ON THE NUCLEAR SHELL MODEL AS BASIS FOR CALCULATING NUCLEAR ENERGY LEVELS

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CHAPTER I

ON THE INDEPENDENT MOTION OF NUCLEAR PARTICLES

The assumption of independent motion[1)] of nuclear particles within the nucleus, which is also referred to as the shell model or quasi-atomic model of the nucleus, has in recent years proved to be a very succesful basis for explaining certain periodicities occurring in the nuclear structure. However, most of the experimental evidence favours the (apparently[2)]) opposite and extreme assumption of strong interaction between the individual particles. This assumption stresses the analogy of the nucleus with a drop of liquid and forms the basis of the concept of the compound nucleus, which very successfully accounts for the most important properties of nuclear reactions.

The assumption of independent motion is valid only if the nucleons rarely suffer strong individual collisions. If the nucleons perform periodic motions, these collisions must be so rare that, on the average, less than one collision occurs per period. At first thought it does not seem as if the nucleons fulfill this requirement, since they are very closely packed on account of the small range of the nuclear forces. In fact, the neutron-proton scattering cross-section at about 20 Mev. (this is the order of the kinetic energy per nucleon inside the nucleus) is known to be of the order of 0.3 barn, which means that the mean free path is less than the nuclear radius, 10^{-13} cm.[3)] If the particles in the nucleus are to move independently of one another, they must for some reason be scattered less frequently than one would expect from the information obtained from the scattering cross-section of an isolated pair of nucleons.

In attempting to interpret nuclear spectra and find quantitative expressions for the positions of the levels, one is forced to approximate the nuclear system by a much simpler system whose properties can be regarded as roughly similar to those of the

1) better: one-particle motion.

2) See later in this chapter.

3) path = 1/ [cross-section X density of particles]

$\approx 5 \times 10^{-14}$ cm, where the density = $3A/4\pi R^3$

$\approx 3 \times 10^{39}/4\pi(1.5)^3$ nucleons per cc.

nucleus. Such a *model* can be based on any one of the two aforementioned assumptions. Although these assumptions are apparently contradictory, the properties exhibited by the nucleus seem to be accounted for partly by the one and partly by the other. On account of this it seems very difficult to choose a satisfactory model for calculating the positions and breadths of the energy levels. Weisskopf and Fermi[1] pointed out, however, that these two assumptions need not necessarily contradict each other as we shall proceed to discuss.

The independent motion model assumes that in the first approximation each nucleon occupies an one-particle state, after the analogy of the electron states in the atom. Thus each nucleon is assumed to move in a fixed potential field, which represents the *average effect* of all the other nucleons. The non-static fluctuations of the interaction due to the close approach of the nearby nucleons are neglected. This does not necessarily imply that the interactions between the individual nucleons are weak. To see this we must remember that a nuclear particle constantly passes other nucleons and therefore through their associated potential wells, which may be so closely spaced that they blend together to form a roughly uniform potential.

Thus we see that strong interaction does not necessarily exclude independent motion. But it still needs explanation why only some of the nuclear properties favour the independent motion picture, while others favour a description in terms of strong interaction between the individual particles. In this connection it is of importance to note that only the properties of the ground states can be predicted successfully by the shell model. Indeed, the occurrence of the *magic numbers*[2] and the variation of the spins with atomic number are all properties of the ground states. On the other hand, the application of the strong interaction models are restricted to problems involving high nuclear excitation, since the compound nucleus formed in a nuclear reaction is always excited. Therefore it seems as if the lowest state (or states) are perhaps better approximated by an independent particle model, while the highly excited states are interpreted in terms of a strong interaction model. This dual

1) V.Weisskopf, Science 113, 101 (1951) and Helv. Phys. Acta 23, 186 (1950);
E. Fermi, *Nuclear Physics*, (Univ. Chicago Press, 1950) p. 167.

2) See Chapter II.

behaviour can probably be accounted for by the Pauli principle. In the ground state all the lowest quantum states are occupied. A collision between nucleons can occur only if there are empty states available into which the nucleons can be transferred. Thus in the ground state scattering will seldom occur and hence it is allowable to speak of independent motion. As the excitation energy increases, more and more of the lowest quantum states become unoccupied and, as a result, the foregoing picture becomes less and less tenable.

It is to be noted that the shell model cannot yet unambiguously predict the variations of the spins with the mass number, and there has been some doubt as to whether the variation of the magnetic moments are directly correlated with any kind of shell structure[1]. Furthermore, it cannot as yet account for all the other properties of the ground states, such as the high quadrupole moments of some nuclei[2]. On the other hand, however, it can interpret some of the properties connected with the lowest excited states, such as the occurrence of *islands of isomerism*, discovered by Feenberg and Hammack and by Nordheim[3], and the small neutron capture cross-section of *magic nuclei*[4], i.e. nuclei with the number of protons and/or neutrons equal to a magic number.

In view of the foregoing discussion it seems unlikely that the assumption of independent motion will ever prove to be a successful basis for calculating the positions of energy levels. However, there are some cases where, in the lowest excited states, there is some reason for expecting the independent model to remain approximately valid. These cases will be discussed in Chapter III.

The attempts, discussed above, to make the supposed one-particle motion of the nuclear constituents plausible by means

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1) See Chapter II, however.

2) See, for example, Helv.Phys.Acta 23, Basel Congress section, p.211 (1950) (discussion of paper by L.Rosenfeld). A promising attempt to reconcile the large quadrupole moments with the shell model has been done by A.Bohr (Phys.Rev.81, 138 (1951)), and J.Rainwater (Phys.Rev. 79, 432 (1950)), who suggested a non-spherically symmetric average potential to allow for the large quadrupole moments. Another attempt in the same direction was done by L.Rosenfeld (Physica 17, 461 (1951).

3) cf. reference 2) on page 15.

4) Hughes, Spatz, Goldstein, Phys.Rev. 75, 1781 (1949).

of the exclusion principle have thus far not been developed quantitatively. However, this problem has been discussed in a semi-quantitative way from a different point of view by Schiff[1], as we shall proceed to discuss. We shall not enter into the mathematical details of this paper.

Schiff, using classical field theory, assumes that the interactions between nucleons arise from mesons which obey a non-linear wave equation. As a first approach he chooses these to be of the neutral scalar type, because then the calculations are relatively simple. In the usual linear theory the Lagrangian density has the form[2]
($c, \hbar, \mu$ = meson mass all equal to unity)

$$L = \tfrac{1}{2}(\partial\varphi/\partial t)^2 - \tfrac{1}{2}(\nabla\varphi)^2 - \tfrac{1}{2}\varphi^2 + f\varphi \ , \qquad (1)$$

where φ is the meson field function, increasing linearly with the nucleon source density f, which in general is a function of the position and time. The source strength is $g = \int f dt$. The field function obeys the following linear equation:

$$\partial^2\varphi/\partial t^2 = \nabla^2\varphi - \varphi + f \ . \qquad (2)$$

Since the meson field amplitude is proportional to the nucleon source strength, the meson fields are superposable and the interaction energy between a number of nucleons is equal to the sum of those of the interacting pairs. Schiff now chooses the non-linearity in such a way that the meson field amplitude increases less rapidly than linearly with the nucleon source strength. Then the change in meson amplitude produced by the addition of a nucleon is less when many nucleons are already present than when only a few are present.

Hence, within nuclei, the two-nucleon interaction is strongly reduced compared with the two-nucleon interaction in empty space. Thus the non-linearity can be expected to smooth out the fluctuations in the average potential and hence leads to the one-body potential and shell structure.

This is also the sort of effect needed to account for saturation, i.e. the close proportionality of the total binding energy (minus the Coulomb energy) with mass number. Since the

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1) L.I.Schiff, Phys.Rev. 84, 1, 10 (1951).

2) cf. G.Wentzel, "Quantum Theory of Fields", (Interscience Publ., N.Y., 1949), Chapter II.

number of interacting pairs are $A(A - 1)/2$, where A is the atomic number, the (negative) potential energy would be expected to increase proportionally to A^2; the (positive) kinetic energy increases as $A^{5/3}$, as is well known from Fermi statistics. Since this will lead to the collapse of heavy nuclei, the potential energy must increase less rapidly than A^2, namely as $A^{5/3}$ or less. The non-linearity considered here produces an effect of this sort.

The non-linearity is introduced by writing, instead of (1),

$$L = \tfrac{1}{2}(\partial\varphi/\partial t)^2 - (\nabla\varphi)^2 - G(\varphi) + fF(\varphi) \; ; \qquad (3)$$

the wave equation now is:

$$\partial^2\varphi/\partial t^2 = \nabla^2\varphi - \partial G/\partial\varphi + f\partial F/\partial\varphi \; . \qquad (4)$$

F is the non-linear function which couples the meson field with the nucleons; G is another non-linear function of φ. F and G approach φ and $\tfrac{1}{2}\varphi^2$ respectively as the field becomes weaker and weaker. For attainable free meson beams the density is so small that the non-linearity is not significant.

In order to simplify matters only one of the functions G and F is assumed non-linear. In the case where the non-linearity is introduced in the term representing the coupling between mesons and nucleons, i.e. $G = \tfrac{1}{2}\varphi^2$ and $F(\varphi)$ to be specified, Schiff's calculations show that the theory does not lead to saturation. If the non-linearity is introduced in the meson field itself, i.e. $F = \varphi$ and $G(\varphi)$ to be specified, a reasonable theory can be constructed when f, and hence φ, depend only on the spatial coordinates. Schiff assumes, for the purpose of calculation, that $G = \tfrac{1}{2}\varphi^2 + \tfrac{1}{4}\alpha\varphi^4$, where α is a constant to be determined from experimental data; this expression for G is in agreement with the fact that φ must increase less rapidly than linearly with f and it leads to saturation. Putting this value in the non-linear equation and taking convenient trial solutions for φ and f, the energy of a free nucleon has been calculated by means of the variational method. The same has been done for a nucleon embedded in nuclear matter; now, however, the infinite self-energy terms, appearing in the case of the free nucleon, cancel out. The latter calculation has been extended to the case of two-nucleon interaction in empty space and in nuclear matter. Comparison of the interaction potentials in these cases shows that in the second case the interaction energy decreases more rapidly with increasing separation than in the first. This indicates that there is a suppression of the two-nucleon interaction within nuclear matter.

