we don't have the exact data of Bhagat's experiments a further quantitative analysis is not opportune, but it is likely that also these results may be explained by assuming the validity of equation (20).

7. *Some reflections on the equations of motion for turbulent flow.* Our experimental results previously described indicate that the turbulent behaviour of He II is governed by the Navier Stokes equation:

$$\rho \frac{d\boldsymbol{v}_n}{dt} = -\text{grad}\, p + \eta_n \nabla^2 \boldsymbol{v}_n \tag{28}$$

independent of the relative velocity.

As a matter of fact, our results deal with time averaged solutions of such an equation corresponding to a quasi stationary flow. The terms including $\partial/\partial t$ should vanish and in fact our starting point should be a time averaged equation of the form:

$$\overline{\rho(\boldsymbol{v}_{\mathrm{n}}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{n}}}^{t} = -\,\overline{\mathrm{grad}\,p}^{t} + \overline{\eta_{\mathrm{n}}\nabla^2\boldsymbol{v}_{\mathrm{n}}}^{t}. \tag{29}$$

On the other hand, we should consider eq. (17) as one of the two fundamental equations of the two fluid model:

$$\rho_{\mathrm{s}}\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{s}}}{\mathrm{d}t} + \rho_{\mathrm{n}}\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{n}}}{\mathrm{d}t} = -\,\mathrm{grad}\,p + \eta_{\mathrm{n}}\nabla^2\boldsymbol{v}_{\mathrm{n}}. \tag{17}$$

After averaging over the time we deduce from (17):

$$\overline{\rho_{\mathrm{s}}(\boldsymbol{v}_{\mathrm{s}}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{s}}}^{t} + \overline{\rho_{\mathrm{n}}(\boldsymbol{v}_{\mathrm{n}}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{n}}}^{t} = -\,\overline{\mathrm{grad}\,p}^{t} + \overline{\eta_{\mathrm{n}}\nabla^2\boldsymbol{v}_{\mathrm{n}}}^{t} \tag{30}$$

Comparing the left hand side of eq. (30) with that of eq. (29), we arrive at a condition:

$$\overline{(\boldsymbol{v}_{\mathrm{s}}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{s}}}^{t} = \overline{(\boldsymbol{v}_{\mathrm{n}}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{n}}}^{t} \tag{31}$$

which should be satisfied to make the equations consistent and moreover in agreement with our exprimental results (For a laminar flow through a tube $(\boldsymbol{v}_{\mathrm{n}}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{n}}$ equals zero). In order to understand condition (31) more fully we consider the instantaneous velocities $\boldsymbol{v}_{\mathrm{n}}$ and $\boldsymbol{v}_{\mathrm{s}}$ to be build up from a constant drift velocity $\boldsymbol{v}^{\mathrm{D}}$ equal to the mean velocity (averaged over time and cross section of the capillary) and a time dependent velocity $\boldsymbol{v}^{t}$. Thus:

$$\boldsymbol{v}_{\mathrm{s}} = \boldsymbol{v}_{\mathrm{s}}^{\mathrm{D}} + \boldsymbol{v}_{\mathrm{s}}^{t}$$

and

$$\boldsymbol{v}_{\mathrm{n}} = \boldsymbol{v}_{\mathrm{n}}^{\mathrm{D}} + \boldsymbol{v}_{\mathrm{n}}^{t} \tag{32}$$

with the additional conditions:

$$\overline{\boldsymbol{v}_{\mathrm{s}}^{t}}^{t} = 0$$

and

$$\overline{\boldsymbol{v}_{\mathrm{n}}^{t}}^{t} = 0. \tag{33}$$

In first approximation the temperature along the capillary should be considered as constant and hence the drift velocities are constant along the capillary.

After substituting eqs. (32) and (33) into eq. (31) we arrive at the condition:

$$\overline{(\boldsymbol{v}_{\mathrm{s}}^{t}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{s}}^{t}}^{t} = \overline{(\boldsymbol{v}_{\mathrm{n}}^{t}\cdot\mathrm{grad})\,\boldsymbol{v}_{\mathrm{n}}^{t}}^{t}. \tag{34}$$

We may remark that the condition does not contain the drift velocities and that a coupling of $\boldsymbol{v}_{\mathrm{n}}^{t}$ and $\boldsymbol{v}_{\mathrm{s}}^{t}$ according to eq. (34) only is sufficient to

satisfy (31). It is not possible to work out equation (34) generally without further information. But in any case we may state that one possible solution is given by:

$$\boldsymbol{v}_{\mathrm{n}}^{t} = \boldsymbol{v}_{\mathrm{s}}^{t}. \tag{35}$$

Although there are of course other solutions eq. (35) has the merit of having a simple physical meaning if some kind of mutual friction exists. From the existing heat conduction experiments, using eq. (15) with the acceleration term equated to zero, it is found that the mutual friction constant A in the turbulent region, within the limits of accuracy, does not show a deviation from the value determined in a laminar flow. It is therefore possible that the acceleration of the superfluid has to be attributed to a mutual friction in addition to the mutual friction depending only on the drift velocities of both fluids.

