Particles and Fields

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1. Atoms

Atoms are the smallest electrically neutral units of chemical elements. They consist of a positively charged massive nucleus, surrounded by a cloud of light negatively charged electrons.

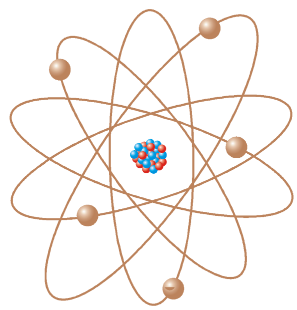

Fig. 1: Classical image of an atom (not to scale)

A simple estimate of the physical size and mass of atoms is based on Avogadro's number, the number of molecules in one mole of a pure chemical substance:

$$N_A = 6.022 \times 10^{23}. \tag{1}$$

At normal temperature and pressure: $T = 273$ K and $p = 1$ atm, the volume of one mole of pure gas is 22.41 liter. By weighing a known volume of gas one can then determine the mass of a single molecule. For inert gases such as helium (He) or neon (Ne), which do not form compound molecules, the molecular mass equals the atomic mass. For simple gases like hydrogen (H_2), nitrogen (N_2) and oxygen (O_2), the atomic mass is half the molecular mass. The smallest and lightest atom is that of ordinary hydrogen. Its mass is

$$m_H = 1.674 \times 10^{-27}\,\text{kg}, \tag{2}$$

and the diameter of its electron cloud is about 1 Å (0.1 nm). The charge of the hydrogen nucleus, the *proton* is

$$e = 1.602 \times 10^{-19}\,\text{C}. \tag{3}$$

The neutral hydrogen atom contains a single electron, with a charge opposite in sign and precisely equal in magnitude to that of the proton. The electron is very light:

$$m_e = 0.911 \times 10^{-30}\,\text{kg}. \tag{4}$$

This follows from a measurement of the charge-to-mass ratio e/m_e, as obtained for example by observing the trajectory of an electron in a known magnetic field. Specifically, a non-relativistic electron with velocity v in a constant magnetic field B moves in a circle of radius

$$R = \frac{m_e v}{eB} \quad \Leftrightarrow \quad \frac{e}{m_e} = \frac{v}{BR}. \tag{5}$$

The velocity is related to the period by $v = 2\pi R/T = \omega R$, where ω is the angular velocity. Hence

$$\frac{v}{R} = \frac{2\pi}{T} = \omega \quad \Rightarrow \quad \frac{e}{m_e} = \frac{\omega}{B}, \tag{6}$$

and e/m_e can be determined by measuring ω as a function of B. The value of the electronic charge e (3), which has to be measured separately, then provides the value of m_e as given in eq. (4).

Comparing the masses of the proton and electron, one finds that more than 99.9% of the atomic mass of hydrogen resides in the proton, with

$$m_p = 1836\, m_e. \tag{7}$$

The atoms of heavier elements are distinguished by nuclear charges which are integer multiples of the proton charge, balanced by an equal number of negative electronic charges. This suggests that an atomic nucleus contains a characteristic number of protons, and is surrounded by an equal number of electrons. However, the atomic masses of elements heavier than hydrogen are substantially larger than accounted for by the number of protons required to explain the nuclear charge. The extra mass is due to the presence of neutrons, electrically neutral particles in the nucleus with almost the same mass as the proton:

$$m_n = 1.675 \times 10^{-27}\,\text{kg} = 1839\, m_e. \tag{8}$$

It is customary to refer to the particles making up the atomic nucleus as *nucleons*; in this terminology the proton is the positively charged nucleon, and the neutron is the neutral nucleon. A general nucleus is therefore identified by two numbers: the number of protons Z, which determines the nuclear charge; and the total number of nucleons A, which is a measure for the nuclear mass. The number of neutrons then equals $A - Z$.

The chemical properties of atoms are determined predominantly by the electron cloud. Therefore the chemical properties are practically the same for atoms with the same nuclear charge Z, even if the number of neutrons is different. Such variations in A for atoms with the same Z do indeed occur in nature; in that case we speak of *isotopes* of the same chemical element. Even for hydrogen isotopes are known. The most abundant form of hydrogen ^{1}H has $A = Z = 1$; however, there is a stable isotope ^{2}H with $A = 2$ and $Z = 1$. This isotope is called *deuterium* and its nucleus contains one proton and one neutron. There is even a third isotope ^{3}H, *tritium*, with $A = 3$, but it is unstable: it decays slowly, with a half-life of 12.3 years, into the lightest helium isotope ^{3}He. In the process a neutron in the hydrogen nucleus is transformed into a proton to produce helium. The mechanism behind this transformation will be described in detail later on.

The size of the nucleus is very small compared to that of the atom as a whole, as determined by the radius of the electron cloud; typically they differ by a factor of 10^5. This implies, that most of the volume of the atom is occupied by the cloud of very light and very tiny electrons. In fact as the number of electrons is limited, and as they are almost point-like, we infer that atoms consist mostly of empty space.

Exercise 1.1
A particle with charge q in a magnetic field $\mathbf{B}$ is subject to a Lorentz force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

a. If the magnetic field is constant: $\mathbf{B} = (0, 0, B)$, and if $v_z = 0$, prove that the particle moves in a circle in the x-y-plane, perpendicular to the magnetic field, with radius

$$R = \frac{mv}{qB},$$

where m is the particle mass, and v is the tangential velocity.
b. Show that the angular velocity is given by

$$\omega = \frac{2\pi}{T} = \frac{qB}{m},$$

independent of the velocity. Compute ω for an electron in a magnetic field of B =1 T.
c. What happens if the initial velocity component $v_z \neq 0$?

Units

In particle physics energies are traditionally expressed in *electronvolts* (eV), or decimal multiples like keV, MeV of GeV. By definition 1 eV is the energy gained by a particle with unit elementary charge when it is accelerated through a voltage difference of 1 V. In terms of SI units this is

$$1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}. \tag{9}$$

The use of these units helps to avoid very small numbers; at the present stage of development, particle energies in accelerator laboratories (Tevatron, LHC) actually reach the TeV scale (1 TeV = 10^{12} eV). In cosmic ray physics the largest energies measured are in the range of 10^{20} eV, which brings us back to the scale of joules. On the other hand, the lightest particles we know (neutrinos) seem to have masses in the range of meV's. Thus particle physics spans an energy range of at least 23 orders of magnitude in energy.

Because of the relativistic relation between mass and energy, it is also customary to express masses in energy units. This works as follows: a particle of mass m has a rest energy mc^2; this rest energy can be expressed in units of eV. Often the mass is then quoted in units of eV/c^2, or one of its equivalents. For example, in these units the electron has a mass

$$m_e = 0.5110\,\text{MeV}/c^2, \tag{10}$$

whilst the proton and neutron have masses

$$m_p = 0.9383\,\text{GeV}/c^2, \qquad m_n = 0.9396\,\text{GeV}/c^2. \tag{11}$$

Especially in theoretical derivations it is often useful to chose natural units in which the velocity of light and Planck's reduced constant take unit values:

$$c = \hbar = 1. \tag{12}$$

With these conventions mass and energy are expressed in the same units (eV), whilst time and length have the inverse dimension, and are expressed in eV^{-1}; for example, the typical nuclear length scale is 1 GeV^{-1}, converted to conventional units as

$$\frac{\hbar c}{1\,\mathrm{GeV}} = 0.1973\,\mathrm{fm}. \tag{13}$$

It was first observed by Planck, that one can actually construct a fully natural system of units by also assigning a unit value to Newton's constant: $G = 1$. The natural unit of mass then is the Planck mass

$$M_P = \sqrt{\frac{\hbar c}{G}} = 1.221 \times 10^{28}\,\mathrm{eV}/c^2 = 2.176 \times 10^{-8}\,\mathrm{kg}. \tag{14}$$

which is also the unit of energy, whilst time and length are expressed in inverse Planck masses. For example, as $\hbar = c = 1$, the Planck unit of length is

$$l_P = \frac{1}{M_P} = \frac{\hbar}{M_P c} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}, \tag{15}$$

in conventional SI units.

Exercise 1.2
a. Compute the time in seconds corresponding to 1 GeV^{-1} in natural units.
b. Derive the conventional expressions for the Planck time and the Planck energy, and compute their values in SI units.

2. Subatomic forces

Within an atom various forces are at work. The main force which binds the electrons to the nucleus are the electric Coulomb forces, which are attractive between charges of opposite sign. For hydrogen, the magnitude of the force is

$$F = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{a^2} \simeq 10^{-7}\,\mathrm{N}, \tag{16}$$

where the radius a of the electron cloud is taken to be 0.5 Å.

The forces which keep the protons and neutrons bound in the nucleus are very different from and much stronger than the Coulomb forces acting between electric charges. Indeed, neutrons have no charge, and therefore no Coulomb field. Nevertheless, a single proton and neutron can form a stable bound state, the deuterium nucleus, sometimes refered to as the deuteron (D). Furthermore, as the distance between nucleons in the nucleus is about 10^{-5} times the distance between the electrons and the nucleus, the repulsive Coulomb forces between protons are about 10^{10} times stronger than the attractive Coulomb forces (16) acting on the electrons. If only electric forces were at work in the nucleus, the nucleons would immediately fly apart and no stable nuclei other than simple hydrogen could exist.

Thus nuclear forces are a separate kind of fundamental interaction, distinguished from and much stronger than the electromagnetic and gravitational forces we are familiar with in the macroscopic world. The range of the nuclear forces must be very small, as we do not see any evidence of them in phenomena on scales exceeding that of the atomic nucleus. In contrast, electric and magnetic fields can exist over large macroscopic distances, as testified by the discharges of lightning in the atmosphere, or the magnetic fields surrounding many astrophysical bodies, like planets and stars.

The transformations between different type of particles, such as that of a neutron into a proton in the decay of tritium, indicate that there is yet another type of interaction at work between subatomic particles. As these transformation processes are generally very slow compared to the characteristic time scale of the nuclear forces, they are called the *weak interactions*. Weak interactions do not give rise to the binding of particles of any type, but they are connected with –though different from– the electromagnetic interactions in a very subtle way. We return to this interconnection in a later chapter. Weak interactions play a role in nuclear fusion, e.g. in the interior of stars; as such they are of great importance for the evolution of matter and life in the universe.

Exercise 2.1

From the Coulomb force (16) and the mass of the electron (4), compute the velocity of an electron in a circular orbit of radius $a = 0.5$ Å. Express this in terms of the velocity of light: $c = 3 \times 10^8$ m/s.

Binding energies
An alternative, and sometimes preferable, way to characterize the strenght of interactions is in terms of energies rather than forces. An example is provided by the binding energy of bound states, the difference between the rest energy of the composite particle and the total rest energy of its constituents in a state as free particles, far apart.

In the case of the hydrogen atom, the binding energy of the electron and proton is the minimal energy required to ionize the hydrogen atom: $\Delta E = 13.6$ eV. This is a typical atomic energy scale. However, similar interaction energies characterize the bound state of an electron with other positively charged particles we will meet later, such as the positron or the anti-muon, even though these particles are much lighter than the proton. The point is, that the binding energy is provided by the Coulomb interaction, which is determined by the charges, not the masses of the particles involved. When the charges are similar, the binding energies are similar.

In contrast, the minimal energy necessary to split a deuteron into a free proton and separate free neutron, is $\Delta E = 2.4$ MeV. This energy scale is typical for atomic nuclei, and about 10^5 times larger than electron binding energies. In the case of α-particles –He-nuclei consisting of two protons and two neutrons– the energy required to remove any one of the nucleons is about 7 MeV, and it does not make much of a difference whether this concerns a proton or a neutron. This shows again that the main interactions between nucleons are not of Coulomb type, and are determined by something else than electric charges.

According to the theory of special relativity, the rest energy of a particle is related to its mass by the well-known relation $E = mc^2$. Hence binding energies are in principle measurable by determining the difference between the mass M of the bound state, and the sum of the masses m_i of its free constituents:

$$\Delta E = \sum_{i=1}^{N} m_i c^2 - Mc^2. \tag{17}$$

The strength of an interaction can then be characterized by taking the ratio of the binding energy and the mass of the bound state. For hydrogen this is

$$\left.\frac{\Delta E}{Mc^2}\right|_{ep} = \frac{13.6}{0.938} \times 10^{-6} = 1.45 \times 10^{-8}; \tag{18}$$

whilst for the deuteron one finds

$$\left.\frac{\Delta E}{Mc^2}\right|_{pn} = \frac{2.4}{1876} = 1.28 \times 10^{-3}. \tag{19}$$

Exercise 2.2
Compute the gravitational binding energy of Jupiter and the sun, and determine the ratio $\Delta E/Mc^2$ of this system. *Rem:* assume that the orbit of Jupiter is circular.

3. Scattering cross sections

When discussing the size of atomic and subatomic particles, it is necessary to give some operational meaning to this notion. Especially in quantum mechanics, where according to circumstances objects behave either as a particle or as a wave, it is not directly clear how to determine their dimensions in a meaningful way. In fact, it can be asked more generally how the properties of subatomic particles are established, including their mass, charge, spin, or their possible substructure.

An important tool for answering such questions is provided by scattering experiments. In essence this a sophisticated form of the way we collect information about the moon: the sun emits light, the moon is in the path of a fraction of the light rays and reflects some of this light in the direction of the earth. By observing the scattered light here on earth, and knowing the distance to the moon, we can reconstruct its size and surface properties.

Now the sun does not only emit light, but also the weakly interacting particles known as neutrinos. In principle we could also study the moon by collecting the neutrinos scattered by the moon towards the earth. Unfortunately, the moon is almost transparent to neutrinos: most of them just continue straight through the moon and only very few are deflected by a collision with one of the nuclei in the moon's material. Hence even if we could observe the neutrinos in an efficient way, it would take a very long time before we had collected enough of them to determine the moon's size and composition. On the other hand, precisely because neutrinos propagate easily through the moon's entire volume, and most neutrinos would be scattered by some collision in its interior, we would observe not only the moon's surface properties, but also obtain information about its internal structure.

From this example we can draw several important lessons:

- scattering particles from an object provides information about its size and structure;
- scattering different kinds of particles from the target can provide different kinds of information;
- the efficiency of the scattering of a particular type of particles from the target determines the amount of information that can be collected within a reasonable time.