Since this theory employs mesons of the neutral scalar type, one cannot expect it to give results in quantitative agreement with experiment. However, it points to a possible correlation between nuclear structure and meson theory. Further developments of this theory may perhaps lead to the solution of various nuclear problems.

CHAPTER II

ON SOME ASPECTS OF NUCLEAR SHELL STRUCTURE WITH STRONG SPIN - ORBIT COUPLING

The existence of closed shells in nuclei is indicated by the particular stability and abundance of nuclear systems with certain numbers of protons or neutrons[1]. Several schemes based on the independent particle model had been proposed almost at the same time to account for these so-called *magic* numbers. The evidence was brought forward by Feenberg and Hammack, Nordheim, Mayer and Haxel, Jensen and Suess (all in 1949) who considered the correlations between the shell structure proposed and the nuclear spins, magnetic moments, quadrupole moments, isomerism and beta-decay[2]. The independent particle picture with the assumption of strong spin-orbit coupling, proposed independently by Mayer and Jensen, *et al.* has thus far proved to be the most successful in explaining various properties of nuclei.

As said before, the single-particle model assumes that each nucleon moves independently in a fixed potential which represents the average effect of all the other nucleons. The total wave function is then simply a linear combination of the products of the single-particle wave functions. In order to calculate these functions a definite potential must be chosen. The potentials employed most often are the oscillator and square well potentials, both of which have the advantage of giving simple functions.

The wave function of a nucleon in the oscillator potential ($U \sim r^2$) can be characterized by 3 quantum numbers n_1, n_2 and n_3,

1) W.Elsasser, J.de Phys. et Rad. 5, 625 (1934);
E.Wigner, Phys.Rev. 51, 947 (1937);
H.E.Suess, Zeits.f.Naturf. 2a, 604 (1947);
M.G.Mayer, Phys.Rev. 74, 235 (1948).

2) E.Feenberg, K.G.Hammack, Phys.Rev. 75, 1877 (1949);
L.W.Nordheim, Phys.Rev. 75, 1894 (1949);
M.G.Mayer, Phys.Rev. 75, 1969 (1949); Phys.Rev. 78, 16 (1951);
O.Haxel, J.H.D.Jensen, H.E.Suess, Naturwiss. 12, 376 (1948);
Phys.Rev. 75, 1766 (1949); Z.Phys. 128, 301 (1950).
For the correlation between magic numbers and quadrupole moments, see W.Gordy, Phys.Rev. 76, 139 (1949) and R.D.Hill, Phys.Rev. 76, 998 (1949).

corresponding to the three normal vibrations in the x,y,z-directions. The total energy, measured from the bottom of the well, is $h\nu(n+3/2)$, where $n = n_1 + n_2 + n_3$ and ν is the classical oscillator frequency. Since the eigenfunctions belonging to this energy are proportional to the product of three Hermitian functions with the sum of their orders equal to n, there is a $(n+1)(n+2)$-fold degeneracy, if the spin degeneracy (factor 2) is also taken into account. Because of the odd or even nature of a given Hermitian function, each level has a definite parity associated with it. Since the oscillator potential is a central field, we can expand the product of the 3 Hermitian functions in series of spherical harmonics:

$$H_{n_1}(x)\, H_{n_2}(y)\, H_{n_3}(z) = \sum_{l,m} f_l^n(r)\, P_l^m(\cos\theta) e^{im\varphi}$$

The left-hand side has the parity of n, while each term on the right-hand side has the parity of the corresponding l. Therefore only l values with the same parity as n can appear on the right.

Different l's compatible with a given n are obtained as follows. For n even, we have (where, in the sum Σ', l has the values 0,2,4 ......)

$$(n+1)(n+2) = \Sigma' 2(2l+1)$$
$$= \sum_{l'=1}^{m} 2(4l'-3) = 2m(2m-1)$$

The value of m in which we are interested is obviously $m = (n+2)/2$. Similarly for n uneven the number of different l's is $(n+1)/2$. The highest l in both cases is $l = n$; in fact if n is even, for example, then $m = (n+2)/2$ is the maximum value of $l' = (l+2)/2$.

It is very improbable that U(r) will vary exactly as r^2. Any deviation of U(r) from harmonicity will result in a splitting into separate levels of definite angular momentum. If the potential varies faster than r^2, the levels will split in such a way that for the l's of a given n, increasing l means increasing stability[1].

We thus arrive at the general sequence of the lower levels in a square well (U(r) varies as r^∞, so to speak). The relative position of these levels is obtained from the requirement that the wave solution inside the well must be joined with equal

1) For a proof of this, see e.g. Fermi's book (p.169), referred to on page 10.

value and derivative at the nuclear radius R to the outside wave function.

As is well known, the radial part of the wave solution for the Schrödinger equation inside the well, multiplied by r, is

$$C_1 r^{1/2} J_{l+1/2}(k'r) \tag{1}$$

and that outside the well

$$C_2 r^{1/2} H_{l+1/2}(ikr)\,, \tag{2}$$

where $k = \left(\frac{2m}{\hbar^2}E\right)^{1/2}$ and $k' = \left(\frac{2m}{\hbar^2}(D - E)\right)^{1/2}$, E being the kinetic kinetic energy, m the reduced mass of the particle and D the depth of the well. H is that Hankel function which vanishes for the large positive imaginary values of the argument. For an infinitely high wall

$$J_{l+1/2}(k'R) = 0, \tag{3}$$

which at once determines the positions of the energy levels in terms of R. For a well of finite depth the condition of continuity gives

$$\frac{\partial}{\partial R}\log\left\{R^{1/2} J_{l+1/2}(k'R)\right\} = \frac{\partial}{\partial R}\log\left\{R^{1/2} H_{l+1/2}(ikR)\right\}, \tag{4}$$

which can be shown to be equivalent to

$$J_{l+3/2}(k'R)\Big/J_{l+1/2}(k'R) = H_{l+3/2}(ikR)\Big/H_{l-1/2}(ikR) \tag{5}$$

which determines E in terms of R and D.

A change in the depth D does not produce a change in the sequence of the levels, but only in their number. Comparison of the positions of the levels of a well with infinite depth with one having a finite depth, but same radius, shows that the general level pattern is approximately the same, but that the levels of the latter are depressed as compared with the corresponding levels of the former and that this depression increases as the level lies higher[1].

On account of the nucleon possessing an intrinsic spin, each level with an orbital momentum l will give rise to two states of total angular momentum $l + 1/2$ and $l - 1/2$. The specific assumption

1) cf. H. Margenau, Phys. Rev. 46, 613 (1934).

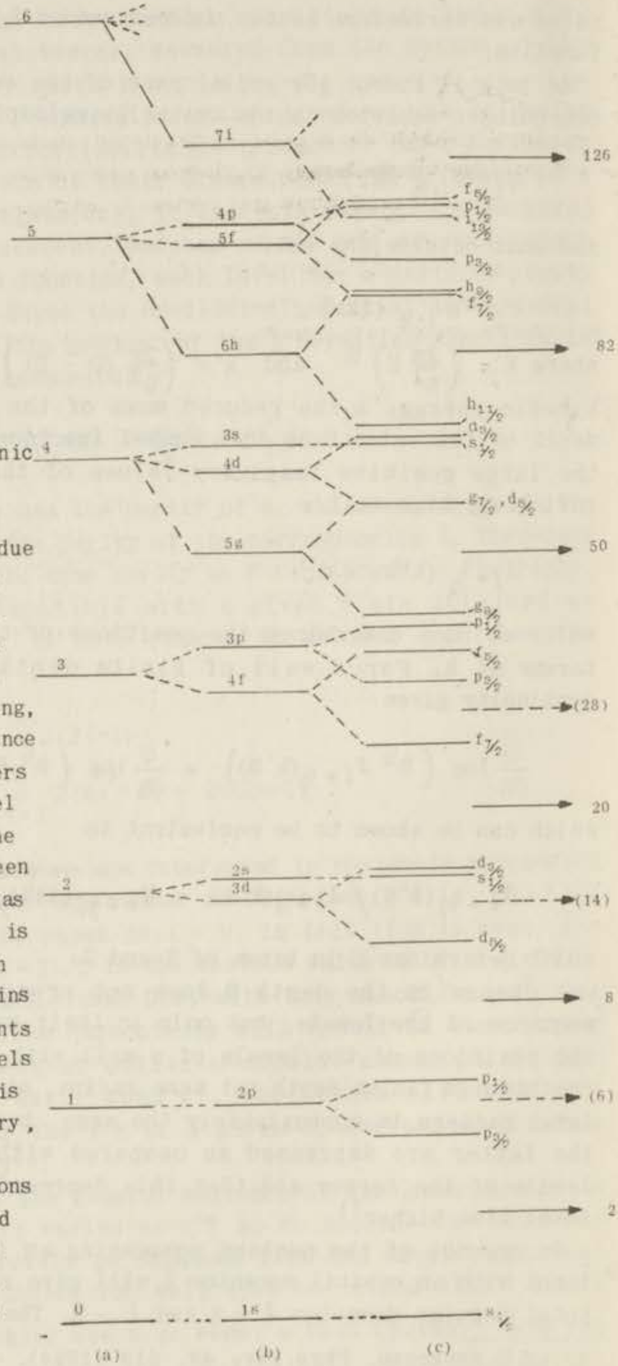

Figure 1

The Mayer-Jensen scheme. (a), The levels of the harmonic oscillator; (b), splitting of the oscillator levels due to the departure of the potential in the direction of a square well; spin-orbit splitting, showing the occurrence of the magic numbers Up to the 5g-level approximately, the level order between the magic numbers as indicated in (c) is suggested by an analysis of the spins and magnetic moments for the higher levels the given order is somewhat arbitrary, being probably different for protons and neutrons and changing as the filling-up of a shell proceeds of the Mayer-Jensen scheme is that there is a strong spin-orbit coupling which has the effect of depressing the $l + \frac{1}{2}$ level and raising the $l - \frac{1}{2}$ level. This coupling increases with increasing orbital momentum and, since for a given n the state with highest l lies lowest (if the potential varies faster than r^2), the splitting of the lowest level may be so large that the $l + \frac{1}{2}$ state belonging to this level may come nearer to the group of levels belonging to n - 1 than to the nearest level of its own group. Since the $l - \frac{1}{2}$ state belonging to the same l has been shifted upwards, the splitting may give rise to an energy level pattern which shows a comparatively large gap at the "original" position of the unsplit state under consideration. The filling up of the levels up to such a gap would then be expected to lead to a configuration which is very stable compared with other neighbouring configurations. The foregoing shows that the number of particles corresponding to such a stable configuration is given by