III. TURBULENCE IN LIQUID ^{3}He–^{4}He MIXTURES

1. *Apparatus and method.* With an apparatus as was described in 1958 by K. W. Taconis [23]), measurements were performed on flow of liquid ^{3}He–^{4}He mixtures in the He II region. It should be pointed out that this was an experiment to study the characteristics of the two fluid model for mixtures in a rather qualitative way only. For completeness, however, we thought it would be useful to give some results of these measurements here.

The apparatus shown in fig. 12 consists of two glass vessels A and B communicating with each other by the capillaries a and b. Vessel A and the two capillaries are insulated by a vacuum jacket. A heating coil H wound around a copper wire and a "de Vroomen-type" carbon thermometer Th_A were soldered into two platinum tubes P which were previously inserted through the glass bottom of the vessel A in such a way that heater and thermometer were in intimate contact with the liquid mixture in A. Another thermometer Th_B was put in vessel B. The vessels were connected to a differential oil manometer M_1 outside the cryostat. Vessel B was also connected to a separate oil manometer M_2 with which the pressure difference with the surrounding helium bath could be measured. The mixture was condensed to the extent that the liquid level was observed half way up the vessel A. The level in B was observed to attain the same height.

When the temperature in A is increased by the heater H, the superfluid flows from B to A and hence a fountain pressure difference is built up; simultaneously a normal fluid flow takes place from vessel A to vessel B. This normal fluid flow is accompanied by a flow of ^{3}He which effects an increase of the ^{3}He concentration in vessel B. Moreover, at the same time the vapour pressure difference resulting from the rise of the pressure corresponding with the increase of the temperature in vessel A, diminished

by the rise of the pressure due to the increase of the concentration in vessel B, leads to a gas flow through the capillary a from vessel A to vessel B. The vapour is relatively rich in ^{3}He and consequently this also causes an increase of the ^{3}He concentration in vessel B. Due to the arising concentration difference, a very large osmotic pressure is created opposing the large fountain pressure and the normal fluid flow is reversed as soon as the

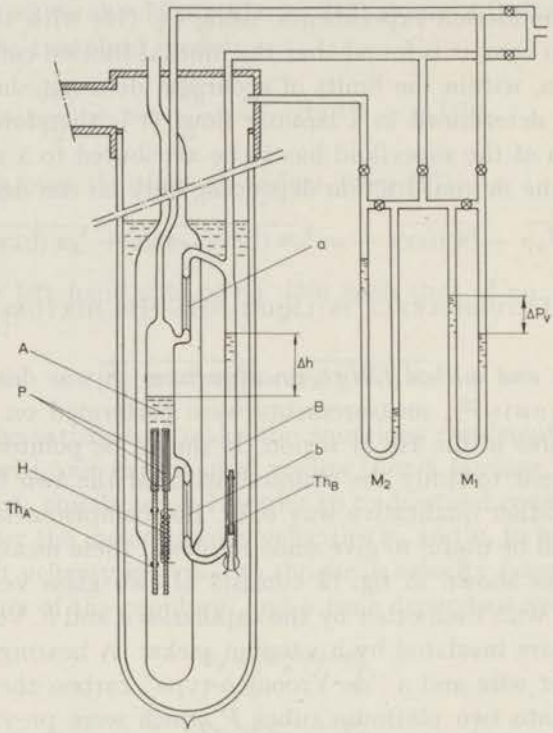

Fig. 12. Flow apparatus for ^{3}He–^{4}He mixtures.

osmotic pressure surpasses the fountain pressure. After some time, a stationary state is reached with a level difference as shown in fig. 12, and it is remarkable that a heat flow has been created from a lower temperature in vessel B to a higher temperature in vessel A (see also next section).