In a similar way we can study the properties of atomic and subatomic particles by bombarding them with other particles, and study how these are scattered in various directions. The information we are after is encoded in a quantity known as the *scattering cross section.*

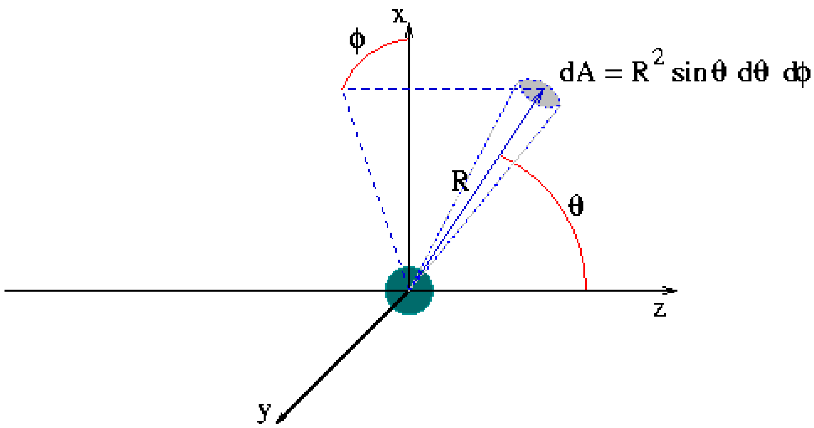

Fig. 3.1: Geometry of scattering of test particles by a target

Suppose we have a homogeneous beam of test particles moving in the z-direction towards a target or *scattering center*; by homogeneous we mean, that the number of particles per unit of area and per unit of time passing through any part of the transverse x-y-plane is constant. We denote this flux of incoming particles by I_{in}:

$$I_{in} = \# \text{ particles passing per unit of time and area in the } x\text{-}y\text{-plane.} \tag{20}$$

We assume that the test particles scatter elastically, i.e., any incoming particle becomes an outgoing particle, no particles are absorbed or transformed by the target, and energy is conserved. Some particles will pass the target undeflected, but some will be scattered into other directions, characterized by the angle θ with the z-axis, and the angle φ with the x-axis in the transverse plane; see fig. 3.1. The particles scattered in directions between $(\theta, \theta + d\theta)$ and $(\varphi, \varphi + d\varphi)$ define a cone; at distance R from the scattering center this cone has a opening area

$$dA = R^2 \sin\theta \, d\theta d\varphi \equiv R^2 d\Omega. \tag{21}$$

Here the spherical angle $d\Omega = \sin\theta \, d\theta d\varphi$ is the opening area of the same cone measured on the unit sphere ($R = 1$). Denote the flux of outgoing particles at distance R in this cone by I_{out}:

$$I_{out}(R, \theta, \varphi) = \# \text{ particles passing per unit of time and spherical area at distance } R. \tag{22}$$

The total number of particles scattered per unit of time in all directions inside the cone is

$$dN_{out}(\theta, \varphi) = I_{out}(R, \theta, \varphi) \, dA = I_{out}(R, \theta, \varphi) \, R^2 d\Omega. \tag{23}$$

In the absence of external fields or other interactions the outgoing particles at sufficient distance from the scattering center are free particles, moving in straight lines; the number of outward-bound particles inside the cone is then the same at any distance R; in other words: dN_{out} is a function of the angles (θ, φ) only, and I_{out} is proportional to $1/R^2$. Now the fraction of particles scattered inside the cone is of course proportional to the flux of incoming particles: twice as many incoming particles per unit of time implies twice as many particles scattered in any direction (provided the beam of incoming particles remains homogeneous):

$$dN_{out}(\theta, \varphi) = d\sigma(\theta, \varphi) I_{in}. \tag{24}$$

The proportionality factor $d\sigma(\theta, \varphi)$ is in general a function of the angles (θ, φ), as the fraction of particles scattered in different directions may be different. By definition it has the dimensions of an area. The *differential scattering cross section* for elastic scattering is defined as the ratio[1]

$$\frac{d\sigma}{d\Omega}(\theta, \varphi) = \frac{1}{I_{in}} \frac{dN_{out}}{d\Omega}(\theta, \varphi). \tag{25}$$

The relation (24) can be interpreted to mean, that $d\sigma(\theta, \varphi)$ is the effective area of the target responsible for scattering test particles into the cone with spherical opening angle $d\Omega$ in the direction (θ, φ).

By integration over the full solid angle 4π we get the *total cross section* for elastic scattering:

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, \frac{d\sigma}{d\Omega}. \tag{26}$$

It is the effective area of the target for scattering test particles in any direction. Observe the importance of the the qualification *effective* in this statement. Indeed, the efficiency with which the target scatters different kind of test particles may be very different, as the example of light versus neutrinos scattered by the moon illustrates. Therefore the differential and total cross sections of the target depend on the kind of test particles used, and are not a purely intrinsic property of the target.

Besides elastic scattering, there also exist *inelastic* scattering processes, where the energy of the scattered particles, and/or their number and/or particle type are not conserved. An example is the collision of an electron with an atom, knocking out another electron from the atom and leaving it behind in an ionized state. One can then generalize the notion of a cross section by defining an *inelastic* differential scattering cross section for this process, as being the fraction of incident electrons per unit area knocking out an electron of specific energy in a specific direction.

[1] More precisely: its limit when the cone becomes very narrow.

4. Elastic scattering of classical particles

If the interaction of the test particles with the target is known, it is possible to derive more detailed expressions for the scattering cross section for specific processes. In this section we illustrate this for elastic scattering by particles obeying classical mechanics. Later we perform a similar calculation in quantum mechanics.

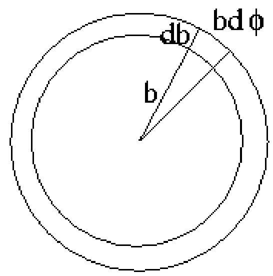

Fig. 4.1: Impact parameter and scattering area

Fig. 4.1 provides a view of the scattering geometry in the direction of the beam of test particles (taken to be the z-direction). The incident particles at a distance from the central axis through the scattering center in the range $(b, b+db)$ form a ring of width db; the distance b is refered to as the *impact parameter*. Let the target be simple enough[2] that the section of the ring within the wedge with angular range $(\varphi, \varphi + d\varphi)$ represents the area element $d\sigma(\theta, \varphi)$ accounting for *all* particles scattered into the cone with opening angle $d\Omega$ in the direction (θ, φ), and not contributing to scattering in any other direction. Then by definition

$$d\sigma(\theta, \varphi) = |b\, db\, d\varphi| = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right| \sin\theta\, d\theta\, d\varphi, \tag{27}$$

which implies

$$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right|. \tag{28}$$

To evaluate this quantity for a specific model, one only needs to specify the relation between impact parameter and scattering angles $b(\theta, \varphi)$. We will do this for the scattering of point particles by a solid sphere, and for the scattering of point charges by a Coulomb potential, i.e. another point charge acting as scattering center.

[2] If not, we may have to sum over various domains of the impact parameter b and angle φ.

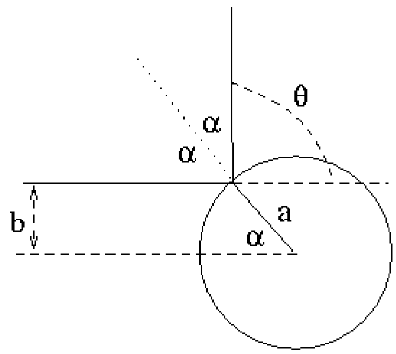

Fig. 4.2: Scattering of point particles by a solid sphere.

a. Elastic scattering by a solid sphere

As an example of this description of scattering, we consider elastic scattering of point masses by a solid sphere of radius a, as sketched in fig. 4.2. In this case, as in many others, the scattering kinematics is the same for all azimuth angles φ, so $b(\theta)$ is a function of θ only. We calculate this quantity as follows. Let α be the angle between the direction of motion of the incident particle and the normal to the sphere at the point of impact; as we consider elastic scattering, the angles of impact and reflection w.r.t. the normal are equal, and therefore the scattering angle $\theta = \pi - 2\alpha$. As shown in fig. 4.2 the impact parameter then is

$$b = a \sin \alpha = a \sin \left(\frac{\pi - \theta}{2} \right) = a \cos \frac{\theta}{2}. \tag{29}$$

As a result

$$\frac{db}{d\theta} = -\frac{a}{2} \sin \frac{\theta}{2}, \tag{30}$$

and by substitution into eq. (28) we obtain for the differential and total cross section the expressions

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4}, \qquad \sigma = \pi a^2. \tag{31}$$

Thus, for elastic scattering of point particles by a solid sphere the differential cross section is constant (independent of the angles), and the total cross section equals the geometric cross section. Note, that this computation would also apply to the scattering of sun light (photons) by the moon, if the moon were a perfectly smooth sphere and a perfect light reflector!

Exercise 4.1
a. Show that the integrated cross section for angles $\theta \geq \delta$:

$$\sigma(\theta \geq \delta) = \int_0^{2\pi} d\varphi \int_\delta^\pi d\theta \, \sin\theta \, \frac{d\sigma}{d\Omega} = \frac{\pi a^2}{2} \left(1 + \cos\delta\right).$$

b. Compute the ratio of integrated cross sections

$$\sigma(\theta \geq 45^\circ) : \sigma(\theta \geq 90^\circ) : \sigma(\theta \geq 135^\circ)$$

b. Coulomb scattering

An important application of classical scattering theory is the scattering of two point charges interacting via the Coulomb force. This calculation was performed by Rutherford in 1909, and was instrumental in the discovery of the atomic nucleus. Therefore the differential cross section for Coulomb scattering is also refered to as the *Rutherford cross section.*

The starting point is the Coulomb interaction of two particles with masses (m_1, m_2) and charges (q_1, q_2); using Newtons third law of motion, the force acting on the particles is

$$\mathbf{F} = -m_1 \ddot{\mathbf{r}}_1 = m_2 \ddot{\mathbf{r}}_2 = \frac{\kappa}{r^2}\,\hat{\mathbf{r}}, \qquad \kappa = \frac{q_1 q_2}{4\pi\epsilon_0}, \tag{32}$$

where

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \tag{33}$$

and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$. As the force depends only on the relative position of the particles, it is useful to work in the CM frame defined as follows. The center of mass (CM) is located at the position $\mathbf{R}$ given by

$$M\mathbf{R} = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2, \qquad M = m_1 + m_2. \tag{34}$$

Eq. (32) implies, that the center of mass moves uniformly:

$$\ddot{\mathbf{R}} = 0 \quad \Rightarrow \quad \dot{\mathbf{R}} = \text{constant}. \tag{35}$$

Then one can choose a special frame in which the center of mass is at rest: $\mathbf{R} = 0$; this is the CM frame.

From eq. (32) and the definition of $\mathbf{r}$ (33) it also follows, that the relative acceleration of the particles is equal to that of a single particle of reduced mass μ:

$$\mu \ddot{\mathbf{r}} = \frac{\kappa}{r^2}\,\hat{\mathbf{r}}, \qquad \mu = \frac{m_1 m_2}{M}. \tag{36}$$

The solution of this equation is a standard problem of classical mechanics. We use the conservation laws for energy and angular momentum in terms of the polar co-ordinates in the CM frame:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{37}$$

The energy in the center of mass system is

$$\varepsilon = \frac{\mu}{2}\,\dot{\mathbf{r}}^2 + \frac{\kappa}{r} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + \frac{\kappa}{r}, \tag{38}$$

whilst the angular momentum is

$$\mathbf{l} = \mu\mathbf{r} \times \dot{\mathbf{r}}. \tag{39}$$

Conservation of angular momentum implies that both the size and direction of $\mathbf{l}$ are constant. Therefore the motion takes place in the plane perpendicular to $\mathbf{l}$; we choose our co-ordinates such that this plane is the x-y-plane. Then

$$\theta = \frac{\pi}{2}, \qquad \dot{\theta} = 0. \tag{40}$$

In that frame the angular momentum is

$$\mathbf{l} = (0, 0, l), \qquad l = \mu(x\dot{y} - y\dot{x}) = \mu r^2 \dot{\varphi}, \tag{41}$$

and the energy is

$$\varepsilon = \frac{\mu}{2}\,\dot{r}^2 + \frac{l^2}{2\mu r^2} + \frac{\kappa}{r}. \tag{42}$$

Rearranging the terms we get an expression for the radial velocity:

$$\dot{r}^2 = \frac{2\varepsilon}{\mu} - \frac{2\kappa}{\mu r} - \frac{l^2}{\mu^2 r^2}. \tag{43}$$

Combining equations (41) and (43) a relation for the shape of the orbit $r(\varphi)$ can be derived:

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{\dot{r}}{\dot{\varphi}}\right)^2 = \frac{2\mu\varepsilon}{l^2}\,r^4 - \frac{2\mu\kappa}{l^2}\,r^3 - r^2. \tag{44}$$

Now it is convenient to switch to a new variable

$$s = \frac{1}{r} + \frac{\mu\kappa}{l^2}. \tag{45}$$

Then eq. (44) reduces to

$$\left(\frac{ds}{d\varphi}\right)^2 = \lambda^2 - s^2, \qquad \lambda^2 = \frac{\mu^2\kappa^2}{l^4}\left(1 + \frac{2\varepsilon l^2}{\mu\kappa^2}\right). \tag{46}$$

It follows, that

$$s = \lambda \cos(\varphi - \varphi_0), \tag{47}$$

where we take the positive root for λ, and φ_0 is an arbitrary constant of integration. When $\kappa > 0$ (charges of equal sign), the solution for $r(\varphi)$ then is

$$r(\varphi) = \frac{l^2/\mu\kappa}{e\cos(\varphi - \varphi_0) - 1}, \qquad e = \sqrt{1 + \frac{2\varepsilon l^2}{\mu\kappa^2}} \geq 1. \tag{48}$$

This is the equation for a hyperbola with one focal point in $r = 0$. For all values of φ_0 the shape is the same, but the hyperbola is rotated clockwise over an angle φ_0 w.r.t. the x-axis; in the following we take $\varphi_0 = 0$.