$$\sum_{n=0}^{N} (n+1)(n+2) + 2(N+1)$$

since the maximum value of l for a given n is equal to n, and the degeneracy of the $l + \frac{1}{2}$ state is $2(l + 1)$. This expression is equal to

$$\tfrac{1}{3}(N + 1)(N^2 + 2N + 6),^{1)}$$

which for N = 0, 1, 2, 3, 4, 5, 6 has the values 2, 6, 14, 28, 50, 82, 126. Although all these numbers correspond in reality to comparatively stable configurations, the stability is less striking as the numbers are smaller. This fact is again in accord with the foregoing scheme. As n decreases, the maximum value of l in each n also decreases, and hence for low n the spin-orbit coupling may be insufficient to bring about the level grouping just described. Then the numbers corresponding to maximum stability are simply those corresponding to the successive filling up of the groups of levels belonging to the successive quantum numbers n = 0, 1, 2, ... They are

$$\sum_{n=0}^{N} (n+1)(n+2) = \tfrac{1}{3}(N+1)(N+2)(N+3),$$

1) This formula was first stated empirically by E. Bagge, Naturwiss. 12, 375 (1948).

i.e. 2, 8, 20, ...Thus the occurrence of the magic numbers 2, 8, 20, 50, 82, 126 is readily interpreted by this model.

It is to be noted that most of the evidence concerning the existence of closed shells is rather indirect. The most direct test would be to consider the nuclear masses, especially those with proton or neutron numbers in the neighbourhood of the magic numbers. At present the information about nuclear masses presents a somewhat confusing picture, pointing to the possible existence of other magic numbers besides those mentioned above and to the possible non-existence of the number 20[1]. However, definite conclusions cannot be made yet, because of the scarcity of nuclear mass data in some mass number regions. It seems however as if these data definitely point to the non-occurrence of the number 20. In this connection computations on nuclear masses were made by Low and Townes[2], who found no indication for the stability of nuclei with 20 protons or neutrons. The same conclusions were made by others[3]. The closing of a shell at the number 20 therefore seems questionable, this being the more so if we consider that the large isotopic spread of Ca[4], which has been the most important reason for chosing this number as a magic number, may also be accounted for by the exceptional stability of Ca^{40} and the existence of a closed neutron shell at 28, which makes Ca^{48} stable.

Obviously the possible non-occurence of a closed shell at 20 offers no serious threat to the Mayer-Jensen theory, as long as there is one at 28. As regards the higher magic numbers, the only encouraging fact is that the nuclear mass data do not seem to contradict the occurrence of these numbers. Definite statements about this fact and the possible occurrence of other

1) H.E.Duckworth, R.S.Preston, Phys.Rev. 82, 468 (1951).
See also: Duckworth, Kegley, Olson, Stanford, Phys.Rev. 83, 114 (1951).

2) W.Low, C.H.Townes, Phys.Rev. 80, 608 (1950).

3) J.Sanada, Y.Yoshizawa, Phys.Rev. 83, 663 (1951);
A.H.Wapstra, Phys.Rev. 84, 838 (1951). See also ref.1 on this page.

4) Not te be confused with the number of isotopes. Ca, with 20 protons, has 6 isotopes, which is not too unusual for this region of the periodic table. The difference in mass number between its heaviest and lightest isotopes is 8 mass numbers, which is quite outstanding, since this difference does not exceed the number 4 for other elements in this neighbourhood.

numbers will have to await the results of further measurements over the whole mass number region.

The theoretical basis underlying the assumption of strong spin-orbit coupling is at present somewhat confused, in spite of the fact that it has already been subjected to a critical study for quite a number of years.

Following Inglis[1], one may, as a first approach, describe the spin-orbit coupling of a nucleon in the same way as in the atomic case. Classically the spin-orbit interaction energy of an electron consists of two terms. First, the magnetic term arises from the fact that, in a frame of reference in which the electron is momentarily at rest and the nucleus moves, the electron is subjected to a magnetic field **H**. The interaction energy between the electron magnetic moment $\boldsymbol{\mu}$ and **H** is

$$-\boldsymbol{\mu}\cdot\bar{\mathbf{H}} = \frac{e}{mc}\,\mathbf{S}\cdot\bar{\mathbf{H}}\ ,$$

where $\bar{\mathbf{H}}$ denotes the time average and **S** the spin. For $e\,\mathbf{H}/mc$, which is the angular velocity $\boldsymbol{\omega}_L$ of the momentary spin-precession, we may write

$$\boldsymbol{\omega}_L = \frac{e}{mc}\,\mathbf{H} = -\frac{e}{mc^2}\,\mathbf{v}\times\mathbf{E} \simeq \mathbf{v}\times\mathbf{a}/c^2\ ,$$

where **v** is the velocity and **E** the electric field strength in the system where the nucleus is at rest; **a** is the corresponding acceleration. In this system, however, the spin axis will at the same time perform the momentary so-called Thomas precession[2]

$$\boldsymbol{\omega}_T = -\mathbf{v}\times\mathbf{a}/2c^2 = -\tfrac{1}{2}\boldsymbol{\omega}_L\ ^{2)}.$$

This is a purely kinematical effect, which Thomas derived from relativity considerations and which exists whatever the cause of the acceleration **a**.

This precession gives rise to an energy term $\boldsymbol{\omega}_T\cdot\mathbf{S}$ Hence we conclude that the total interaction energy is equal to half the average magnetic interaction energy, thus introducing the well-known Thomas factor ½ in the total spin-orbit energy. Since $\mathbf{v}\times\mathbf{a}$ is proportional to the orbital angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$, the energy has the well-known form $\mathbf{s}\cdot\mathbf{l}\times$ constant.

1) D.R. Inglis, Phys. Rev. 50, 783 (1936).

2) See for example, D.R. Inglis S. Dancoff, Phys. Rev. 50, 784 (1936).

The simple relation $\omega_T = -\frac{1}{2}\omega_L$ is due essentially to the fact that the magnetic field at the electron arises from the same field which causes the acceleration. In the nuclear case this is not true. Here the nucleon is kept in its orbit by non-electric binding forces; the electric forces are relatively unimportant, being repulsive for protons and totally absent for neutrons. If the electric field were sufficiently strong (and of the right sign) to keep the proton in its orbit, as is the case with the electron, the magnetic effect would predominate over the relativistic effect in the proton case; since this is not so, the relativistic term may be expected to dominate[1]. The same is true for the neutron. Thus the interaction energy has a different sign from the analogous term in the atomic case. Hence the nuclear doublets for protons and neutrons may be expected to be inverted as compared with atomic doublets; i.e. the $l + \frac{1}{2}$ state has the lowest energy.

Although in this way the required order of the level splitting is obtained as far as the Mayer-Jensen scheme is concerned, the magnitude of the splitting is probalby too small, as was shown by Dancoff[2]. By assuming that the ground and excited level of He^5 at about .25 Mev. above the ground state forms a doublet arising from spin-orbit coupling of the odd neutron in a p-orbit, and making a rough estimate of the p-orbit Thomas splitting, he found that the latter gives rise to a doublet which is several orders of magnitude too small. In passing, it should be noted that, to date, it seems more likely that the excited level in question does not exist at all[3].

However, stronger spin-orbit interaction is not irreconcilable with present field theories of nuclear forces. Thus the vector meson theory, with both vector and tensor coupling between meson field and nucleons, can give large spin-orbit interaction if the *non-static* interaction term to the first order of approximation in the nucleon velocities is taken into account[4]. In this connection calculations have been made by Gaus[5] which intend to

1) It is of course assumed that the nuclear forces do not give rise to a magnetic field.

2) S.M.Dancoff, Phys.Rev. 58, 326 (1940).

3) S.Bashkin, F.P.Mooring, B.Petree, Phys. Rev. 82, 378 (1951).

4) L.Rosenfeld, Kgl. Dansk.Vid. Selsk. Math.-Fys. Medd.23, no. 13 (1945): See also his book *Nuclear Forces*,North-Holland Publ. Co. (1948).

5) H.Gaus,Zeits.f.Naturf. 4a, 721 (1949). See also: J.Keilson, Phys. Rev. 82, 759 (1951).

show that this type of interaction can give a sufficiently large splitting.

The most important test for the validity of a given theory of nuclear structure is its capability of explaining the variations of the nuclear spin of even-odd nuclei with the mass numbers (the spins of even-even nuclei are zero). To be sure, these variations, together with the occurrence of the magic numbers, originally served as a guide in the search for a suitable theory. Of the theories mentioned earlier the Mayer-Jensen theory has proved to be the most successful in explaining these variations. There are a few exceptional cases which do not fit into the Mayer-Jensen scheme; however, some of these deviations have been satisfactorily accounted for in the framework of this theory[1].