2. *Numerical evaluation and analysis of the flow phenomenon.* In the stationary state the number of moles of ^{3}He flowing through the capillary a has to be equal to that flowing through capillary b. If we realize that the concentration of the vapour flowing through the capillary a equals the vapour concentration of vessel A we may write for the number of moles per second passing a cross section of one of the capillaries:

$$\dot{N}_3 = \frac{\pi r_b^2 \bar{v}_{3b} \rho_l X_l}{M_l} = \frac{\pi r_a^2 \bar{v}_{3a} \rho_V X_{V,A}}{M_V} \tag{36}$$

in which:

r_a and r_b are the radii of the capillaries a and b,

$\bar{v}_{3a}$ and $\bar{v}_{3b}$ the mean velocities of the ^{3}He averaged over the considered cross-sections,

ρ_l and ρ_v the densities of the liquid and the vapour in g cm^{-3},

$X_{v,A}$ and M_v the molar concentration and the molar weight of the vapour in vessel A,

X_l and M_l The molar concentration and the molar weight of the liquid passing the considered cross section of capillary b.

From the vapour pressure and temperature measurements in both vessels A and B the concentrations of the liquid and the vapour can be derived from the well known P–T–X relation [24][25][26]) of these mixtures. As the flow of the gas mixtures in capillary a is laminar and the viscosities of the vapour mixtures are known from the work of Becker, Misenta and Schmeissner [27]), we can derive the velocity of the ^{3}He in the capillary a from the measurement of the vapour pressure difference Δp_v using Poiseuille's law:

$$\bar{v}_{3a} = \frac{r_a^2}{8\eta_v l_a} \Delta p_v$$

where l_a is the length of the capillary a. Consequently $\dot{N}_3$ can be calculated. According to the energy balance applied as a check to vessel A, the amount of heat supplied to vessel A equals the heat needed to evaporate liquid of the vapour concentration $X_{v,A}$, diminished by the energy supplied by the heat flow through the capillary b. The calculation showed us that it was necessary to include a correction to the pressure difference across the vapour capillary, known as the Hagenbach correction. This correction arises from the decrease in the pressure due to accelerated motion at the entrance of the capillary a. The effective pressure difference is obtained after the subtraction of $\rho_v \bar{v}_{3a}^2$. The proper value of $\dot{N}_3$ is obtained after applying this correction.

From the fact that the velocity of the ^{3}He in the liquid equals the velocity of the normal fluid, the Reynolds number in capillary b is known to be:

$$Re_v = \frac{2 r_b \bar{v}_{3b} \rho_l}{\eta_l}$$

and with eq. (36):

$$Re_v = \frac{2\dot{N}_3 M_l}{\pi r_b \eta_l X_l} \,. \tag{37}$$

It appeared that the value of Re_v was high enough to expect turbulence. We therefore calculated the corresponding value of Re_p according to the scheme given in the second part of this chapter. Re_p is determined by the

measurement of the hydrostatic pressure difference across the capillary b:

$$\Delta p_1 = \rho_1 g \Delta h - \Delta p_v$$

where Δh and Δp_v denote the level difference and vapour pressure difference respectively. Re_p is thus determined by:

$$Re_p = \frac{r_b^3 \rho_1}{4\eta_1^2 l_b} (\rho_1 g \Delta h - \Delta p_v) \tag{38}$$

in which l_b denotes the length of the capillary b. We may point out that Re_p according eq. (38) is an average over the length of the capillary because grad p is replaced by $\Delta p/l$. In order to obtain the corresponding value of Re_v one should consider the conditions in the capillary in more detail, because it seems likely that $\bar{v}_3$ may vary during the flow from B to A as the concentration difference and the temperature difference are unusually large: *e.g.* $X_B - X_A \sim 8\%$ and $T_A - T_B \sim 0.2°K$. Using the conditions that the entropy flux and the ^3He flux are constant along the channel b, it is found that the temperature and the ^3He concentration are related to each other. Along the capillary, the temperature and the ^3He concentration both increase, both decrease or both remain constant. This seems to contradict the observations, as it is found that the liquid concentration in vessel B is higher than the liquid concentration in vessel A, and the temperature of vessel B is lower than the temperature of vessel A. This apparent discrepancy can be explained as the following considerations show. If we assume that going from B to A the temperature increases along the capillary, the condition that the entropy flux remains constant along the capillary requires that also the ^3He concentration increases going from B to A. This, however, means that the fountain pressure gradient as well as the osmotic pressure gradient in the capillary b would drive the normal fluid (including the ^3He) in the opposite direction of the observed flow. We may therefore conclude that temperature and concentration do not increase going from B to A.