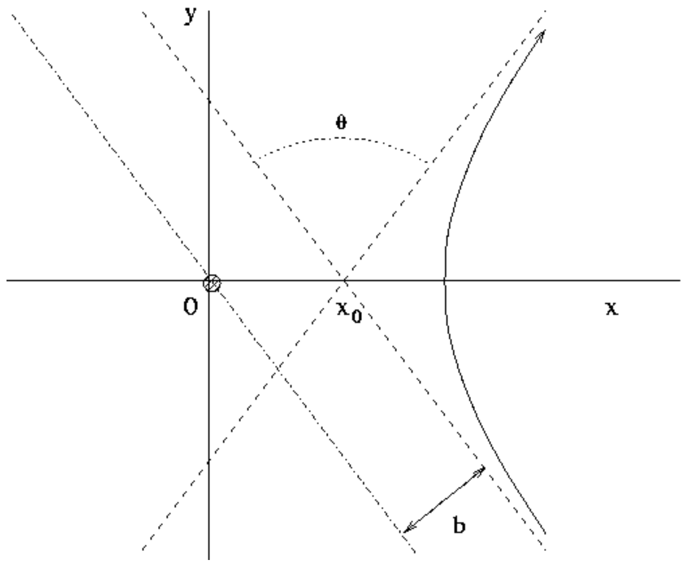

Fig. 4.3: Hyperbolic orbit, with midpoint at $x_0 = \kappa/2e\varepsilon$.

The impact parameter in this case is the perpendicular distance from the asymptote to the parallel line passing through the center of the Coulomb force; as in our case the Coulomb force is repulsive, the scattering center $r = 0$ is the far (external) focal point; see fig. 4.3. The angular momentum evaluated at the start of the scattering process: $t \rightarrow -\infty$, $r \rightarrow \infty$, is

$$l = \mu b v_\infty, \qquad v_\infty = \lim_{r\rightarrow\infty} \sqrt{\dot{r}^2} = \sqrt{\frac{2\varepsilon}{\mu}}. \tag{49}$$

It follows, that

$$b = \frac{l}{\sqrt{2\mu\varepsilon}}. \tag{50}$$

Now in both asymptotic regions of the orbit (initial and final), we have $r \rightarrow \infty$, and therefore at $t \rightarrow \pm\infty$:

$$\cos\varphi_\infty = \cos\varphi_{-\infty} = \frac{1}{e}, \tag{51}$$

as the x-axis has been chosen as the symmetry axis of the hyperbola by setting $\varphi_0 = 0$. It follows, that

$$\pi - \theta_{CM} = \varphi_\infty - \varphi_{-\infty} = 2\varphi_\infty, \tag{52}$$

and

$$\begin{aligned} \text{cotan}\,\frac{\theta_{CM}}{2} &= \tan\left(\frac{\pi}{2} - \frac{\theta_{CM}}{2}\right) = \tan\,\varphi_\infty \\ &= \sqrt{e^2 - 1} = \frac{2b\varepsilon}{\kappa}. \end{aligned} \tag{53}$$

By substitution into eq. (28) it is now straightforward to evaluate the differential cross section:

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \left(\frac{\kappa}{4\varepsilon}\right)^2 \frac{1}{\sin^4\theta_{CM}/2}. \tag{54}$$

With κ given by eq. (32) this is the Rutherford differential cross section for Coulomb scattering. Observe, that in contrast to the case of scattering by solid spheres, the Rutherford cross section is not constant as a function of θ_{CM}, but decreases rapidly towards the value $(\kappa/4\varepsilon)^2$ for $\theta_{CM} \to \pi$, whilst it diverges for small angles in the forward direction. Integrating the differential cross section over all angles θ_{CM} larger than a given angle δ: $\delta \leq \theta \leq \pi$, and over all values of φ, we get

$$\sigma_{CM}(\theta \geq \delta) = \int_0^{2\pi} d\varphi \int_\delta^\pi d\theta \, \sin\theta \, \frac{d\sigma}{d\Omega} = \frac{\pi\kappa^2}{4\varepsilon^2} \, \frac{1+\cos\delta}{1-\cos\delta}. \tag{55}$$

From this formula it is clear, that the number of particles scattered from a beam per unit of time is finite if we require the scattering angle to be larger than some $\delta > 0$. However, if we count particles scattered over arbitrarily small angles, then even particles with a very large (infinite) impact parameter are scattered; in other words, if the beam has infinite width, an infinite number of particles up to arbitrarily large impact parameters is scattered by the Coulomb potential. For this reason the Coulomb potential is said to have an *infinite range.*

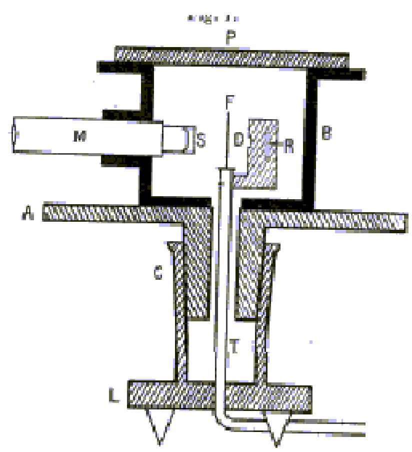

Fig. 4.4: Sketch of Rutherford and Geiger's experiment.

In 1909 Rutherford and Geiger (assisted by a physics student, Marsden) investigated the structure of atoms by scattering 5 MeV α-particles (^{4}He nuclei) on a target consisting of gold atoms; see fig. 4.4. In this experiment a source of α particles R is placed in a vacuum chamber with a target consisting of a gold foil F. The scattered α particles are observed by their impact on a ZnS scintillation screen S in front of the microscope M.

During the experiment they counted the numbers of α-particles scattered into different directions, and compared the results with the prediction (54) for scattering of point charges by a Coulomb interaction. From this analysis Rutherford concluded that the gold atoms contain a small massive nucleus behaving as a point charge up to distances 10,000 times smaller than the atom itself.

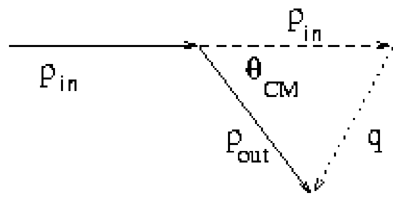

Fig. 4.5: Momentum transfer in Coulomb scattering.

Momentum transfer
In Coulomb scattering the particles in their initial and final state are free particles:

$$r \to \infty \quad \Rightarrow \quad V(r) = \frac{\kappa}{r} \to 0. \tag{56}$$

Therefore the energy of the incoming and outgoing particles is purely kinetic. Now as shown in exc. 3.2, in the CM frame the momentum is

$$\mathbf{p} \equiv \mu\dot{\mathbf{r}} = \mathbf{p}_2 = -\mathbf{p}_1. \tag{57}$$

Energy conservation in the CM frame then implies, that the magnitudes of the initial and final momentum $\mathbf{p}$ are equal:

$$2\mu\varepsilon = \mathbf{p}_{in}^2 = \mathbf{p}_{out}^2. \tag{58}$$

It follows that the Coulomb interaction can only change the direction of the momentum, as shown in fig. 4.5. Denoting the change in momentum by $\mathbf{q}$, it follows that

$$\mathbf{p}_{out} = \mathbf{p}_{in} + \mathbf{q} \quad \Rightarrow \quad \mathbf{q}^2 = (\mathbf{p}_{out} - \mathbf{p}_{in})^2 = \mathbf{p}_{in}^2 + \mathbf{p}_{out}^2 - 2\mathbf{p}_{in} \cdot \mathbf{p}_{out} \tag{59}$$

Combining these results we then find

$$\mathbf{q}^2 = 2\,\mathbf{p}_{in}^2\,(1 - \cos\theta_{CM}) = 8\mu\varepsilon\sin^2\frac{\theta_{CM}}{2}. \tag{60}$$

Therefore the Rutherford cross section can be rewritten as

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{4\kappa^2\mu^2}{|\mathbf{q}|^4}. \tag{61}$$

In this form the explicit dependence on the energy ε has disappeared. Observe, that the differential cross section decreases with increasing momentum transfer, as this corresponds to increasing scattering angle. For scattering of nuclei with charge number $Z_{1,2}$, the explicit expression becomes

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = 4Z_1^2Z_2^2\alpha^2\,\frac{(\mu\hbar c)^2}{|\mathbf{q}|^4}. \tag{62}$$

Exercise 4.2
a. Show that the lab and CM co-ordinates are related by

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{M}\,\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{M}\,\mathbf{r}.$$

b. From this result, derive eq. (36).
c. Prove that in the CM frame $\mathbf{R} = 0$ the total momentum vanishes:

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = 0,$$

and that as a result

$$\mathbf{p}_2 = -\mathbf{p}_1 = \mu\dot{\mathbf{r}}.$$

d. Using the definition of momentum transfer (59), show that

$$\mathbf{p}_{1\,out} = \mathbf{p}_{1\,in} - \mathbf{q}, \qquad \mathbf{p}_{2\,out} = \mathbf{p}_{2\,in} + \mathbf{q}.$$

Exercise 4.3
a. Prove that the energy ε, eq. (38), and angular momentum $\mathbf{l}$, eq. (39), are constants of motion:

$$\frac{d\varepsilon}{dt} = 0, \qquad \frac{d\mathbf{l}}{dt} = 0.$$

b. Derive the expressions for ε and $\mathbf{l}$ in polar co-ordinates.

Exercise 4.4
a. Derive eq. (44) for the shape of the orbit $r(\varphi)$.
b. Perform the variable substitution (45), and prove eq. (46).
c. Check the solution (48) for $\kappa > 0$ (charges of equal sign); what happens for charges of opposite sign?

Exercise 4.5
a. From eqs. (50) and (51) and the definition of e, derive the result (53).
b. Substitute the result for b into eq. (28) to prove the result (54) for the Rutherford cross section.
c. By performing the angular integrals, derive eq. (55).
d. Compute the ratio of the cross sections

$$\sigma_{CM}(\theta \geq 45°) : \sigma_{CM}(\theta \geq 90°) : \sigma_{CM}(\theta \geq 135°)$$

and compare this with the result of exerc. 3.1 for the scattering by a solid sphere.

c. Thomson scattering

In the previous sections we have discussed the scattering of charged particles by Coulomb forces. However, charged particles can also scatter radiation. The principle is simple: when an electro-magnetic wave impinges on a charged particle, the particle vibrates in response to the electric component of the wave. However, an oscillating charge emits radiation, hence the energy it has absorbed out of the electro-magnetic wave is re-emitted as radiation in other directions. If we denote the intensity (= energy flux) of the incident and scattered radiation by Φ_{in} and Φ_{out}, the fraction of radiation scattered through a spherical surface element $dA = R^2 d\Omega$ in the direction (θ, φ) defines the differential scattering cross section:

$$\frac{d\sigma}{dA}(\theta,\varphi) = \frac{1}{R^2}\frac{d\sigma}{d\Omega}(\theta,\varphi) = \frac{\Phi_{out}(\theta,\varphi)}{\Phi_{in}}. \tag{63}$$

J.J. Thomson was the first to calculate this cross section for scattering of an unpolarized plane wave by a free non-relativistic point charge q with mass m according to classical electrodynamics. The derivation is reproduced in appendix C. The result is

$$\frac{d\sigma}{d\Omega} = \frac{\rho^2}{2}\left(1+\cos^2\theta\right), \tag{64}$$

where ρ is the *electro-magnetic radius* of the particle, defined as

$$\rho = \frac{q^2}{4\pi\varepsilon_0 mc^2}. \tag{65}$$

Note that the electro-magnetic radius is inversely proportional to the mass. The total cross section becomes

$$\sigma_T = \int d\Omega\,\frac{d\sigma}{d\Omega} = \frac{8\pi}{3}\,\rho^2. \tag{66}$$

This result implies, that although their charges have the same value, free protons scatter radiation less efficiently than free electrons, by a factor

$$\frac{\sigma_p}{\sigma_e} = \frac{m_e^2}{m_p^2} = 0.3\times10^{-6}. \tag{67}$$

For an electron the electro-magnetic radius has the value

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = 2.817\times10^{-15}\,\mathrm{m}. \tag{68}$$

It is also known as the *classical electron radius*. The corresponding total cross section is

$$\sigma_e = 0.665\times10^{-28}\,\mathrm{m}^2. \tag{69}$$

One can think of these numbers as the size of an electron as seen by electro-magnetic radiation. As the electron cross section sets the scale for the electro-magnetic cross section of other charged particles, a corresponding unit of area the *barn*, has been introduced in particle physics:

$$1\,\text{barn} = 10^{-28}\,\text{m}^2. \tag{70}$$

Exercise 4.6
Consider a gas with a density of N molecules per unit of volume; let n be the number of electrons per molecule. Run a light beam of intensity Φ through the gas, of sufficiently short wavelength that the electrons respond to the beam as if they were free particles. Show that due to scattering by the electrons the intensity of the light beam decreases as a function of distance x traveled through the gas according to the formula

$$\frac{d\Phi}{dx} = -nN\sigma_e\Phi.$$

Solve this equation, and show that Φ decays exponentially over a typical path length

$$l = \frac{1}{nN\sigma_e}.$$

Evaluate this length for hydrogen (H_2) at standard temperature and pressure ($T = 273$ K and $p = 1$ atm).

5. Quantum theory

So far we have discussed atoms and subatomic particles like electrons and nucleons, in classical language. However, the proper formalism for treating the properties and interactions of these particles is quantum theory. There are different levels of quantum theory, appropriate for different ranges of phenomena. The structure and interactions of systems consisting of a finite number of non-relativistic particles are generally well described by the Schrödinger equation with a suitably chosen potential, which may depend on the position and spin of the particles involved. The wave function $\Psi(t; \mathbf{r}_1, ..., \mathbf{r}_n; s_1, ..., s_n)$ represents the probability amplitude for the particles to be in a specific configuration, characterized by their positions $\mathbf{r}_i$ and spins s_i at time t. For example, an electron in a hydrogen atom moves in the Coulomb potential of the nucleus, and its wave function is a solution of the equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m_e} \Delta + V(\mathbf{r}) \right) \Psi, \qquad V(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}. \tag{71}$$

The stationary states of the system are specified by wave functions of the factorized form

$$\Psi(t, \mathbf{r}) = e^{-iEt/\hbar}\, \psi_E(\mathbf{r}), \tag{72}$$

with $\psi_E(\mathbf{r})$ a normalizable solution of the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m_e} \Delta + V(\mathbf{r}) \right) \psi_E = E\psi_E. \tag{73}$$

The eigenvalue E represents the energy of the electron in the stationary state ψ_E. In general it can not take arbitrary values. For the bound states of the electron in the hydrogen atom the allowed values are

$$E_n = -\frac{\alpha^2}{2n^2}\, m_e c^2, \qquad n = 1, 2, 3, ... \tag{74}$$

Here α is a dimensionless number known as the fine-structure constant:

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 0.007297 \approx \frac{1}{137}, \tag{75}$$

and $m_e c^2$ is the rest energy of the electron. These energy levels were explained long before the development of quantum mechanics in a semi-empirical model by Niels Bohr. His derivation, based on the old quantum theory of Planck and Einstein, is reproduced in appendix B.