In explaining these variations the Mayer-Jensen theory (and also the other less successful theories) avails itself of a further assumption, namely that the spin and magnetic moment of an even-odd nucleus are due entirely to the odd nucleon. This, again, is based on the empirical fact mentioned above that, with no known exception, the even-even nuclei have spins and magnetic moments zero. If this assumption were true, the magnetic moments, as a function of the spin, would be expected to lie on two pairs of lines known as the Schmidt lines[2], corresponding to the values expected for a single proton or neutron with spin $l + \frac{1}{2}$ or $l - \frac{1}{2}$.

In reality the magnetic moments usually lie between these lines although there is a definite tendency toward grouping near these lines, thus enabling one to ascribe a definite orbital angular momentum to the nucleus considered. Hence the Mayer-Jensen theory, built on the assumption that the Schmidt curves are strictly true, cannot be taken seriously, unless the deviations from the Schmidt curves can be explained by a modification of the theory which does not destroy its basic features. In such a modification the assignment of l and j to the odd nucleon obviously must be preserved. (j is the total spin).

In order to account for these deviations several explanations have been offered. The first one was proposed by Nordheim[3], who suggested that the ground state may actually be a mixture of a

1) cf. L.Rosenfeld, Physica 17, 461 (1951); D.Kurath, Phys. Rev. 80, 98 (1950).

2) T.Schmidt, Zeits. f. Phys. 106, 358 (1937).

3) See ref.2) on page 15. See also A. Bohr, Phys.Rev. 81,134 (1951).

state $l + \frac{1}{2}$ and a state $l - \frac{1}{2}$. However, a considerable mixing of such states is required, which is rather unlikely if the Mayer-Jensen scheme is to be valid, seeing that the states which must be mixed differ widely in energy. Furthermore, l then ceases to be a good quantum number for the odd nucleon and hence for the core; hence the interaction between odd nucleons and core cannot be represented by the usual central type of force, as assumed by the Mayer-Jensen theory.

Another interpretation has been offered by Foldy and Milford[1]. Since l and j of the odd nucleon are to remain constants of motion, they will suffer changes in direction only and not in magnitude, assuming of course that there are enough unoccupied quantum numbers for the odd nucleon. These changes are transferred to the core, which thus acquires orbital angular momentum. By assuming that tidal forces are responsible for this transfer of orbital angular momentum, they calculated the deviations from the Schmidt dcurves and obtained results in the right direction.

This picture fails, among others, to give any deviations from the Schmidt curves for nuclei with total spin $\frac{1}{2}$. Yet in this respect most such nuclei do not behave differently from others.

A further explanation has been offered independently by Bloch, de-Shalit and Miyazawa[3]. Besides giving a reasonable account of the empirical facts, it has the additional advantage of being compatible with a strict adherence to the shell model, which is not the case with the two aforementioned theories. They assumed that, on account of its binding to the core, the intrinsic magnetic moment of the odd nucleon is partly suppressed("quenched"); the amount of deviation from the "usual" values, $\mu_p = 2.79$ and $\mu_n = -1.91$ nuclear magnetons, being readily calculated from the Schmidt formulas, once the orbital moment is assumed known.

By drawing a curve of the amount of quenching against the proton or neutron number, Bloch found that the amount of quenching for both protons and neutrons varies in a regular way and is least towards the completion of a shell. The regularity of the variation seems to corroborate the assumption of quenching, although it is not yet clear why the quenchings should vary in this particular way.

There are other indications which point to an explanation in

1) L.L.Foldy, F.T.Milford, Phys.Rev. 80, 751 (1950).

2) F.Bloch, Phys.Rev. 83, 839 (1951);
A.de-Shalit, Helv.Phys. Acta 24, 296 (1951);
H.Miyazawa, Prog.Theor.Phys. 6, 263 (1951).

terms of quenching. Nearly all the magnetic moments of the odd nuclei fall in one of the two regions bounded by the Schmidt curves and the so-called Dirac-lines; the latter are obtained in the same way as the Schmidt curves except that μ_p and μ_n are now equal to the values predicted by the Dirac equation, namely 1 and 0 nuclear magnetons, respectively.

Another indication in favour of such an explanation is furnished by the magnetic moments of *self-conjugated* nuclei such as Li^6, B^{10}, N^{14} and Na^{22}. If we assume that the odd proton and neutron are in the same state and have the same spin directions, i.e. oppositely directed intrinsic magnetic moments, one can expect the amount by which the moment of each is suppressed to be equal and opposite, so that these will cancel each other. Hence the experimental values can be expected to be approximately equal to the sum of the two free-nucleon values, as calculated by the Schmidt formulas. The following table shows this to be the case indeed.

Nucleus	Spin	Assumed state and spin of odd nucleons	magnetic moments calculated	observed
Li^6	1	even, ½	0.88	0.82
B^{10}	3	odd, 3/2	1.88	1.80
N^{14}	1	odd, ½	0.37	0.40
Na^{22}	3	odd, 3/2	1.88	1.80

As regards the theoretical interpretation of the quenching, meson theory seems to offer the most obvious starting-point. From the fact that the deviation of the intrinsic magnetic moment of a *free* nucleon from the value predicted by the Dirac theory (anomalous moment) can be accounted for qualitatively as arising from the presence of a virtual meson cloud around the nucleon, it seems likely that in general the variation of the intrinsic magnetic moment (and hence also the total magnetic moment) can be explained as arising from the variation of the anomalous part on account of the modification of the meson field due to the odd nucleon being embedded in nuclear matter.

Schiff, in his non-linear meson theory of nuclear forces discussed in Chapter I, has already touched upon this matter. According to this theory the meson field of a nucleon is expected to be much less when the nucleon is within the nucleus than when it is in empty space. That this may lead to a reduction of the anomalous magnetic moment is quite conceivable. However, a definite statement on this point will have to await the results of similar investigations of the other meson theories besides the scalar one considered by Schiff.

Miyazawa suggested that the suppression of the intrinsic magnetic moment may be related to the Pauli principle. His argument runs as follows. Consider a Fermi gas composed of nucleons with momentum up to P. In this nucleus the nucleons can undergo virtual transitions only to unfilled momentum states. Consequently, for a nucleon with momentum zero, say, the virtual mesons with momentum less than P are lacking. This modification of the meson field gives rise to a change in the anomalous part of the intrinsic moment. For a free nucleon it is likely that the virtual mesons with low momentum contribute most to the anomalous magnetic moment, for otherwise the nucleon recoil current in the case of the neutron (it is zero for the proton) would become so large that the approximate equality of the anomalous parts of the proton and neutron magnetic moments ($\mu_p - 1 \approx |\mu_n|$) are destroyed.

Hence for a nucleon in a nucleus a considerable decrease is expected in the anomalous magnetic moment. Miyazawa's calculation shows that the anomalous moment of the bound nucleon is about half that of a free nucleon.

Note:

I am greatly indebted to P.F.A.Klinkenberg for the following remarks on the Bloch curve.

(1) The same type of curve is obtained if, instead of plotting the defect of the intrinsic magnetic moment against the number of protons or neutrons, simply the differences between the observed moments and the theoretical Schmidt values are plotted against the proton or neutron number.

(2) In Bloch's figure the neutron points nearly always lie below the proton points; and hence they probably lie on different curves, and not on one as suggested by Bloch.

The first remark implies that the correlation between magic numbers and the periodicity in the deviation of the intrinsic magnetic moments cannot be used as an argument in favour of the assumption of the quenching of the odd nucleon's intrinsic moment. This statement is somewhat more general than that of Bloch and, like that of Bloch, it contradicts the assertion, which is often made, that the shell structure is in no way reflected in the variation of the magnetic moments.

The second remark implies that an eventual reduction of the intrinsic moment is different according as the odd nucleon is a proton or a neutron. Klinkenberg points out that there is more reason for believing that the magnetic moments of the odd proton and neutron undergo the same percentage change, instead of changing by the same amount. Indeed, the experimental Schmidt curves can be described best by taking μ_p and μ_n equal to 1.5 and -1.0 nuclear magnetons, respectively; i.e. both are changed by approximately 47 %.

CHAPTER III

ON THE ENERGY LEVELS OF SOME LIGHT NUCLEI

In Chapter I we mentioned that, although the assumption of independent motion is probably an unsuitable basis for calculating energy levels, there may be some cases where, in the lowest states, the approximation of independent motion may be a reasonably good one. In nuclei which, according to the shell model, are of the closed-shell-plus-one type, the odd nucleon starts a new orbit, so that on the average it keeps somewhat on the outside of the closed shell (or core). The latter acts as the source of a field which, as a first approximation, we shall assume to be a central square well in the case of a neutron and the same with the Coulomb potential added to it in the region $r > R$ in the case of a proton. The lowest excited states of this kind of nucleus may probably be explained as one-nucleon states, determined by the field of the unexcited core[1].

The order of magnitude of the energy required to excite the core to its lowest state of excitation may be judged, for light nuclei, from the fact that the lowest excited state of C^{12} ($p_{3/2}$ shell closed according to the Mayer-Jensen model) is 4.47 Mev.[2], and that of O^{16} ($p_{1/2}$ shell closed) is 6.05 Mev. above the ground state. For those light nuclei belonging to the type under consideration the energy levels less than (say) 4 Mev. above the ground state may therefore be tentatively interpreted as one-nucleon states of excitation. As the atomic number increases, the lowest excited state of the core gets roughly lower and lower on the energy level diagrams, so that we must expect this interpretation of the lowest levels to become more and more uncertain. Already for Ne^{20} (2s shell probably closed) the lowest excited state is only 1.5 Mev. above the ground state.