On the other hand, if we start from the assumption that the ^3He concentration along the capillary remains constant, then for the same reason as mentioned above, also the temperature should be constant along the capillary. This implies that there is neither an osmotic force nor fountain force acting on the superfluid inside the capillary, hence there is no force opposing the mutual friction force on the superfluid. In this case the superfluid should flow in the same direction as the normal fluid does. Moreover at the exit of the capillary in vessel A a strong decrease of the concentration and a strong increase of the temperature appear. As the osmotic pressure difference exceeds the fountain pressure difference, the effective pressure gradient drives the superfluid out of vessel A into the capillary. The transfer of ^4He to the exit of the capillary from both sides results however in a decrease

of the concentration near the exit. Hence the initial condition of constant concentration and temperature along the capillary cannot be maintained and the process will continue until a stationary condition is reached with an osmotic pressure gradient and a fountain pressure gradient at the exit of the capillary which nearly compensate each other. We may finally conclude that going from B to A the concentration contineously decreases along the capillary from a value $X_b(\sim 10\%)$ to a value $X_e(\sim 5\%)$ at the end of the capillary and near the exit in vessel A further decreases to the mean concentration X_A of vessel A $(\sim 2\%)$. Consequently going from B to A the temperature must continuously decrease along the capillary from

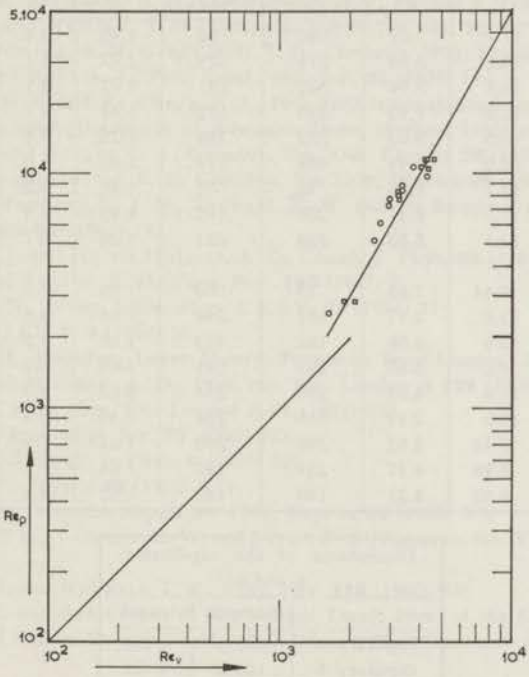

Fig. 13. Flow characteristic for the ^{3}He–^{4}He mixtures.
○: $T_B = 1.709°K$ □: $T_B = 1.510°K$

a temperature $T_B(\sim 1.7°K)$ to such a temperature $T_e(\sim 1.5°K)$ that the temperature increase $T_A - T_e$ $(\sim 0.4°K)$ at the exit corresponds to the concentration decrease $X_e - X_A$ $(\sim 3\%)$ at the exit according to the aforementioned condition with respect to the osmotic pressure gradient and fountain pressure gradient.

It is clear that the temperature and the concentration at the entrance of the capillary in vessel B are equal to the measured temperatures and concentrations of vessel B. The temperature and concentration at the end of the capillary in vessel A, however, should be calculated from the conditions that the entropy flow density and the ^{3}He flow density are constant

along the capillary and from the equilibrium condition between the osmotic pressure and fountain pressure near the exit of the capillary in vessel A. Now at both ends of the capillary Re_v can be calculated from eq. (37). In a first approximation we may assume grad p to change linearly along the capillary. According to eq. (20) the value of Re_v corresponding to the average value of Re_p calculated from eq. (38) must be obtained from averaging $Re_v^{1.75}$ over the length of the capillary.

TABLE III

T_B °K	T_A °K	X_B %	X_A %	Δp_V dyn/cm²	Δp_1 dyn/cm²	T_e °K	X_e %	$10^{-3}Re_v$	$10^{-3}Re_p$
1.5099	1.601	11.2	7.67	100	58	1.46	9.4	1.88	2.82
	1.787	14.6	3.94	519	279	1.27	8.1	4.34	11.4
	1.786	14.6	3.94	520	281	1.27	8.1	4.62	11.5
	1.648	12.5	7.10	215	170	1.41	9.7	3.27	7.64
	1.704	13.5	5.83	322	246	1.35	9.4	4.01	10.6
	1.740	14.0	4.99	404	274	1.30	9.0	4.33	11.6
	1.591	10.2	7.28	96	56	1.46	9.2	2.11	2.82
	1.642	11.3	6.51	205	175	1.42	8.9	3.28	8.34
	1.691	12.1	5.53	328	251	1.38	8.7	4.28	11.45
1.7088	1.764	9.44	7.63	75	60	1.69	9.0	1.60	2.55
	1.960	13.0	2.97	531	244	1.56	8.6	3.31	8.45
	1.810	10.9	6.86	152	133	1.66	9.3	2.51	5.19
	1.856	11.6	5.67	227	191	1.64	9.2	2.93	7.22
	1.881	11.8	4.97	290	214	1.62	9.0	3.22	8.00
	1.935	10.0	2.17	546	234	1.50	5.5	4.30	9.60
	1.893	9.42	2.93	396	250	1.61	6.8	3.74	10.62
	1.843	8.95	4.27	237	198	1.64	7.3	3.32	8.69
	1.803	8.53	5.27	149	142	1.67	7.4	2.70	6.13