In the regime of relativistic particles, quantum mechanics formulated in terms of a Schrödinger equation for a finite number of particles is no longer applicable, and one has to take recourse to relativistic quantum field theory (QFT), of which ordinary quantum mechanics represents the non-relativistic limit. An important difference between relativistic quantum field theory and non-relativistic quantum mechanics is, that in quantum field theory the number of particles is generally not conserved: particles can be created and annihilated in various ways; only the total energy and momentum are conserved. Indeed, in the relativistic context mass represents a specific form of energy; therefore the particle number can be changed by converting some mass into other forms of energy, for example into kinetic energy for other particles present in the interaction process, or into newly created particles with different masses. In a later section we provide a more complete description of the essentials of quantum field theory.

Exercise 5.1
Check that the fine-structure constant α is a dimensionless number.

Potential scattering

In addition to the bound states of negative energy, charged particles can also be in scattering states of positive energy. In such a state the initial and final particles are free particles of well-defined energy, but in between there is an interaction process in which momentum can be transfered, as we have seen in classical Coulomb scattering in par. 3.b. In this section we show how to compute this kind of elastic scattering process in non-relativistic quantum mechanics. Specifically we show, that to first approximation such a computation reproduces the Rutherford scattering cross section for charged particles.

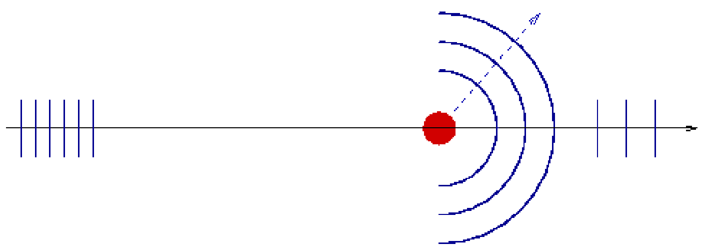

Fig. 5.1: quantum scattering

The set-up of the analysis is very close to the classical one. We start with a beam of mono-energetic particles coming in along the z-axis; the particles enter a region in which there is a non-vanishing potential $V(\mathbf{r})$, associated with a scattering center in the origin. Finally, there may be a part of the beam which continues without scattering, and a part which is deflected in all directions. The quantum character of the process is reflected in all three stages. First, the incoming beam is described by a plane wave

$$\psi_{in} = A_k \, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{76}$$

with the wave vector related to the momentum by $\mathbf{p} = \hbar\mathbf{k} = \hbar(0, 0, k)$, and the amplitude $|A_k|^2$ is determined by the particle density in the incoming beam (the number of particles per unit of volumein the beam). Next, the free particles in the deflected wave, at sufficient distance from the scattering center, are described by a spherical wave

$$\psi_{out} = B_k(\theta, \varphi) \, \frac{e^{ikr}}{r}. \tag{77}$$

The scattering cross section is determined by the probability for a particle in the initial plane wave to end up as a spherical wave in the direction (θ, φ). By the elementary rules of quantum mechanics, this probability measured at a distance R from the scattering center is given by

$$\frac{d\sigma}{dA} = \frac{1}{R^2} \frac{d\sigma}{d\Omega} = \frac{1}{R^2} \left| \frac{B_k(\theta, \varphi)}{A_k} \right|^2, \tag{78}$$

or

$$\frac{d\sigma}{d\Omega} = |f_k(\theta, \varphi)|^2, \qquad f_k(\theta, \varphi) = \frac{B_k(\theta, \varphi)}{A_k}. \tag{79}$$

The quantity $f_k(\theta, \varphi)$ is called the *scattering amplitude*. In the remainder of this section we show how to compute this amplitude for scattering from a general potential.

The starting point for the derivation is the time-independent Schrödinger equation for a particle of mass m in the potential $V(\mathbf{r})$:

$$\left(\Delta + k^2\right) \psi_E = \frac{2m}{\hbar^2} V \psi_E, \tag{80}$$

where k^2 is related to the energy of the free incoming particles by

$$E = \frac{\hbar^2 k^2}{2m}. \tag{81}$$

The wave functions ψ_E are positive-energy eigenfunctions of the full Schrödinger equation with the energy eigenvalue equal to the energy of the free incoming particles, described by the incoming plane wave. By energy conservation this must equal the energy eigenvalue of the outgoing spherical waves at distances where $V(\mathbf{r}) \rightarrow 0$, such that the right-hand side of eq. (80) vanishes; for details, see exercises (4.2) and (4.3). We compute the wave function ψ_E by a method of successive approximations, known as the Born series. These approximations are based on the solution of the inhomogeneous linear partial differential equation

$$-\left(\Delta + k^2\right) \phi = \rho, \tag{82}$$

where $\rho(\mathbf{r})$ is some prescribed function acting as a source for the field $\phi(\mathbf{r})$. The solution can be constructed using the *Green's function* $G_k(\mathbf{r}, \mathbf{r}')$, which is a solution of the special inhomogeneous equation for a δ-function source:

$$-\left(\Delta + k^2\right) G_k(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'). \tag{83}$$

Once we have the Green's function, a particular solution of (82) is obtained by taking

$$\phi_k^{(0)}(\mathbf{r}) = \int d\mathbf{r}'\, G_k(\mathbf{r}, \mathbf{r}')\, \rho(\mathbf{r}'). \tag{84}$$

The general solution differs from this special solution at most by a solution of the homogeneous (free) equation, i.e. a plane wave. In appendix D it is shown, that the Green's function is given by

$$G_k(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{e^{ik|\mathbf{r}' - \mathbf{r}|}}{|\mathbf{r}' - \mathbf{r}|}. \tag{85}$$

The results (84) and (85) allow us to transform the differential equation (80) into an integral equation

$$\psi_E = Ce^{i\mathbf{k}\cdot\mathbf{r}} - \frac{2m}{\hbar^2}\int d\mathbf{r}'\, G_k(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi_E(\mathbf{r}'). \tag{86}$$

The Born series is obtained by reinserting the equation into itself:

$$\begin{aligned}\psi_E &= Ce^{i\mathbf{k}\cdot\mathbf{r}} - \frac{2Cm}{\hbar^2}\int d\mathbf{r}'\, G_k(\mathbf{r},\mathbf{r}')V(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'} \\ &\quad + \frac{4Cm^2}{\hbar^4}\int d\mathbf{r}' G_k(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\int d\mathbf{r}'' G_k(\mathbf{r}',\mathbf{r}'')V(\mathbf{r}'')e^{i\mathbf{k}\cdot\mathbf{r}''} + \ldots\end{aligned} \tag{87}$$

The series can be given a physical interpretation as follows: the first term is the unscattered plane wave; the second term represents the wave after a single scattering from the potential; the next term repesents double scattering, etc. The first-order Born approximation is obtained by terminating the series after the term for single scattering. Explicitly:

$$\psi_E \approx C\left(e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int d\mathbf{r}'\,\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\,V(\mathbf{r}')\,e^{i\mathbf{k}\cdot\mathbf{r}'}\right). \tag{88}$$

It remains to evaluate the integral in (88). In fact, we are predomininantly interested in the asymptotic form, i.e. terms that do not vanish faster than $1/r$. We can find these for a quite general potential $V(\mathbf{r})$ by making the expansion

$$|\mathbf{r}'-\mathbf{r}| = \sqrt{(\mathbf{r}'-\mathbf{r})^2} = r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \mathcal{O}[1/r], \tag{89}$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction:

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{90}$$

Inserting this expansion both in the denominator and in the exponential, we get

$$\psi_E(\mathbf{r}) = C\left(e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\,\frac{e^{ikr}}{r}\int d^3\mathbf{r}'\,e^{i(\mathbf{k}-k\hat{\mathbf{r}})\cdot\mathbf{r}'}\,V(\mathbf{r}')\,+\mathcal{O}[1/r^2]\right). \tag{91}$$

Asymptotically this is of the desired form

$$\psi_E = C\left(e^{i\mathbf{k}\cdot\mathbf{r}} + f_k(\theta,\varphi)\frac{e^{ikr}}{r}\right), \tag{92}$$

with the scattering amplitude given in terms of the Fourier transform of the potential by

$$f_k(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}'\,e^{i(\mathbf{k}-k\hat{\mathbf{r}})\cdot\mathbf{r}'}\,V(\mathbf{r}'). \tag{93}$$

It should be noted, that the whole procedure is well-defined only if the potential is mathematically well-behaved; in particular for large r it should vanish sufficiently fast. Details can be found in handbooks of quantum mechanics.

Exercise 5.2
Show that for $r > 0$ the spherical wave

$$\psi_E = B \, \frac{e^{ikr}}{r},$$

is a solution of the free time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \, \Delta\psi_E = E\psi_E,$$

with energy eigenvalue $E = \hbar^2 k^2 / 2m$.

Exercise 5.3
The wave function $\psi_E(\mathbf{r})$ of the form (92) represents a solution of the time-independent Schrödinger equation with a potential $V(\mathbf{r})$. It is turned into a solution of the full time-dependent Schrödinger equation by multiplication with the time-dependent phase factor

$$\Psi(t, \mathbf{r}) = e^{-iEt/\hbar} \, \psi_E(\mathbf{r}) = C \left(e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + f_k(\theta, \varphi) \frac{e^{i(kr - \omega t)}}{r} \right),$$

where $E = \hbar\omega$. Taking $\mathbf{k} = (0, 0, k)$ with $k > 0$, show that the first term on the right-hand side represents a wave propagating in the positive z-direction, whilst the second term represents a spherical wave propagating in the outward (positive r) direction.

6. Yukawa-Coulomb scattering

The general formalism developed above wil now be used to compute the scattering amplitude for a potential of the form

$$V(\mathbf{r}) = \frac{\kappa e^{-\lambda r}}{r}, \tag{94}$$

where κ and λ are constants. At large distances $\lambda r \gg 1$ this potential decreases faster than exponentially, whilst at small distances $\lambda r \ll 1$ it behaves as a Coulomb potential. In fact, in the limit $\lambda \rightarrow 0$ it reduces to the Coulomb potential. The general form (94) including the exponential is known as the Yukawa potential.

Inserting this potential into eq. (93) gives the scattering amplitude

$$\begin{aligned} f_k(\theta, \varphi) &= -\frac{m\kappa}{2\pi\hbar^2} \int d^3\mathbf{r}' \, \frac{1}{r'} \, e^{i(\mathbf{k}-k\hat{\mathbf{r}})\cdot\mathbf{r}' - \lambda r'}. \\ &= -\frac{m\kappa}{\hbar^2} \int_{-1}^{+1} d\cos\theta \int_0^\infty dr' \, r' e^{(i|\mathbf{k}-k\hat{\mathbf{r}}|\cos\theta - \lambda)r'} \\ &= \frac{im\kappa}{\hbar^2 |\mathbf{k}-k\hat{\mathbf{r}}|} \int_0^\infty dr' \left(e^{-(\lambda - i|\mathbf{k}-k\hat{\mathbf{r}}|)r'} - e^{-(\lambda + i|\mathbf{k}-k\hat{\mathbf{r}}|)r'} \right) \\ &= -\frac{2m\kappa}{\hbar^2} \frac{1}{(\mathbf{k}-k\hat{\mathbf{r}})^2 + \lambda^2}. \end{aligned} \tag{95}$$

Now the momentum of the outgoing particles in the radial direction is $\mathbf{p}_{out} = \hbar k \, \hat{\mathbf{r}}$. The momentum transfer in the scattering process then is

$$\mathbf{q} = \mathbf{p}_{in} - \mathbf{p}_{out} = \hbar \left(\mathbf{k} - k\hat{\mathbf{r}} \right). \tag{96}$$

As a result the scattering cross section takes the simple form

$$\frac{d\sigma}{d\Omega} = \left| f_k(\theta, \varphi) \right|^2 = \frac{4m^2\kappa^2}{\left(\mathbf{q}^2 + \hbar^2\lambda^2 \right)^2}. \tag{97}$$

In the Coulomb limit $\lambda \rightarrow 0$, with m replaced by the reduced mass μ, this reproduces exactly the Rutherford cross section (61). Therefore the classical Rutherford expression equals the quantum mechanical result for large distances in the limit of single scattering (first-order Born approximation).

Exercise 6.1
Perform explicitly the steps in the derivation (95).