According to the Mayer-Jensen scheme, the numbers 2, 6, 8, 14 mark the completion of particularly stable structures. If another nucleon is added to such a system, its comparatively

1) This means that in spite of all differences, there should be a certain analogy to the lowest excited states of the alkali-type of atoms, where only the valence electron is excited.

2) cf. Hornyak, Lauritsen, Morrison and Fowler, Rev. Mod. Phys. 22, 291 (1950).

wide separation from the rest of the nucleus will result in its binding energy being somewhat smaller than that of the other constituent nucleons, which is about 8 Mev. However, because of the low mass number, such nuclei are of the type alpha-nucleus-plus-one-nucleon, so that the binding energy of the odd nucleon is in reality considerably smaller than that of the others, because of the additional effect of alpha-grouping[1)] in the case of the latter. Hence the occurrence of very low binding energy in itself is no indication for the beginning of a new shell, as is often believed[2)]; such a beginning is indicated rather by an irregularity in the variation with the mass number of the odd-nucleon binding energy in nuclei of the type 4n + 1. In this way the number 8 was obtained[3)]. The binding energies do not provide any evidence for the closing of a shell at 6. The number 2 follows from the particular stability of the alpha particle; it is to be noted, however, that the stability of the alpha particle cannot be used as an argument in favour of the shell model, because it can also be explained in other ways.

In the rest of this chapter we shall be concerned with the lower levels of nuclei with 6 or 8 protons (and neutrons) in the core and one proton or neutron on the outside. The low energy regions of C^{13}, N^{13}, O^{17}, F^{17} have recently been extensively investigated[3)], so that the experimental results can more or

1) For light nuclei there is considerable evidence (e.g. high stability of "alpha" nuclei, binding energy per nucleon in alpha particle is only somewhat less than the average binding energy in these nuclei) which makes it reasonable to approximate the nucleus by a model in which the nucleons are assumed to group together into comparatively loosely bound alpha "clusters." This tendency to alpha clustering (to be distinguished from the older picture of the alpha particles retaining their indivuduality in the nucleus) needs not necessarily be at variance with the shell model; one can even find arguments to show that the shell model favours such a clustering (see e.g. L.Rosenfeld, ref. 2, page 11) However, it is still far from being clear how the "alpha-particle" model fits into the shell model.

2) See, for example, M.Verde, Helv.Phys.Acta 23, 501 (1950).

3) cf H.A.Bethe, R.F.Bacher, Rev.Mod.Phys. 8, 82 (1936), § 33.

4) a.G.Goldhaber, R.M.Williamson, Phys.Rev.82, 495 (1951); b.J.Rotblat, Phys.Rev.83, 1271 (1951); c.R.A.Laubenstein, M.J.W.Laubenstein, L.J.Koester, R.C.Mobley, Phys.Rev.84, 12 (1951); d.R.A.Laubenstein, M.J.W.Laubenstein, Phys.Rev.84, 18 (1951).

less be relied upon to give a complete picture of the nuclear level patterns. This is supported by the fact that the level structures of the mirror nuclei in both cases appear to be very similar, as one would expect.

The ground state of C^{13} has spin ½ [1] and magnetic moment .7023 [2] nuclear magnetons which, according to the Schmidt formulas, corresponds to a p-state of the odd neutron, in agreement with the Mayer-Jensen scheme. Its lowest excited levels are at 3.08 [3], 3.68 and 3.88 Mev. above the ground state (all bound levels); the fact that there exists a doublet at about 3.7 Mev., has only recently been established by Rotblat, *et al.*[4] from the reaction C^{12} (d,p) C^{13}.

Rotblat has also obtained the parities and possible spins[5] of these states by studying the angular distribution of protons from the reaction just mentioned[6]. The assignments of Rotblat are based on the curves calculated by Butler[7] for the angular distribution in (d,p) and (d,n) reactions. They are:

3.08 Mev.: even parity and spin ½, i.e. $s_{1/2}$;

3.68 Mev.: odd parity and spin ½ or 3/2;

3.88 Mev.: even parity and spin 3/2 or 5/2.

The assignment to the 3.08 Mev. level is in agreement with the observation by Thomas[8] that the gamma-ray radiation from this level to the ground state is an electrical dipole radiation.

As regards the energy levels of N^{13}, recently much research

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1) F.A.Jenkins, Phys.Rev. 74,355 (1948).

2) H.L.Poss, Phys.Rev. 75, 600 (1949);
R.H.Hay, Phys.Rev. 60, 75 (1941).

3) R.Malm, W.W.Buechner, Phys.Rev. 81, 519 (1951).

4) Rotblat, Burrows, Powell, see ref. 4) b. on page 29.

5) Due to the invariance of the nuclear system with respect to spatial rotations and inversion of the spatial coordinates, each state must be characterized by quantum "numbers" giving the total spin and parity, respectively. As regards the latter, the wave function describing the state can have either odd or even parity; in the first case the function changes sign on inversion, in the second case it does not (magnitude unchanged in both cases).

6) In addition to the aforementioned paper by Rotblat, see also Nature 167, 1027 (1951).

7) S.T.Butler, Proc.Roy.Soc. 208, 559 (1951).

8) R.G.Thomas, Phys. Rev. 80, 138 (1950).

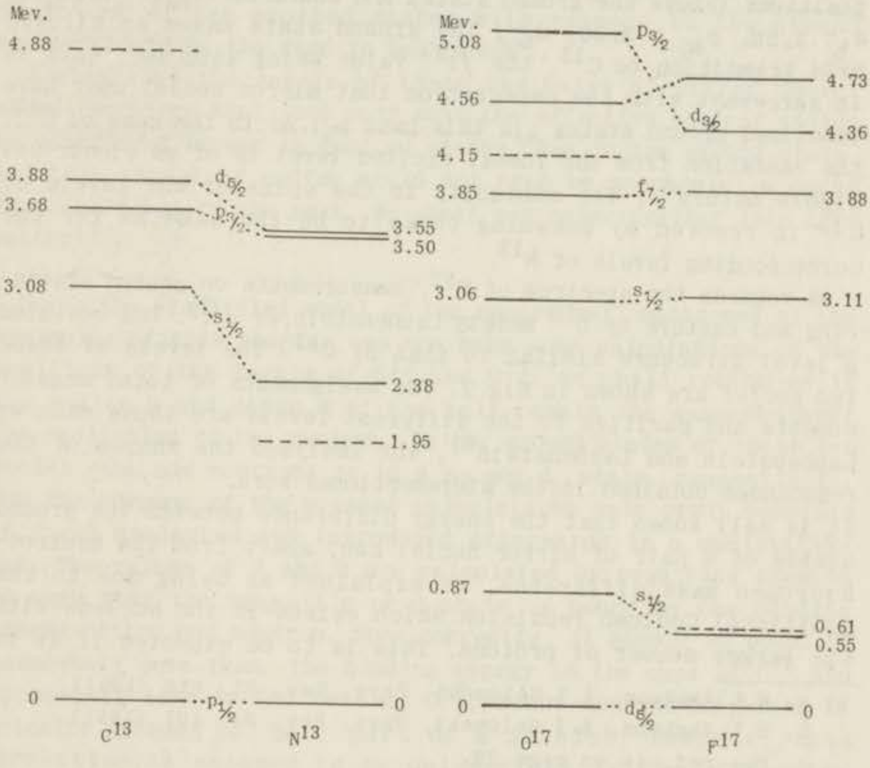

Figure 2

The lowest energy levels of C^{13}, N^{13}, O^{17} and F^{17}. In each case the broken line gives the dissociation energy of the odd nucleon. The half-widths (in Mev.) of the virtual levels are measured or estimated to be (see the corresponding literature):

N^{13} : $s_{1/2}$, 0.033; $p_{3/2}$, 0.042; $d_{5/2}$, 0.040;

F^{17} : $s_{1/2}$, 0.019, $f_{7/2}$, .003; $d_{3/2}$, 0.5; $p_{3/2}$, 0.24;

O^{17} : $p_{3/2}$, 0.04, $d_{3/2}$, 0.10.

has been done on the elastic scattering of protons by C^{12}. Analysis[1] of the experimental data obtained by Goldhaber and Williamson[2], based on the technique developed by Critchfield and Dodder[3] and extended to this kind of problem by (R.A.) Laubenstein[4], shows that these levels have the following (virtual) positions (above the ground state) and characteristics: 2.38 Mev $s_{1/2}$; 3.50, $p_{3/2}$; 3.60, $d_{5/2}$. The ground state makes an allowed beta transition to C^{13}, the ft[5] value being 4500 sec; this is in agreement with the expectation that mirror nuclei must have identical ground states (in this case $p_{1/2}$). As in the case of C^{13}, the radiation from the lowest excited level is of an electrical dipole nature[6]. The ambiquity in the spins of the levels of C^{13} is removed by assuming these to be the same as for the corresponding levels of N^{13}.

As regards the spectrum of F^{17}, measurements on proton scattering and capture by O^{16} made by Laubenstein, *et al.*[7] has revealed a level structure similar to that of O^{17}. The levels of these two nuclei are shown in Fig. 2. The assignments of total angular momenta and parities to the different levels are those made by Laubenstein and Laubenstein[8], who analysed the shapes of the resonances obtained in the aforementioned work.