Dimensions of the capillaries a and b		
	diameter	length
capillary a	253 μ	2 cm
capillary b	156 μ	2 cm

The results of the measurements and the calculations are given in table III and shown in fig. 13. Once again it is to be seen that also for mixtures in the case of turbulence the flow phenomena are satisfactory described by eq. (20). It should be mentioned that these measurements were performed three years ago and were not understood as laminar flow was assumed (using density ρ_n). However the obtained viscosity was so dependent on the velocity that turbulence was indicated. This was in fact the reason to start our work on the viscosity [28]) of mixtures and the research on turbulence.

Finally we remark that from the equation of motion of the superfluid the value of the mutual friction constant A was estimated to be about 70 cm s g^{-1}. This value is of the same order of magnitude as the value

found for pure ^{4}He which indicates that the influence of the presence of ^{3}He on the value of A might be small.

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CHAPTER III

SUPERFLUID FLOW IN WIDE CAPILLARIES

Summary

A method is described to detect the phenomenon of mutual friction in a superfluid flow with the normal fluid at rest. It was found, as was expected, that the mutual friction varied with a third power of the superfluid velocity. The critical velocities of onset of mutual friction have been measured as a function of the temperature.

1. *Introduction.* We discussed the results of several types of heat flow experiments in wide capillaries [1]) in chapter II. One of the important conclusions of the experiments was that the superfluid showed no interaction with the wall of the capillary. Even at very high velocities only mutual friction was found as suggested originally by Gorter and Mellink [2]). As we plan now to study again the onset and characteristics of the mutual friction, we like to discuss here some preliminary work of a few years ago in this field [3]). In these experiments a wide capillary was put in series with a superleak, and surrounded by vacuum. A stationary flow of helium II created through such a capillary can only exist as a superfluid flow, because the superleak prevents normal fluid flow. The motion of the superfluid and the normal fluid respectively are described by the equations [2]):

$$\rho_s \frac{d\boldsymbol{v}_s}{dt} = -\frac{\rho_s}{\rho} \operatorname{grad} p + \rho_s S \operatorname{grad} T - \boldsymbol{F}_{sn} \tag{1}$$

and

$$\rho_n \frac{d\boldsymbol{v}_n}{dt} = -\frac{\rho_n}{\rho} \operatorname{grad} p - \rho_s S \operatorname{grad} T + \eta_n \nabla^2 \boldsymbol{v}_n + \boldsymbol{F}_{sn} \tag{2}$$

where according to Gorter and Mellink F_{sn} denotes the mutual friction force per unit volume acting between both fluids and the other symbols have the standard meaning [1]).

If we assume stationary laminar flow of the superfluid without normal fluid flow the addition of these two equations gives:

$$\operatorname{grad} p = 0 \tag{3}$$

and the first equation is transformed into:

$$F_{sn} = \rho_s S \operatorname{grad} T \tag{4}$$

It is found from heat conduction experiments that mutual friction appears to occur at relative velocities $v_s - v_n$ higher than a certain critical value of the relative velocity. Below the critical velocity no mutual friction is observed. We were especially interested in the onset of mutual friction and therefore we intended to measure the temperature gradient effected by mutual friction according to (4).