7. Form factors

The Rutherford formula and its Yukawa generalization (97) describe scattering by the potential of a single point charge. We now generalize this result to the case of an extended charge distribution. Consider a total charge Z, distributed over a finite volume, described by a density function $Z\rho(\mathbf{r})$, where ρ is normalized to unity:

$$\int d^3\mathbf{r}\,\rho(\mathbf{r}) = 1. \tag{98}$$

The potential at point $\mathbf{r}$ due to the charge element $Z\rho(\mathbf{r}')d^3\mathbf{r}'$ at the point $\mathbf{r}'$ is then given by the superposition

$$V(\mathbf{r}) = Z\kappa \int d^3\mathbf{r}'\,\rho(\mathbf{r}')\,\frac{e^{-\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{99}$$

By substitution into the Born formula (93) we can again compute the scattering amplitude

$$f_k(\theta,\varphi) = -\frac{mZ\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}'\,e^{i\mathbf{q}\cdot\mathbf{r}'/\hbar}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')\,\frac{e^{-\lambda|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|}, \tag{100}$$

where as before we use the momentum transfer $\mathbf{q} = \hbar(\mathbf{k} - k\hat{\mathbf{r}})$. By interchanging the order of integration this becomes

$$\begin{aligned} f_k(\theta,\varphi) &= -\frac{mZ\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')\int d^3\mathbf{r}'\,e^{i\mathbf{q}\cdot\mathbf{r}'/\hbar}\,\frac{e^{-\lambda|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|} \\ &= -\frac{mZ\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')e^{i\mathbf{q}\cdot\mathbf{r}''/\hbar}\int d^3\mathbf{r}'\,e^{i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r}'')/\hbar}\,\frac{e^{-\lambda|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|}. \end{aligned} \tag{101}$$

Now we can shift the argument of the last integral by defining $\mathbf{R} = \mathbf{r}' - \mathbf{r}''$; clearly, the result no longer depends on $\mathbf{r}''$. Therefore the expression factorizes into two independent integrals. If we decompose $\mathbf{R}$ into spherical co-ordinates:

$$\begin{aligned} f_k(\theta,\varphi) &= -\frac{mZ\kappa}{\hbar^2}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')e^{i\mathbf{q}\cdot\mathbf{r}''/\hbar}\int_{-1}^{+1} d\cos\theta\int_0^\infty dR\,R\,e^{iqR\cos\theta-\lambda R} \\ &= -\frac{2mZ\kappa}{\mathbf{q}^2+\hbar^2\lambda^2}\,F(\mathbf{q}), \end{aligned} \tag{102}$$

where the form factor $F(\mathbf{q})$ is the Fourier transform of the charge distribution:

$$F(\mathbf{q}) = \int d^3\mathbf{r}\,\rho(\mathbf{r})\,e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}. \tag{103}$$

For the scattering cross section this result implies

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{point} |F(\mathbf{q})|^2, \tag{104}$$

where the label *point* refers to the scattering cross section for a point-like charge of strength Z, e.g. the Rutherford cross section for Coulomb scattering. It follows, that experimentally one can determine the charge distribution by measuring the differential cross section and comparing it with the point-like cross section:

$$|F(\mathbf{q})|^2 = \frac{(d\sigma/d\Omega)_{exp}}{(d\sigma/d\Omega)_{point}}. \tag{105}$$

The expression for $F(\mathbf{q})$ simplifies further in the case of a spherically symmetric charge distribution. Defining the wave number ξ by $\hbar\xi = |\mathbf{q}|$, and expanding in polar co-ordinates eq. (103) becomes

$$\begin{aligned} F(q = \hbar\xi) &= 2\pi \int_{-1}^{+1} d\cos\theta \int_0^\infty dr\, r^2 \rho(r)\, e^{i\xi r\cos\theta} \\ &= 4\pi \int_0^\infty dr\, r^2 \rho(r)\, \frac{\sin\xi r}{\xi r}. \end{aligned} \tag{106}$$

By expanding the sine function, this can be rewritten as

$$F = 4\pi \int_0^\infty dr\, r^2\, \rho(r) \left(1 - \frac{1}{3!}\,(\xi r)^2 + ...\right) = 1 - \frac{1}{6}\,\xi^2\,\overline{r^2} + ..., \tag{107}$$

where $\overline{r^2}$ is the expectation value of the radius squared of the charge distribution:

$$\overline{r^2} = \int d^3\mathbf{r}\, r^2\, \rho(r). \tag{108}$$

Therefore this quantity can be extracted directly from the measured cross section by determining the slope of F at zero momentum transfer:

$$\overline{r^2} = -\,6\,\frac{dF}{d\xi^2}\bigg|_{\xi^2=0}. \tag{109}$$

As an example, consider the exponential charge distribution

$$\rho = \frac{1}{8\pi a^3}\, e^{-r/a}. \tag{110}$$

For the form factor the first line of eq. (106) gives

$$F = \frac{1}{4a^3} \int_{-1}^{+1} d\cos\theta \int_0^\infty dr\, r^2 e^{i\xi r\cos\theta - r/a}. \tag{111}$$

This integral is very similar to the one in (102), and so is the result:

$$F = \frac{1}{(1 + a^2\xi^2)^2} = 1 - 2a^2\xi^2 + 3a^4\xi^4 + ... \tag{112}$$

By differentiation w.r.t. ξ^2 and applying eq. (109) it follows, that the mean squared radius is

$$\overline{r^2} = 12a^2. \tag{113}$$

Exercise 7.1
Prove eq. (107) by showing that

$$\overline{r^2} = \int d^3\mathbf{r}\, r^2 \rho(r) = 4\pi \int_0^\infty dr\, r^4 \rho(r).$$

Exercise 7.2
Compute the integrals in (111) to prove the result (112).

Exercise 7.3
a. Compute $F(q)$ for a homogeneously charged sphere of radius a:

$$\rho = \begin{cases} \dfrac{3}{4\pi a^3}, & r < a; \\ 0, & r > a. \end{cases}$$

b. From the result, derive that

$$\overline{r^2} = \frac{3}{5}\, a^2.$$

8. Nuclei

Atomic nuclei are bound states of protons and neutrons. The number of protons Z determines the chemical identity of the element, whilst the total number of nucleons A determines which isotope of the element the nucleus represents. The general notation for an isotope of element X with A nucleons and charge number Z is ${}^{A}_{Z}\mathrm{X}$; for example, the element helium has two naturally occurring isotopes, ${}^{4}_{2}\mathrm{He}$ and ${}^{3}_{2}\mathrm{He}$. The first and most abundant of these nuclei, with two protons and two neutrons, is also known for historic reasons as the α-particle.

As the mass of the proton and neutron are nearly equal, nuclei with the same number of nucleons have nearly equal masses. However, the actual masses are not only determined by the values of A and Z, but also by the nuclear binding energy; the mass $M(A, Z)$ of the nucleus ${}^{A}_{Z}\mathrm{X}$ is therefore made up of three contributions:

$$M(A, Z)c^2 = Zm_pc^2 + (A - Z)m_nc^2 - A\,\varepsilon(A, Z), \tag{114}$$

where $\varepsilon(A, Z)$ is the average binding energy per nucleon of the element: $\varepsilon(A, Z) = \Delta E/A$. This average binding energy is to first approximation independent of Z, hence the same for protons and for neutrons.

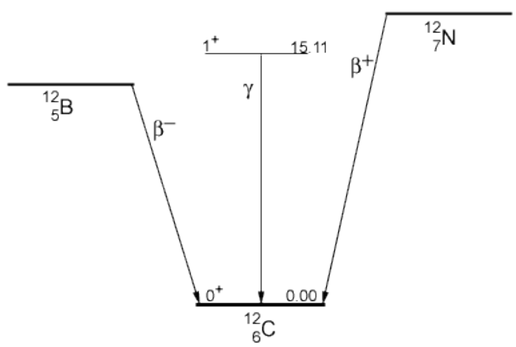

Fig. 8.1: Energy levels of nuclei with $A = 12$.

Actually, a more precise statement is that in nuclei of the same A and *other quantum numbers, such as angular momentum*, the binding energy of nucleons is to good approximation independent of Z. This is illustrated in fig. 8.1, which shows the lowest energy levels of three nuclei with $A = 12$. Clearly, the ground state of ${}^{12}_{6}\mathrm{C}$ represents the absolute minimum of energy. However, the carbon nucleus has an excited state which is nearly degenerate in energy with the ground states of ${}^{12}_{5}\mathrm{B}$ and ${}^{12}_{7}\mathrm{N}$. This excited state of ${}^{12}_{6}\mathrm{C}$ differs from the ground state in its angular momentum, but the angular momentum of the excited state is the same as that of ${}^{12}_{5}\mathrm{B}$ and ${}^{12}_{7}\mathrm{N}$ in their ground states. Furthermore, the actual energies of the three nearly degenerate states have to be corrected for the repulsive Coulomb interactions of the protons, which tend to lower the binding energy for increasing Z for the same A. Therefore these observations support the conclusion, that up to small corrections the binding energies of nucleons in isotopes with equal A and the same angular momentum are equal. This is further evidence that the interactions responsible for nuclear binding are charge independent, and hence not of electro-magnetic origin.

The binding energy per nucleon does depend on the total number of nucleons A. This is illustrated in the chart in fig. 8.2, which shows the binding energy per nucleon as a function of A. Observe, that for most light elements the binding energy per nucleon tends to increase with increasing A, the most important exception being ${}^4_2$He (α-particle), which is an exceptionally stable light nucleus. For nuclei with $A > 62$ the binding energy per nucleon decreases again. This observation implies that light nuclei, such as H or Li, can release energy by fusion, which increases the A of the product nuclei, whereas heavy nuclei like U or Ra can release energy by fission, reducing A compared to the parent nuclei.

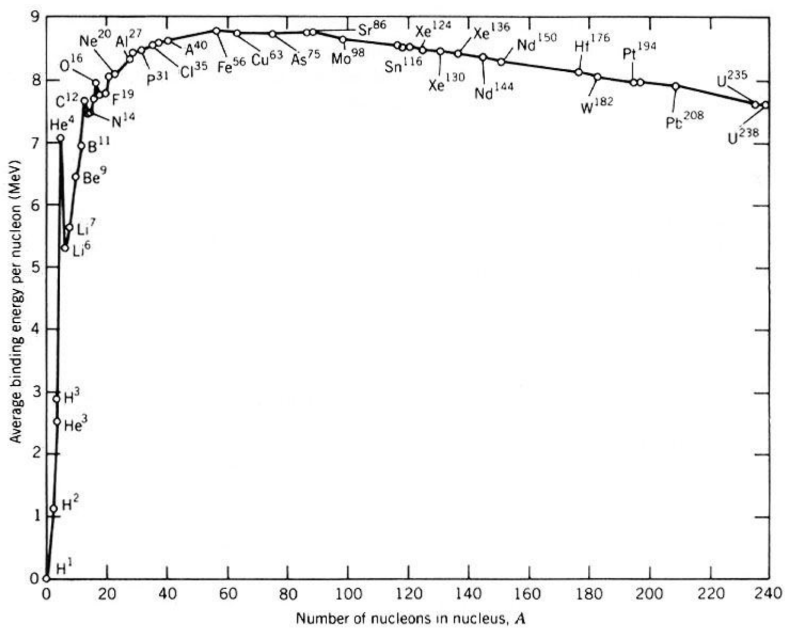

Fig. 8.2: Chart of nuclear binding energies

Nuclei of ${}^{56}_{26}$Fe and the rare ${}^{62}_{28}$Ni isotope have the largest binding energy per nucleon:

$$\varepsilon_X = 8.8\,\mathrm{MeV}, \qquad X = (\mathrm{Fe,\ Ni}).$$

Therefore these elements are the most stable in nature. In particular iron (${}^{56}_{26}$Fe) forms the natural end point of fusion in the interior of massive stars. If the core of a star in the course of its life builds up a sufficient amount of iron through fusion of lighter elements, it can no longer resist the pressure of its own weight by releasing energy through fusion into heavier nuclei. Such a star can then implode in a spectacularly energetic event, known as a supernova. The result is usually a very compact massive star consisting almost entirely of neutrons: a bound state of about 10^{57} nucleons in a spherical volume with a radius of 10-15 km. Its binding energy is not only provided by nuclear forces but also by gravity, with the effect that the binding energy per nucleon is higher than in iron.

9. Nuclear form factors

The existence of atomic nuclei was established by Geiger and Rutherford in scattering experiments with α-particles, discussed in sect. 4.b. Modern experiments scattering electrons off nuclei give more detailed information, in particular about the charge distribution.

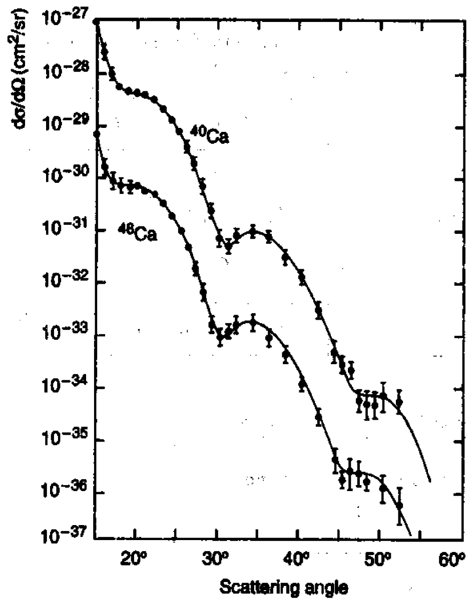

Fig. 9.1: Nuclear form factor of Ca isotopes.

Fig. 9.1 shows the differential cross section for elastic scattering of two calcium isotopes. Eq. (104) states that this cross section is a product of the Rutherford cross section and the absolute square of the form factor:

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{point} |F(\mathbf{q})|^2.$$

The Rutherford cross section for the scattering of point-like charges decreases smoothly as a function of scattering angle or momentum transfer. The equal total charges of the two calcium isotopes lead to different absolute cross sections, because the charge is more diluted in the case of ${}^{48}_{20}\mathrm{Ca}$ than in the case of ${}^{40}_{20}\mathrm{Ca}$, which contains fewer neutrons.

Furthermore, the nuclear cross sections in fig. 9.1 show on top of the Rutherford-like decrease an oscillating behaviour. Such oscillating behaviour is typical for a charge distribution which is constant up to some radius a, after which it drops steeply to zero, as in exercise 7.3. This behaviour is confirmed by measured nuclear mass distributions, as shown in fig. 9.2. The resulting picture of the atomic nucleus is that of a kind of droplet of nuclear matter with a rather sharply defined surface. The typical radius of the nucleus in this droplet model is in the range 3-7 fm.

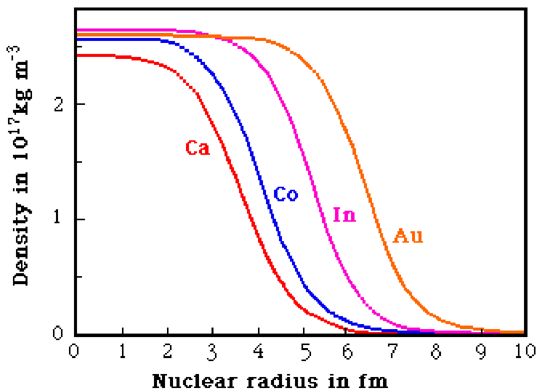

Fig. 9.2: Nuclear density profiles.

10. Radioactivity

In nature only a limited number of isotopes are found to exist, in which the number of protons and neutrons is more or less balanced: $A = 2\,Z$ in the light elements deuterium and helium, whilst there is a moderate excess of neutrons in heavy nuclei: $A \approx 2.5\,Z$ in uranium and other elements near the end of the periodic table. Moreover, elements found in our natural environment all have $Z \leq 92$. Very heavy elements or elements with unfavorable ratios $Z : A$ are unstable, and spontaneously transform into more stable nuclei by various processes. These transformations are accompanied by the emission of different kinds of particles; collectively these processes are refered to as radioactivity.

Spontaneous fission
Fig. 8.2 shows that elements with more than about a hundred nucleons can free internal energy by fission into lighter elements: the average binding energy per nucleon of the lighter nuclei is larger than in the original parent nucleus. However, in most cases this fission process goes extremely slow, as there is an energy barrier between the initial and final state and spontaneous fission can proceed only through quantum tunneling. The spontaneous decay of gold or mercury takes longer than the life-time of the universe, and therefore these elements can still be found with some abundance in rocks and minerals.