It is well known that the energy difference between the ground states of a pair of mirror nuclei can, apart from the neutron-hydrogen mass difference, be explained as being due to the additional Coulomb repulsion which exists in the nucleus with the larger number of protons. This is to be expected if it is

1) a. H.L.Jackson, A.I.Galonski, Phys. Rev. 83, 876 (1951);
   b. H.L.Jackson, A.I.Galonski, Phys. Rev. 84, 401 (1951).
   c. See ref. 4a on page 29.
2) See ref. 4a on page 29.
3) C.L.Critchfield, P.C.Dodder, Phys. Rev. 76, 602 (1951).
4) See, for example, ref. 4d on page 29.
5) The product ft is taken as a measure for the degree of forbiddenness of a beta transition; the value given above represents a so-called super-allowed transition, this being an allowed transition between nuclei having similar nuclear wave functions, i.e. mirror nuclei. See, for example, Fermi's book (Chapter IV), referred to on page 10.
6) W.A.Fowler, C.C.Lauritsen, T.Lauritsen, Rev.Mod.Phys. 20, 236 (1948).
7) See ref. 4c on page 29.
8) See ref. 4d on page 29.

assumed that the force between two protons is the same as that between two neutrons, except for the Coulomb repulsion. Because of this assumption it seems plausible to expect that the energy difference between the corresponding excited states can be accounted for in the same way, so that, if the radii of the excited states are the same as in the ground states, the positions of the excited states with respect to the ground states should be the same in both nuclei.

A glance at the levels of these two pairs of mirror nuclei shows therefore that, not only must the effective nuclear radius be considered larger in excited states than in the ground state, but this change of radius would not even be monotonic in going from one level to the next. We shall not enter further into this matter[1].

Using the simplified model of the square well mentioned at the beginning of this chapter, we now make some calculations on the positions of the levels of C^{13} and O^{17}. We shall reckon as if the radius R and depth D of the well remain the same whatever the excitation of the nucleus. In the ground states of these two nuclei (the odd neutron) is in a $p_{1/2}$ and $d_{5/2}$ state, respectively. For the purpose of the present calculations spin orbit coupling is first neglected and introduced afterwards in a qualitative way. The values of R and D are calculated by requiring them to be such that the unsplit p or d state is bound by the binding energy of the odd neutron. More correctly, it should be bound by (somewhat) more than the binding energy in the case of C^{13} and (somewhat) less in the case of O^{17}, since the ground state is actually assumed to be part of a doublet. However, this correction is assumed to be unimportant in the following discussions.

The bound levels are given by formula (5), Chapter II. We have

s: $\tan y + y/x = 0$ (1)

p: $1/y^2 - \cot y/y + (x\ \ 1)/x^2 = 0$ (2)

d: $3/y^2 + 3/x^2 - 1/(1 - y \cot y) + 1/(x+1) = 0$ (3)

f: $5/y^2 + 5/x^2 + (y - \tan y)/(3 \tan y - 3y - y^2 \tan y) + (x+1)/(x^2+3x+3) = 0$, (4)

where $x = kR = (\frac{2m}{\hbar^2} E)^{1/2} R$ and $y = k'R = \left\{\frac{2m}{\hbar^2} (D - E)\right\}^{1/2} R$,

where m is the reduced mass of

1) The reader is referred to the paper: "On the relative displacement of corresponding energy levels of C^{13} and N^{13}" by J.B.Ehrman, Phys.Rev. 81, 412 (1951).

the neutron. In the ground state E is known, hence we get a relation between the unknowns R and D. In calculating the unoccupied bound states, R and D are assumed to be related in the same way. Thus the positions of the levels can be given in terms of one parameter, which can be adjusted to obtain the best possible agreement with the actual positions of the levels.

For the purpose of the present considerations we define the virtual states in the following way. Consider a beam of nucleons incident on a square well. The radial wave equations are

$$d^2G/dr^2 + \left[k^{*2} - l(l+1)/r^2\right] G = 0 \quad (r<R) \tag{5}$$

and

$$d^2G/dr^2 + \left[k^2 - l(l+1)/r^2\right] G = 0 \quad (r>R), \tag{6}$$

where $k^* = \left\{\frac{2m}{\hbar^2}(D-E)\right\}^{1/2}$ and $k = \left(\frac{2m}{\hbar^2} E\right)^{1/2}$, with E the kinetic energy (in the centre-of-mass system) of the incident particle. G is the Schrödinger radial wave function, multiplied by r.

The solutions are respectively[1]

$$G_1 = A_l(\tfrac{1}{2}\pi k^* r)^{1/2} J_{l+1/2}(k^* r) \tag{7}$$

and

$$G_2 = (\tfrac{1}{2}\pi kr)^{1/2} \left[\cos \eta_l \, J_{l+1/2}(kr) + (-1)^l \sin \eta_l \, J_{-l-1/2}(kr)\right]. \tag{8}$$

The coefficient of the Bessel function in (7) has been arbitrarily chosen so as to simplify the calculations somewhat. The reason for the particular choice of the coefficients in (8) is that, for large kr,

$$J_{l+1/2}(kr) \longrightarrow \left(\frac{2}{\pi kr}\right)^{1/2} \sin(kr - \tfrac{1}{2}l\pi)$$

$$J_{l-1/2}(kr) \longrightarrow (-1)^l \left(\frac{2}{\pi kr}\right)^{1/2} \cos(kr - \tfrac{1}{2}l\pi),$$

so that

$$G_2 \longrightarrow \sin(kr - \tfrac{1}{2}l\pi + \eta_l),$$

---

1) cf. Mott and Massey "Theory of atomic collisions" (Clarendon Press, 1949), Chapter II, § 3.

as required by the asymptotic behaviour of equation (6). η_l is calculated from the condition that, for $r = R$, the wave functions should be joined with equal values and derivatives:

$$\tan \eta_l = (-1)^{l-1} M'_l / N'_l \quad , \tag{9}$$

where

$$\left.\begin{aligned} M'_l &= kJ_{l+\frac{1}{2}}(k^*R)J'_{l+\frac{1}{2}}(kR) - k^*J_{l+\frac{1}{2}}(kR)J'_{l+\frac{1}{2}}(k^*R) \\ &\text{and} \\ N'_l &= kJ_{l+\frac{1}{2}}(k^*R)J'_{-l-\frac{1}{2}}(kR) - k^*J_{-l-\frac{1}{2}}(kR)J'_{l+\frac{1}{2}}(k^*R), \end{aligned}\right\} \tag{10}$$

where J' means the derivative with respect to the argument. By making use of the relations

$$xJ'_{l+\frac{1}{2}}(x) = xJ_{l-\frac{1}{2}}(x) - (l+\tfrac{1}{2})J_{l+\frac{1}{2}}(x)$$

$$xJ'_{-l-\frac{1}{2}}(x) = -(l+\tfrac{1}{2})J_{-l-\frac{1}{2}}(x) - xJ_{-l+\frac{1}{2}}(x) \; ,$$

(9) and (10) can be transformed into the following form:

$$\tan \eta_l = (-1)^l M_l/N_l \tag{11}$$

where

$$\left.\begin{aligned} M_l &= kJ_{l+\frac{1}{2}}(k^*R)J_{l-\frac{1}{2}}(kR) - k^*J_{l+\frac{1}{2}}(kR)J_{l-\frac{1}{2}}(k^*R) \\ &\text{and} \\ N_l &= kJ_{l+\frac{1}{2}}(k^*R)J_{-l+\frac{1}{2}}(kR) + k^*J_{-l-\frac{1}{2}}(kR)J_{l-\frac{1}{2}}(k^*R). \end{aligned}\right\} \tag{12}$$

A_l in (7) is a constant depending on the incident energy. According to (7) and (8),

$$A_l = \left(\frac{k}{k^*}\right)^{\frac{1}{2}} \left[\cos \eta_l \, J_{l+\frac{1}{2}}(kR) + (-1)^l \sin\eta_l J_{-l-\frac{1}{2}}(kR)\right] \Big/ J_{l+\frac{1}{2}}(k^*R). \tag{13}$$

A_l^2 is taken as a measure for the probability of the neutron entering the nucleus. If A^2_l possessess a sharp maximum at some energy, the l-th partial wave is in resonance with the scattering potential, and we say there is a virtual state at this energy.

In calculating the positions of the virtual levels, R and D are assumed to be related as before.

For low incident energy ($kR \ll 1$) we can make use of the following approximations:

$$J_{l+\frac{1}{2}}(kR) = \left(\frac{2kR}{\pi}\right)^{\frac{1}{2}} \frac{(2kR)^{l}\, l\,!}{(2l+1)\,!}$$

$$J_{-l-\frac{1}{2}}(kR) = \left(\frac{2kR}{\pi}\right)^{\frac{1}{2}} \frac{(2kR)^{-l-1}\, 2(2l)\,!}{l\,!} \left\{1 + \frac{(kR)^2}{4l-2}\right\}(-1)^{l},$$

so that A_l becomes:

$$A_l = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2kR)^{l+1}\, l\,! \Big/ 2(2l)\,!\, (k^*R)^{\frac{3}{2}}\, J_{l-\frac{1}{2}}(k^*R).$$

Thus, as the energy tends to zero, A_l in general also tends to zero, as is to be expected. The exceptional case when $J_{l-\frac{1}{2}}(k^*R)$ also tends to zero corresponds to the existence of a level with quantum number l at zero energy.

The virtual states defined above are the continuations of the bound states defined previously. To see this we imagine R decreasing steadily, while D remains constant. The bound levels will all move towards the *edge* of the well. As a level approaches zero binding energy, $J_{l-\frac{1}{2}}(k^*R)$ will tend to zero, according to equation (5), Chapter II (the right side of this equation tends to ∞). Thus, in the neighbourhood of the zero energy the position of this level (bound or virtual) as a function of R will be given by the formula

$$J_{l-\frac{1}{2}}(k^*R) = 0,$$

where, in $k^* = \left\{\frac{2m}{\hbar^2}(D + E)\right\}^{\frac{1}{2}}$, E can now be either positive or negative.

Thus the transition of a state from being bound to being virtual occurs continuously. Hence, for a given R and D, the order of succession of the bound and virtual levels combined is the same as when all these levels are bound.