Gorter and Mellink put the mutual friction force per unit volume:

$$F_{sn} = A\rho_s\rho_n(v_s - v_n)^3 \tag{5}$$

and in the case of $v_n = 0$:

$$F_{sn} = A\rho_s\rho_n v_s^3 = \rho_s S \operatorname{grad} T. \tag{6}$$

2. *Apparatus, method and results.* The apparatus used in the first experiment is shown in fig. 1. The capillary C (290μ in diameter and 4 cm in length) is connected to a superleak S. The other side of the superleak was connected to a vessel B in which a heater H was placed. The superleak and the capillary are both insulated by a vacuum jacket. A manometer tube M is mounted between the capillary and the superleak. The whole apparatus is movable vertically. The apparatus was immersed in the helium bath in

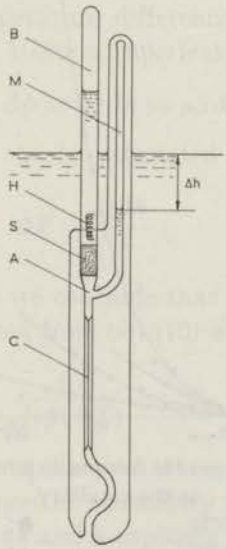

Fig. 1. Apparatus.

such a way that one level appeared in vessel B above the heater H and another level in manometer tube M (see fig. 1). Superfluid flow is effected by supplying heat in vessel B with the heater H. The level in vessel B will move upwards gradually due to the fountain effect. We measured the flow rate in a stationary condition by following the level in vessel B with a

cathetometer, while the level difference Δh remains constant. The superfluid velocity v_s in capillary C can be calculated from

$$v_s = \frac{\rho}{\rho_s} \frac{R_B^2}{r^2} v_B \tag{7}$$

in which

R_B is the radius of vessel B,
r the radius of the capillary C,
v_B the velocity of the level in B.

At the same time we measured the level difference Δh between the level in the manometer M and the bath level. This level difference is a measure for the temperature difference across the capillary C. The temperature gradient arises from the mutual friction in the capillary. The normal fluid is dragged along with the superfluid in the direction of A (see fig. 1). This gives an increase of the temperature in A, which is observed by the increase of the vapour pressure in M with respect to the vapour pressure of the bath. The level difference Δh is plotted in fig. 2 as a function of the superfluid velocity

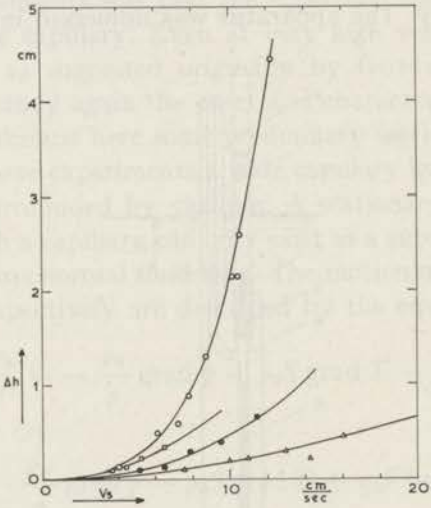

Fig. 2. The level difference Δh in cm He column as a function of the superfluid velocity in the capillary.

○: $T = 1.89°K$ ●: $T = 1.48°K$
□: $T = 1.60°K$ △: $T = 1.23°K$

v_s. The temperature difference ΔT across the capillary can be calculated from Δh, which level difference is effected by the vapour pressure only; because there can not exist a pressure gradient in the capillary. The measurements were performed at 4 different temperatures and it was found that they can be represented by a third power dependence at a temperature of 1.9°K. At lower temperatures the power may slightly differ from the

third power. From the third power dependence of Δh on v_s, we can calculate the value of A from eq. (6). A vapour pressure difference Δh corresponds to a temperature difference

$$\Delta T = \frac{\rho g \Delta h}{(\partial p_v / \partial T)} \tag{8}$$

and with eq. (6):

$$A = \frac{S g \Delta h}{(\rho_n/\rho)(\partial p_v/\partial T)\, l v_s^3}. \tag{9}$$

TABLE

The mutual friction constant A				
T (°K)	1.89	1.60	1.48	1.23
A (cm s g^{-1})	130	135	85	64

The values calculated with eq. (9) are given in the table. The critical velocity appeared to be so low that it was impossible to detect the appearance of the temperature gradient. In order to obtain data on the critical velocity the following consideration was very useful:

A temperature difference measurement based on the fountain effect is more sensitive than a temperature difference measurement based on the vapour pressure difference. Using a superleak, from London's relation:

$$\Delta p = \rho g \Delta h = \rho S \Delta T$$

the temperature difference can be calculated from:

$$\Delta T = \frac{g \Delta h}{S}. \tag{10}$$

Comparing this with eq. (8) we conclude that the sensitivity of the temperature measurement calculated from eq. (10) is multiplied by a factor

$$\frac{\rho S}{(\partial p_v / \partial T)} \sim 12.$$

Moreover we can choose a capillary of one meter length instead of one of 4 cm length. This will enlarge the sensitivity by another factor of 25. And therefore if we use a superleak and a capillary of one meter length the sensitivity is enlarged by a factor of 300. We therefore constructed another apparatus (see fig. 3).