However, in some cases a massive nucleus has a larger probability to spontaneously emit a proton or a neutron, or even a more complicated light nucleus such as an α-particle. Indeed, α-particles are the most stable of the light nuclei, making spontaneous fission by emission of α-particles energetically favorable. An example is the fission of a uranium nucleus into a thorium nucleus and an α-particle:

$$^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}. \tag{115}$$

Among the strongest emitters of α-particles in nature are radium (Ra) and polonium (Po), first isolated by Marie Curie in her famous experiments in 1898. The emissions in this type of radioactive process were originally known as α-rays, from which the α-particles derive their name.

Most of the naturally found helium on earth is produced by α-decay of radioactive elements. In fact, α-particles were first identified as helium by Rutherford and Royds in 1909 by collecting the α-particles produced by a radioactive substance in the top section of an inverted glass tube sealed off below by mercury, and studying their spectral lines.

β-radioactivity
A completely different kind of nuclear transformation process is the transformation of a neutron into a proton or vice versa, as in the transformation of tritium into helium discussed in sect. 1:

$$^3_1\mathrm{H} \rightarrow {}^3_2\mathrm{He}.$$

In this process Z changes, whilst A remains the same. Such processes can happen spontaneously if the neutron in the final state has a sufficiently lower energy than the proton in the initial state, e.g. because of lower electrostatic Coulomb energy, as is the case here.

However, by itself these transformation processes would seem impossible, as they violate the conservation of electric charge. Indeed, the transformation is necessarily accompanied by the emission of charged particles: ordinary electrons in the case of neutrons changing into protons, and positively charged electrons, or *positrons*, when protons change into neutrons. In early studies of radioactivity these emissions were called β-rays, and therefore the electron and positron are also called β-particles. They were first recognized as electrons because their charge-to-mass ratio e/m is identical to that of the electron.

The simplest β-decay process is that of the neutron: a single neutron is unstable and decays after almost 15 minutes into a proton, an electron and a third, almost massless and electrically neutral particle, the *anti-neutrino*:

$$n \rightarrow p^+ + e^- + \bar{\nu}. \tag{116}$$

For this reason no free neutrons are found in nature. All neutrons in the universe reside in stable nuclei like deuterium or α-particles, where the process (116) is forbidden by energy conservation. However, in many nuclei the β-decay of a neutron is energetically allowed, making them unstable. An example is the process

$$^{12}_5\mathrm{B} \rightarrow {}^{12}_6\mathrm{C} + e^- + \bar{\nu}, \tag{117}$$

illustrated in fig. 8.1. As the diagram shows, the groundstate energy of $^{12}\mathrm{C}$ is considerably lower than that of $^{12}\mathrm{B}$.

Exercise 10.1
a. Show that the energy released in the β-decay process

$$^A_Z\mathrm{X} \rightarrow {}^A_{Z+1}\mathrm{X}' + e^- + \bar{\nu},$$

is positive if

$$\varepsilon(A, Z+1) \geq \varepsilon(A, Z) + \frac{1}{A}\left(m_p + m_e + m_\nu - m_n\right) c^2.$$

b. Estimate the binding energies of nucleons in ${}^3_1$H and ${}^3_2$He and calculate an upper limit for the mass of the neutrino.
Hint: use the following masses

particle	mass (MeV/c^2)
e	0.511
p	938.272
n	939.556
^{3}H	2808.920
^{3}He	2808.391

11. Exponential decay and Poisson statistics

By experimenting with different uranium salts, Marie Curie discovered that the intensity of α-emission in the decay of uranium was proportional to the number of uranium atoms in a substance, and independent of its chemical composition. Apparently the rate at which radioactive nuclei decay is an intrinsic nuclear property, a characteristic number for each isotope of each element. This can be interpreted statistically to imply, that the probability for a specific nucleus to decay depends only on the type of nucleus, not on any environmental factor. In fact, such a statistical interpretation is imposed by quantum mechanics, which allows us only to calculate the *probability amplitude* for a certain process from first principles.

Suppose a substance A is converted at a constant rate into a substance B, with the total number of particles of type A and type B constant: each particle of A gives rise to one and only one particle of type B. Then the fraction of particles A converted in a short time interval Δt is proportional to the time interval, and the proportionality factor is independent of environmental factors or the number of particles present:

$$\frac{\Delta N_A}{N_A} = -\lambda\,\Delta t, \tag{118}$$

with λ a constant, characteristic of substance A. It follows that the number of particles A decreases exponentially:

$$N_A(t) = N_A(0)\,e^{-\lambda t}, \tag{119}$$

where $N_A(0)$ is the number of particles of A present at time $t = 0$. Of course, the number of particles B increases at the same rate:

$$N_B(t) = N_B(0) + N_A(0)\left(1 - e^{-\lambda t}\right). \tag{120}$$

Now this process has to be interpreted statistically: it holds for large numbers of particles, but at the level of an individual particle we can only state the probability that this particle will survive for a certain time. To reproduce the above decay law, the probability for a particle of type A to survive after a time t is

$$P_A(t) = \frac{N_A(t)}{N_A(0)} = e^{-\lambda t}. \tag{121}$$

This implies, that the number of particles decaying in a specific period of time is governed by Poisson statistics. The Poisson distribution gives the probability to observe n events in a time t for statistically independent events, when the probability to observe a single event in a short time Δ is proportional to Δ, whilst the probability to observe more than one event is of order Δ^2. With these assumptions, the probability to observe no events in a finite time t is given by an exponential distribution

$$L_0(t) = e^{-\lambda t}, \tag{122}$$

whilst the probability $L_n(t)$ to observe $n \geq 1$ events in a time t is the n-th moment of this distribution:

$$L_n(t) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}. \tag{123}$$

It is straightforward to check that these probabilities are properly normalized:

$$\sum_{n=0}^{\infty} L_n(t) = 1. \tag{124}$$

The proof of these results is sketched in exercise 11.2.

A related probability density, which can be derived from $P_A(t)$, is the probability $W_A(t)dt$ for a particle to decay in the time interval between t and $t + dt$; it is given by

$$W_A(t)dt = P_A(t) - P_A(t + dt) = \lambda e^{-\lambda t} dt = -dL_0(t). \tag{125}$$

Again the density $W_A(t)$ is properly normalized:

$$\int_0^\infty W_A(t)dt = -\int_0^\infty dL_0(t) = 1. \tag{126}$$

The average life-time of a particle A then is computed by weighing the time it lives with this probability:

$$\langle t \rangle = \int_0^\infty t\, W_A(t)dt = \lambda \int_0^\infty t e^{-\lambda t} dt = \frac{1}{\lambda} \equiv \tau. \tag{127}$$

This average $\tau = 1/\lambda$ is the time after which the number of particles of type A has decreased by $1/e$; it is called the life-time of the particles. It differs from the quantity sometimes quoted as the *half-life* $t_{1/2}$: the time after which 50 % of the original particles has decayed:

$$t_{1/2} = \tau \ln 2. \tag{128}$$

The life-time τ is an average; in an actual experiment the measured values will scatter around this average; this scatter is expressed by the root-mean square deviation:

$$\sqrt{\langle (t - \tau)^2 \rangle} = \frac{1}{\lambda}. \tag{129}$$

An important property of the exponential decay law (121) is, that it holds independent of the choice of initial time. Thus it does not matter how many particles have already decayed: once you know the number of particles A present initially, the number will always decrease in time in the same way. In this sense radioactive decay has no memory.

Exercise 11.1
A radioactive element A decays into element B with a probability per atom per unit of time λ_A. The element B is also radioactive and decays to a stable element C with a probability λ_B per atom and per unit of time.
a. Explain that the change in time of the average number of atoms of kind A and B is given by

$$dN_A(t) = -\lambda_A N_A(t)dt,$$

$$dN_B(t) = -\lambda_B N_B(t)dt + \lambda_A N_A(t)dt.$$

b. Derive an expression for the change $dN_C(t)$ of the average number of atoms C in a time interval dt.
c. At time $t = 0$ we start out with a pure sample of N_0 atoms of type A; calculate the number of atoms of each kind (A, B, C) as a function of time, for the two cases $\lambda_B \neq \lambda_A$ and $\lambda_B = \lambda_A$.
d. Sketch the solutions $N_A(t)$, $N_B(t)$ and $N_C(t)$.

Exercise 11.2
Radioactive decay processes are statistically independent events: the probability of a single atom decaying in a time interval dt is proportional to the length of the interval, independent of its history and independent of the concentration of atoms. Show:
For a sufficiently large number N of a radioactive atoms the probability $L_n(t)$ of a number of n particles decaying in a time t depends only on the decay probability λ of a single atom, and on the length of the time interval t:

$$L_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Hints:
(i) The statistical independence of the decay processes implies that the probability for n_1 decays[3] in a time interval Δ_1 and n_2 decays in a non-overlapping time interval Δ_2 is

$$L(n_1, \Delta_1; n_2, \Delta_2) = L_{n_1}(\Delta_1) L_{n_2}(\Delta_2).$$

(ii) The probability for a single decay in a short time interval Δ is

$$L_1(\Delta) = \lambda\Delta + \mathcal{O}(\Delta^2).$$

(iii) The probability for two or more decays in a short time interval Δ vanishes in the limit $\Delta \to 0$:

$$L_n(\Delta) = \mathcal{O}(\Delta^2), \qquad n \geq 2.$$

a. From these properties, show that

$$L_0(t + \Delta) = L_0(t)L_0(\Delta) = L_0(t)\left(1 - \lambda\Delta + \mathcal{O}(\Delta^2)\right),$$

[3]Properly normalized, such that the sum of all probabilities in any time-interval equals unity.

and prove that in the limit $\Delta \to 0$

$$\frac{dL_0(t)}{dt} = -\lambda L_0(t).$$

b. Next consider the probability for one or more decays to take place in an interval $t + \Delta$; show that

$$L_n(t+\Delta) = \sum_{k=0}^{n} L_k(t) L_{n-k}(\Delta) = L_n(t)\,(1 - \lambda\Delta) + L_{n-1}(t)\,\lambda\Delta + \mathcal{O}(\Delta^2),$$

and by taking the limit $\Delta \to 0$, derive

$$\frac{dL_n(t)}{dt} = -\lambda L_n(t) + \lambda L_{n-1}(t), \qquad n \geq 1.$$

c. With the boundary conditions $L_0(0) = 1$, $L_n(0) = 0$ for $n \geq 1$, solve the differential equations for $L_n(t)$ to derive the results (122), (123):

$$L_0(t) = e^{-\lambda t}, \qquad L_n(t) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}, \qquad (n \geq 1).$$

d. Check the normalization of these probabilities:

$$\sum_{n=0}^{\infty} L_n(t) = 1.$$

Exercise 11.3
Prove the result (129) for the root-mean square deviation of the decay times for an exponential decay law.

12. Neutrinos

The β-decay of the neutron is accompanied not only by the emission of an electron, but also a very light neutral particle, the anti-neutrino. The existence of neutrinos was predicted by Pauli in 1930 on the basis of energy conservation in β-decay. In view of later developments the particle accompanying the electron is now refered to as the *anti-neutrino*. The name neutrino is given to the particle emitted in β-decay process in which a positron is produced:

$$ {}^{A}_{Z}\mathrm{X} \rightarrow {}^{\;\;A}_{Z-1}\mathrm{X}' + e^{+} + \nu. \tag{130} $$

In these processes a proton is converted into a neutron, as in the decay

$$ {}^{12}_{7}\mathrm{N} \;\rightarrow\; {}^{12}_{6}\mathrm{C} + e^{+} + \nu, $$

illusttrated in fig. 8.1. This can only happen in unstable nuclei, as the free proton itself is stable. Neutrinos and anti-neutrinos are very difficult to observe, and it took until 1956 before they were first detected experimentally by Reines and Cowan, using anti-neutrinos from a nuclear reactor at Savannah River.

Pauli's argument was based on the observation that the electrons produced in the decay of the neutron have a continuous energy spectrum. Now if a neutron at rest would decay into two particles, a proton and an electron, the electron would have a fixed energy; indeed energy and momentum conservation imply

$$ E_p + E_e = m_n c^2, \qquad \mathbf{p}_p + \mathbf{p}_e = 0. \tag{131} $$

Using the relativistic energy-momentum relation (355) the electron energy is solved from

$$ \begin{aligned} E_e &= m_n c^2 - \sqrt{\mathbf{p}_p^2 c^2 + m_p^2 c^4} = m_n c^2 - \sqrt{\mathbf{p}_e^2 c^2 + m_p^2 c^4} \\ &= m_n c^2 - \sqrt{E_e^2 - m_e^2 c^4 + m_p^2 c^4}. \end{aligned} $$

Solving for E_e:

$$ E_e = \frac{(m_n^2 - m_p^2 + m_e^2) c^2}{2 m_n} = 1.3\,\mathrm{MeV}. \tag{132} $$

This is in contradiction with the observations, which show a continuous range of energies $m_e c^2 \leq E_e \leq 1.3$ MeV for the electrons produced, as sketched in fig. 12.1.

When there are three particles in the decay, the continuous spectrum is easily explained. In that case the energy and momentum conservation laws in the neutron rest frame become

$$ E_p + E_e + E_\nu = m_n c^2, \qquad \mathbf{p}_p + \mathbf{p}_e + \mathbf{p}_\nu = 0. \tag{133} $$

Together with the 4 energy-momentum relations (355) for the individual particles, there are 7 equations for 12 variables, which leaves sufficient room for a large range of momenta $\mathbf{p}_e$ and $\mathbf{p}_\nu$ in the final state of the electron and the neutrino, and a corresponding range of energies.

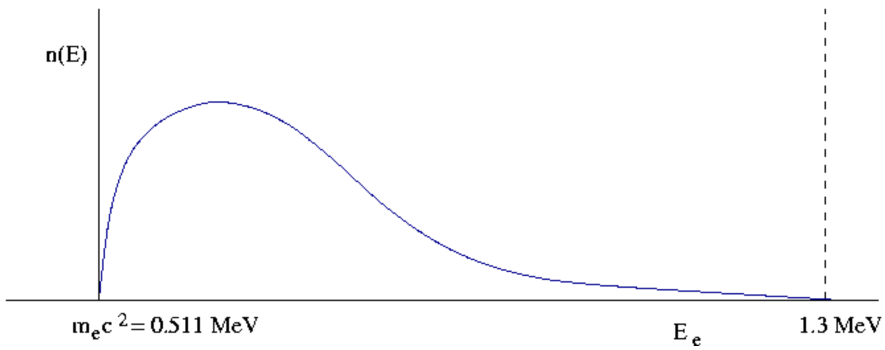

Fig. 12.1: Energy spectrum of electrons from neutron decay.