Therefore, although in the level spectra under consideration a bound level mostly corresponds to a virtual one, there is no difficulty to account for the fact that corresponding levels have identical characteristics. However, the assumption of a square-well potential does not lead to the correct assignments of the excited levels, as may be seen by comparing the assignments in Fig. 1 (b) with those in Fig. 2. In spite of this, we have

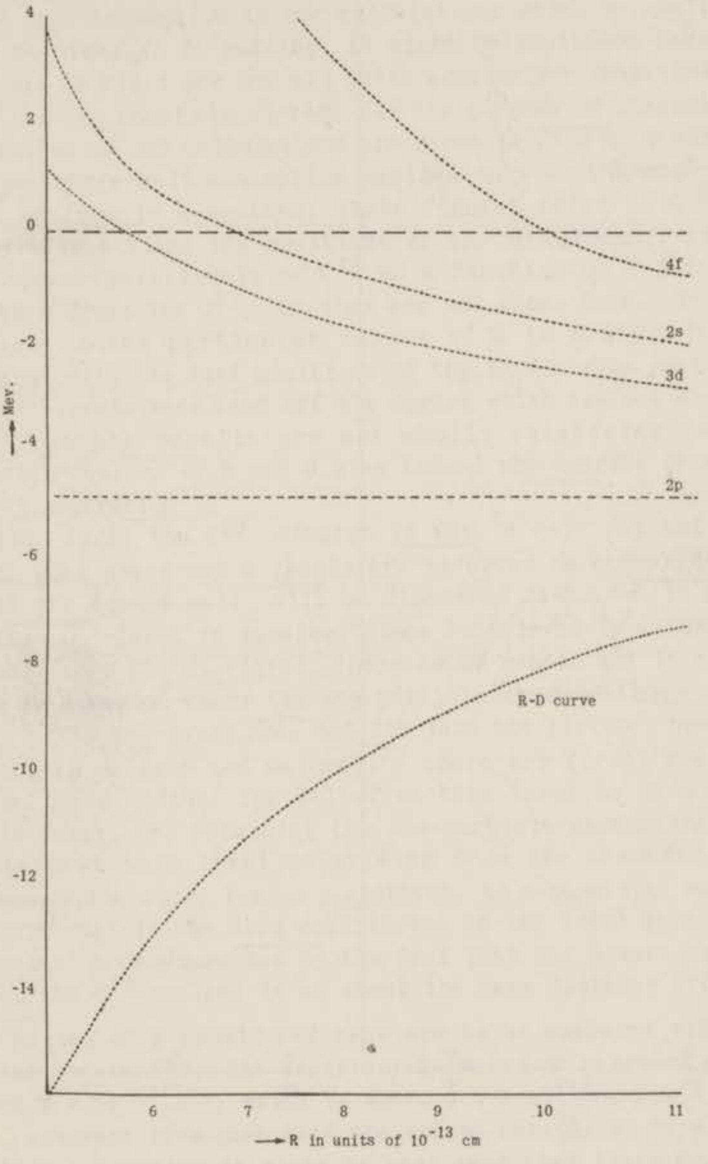

Figure 3

The positions of the 2p, 3d, 2s, 4f levels for C^{13} as a function of R. Dissociation energy of the odd neutron is taken as the zero of energy. The R-D curve is given by formula (2).

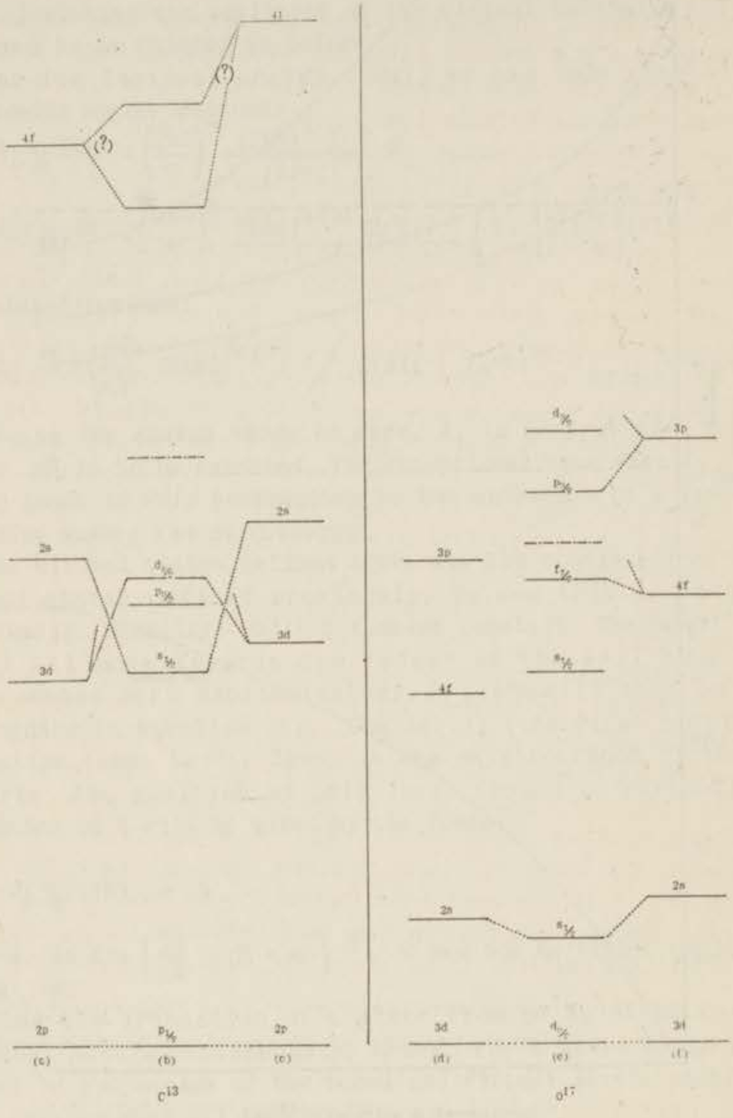

Figure 4

Comparisons with the actual positions of the levels.

(a), $R = 8 \times 10^{-13}$ cm, $D = 9.7$ Mev; (b), Levels of C^{13}-nucleus; (c), $R = 7.5 \times 10^{-13}$ cm, $D = 10.3$ Mev; (d), $R = 9.3 \times 10^{-13}$ cm, $D = 10.4$ Mev; (e), Levels of O^{17}-nucleus; (f), $R = 8 \times 10^{-13}$ cm, $D = 12.4$ Mev.

retained this assumption in our calculations, which we shall now proceed to discuss. In passing, it might be mentioned that the assignments in Fig.2 are not all quite conclusive; nevertheless, we shall assume them to be correct for the purpose of discussion.

The results of our calculations are given in Figs 3, 4 and 5. Since the square-well assumption applies only to the case when the odd nucleon is a neutron, these figures refer to C^{13} and O^{17} only. Fig.3 gives the positions of the 2p (ground), 3d, 2s and 4f square-well levels of C^{13} as a function of R. (Similar curves were drawn for O^{17}, but they are not shown here). In Fig. 4 we have chosen particular values of R in order to make comparisons with the real positions of the levels. (The positions of the O^{17} levels were read off the curves, which are not shown). We see that the results are not wholly unsatisfactory; *reasonable* values of R and D give indeed the correct order of the level separation[1].

Consider first the C^{13} -diagram in Fig. 4 (a), (b) and (c). The fact that the s and d levels are inverted as compared with those of the square well, will be discussed later on. It seems as if the $d_{3/2}$ -level is missing. Since this level is assumed to lie higher than the $d_{5/2}$-level, there is no reason why it should not lie in a region where the one particle approximation ceases to hold[2]. The $p_{3/2}$-level does not fit into the picture; however, according to Jackson and Galonski[3] there are formal reasons, based on level widths, for believing this level to be a many-particle level. Or, retaining the one-particle assumption, one may interpret this level as arising from the transfer of a p-neutron of the core, i.e. a $p_{3/2}$-neutron, to a $p_{1/2}$-state, so that only the *hole* in the core contributes to the total spin. This interpretation is supported by the fact that the lowest excited level of the C^{12}-nucleus is at about the same distance from the

1) The values of R considered here are to be compared with the values obtained from the empirical formula for the ground state $R \doteq 1.5 \times 10^{-13} A^{1/3}$, which is about 3.7×10^{-13}cm for A = 15; i.e. somewhat less than half the values considered here.

2) In this connection it might be mentioned that there probably exists another level (perhaps the missing $d_{3/2}$-level) at 5.7 Mev, according to the diagram of Hornyak, Lauritsen, Morrison and Fowler Rev. Mod. Phys. 22, 291 (1950). The two virtual levels indicated here were taken from Bockelman, Miller, Adair, Barschal, Phys. 84, 69 (1951).

3) See ref. 1 b) on page 32.

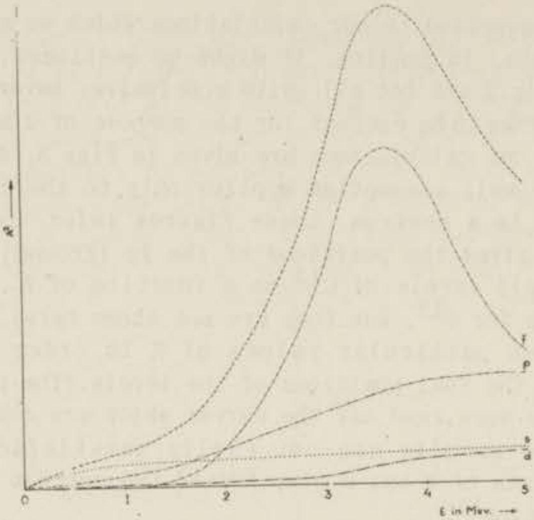

Figure 5

The shape of the C^{13} virtual level given in Fig. 4
($R = 7.5 \times 10^{-13}$ cm, D = 10.3 Mev.).