The capillary C was connected to a conical vessel K through a porous jewellers rouge plug S, which served as a superleak. The capillary (260 μ in diameter and one meter length) and the superleak were insulated again by a vacuum jacket V. Good thermal contact between vessel K and the bath was ensured by two copper rings R. To diminish the effect of radiation,

the whole vessel K was wrapped with cotton cord. The level height in vessel K could be observed in a side tube T. In order to eliminate the effect of capillarity another tube of the same diameter was placed next to tube T in which we could observe the level corresponding with the bath level. The whole apparatus is movable in a vertical direction. The measurements were performed by the following procedure: The vessel K was partially immersed in the helium bath. The temperature of the helium bath was kept constant

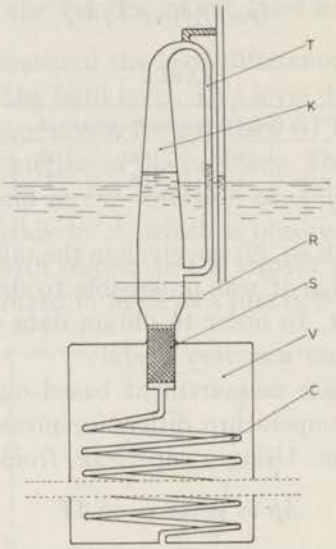

Fig. 3. Apparatus.

within 10^{-5} degrees. By supplying more or less heat with a heater in the helium bath we could regulate the evaporation of the bath. A fall of the bath level with constant velocity and an allied fall of the level inside the conical vessel gives as a result a superfluid flow in the capillary C with a slowly increasing velocity. The positions of the levels in the side tubes were both measured as a function of the time. Initially they both move with the same velocity. A small constant difference is observed due to some radiation into tube T. Suddenly after some time the level in tube T lags behind and shows an extra level difference corresponding with a temperature difference across the superleak. The temperature at the end of the capillary just below the superleak is suddenly decreased and a temperature gradient along the capillary exists according to equation (6). A typical example of a run is given in fig. 4. The critical velocity in the capillary corresponding with the sudden increasing of the level difference in the side tubes is calculated from a formula analogous to eq. (7). The measured critical velocities are shown in fig. 5 which shows a pronounced maximum at $T = 1.5°K$ of 0.5 cm/s. It appears that the magnitude of 0.5 cm/s is of the order we can expect

in the case of a capillary, 260 μ in diameter (see for instance Atkins "Liquid Helium") [4]).

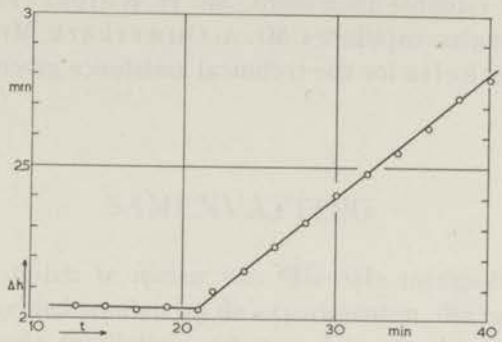

Fig. 4. The level difference Δh as a function of the time.

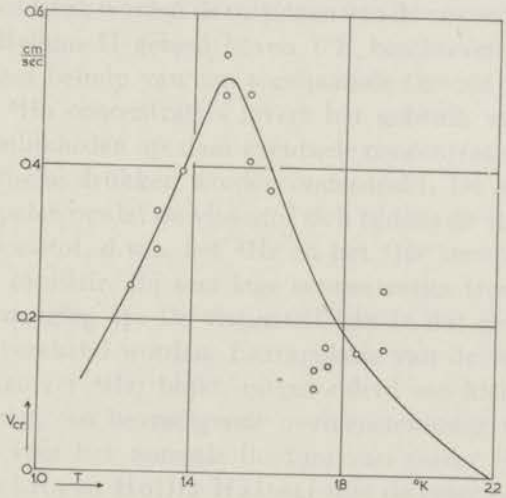

Fig. 5. The critical velocity as a function of the temperature.

The highest temperature gradients measured in the capillary are still smaller than the temperature gradients which can be expected from eq. (6), indicating that the measurements performed above the critical value are situated in a kind of transition region below the third power mutual friction curve [5]).