Another argument for the existence of the neutrino, which was not yet available when Pauli first made his proposal, is that the neutron, the proton and the electron are all fermions (particles obeying Fermi-Dirac statistics), with spin 1/2 in units $\hbar$. As orbital angular momenta are always quantized in integer number of this fundamental unit, the conservation of angular momentum would be violated if β-decay of the neutron would not involve a third spin-1/2 particle. This immediately implies that the neutrino must be a fermion as well.

The mass of the neutrino is not known, but upper limits have been set by various experiments. A firm upper limit is $m_\nu < 1$ eV$/c^2$, but it is likely that the actual mass is still 10-100 times smaller. This indicates that neutrinos are several million times lighter than electrons. Such small masses are not directly measurable. Moreover, as neutrinos have no electric charge their presence can not be established by any electromagnetic interactions. This makes neutrinos extremely difficult to detect.

The first detection of anti-neutrinos was actually made by running the β-decay in a kind of reverse mode, by scattering anti-neutrinos with nuclei and looking for the reaction

$$\bar{\nu} + p^+ \rightarrow e^+ + n. \tag{134}$$

The probability of this process, as expressed by the scattering cross section, is extremely small, but non-zero. Therefore if one has a very intense neutrino source, such reactions can actually be observed in small numbers. Reines and Cowan used the anti-neutrinos from the nuclear reactor in the Savannah River power plant to bombard a tank with cadmium chloride $\mathrm{Cd\,Cl_2}$. Whenever a positron was produced, it would immediately annihilate an electron in the fluid and produce characteristic γ-rays. To make sure the positron was produced in the reaction (134), they also looked for the signal of the neutron being absorbed by a cadmium nucleus under emission of another γ-ray:

$$^{108}\mathrm{Cd} + n \rightarrow {}^{109}\mathrm{Cd} + \gamma. \tag{135}$$

By observing such events at the rate anticipated in view of the number of anti-neutrinos expected to arrive from the nuclear reactor, they established the presence of these anti-neutrinos. An important observation was subsequently made by Davis and Harmer, who established that the reaction

$$\bar{\nu} + n \;\rightarrow\; e^- + p^+, \tag{136}$$

does *not* occur, whilst the similar reaction using neutrinos instead of anti-neutrinos *is* actually observed. This proves that neutrinos and anti-neutrinos are indeed different particles. This difference can be expressed by assigning to electrons and neutrinos a new conserved quantum number, called *lepton number*:

$$L_{e^-} = L_\nu = +1, \tag{137}$$

and to positrons and anti-neutrinos the opposite lepton number:

$$L_{e^+} = L_{\bar{\nu}} = -1. \tag{138}$$

Nucleons are assigned a vanishing lepton number:

$$L_p = L_n = 0.$$

The absence of the reaction (136) can now be explained as it violates the conservation of lepton number, whereas the reaction (134) and

$$\nu + n \;\rightarrow\; e^- + p^+, \tag{139}$$

do respect the conservation of lepton number, and therefore actually occur.

In some sense the conservation of lepton number is like the conservation of electric charge. However, there is an important difference: the conservation of electric charge is closely linked to the properties of the electro-magnetic field and is required by Maxwell's equations. In classical terms, electric field lines can not vanish or end in empty space. No such explanation can be given for the conservation of lepton number, as there is no field connected to it. Nevertheless, although theoretically it might be possible that there are processes in nature which violate lepton number, no such violation has ever been observed in any experiment.

Exercise 12.1

On the basis of conservation laws such as charge, lepton number and baryon number indicate which of the following reactions are possible or impossible, and why:

a. $p^+ + e^- \rightarrow n + \nu$
b. $p^+ + e^- \rightarrow n + \bar{\nu}$
c. $n + e^- \rightarrow p^+ + \nu$
d. $n + e^+ \rightarrow p^+ + \bar{\nu}$
e. $p^+ + \bar{\nu} \rightarrow n + e^+$
f. $p^+ + \nu \rightarrow n + e^+$

13. Anti-matter

Cosmic rays

In the early part of the 20th century it was discovered there is a background of ionizing radiation in our natural environment. Theodor Wulf and Victor Hess established, that this radiation had no terrestrial origin, such as radioactive minerals. In 1912 Hess made a number of rather courageous balloon flights to atmospheric altitudes above 5 km, carrying on board a very sensitive electrometer of a type developed by Wulf. He definitely established the increase of the intensity of this radiation with altitude, and thereby its extraterrestrial origin. Thus cosmic rays were discovered.

In the course of time it became clear, that cosmic rays consist of very energetic particles bombarding the earth. Most of these particles are protons or heavier nuclei, which upon entering the atmosphere collide with a nitrogen or oxygen nucleus. Such collisions then create a shower of secondary particles. This is the origin of the *extended air showers*, correlated bursts of energetic particles raining down on earth over large areas, as discovered by Pierre Auger in the 1930's. For a long period of time, until the advent of modern high-energy accelerators in the 1950's, cosmic rays were the only source of very high-energy particles available to experimentalists. As such they were the source for a number of important discoveries,

Anti-particles

In 1932 Carl Anderson established the existence of positively charged electrons, or *positrons* in cosmic rays.

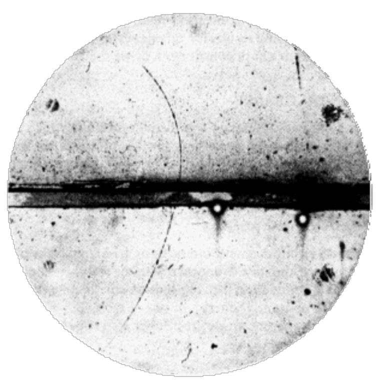

Fig. 13.1: First evidence of the existence of the positron

Anderson studied the tracks of charged cosmic particles in a cloud chamber, placed in a magnetic field. In the magnetic field the tracks of charged particles are curved, the radius of curvature depending on the charge-to-mass ratio q/m and the energy of the particle. Positively and negatively charged particles curve in opposite directions, but to establish this direction one must know on what side the particle entered the cloud chamber. This Anderson determined by placing a slab of lead in the middle of the cloud chamber. During passage throught this slab the particle lost energy, and its radius of curvature decreased. In Fig. 13.1 the track of a positron is seen, entering from below and curving to the left, in the anti-clockwise direction. The charge-to-mass ratio of the particle was equal in magnitude to e/m_e of the electron, but opposite in sign.

Two years earlier Paul Dirac had actually predicted the existence of positrons on the basis of his relativistic electron theory. The positron and electron have opposite charge and can annihilate each other, by simultaneous creation of two gamma rays:

$$e^+ + e^- \rightarrow \gamma + \gamma. \tag{140}$$

The total energy and momentum of the gamma rays equal those of the original charged particles; this energy is at least equal to the combined rest mass of the electron and positron: twice 0.511 MeV. Thus energy and momentum are conserved, as is the total electric charge, but neither the individual charges nor the individual masses of the original particles are preserved.

Later, other particles such as protons, neutrons and neutrinos turned out to have corresponding anti-particles as well; conventionally anti-particles are denoted by an overbar, e.g. $\bar{p}$ and $\bar{n}$ for the anti-proton and anti-neutron, respectively[4]. In all cases particles and anti-particles have the same mass, but opposite charge and other quantum numbers, such as lepton number. We already briefly touched upon this in our discussion of neutrinos, sect. 12. In fact, the existence of anti-nucleons makes it possible to associate a conserved quantum number with nucleons, the *baryon number*, in the same way as the lepton number is associated with electrons and neutrinos. By convention, the baryon number takes a positive value for the proton and neutron:

$$B_p = B_n = +1, \tag{141}$$

and the opposite negative value for the anti-proton and anti-neutron:

$$B_{\bar{p}} = B_{\bar{n}} = -1. \tag{142}$$

The leptons have vanishing baryon number:

$$B_e = B_\nu = 0. \tag{143}$$

It turns out, that in all reactions observed to date the total baryon number is conserved. Like the conservation of lepton number, this is a phenomenological conservation law, not associated with the existence of a Maxwell-type field.

The assignments of baryon and lepton numbers indicates the existence of two separate classes of particles, baryons and leptons, each characterized by their own conserved quantum number.

The existence of anti-particles is a general consequence of relativistic quantum theory. Nowadays positrons and anti-protons can be made in such abundance that they can be

[4]Sometimes it is more convenient to identify particles or anti-particles by their charge, e.g. e^- and e^+ for electrons and positrons, or p^+ and p^- for protons and anti-protons; of course, this does not work for neutral particles.

used in high-energy accelerators to collide with electrons or protons, thereby creating extreme energy densities used to search for other new forms of matter.

Exercise 13.1
Check which of the following processes are allowed or forbidden by the conservation laws of charge, lepton number or baryon number.

a. $p + n \rightarrow \bar{e} + \nu$
b. $p + \bar{n} \rightarrow e + \bar{\nu}$
c. $\bar{n} + \bar{e} \rightarrow p + \bar{\nu}$
d. $\nu + \bar{p} \rightarrow e + \bar{n}$
e. $\bar{p} + n \rightarrow e + \bar{\nu}$
f. $\bar{e} + \nu \rightarrow \bar{p} + n$

14. The photon

The photon was among the first elementary particles known in physics, but its recognition as a particle took quite a long time. It started in 1900 with Planck's discovery of the law of black body radiation and his derivation of this law, based on discretizing the energy of monochromatic radiation emitted by a perfect black body at temperature T in quanta

$$E = h\nu, \tag{144}$$

where ν is the frequency. The next step was taken by Einstein in 1905. He argued that quanta represent the actual state of free radiation, and that this could explain the photoelectric effect: if light falls on a conducting surface, electrons are set free with an energy which depends only on the frequency of the light, not its intensity:

$$E_e = h\nu - P, \tag{145}$$

where P is the fixed energy the electron needs to escape form the surface of the conductor. In contrast, the number of electrons emitted *does* depend on the intensity. This effect could be explained simply if each electron absorbs a single photon, such that it gains a fixed energy $h\nu$; moreover, the number of electrons emitted is then proportional to the number of incident photons per unit of area.

In 1917 Einstein took the next step, assigning to photons a momentum of size $q = h/\lambda$ in the direction of motion:

$$\mathbf{q} = \frac{h\mathbf{k}}{2\pi} \equiv \hbar\mathbf{k}, \tag{146}$$

with $\mathbf{k}$ the wave vector, of magnitude

$$|\mathbf{k}| = \frac{\omega}{c} = \frac{2\pi\nu}{c} = \frac{2\pi}{\lambda}. \tag{147}$$

This implies that the photon is a particle with vanishing rest mass, always moving at the speed of light:

$$E^2 = h^2\nu^2 = \hbar^2\mathbf{k}^2c^2 = \mathbf{q}^2c^2. \tag{148}$$

This is to be compared with the energy-momentum relation for a massive particle with momentum $\mathbf{p}$, like the electron:

$$E^2 = m^2c^4 + \mathbf{p}^2c^2. \tag{149}$$

These energy-momentum relations become the same upon taking $m = 0$.

The final step in the story of the photon was taken by Arthur Compton, and independently by Peter Debije, who analyzed elastic scattering of a photon with an electron in purely kinematical terms, using energy and momentum conservation. If the electron is originally at rest, and we denote the momenta and energies after scattering by a prime, we find

$$\mathbf{q} = \mathbf{q}' + \mathbf{p}', \qquad E_\gamma + m_e c^2 = E'_\gamma + \sqrt{m_e^2c^4 + \mathbf{p}'^2c^2}. \tag{150}$$

If θ denotes the scattering angle of the photon:

$$\mathbf{q} \cdot \mathbf{q}' = |\mathbf{q}||\mathbf{q}'| \cos\theta, \tag{151}$$

then we find from eqs. (146) - (151) that in vacuum

$$\lambda' - \lambda = \frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_e c}\left(1 - \cos\theta\right). \tag{152}$$

This relation was verified by Compton in an experiment; the quantity

$$\lambda_e = \frac{h}{m_e c} \approx 2.426 \times 10^{-12}\,\mathrm{m}, \tag{153}$$

is called the *Compton wave length* of the electron. The Compton experiment definitely etablished the status of the photon as a particle. By conservation of angular momentum in atomic transitions, where photons are emitted or absorbed, it is established that the photon has spin $s = 1$.

Exercise 14.1
Eq. (152) states how the wave length (or energy) of a photon is changed when it scatters off an electron at rest through an angle θ; it does not *predict* the angle or the wave-length shift. The angular dependence depends on further parameters, e.g. the impact parameter. This information is encoded in the differential scattering cross section, in this case the Thomson cross section. The differential cross section therefore contains more information than the scattering kinematics.

a. Consider the photon-electron system in the CM frame, in which the total momentum vanishes. By analyzing the energy-momentum conservation conditions, show that the energies of the photon and electron can not change in the collision, and that only the *direction* of the momenta before and after the collision can (and generally will) be different.
b. Explain why the energy of the electron and photon *can* change in the lab frame, in which the electron is originally at rest. Consider in particular the case $\theta = 180°$.

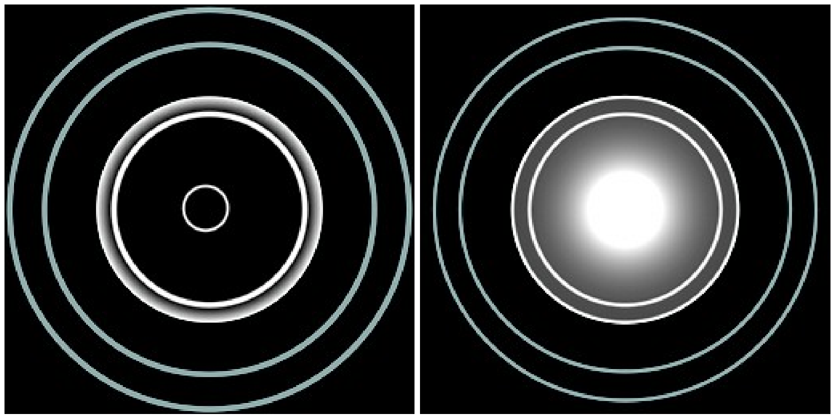

Fig. 15.1: X-ray and electron diffraction patterns from scattering by aluminum

15. Matter waves

The quantum theory of light assigns particle properties (energy and momentum) to light waves. In 1923 Louis De Broglie proposed that matter possesses wave properties. The correspondence is basically the same as for the photon:

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}, \tag{154}$$

with

$$\omega = 2\pi\nu, \qquad |\mathbf{k}| = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \tag{155}$$

Here λ is the wave length in empty space. The relativistic energy momentum relation between particles then takes the form

$$E^2 = m^2c^4 + \mathbf{p}^2c^2 \quad \Leftrightarrow \quad \omega^2 = \frac{4\pi^2c^2}{\lambda_c^2} + \mathbf{k}^2c^2, \tag{156}$$

with

$$\lambda_c = \frac{2\pi\hbar}{mc}. \tag{157}$$

This characteristic wave length is known as the Compton wave length.