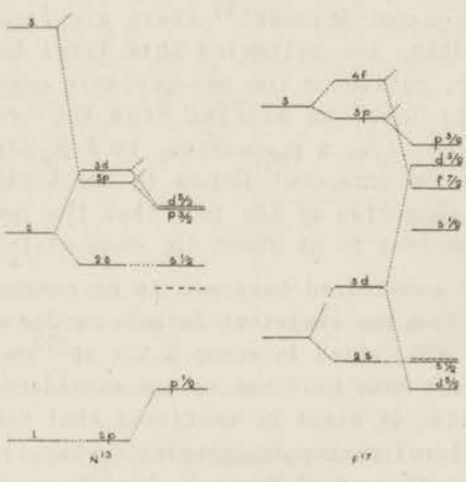

Figure 6

Level diagrams for N^{13} and F^{17}, according to Koester, *et al.*
References: Figs 1 and 2.

C^{12}-ground state as the $p_{3/2}$-level from the C^{13}-ground state.

The one-particle assignment tothe two virtual levels[1] is most probably wrong; in fact, as mentioned at the beginning of this Chapter, the one-particle limit probably lies at about 4 Mev. above the ground state. Another indication against the one-particle assignment is given by Fig.6, which gives the calculated shape of the C^{13}-virtual level (4 f) given in Fig.4 (c). We see that, although the maximum is not destroyed by the contributions of the neighbouring levels 2s and 3p, the state is so broad that one cannot properly speak of a level[2]. Increasing the*strength* of the well (given by DR^2) in order to reduce the level width only gives rise to more bound states than required by experiment.

As regards the levels of O^{17}, the fact that the ground state and the $d_{3/2}$ -state probably form a doublet shows that the foregoing assumptions about the interaction between the odd nucleon and the core are probably inadequate. In fact, such a wide separation of the doublet levels points to the existence of an interaction which is strongly spin-dependent.

It is to be noted however that,except for the ground state and lowest excited state (at 0.87 Mev.), the assignments to the levels of O^{17} are rather uncertain[3]. There is no information available about the nature of the 3.06 and 3.85 Mev.-levels,and it is assumed without more that they have the same characteristics as the 3.11 and 3.88 Mev.-levels of F^{17}, which fortunately are reasonably certain. The assignments to the two higher levels of both O^{17} and F^{17} are not very reliable. Hence the existence of a d-doublet should not yet be taken as a certainty.It is even doubtful whether the 4 higher levels are to be regarded as one particle levels; according to Laubenstein and Laubenstein[4]at least the 3.11 Mev.($s_{1/2}$) and 3.88 Mev. ($f_{7/2}$)-levels of F^{17} are probably many-particle levels)note their comparatively small level widths), and hence one can expect the corresponding levels of O^{17}tobe of the same nature. In view of all this, nothing definite can as yet be said about the validity of the one-particle assumption for these two nuclei and more particularly the square-well model for O^{17}.

The same type of calculation as the foregoing could be done

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1) See ref. 2) on page 39.

2) The actual level widths are 6.96 Mev. .011 Mev.;7.83 Mev.,.12 Mev.

3) See ref. 4 d) on page 29.

4) See ref. 4 d) on page 29.

for N^{13} and F^{17}, if the Coulomb potential is taken into account. Considering the values of R and D we are using, the addition to the square well of a Coulomb potential in the region $r > R$ will not seriously affect the positions of the levels (however, the widths of the low-lying virtual levels will be reduced considerably). For example, if $R = 8 \times 10^{-13}$cm(i.e. D of the order of 10 Mev.), the height of the Coulomb barrier Ze^2/R is about 2 Mev. Hence the square-well method is sufficient for comparing roughly the level positions of one nucleus with those of its mirror.

Such comparisons have been made and they show that, in spite of the fact that a bound level of C^{13} or O^{17} mostly corresponds to a virtual one here (see Fig. 2), fairly good agreement as to the mean level separation can be obtained by using, for example, one and the same value of R for both nuclei (D different, however, since the R-D curve is different for different nuclei). It is to be noted however that, since the higher virtual states of N^{13} and F^{17} lie high above the dissociation energy, the maximum of a resonance in this region is mostly destroyed by the contributions from neighbouring levels, so that such an agreement is probably meaningless for these levels.

For further discussion we assume that as many as possible of the levels under consideration are one-particle levels. The fact that the square well does not give the correct sequence of the levels, does not necessarily mean that the latter cannot be explained in terms of the Mayer-Jensen theory. On the contrary, an inspection of the assignments in Fig 2 shows that the $l + \frac{1}{2}$ level always occurs first on the diagram, exactly as predicted by this theory. Furthermore, this theory does not make any definite assumption about the sequence of the different l - levels belonging to the same oscillator quantum number n, so that any average potential which gives better agreement with the observed level sequence than the square well is by no means unreasonable from the point of view of this theory. Thus, for example, one can explain the inversion of the 3d and 2s-levels in C^{13}.(Fig.6) by imagining a central depression in the floor of the square well.[2] Another possibility is to consider a potential which varies less rapidly than r^2 or a r^2-potential which is less singu-

2) See e.g. Feenberg and Hammack, ref. 2) on page 15. It is interesting to note that Feenberg and Hammack suggested this inversion to account for something else, namely the observed spins and magnetic moments of some light nuclei.

lar at the origin in these cases the sequence of the levels belonging to a given n will be the reverse of that of the square-well.

Using this level sequence, Koester Jackson and Adair[1] have succeeded in giving a reasonable but purely qualitative explanation of the level sequence of the low states of some light nuclei of which the level characteristics are known. The closed-shell-plus-one type of nucleus turns out to be better adapted to this (essentially one-particle) scheme.
Their level diagrams for N^{13} and F^{17} (and hencepresumably also for C^{13} and O^{17}) are given in Fig. 6. We see that the $s_{1/2}$ -level of F^{17} does not fit into their scheme; however, as said before, this is probably a many-particle level. The other levels which are probably also many-particle levels, namely the $p_{3/2}$ -level of N^{13} and the $f_{7/2}$ -level of F^{17}, can also be excluded from this scheme, without appreciably affecting its value.

1) L. J. Koester, H. L. Jackson, L. K. Adair, Phys. Rev. 83, 1250 (1951).

SUMMARY

The assumption of one-particle motion of the nuclear constituents has in recent years proved to be a very successful basis for explaining many ground-state properties of nuclei. This suggests the possibility of making similar assumptions for excited nuclear states. This suggestion is discussed in Chapter I, which moreover contains an attempt to make the supposed one-particle motion plausible.

In Chapter II we have discussed that one-particle model which at present is favoured most, namely the Mayer-Jensen theory. Although the Mayer-Jensen theory rests upon experimental evidence (often rather uncertain) which does not always confirm it and a theoretical basis which is still far from being clearly understood, this theory seems to form a promising starting-point for discussions about nuclear structure and interpretations of nuclear data.

This becomes especially clear in Chapter III, where we have made some calculations on some nuclei for which one might expect the one-particle approximation to hold for the lower excited states. These calculations show that the correct order of level separation follows from a very simple one-particle model, in which the radius of the nuclear field is about twice as large as the conventional radius for non-excited nuclei. Furthermore, the recent assignments to these levels seem to support the Mayer-Jensen hypotheses of large spin-orbit coupling and level inversion in nuclei.

STELLINGS

I Die bestaan van die *wondergetalle* is op sigself nog geen oortuigende bewys dat die atoomkern (grond-toestand) in terme van 'n quasi-atoommodel beskryf kan word nie.

II Aangesien daar genoegsaam aanduiding is uit die bindingsenergieë dat die getal 8 wel deeglik 'n *wondergetal* is, is die vermoedens van Wapstra oor 'n moontlike ander oorsaak van die ekstra stabiliteit van die O^{16}-kern nie gangbaar nie. (Wapstra wou die nie-noodsaaklikheid van die getal 8 as wondergetal aanneemlik maak.)
A.H.Wapstra, Phys. Rev. 84, 838 (1951).

III Dit is nie onwaarskynlik dat 'n hele aantal van die energietoestande van ligte kerne soos gegee deur Hornyak, *et al.* by noukeuriger metings sal blyk te bestaan uit meer dan een dig-bymekaarliggende toestande nie.
W.F.Hornyak, T.Lauritsen, P. Morrison, W.A. Fowler, Rev. Mod. Phys. 22, 291 (1950).

IV Die teorie van Born en Yang oor die kern-skilstruktuur is nie aanvaarbaar tesame met die aanname van spin-baan koppeling nie
L.M.Yang, M.Born, Nature 166, 399 (1950),
L.M.Yang, Proc. Phys. Soc. 64, 632 (1951),
H.R.Paneth, Proc. Phys. Soc. 64, 939 (1951).

V Die aanname van Jastrow oor die wisselwerking tussen 'n proton en 'n neutron (aantrekkende potensiaal wat oorgaan in grote afstoting by kleiner-wordende verwydering in geval van singlettoestande en dieselfde met 'n veel kleiner afstoting in geval van triplet-toestande) is onaanvaarbaar.
R.Jastrow, Phys. Rev. 81, 165 (1951).

VI Die bewering van Bagge dat in die geval waar die singuliere oplossing van die tyd-onafhanklike Schrödinger-vergelyking normeerbaar is, die toestand gegee deur hierdie oplossing ook beset kan word, is onaanvaarbaar.
E.Bagge, Naturwiss. 20, 472 (1951).

VII Dit val te betwyfel of die eenheid van bestraling soos gebruiklik in röntgen-geneeskunde, naamlik die röntgen, werklik 'n juiste maat is vir die biologiese werking van die röntgenstrale.

VIII Die studie van die moderne teoretiese natuurkunde behoort meer aandag te geniet aan die meeste Suid-Afrikaanse universiteite. Veral behoort meer begrip getoon te word vir die fundamentele belang van quantum-meganika vir basiese navorsing, teoreties sowel as eksperimenteel. 'n Verbetering in hierdie opsig sal die basiese navorsing slegs ten goede kom.