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SAMENVATTING

Teneinde viscositeiten te meten van ^3He–^4He mengsels en zuiver ^4He in het Helium II gebied werden bij de experimenten, die in dit proefschrift zijn beschreven wijde capillairen aangewend in afwijking van de meestal gebruikte nauwe spleten. Hierdoor werd een hoge nauwkeurigheid bereikt.

In het eerste hoofdstuk worden de metingen van de viscositeit van ^3He–^4He mengsels in het Helium II gebied boven 1°K beschreven. Deze metingen werden verricht met behulp van een zogenaamde Oswald viscosimeter. Bij niet al te kleine ^3He concentraties levert het gebruik van de isotherme methode geen moeilijkheden op, daar eventuele concentratieverschillen door optredende osmotische drukken worden onderdrukt. De viscositeit is dan gemakkelijk te bepalen omdat de vloeistof zich tijdens de stroming gedraagt als een gewone vloeistof, d.w.z. het ^3He en het ^4He stromen met dezelfde snelheid door het capillair. Bij zeer lage concentraties treedt er tijdens de stroming een ontmenging op. De viscositeit kan in dat geval toch nog op eenvoudige wijze berekend worden. Extrapolatie van de viscositeiten naar concentratie nul (zuiver ^4He) blijkt, uitgezonderd een kleine afwijking bij hogere temperaturen, een bevredigende overeenstemming te geven met de gemeten waarden voor het normale fluidum van zuiver ^4He, die bepaald werden door Heikkila en Hollis Hallett met de roterende cylinder. Dit is een experimentele bevestiging van de uitspraak dat het ^3He deelneemt aan de beweging van het normale fluidum. De bij dit experiment gevonden viscositeiten stemmen op bevredigende wijze overeen met het door Zharkov theoretisch voorspelde verloop van de viscositeit bij lage temperaturen en zeer verdunde mengsels.

De bovengenoemde kleine afwijking gaf ons reden de capillaire methode ter bepaling van de viscositeit van het normale fluidum van zuiver ^4He te verbeteren. Tot nu toe leverde, wegens het optreden van de wederkerige wrijving, de capillaire methode bij ^4He moeilijkheden op indien men met wijde capillairen de viscositeit van het normale fluidum trachtte te bepalen. In het eerste gedeelte van het tweede hoofdstuk wordt een toestel beschreven waarmee deze moeilijkheden zijn geëlimineerd. De metingen werden verricht met drie capillairen van verschillende afmetingen en inderdaad werden de reeds vermelde afwijkingen van de metingen met de roterende cylinder

bevestigd. Het tweede gedeelte van het tweede hoofdstuk is gewijd aan het verschijnsel turbulentie. Het blijkt dat de stroming, onafhankelijk van de snelheid van het superfluidum, beschreven kan worden met behulp van een Reynoldsgetal, dat o.a. de totale dichtheid van de vloeistof en de snelheid van het normale fluidum bevat. Aan de empirische formule van Blasius voor turbulentie van gewone vloeistoffen wordt altijd voldaan indien men dit Reynoldsgetal gebruikt. Dit gedrag kan verklaard worden als men aanneemt dat de snelheidsfluctuaties van het superfluidum en die van het normale fluidum cohaerent zijn. Binnen de grenzen van de meetnauwkeurigheid is aangetoond, dat het superfluidum geen direkte wisselwerking heeft met de wand. Een wisselwerking van het superfluidum met de wand heeft slechts plaats via het normale fluidum. De in het getal van Reynolds voorkomende snelheid van het normale fluidum werd steeds berekend uit de warmtestroomdichtheid. Daar bij een ^{3}He–^{4}He mengsel het ^{3}He zich met de snelheid van het normale fluidum beweegt is het mogelijk het Reynoldsgetal te berekenen uit het ^{3}He transport door het capillair. Een toepassing hiervan vindt men in het derde deel van het tweede hoofdstuk waar een experiment wordt beschreven met ^{3}He–^{4}He mengsels. Het bleek, dat ook in dit laatste geval de klassieke beschrijving voor turbulentie opgaat.

In het derde hoofdstuk zijn twee experimenten beschreven waarin het superfluidum stroomt terwijl het normale fluidum in rust blijft. De wederkerige wrijving is binnen de bereikte meetnauwkeurigheid te beschrijven met een derde macht van de relatieve snelheid van het superfluidum t.o.v. het normale fluidum, zoals Gorter en Mellink dit deden. De kritische snelheden, gemeten in een capillair van 260 μ tonen een maximum bij een temperatuur van 1.5°K.