De Broglies proposal was confirmed in 1927 by the experiments of Davisson and Germer, who scattered electrons from a crystal lattice and found a diffraction pattern as expected for waves; fig. 15.1 shows a comparison of x-ray and electron diffraction patterns. The wave-particle duality is at the heart of quantum mechanics, as discussed in sect. 5. It plays a major role in particle physics; indeed, according to quantum field theory all particles can be interpreted as quanta associated with some type of wave field. To a large extend particles can therefore be distinguished by the kind of wave equation their corresponding fields satisfy.

16. Nucleon structure

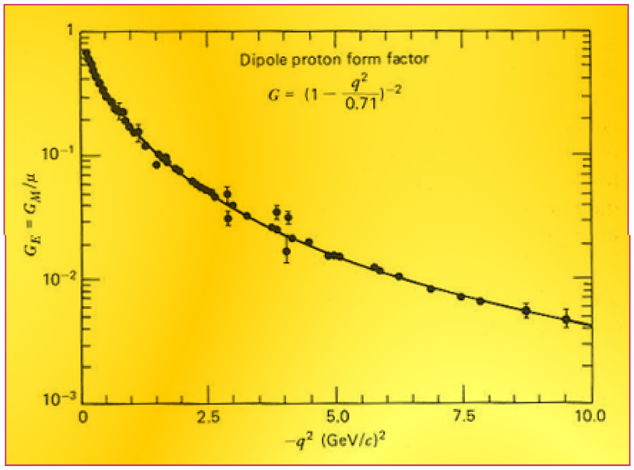

Fig. 16.1: Proton form factor from ep-scattering

In 1960 Robert Hofstadter studied the scattering of electrons, with energies in the range 200-550 MeV, on stationary protons in a hydrogen-rich target. His measurements indicated that protons are not point-like: the differential cross section approximately matches a finite charge distribution described by a form factor of the type (112):

$$F(q) = \frac{1}{(1 + a^2 q^2/\hbar^2)^2}.$$

This form factor corresponds to an exponential charge distribution (110) of characteristic size a, which in the case of the proton has the value

$$a = 0.24\,\mathrm{fm} \quad \Leftrightarrow \quad \frac{\hbar^2}{a^2} = 0.71\,\mathrm{GeV}^2/c^2. \tag{158}$$

Some ten years later Friedmann, Kendall and Taylor carried out a similar experiment with electron energies of about 5 GeV, to study the *inelastic* scattering of electrons and protons. In inelastic processes some of the energy of the electron is not used to transfer kinetic energy to the proton, but to excite internal degrees of freedom of the proton. This becomes evident in the final state of the process, as the proton must emit one or more new particles to get rid of the additional internal energy and return to the ground state. In principle the new particle can be a photon, but more often it is a strongly interacting particle like a pion, which we will encounter shortly.

Fig. 16.2 shows the cross section for electron-proton scattering, at an incident electron energy of 4.88 GeV, as a function of the scattered electron energy at a fixed scattering angle of 10°. The electrons scattering elastically from a proton at rest have a fixed energy after scattering close to 4.5 GeV. These elastically scattered electrons therefore form a very narrow peak, shown in reduced size at the far right end of the spectrum.

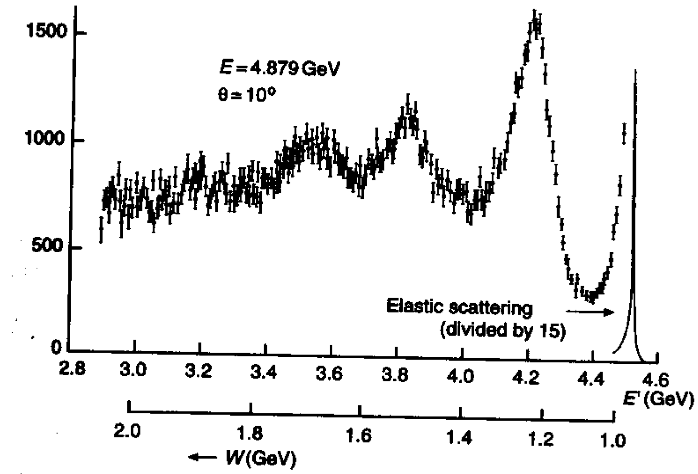

Fig. 16.2: Inelastic electron-proton scattering

If the scattered electron comes out of the collision with a lower energy, some energy has been transfered to excite internal degrees of freedom of the proton. As the figure shows, this happens preferentially at certain energies characterized by the peaks in the distribution. This is a strong indication for internal structure of the proton. In fact, Friedmann et al. also observed some rare scattering over very large angles, just like in Rutherford scattering. This hints at the existence of small point-like objects inside the proton. These objects were refered to by the neutral name of *partons*. Nowadays, we identify them with point-like elementary particles known as *quarks*.

Similar results were found for the neutron, which possesses a finite charge distribution and inelastic scattering peaks as well. The mere fact that electrons interact with the neutron already shows, that the neutron must contain internal charges, as the electron is insensitive to the strong interactions. Of course, the total charge of the neutron vanishes, therefore it must contain equal amounts of positive and negative charge. Nevertheless both the proton and neutron both have a non-zero magnetic moment:

$$\mu_p = 2.79\,\mu_N, \qquad \mu_n = -1.91\,\mu_N, \tag{159}$$

where μ_N is the nuclear magneton, defined as

$$\mu_N = \frac{e\hbar}{2m_p} = 3.152 \times 10^{-14}\,\mathrm{MeV/T}. \tag{160}$$

The proton and neutron magnetic moments (159) result from the spins and angular momentum of their charged constituents, although they are difficult to calculate theoretically from first principles.

Exercise 16.1
An electron with an energy of 4.9 GeV scatters elastically with a proton at rest. Calculate the electron energy if it scatters over an angle $\theta = 10°$, and over angles $\theta = 20°$ and 45° as well.

17. Hadrons

In sect. 9 we established, that the typical size of a nucleus is ~ 5 fm. This gives a rough estimate for the range of the strong interactions between nucleons. If these interactions are mediated by some kind of particle, associated with a nuclear field, this particle must have a Compton wave length comparable to the range of the interactions; this implies that its mass is of the order of 100 MeV/c^2. The existence of such a particle associated with the strong nuclear interactions was proposed by Hideki Yukawa in 1935. By 1947 a number of particles with masses in this range were identified in cosmic rays by Powell, Occhialini and co-workers. Three of them eventually turned out to be strongly interacting particles of the kind Yukawa envisioned. These particles were called pions, and have the following properties

particle	charge (e)	mass (MeV/c^2)
π^+	+1	139.6
π^0	0	135.0
π^-	-1	139.6

In addition, the pions are spinless: $s = 0$, hence they are bosons. As can be guessed from this table, the charged pions π^+ and π^- are each other's anti-particle, with exactly equal masses but opposite electric charges. In contrast, the neutral π^0 has no anti-particle, or rather: it is its own anti-particle, and it is somewhat lighter. This situation is similar to that of the nucleons, which also have different charges, but almost equal masses.

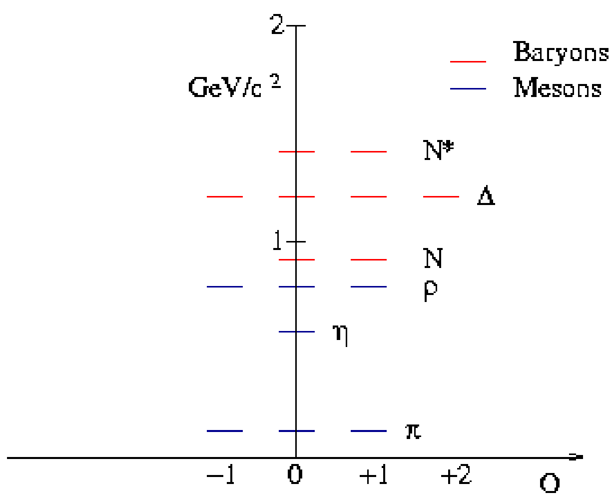

Fig. 17.1: Partial hadron spectrum

After the discovery of the pions, many more strongly interacting particles with considerably larger masses were identified in cosmic ray studies. The advent of modern high-energy accelerators greatly extended the range of masses in which the particle spectrum could be explored and by now hundreds of strongly interacting particles are known[5]. These strongly

[5]For a detailed overview, see: C. Amsler et al., *Review of Particle Physics*, Phys. Lett. B667 (2008); all data are also available in digital format at: http://pdg.lbl.gov/

interacting particles are collectively called *hadrons*. Fig. 17.1 shows the spectrum of some of the lighter hadrons.

Hadrons fall naturally into two classes: the boson class of particles with integer spin $s = 0, 1, 2, ...$, which are called *mesons*; and the fermion class of particles with odd half-integer spin $s = 1/2, 3/2, ...$, known as *baryons*. Obviously, the pions (π) fall into the first class, whilst the nucleons (N) fall into the second class.

The pions are the lightest of all hadrons; the next lightest ones are the neutral η-meson, also with $s = 0$ and mass $m_\eta = 547.5$ MeV/c^2; and the triplet of ρ-mesons with electric charges like the pions, but spin $s = 1$ and an average mass of about 769 MeV/c^2.

The nucleons are the lightest baryons. Next in mass is the quadruplet of Δ-baryons, with charges $(+2, +1, 0, -1)$ and spin $s = 3/2$; the average mass is around 1232 MeV/c^2. Their also exist excited states of the nucleons, the N^* doublet, with the same quantum numbers as proton and neutron, but a considerably larger mass of 1440 MeV/c^2. All these baryons have their distinct anti-particles, the anti-baryons $\bar{N}$, $\bar{\Delta}$ and $\bar{N}^*$, with opposite electric charges.

An important difference between mesons and baryons is, that mesons carry no baryon number, whereas baryons do:

$$\begin{aligned} B_\pi &= B_\eta = B_\rho = 0, \\ B_N &= B_\Delta = B_{N^*} = +1, \\ B_{\bar{N}} &= B_{\bar{\Delta}} = B_{\bar{N}^*} = -1. \end{aligned} \tag{161}$$

Experimentally, in all interactions of these particles the total baryon number is conserved. This has important consequences related to the stability of these particles. For example, the Δ- and N^*-baryons can decay to ordinary nucleons by emission of pions:

$$(\Delta, N^*) \rightarrow N + \pi. \tag{162}$$

This includes e.g. the specific cases:

$$\Delta^{++} \rightarrow p^+ + \pi^+, \qquad N^{*\,0} \rightarrow n + \pi^0 \quad \text{or} \quad N^{*\,0} \rightarrow p^+ + \pi^-. \tag{163}$$

Because these interactions are strong, they happen very fast; indeed, the life-times of the (Δ, N^*)-baryons are of the order of a few times 10^{-24} sec. It is not difficult to see, that these decays obey the conservation of baryon number. However, the ordinary nucleons (p^+, n) are the lightest baryons, hence there are no lighter ones for them to decay into. Therefore, to the extent that the conservation of baryon number is an absolute and exact conservation law, the nucleons must be stable. Experiments looking for proton decay have established, that the proton life time is at least about 10^{30} years, or 10^{20} times the life-time of the visible universe.

For mesons the situation is slightly different. As the π^0, η and ρ^0 are their own antiparticle, they must have $B = 0$. Now the charged mesons have the same baryon number as their neutral counterparts: $B = 0$. Therefore the conservation of baryon number never forbids their decay into other particles. Even so, there are different modes of decay which come into play. Indeed, as the strong interactions are by nature fast, the dominant decay mode of heavier mesons is usually through pion emission; for example, the ρ-mesons decay almost exclusively into pions in about 4×10^{-24} sec:

$$\rho^{\pm} \rightarrow \pi^{\pm} + \pi^{0}, \qquad \rho^{0} \rightarrow \pi^{+} + \pi^{-} \quad \text{or} \quad \rho^{0} \rightarrow \pi^{0} + \pi^{0}. \tag{164}$$

But the pions themselves, being the lightest of all hadrons, can not decay by strong interactions into lighter hadrons. Therefore they decay by electromagnetic or weak interactions. Indeed, the neutral π^0 decays predominantly into a pair of photons after about 10^{-16} sec, whilst the charged pions decay by a form of β-decay into a lepton/anti-lepton pair, which takes 2.6×10^{-8} sec. These times scales, although very short, are many orders of magnitude longer than those of the strong (hadronic) decays of the heavier mesons and baryons. Thus, although pions are not protected by baryon number conservation and they are not absolutely stable, they are much more stable than other mesons, with life times 10^8 or 10^{16} times longer than typical in hadronic interactions.

Exercise 17.1
a. Consider the decay $N^* \rightarrow N + \pi$; the decay time is $\Delta t = 2 \times 10^{-24}$ sec. Compute the difference in rest energy ΔE of the N^*- and N-baryons, and show numerically that

$$\Delta E \Delta t \approx \hbar.$$

b. Write down all the possible ways the (N^{*+}, N^{*0})-baryons can decay into a Δ-baryon and a pion.
c. Argue on the basis of Heisenberg's time-energy uncertainty relation whether the decay of N^* into Δ will be faster or slower than the decay into N.

Exercise 17.2
a. In a nuclear interaction a charged pion is produced with an energy of 3 GeV. Its life time in the rest frame is 2.6×10^8 sec. How far will it travel on average before it decays?
b. And what is the expected distance covered by a ρ-meson of 30 GeV?

18. Quarks

The large number of hadrons known, combined with the observed substructure of the nucleons, indicates that hadrons are not simple elementary particles, but are actually bound states of point-like constituents: *quarks*, and/or their anti-particles. Quarks were introduced in the 1960's to explain the known facts about the hadron spectrum by Gell-Mann and Ne'eman. There must be at least two kinds of them, u and d, together with the anti-quarks $\bar{u}$ and $\bar{d}$, distinguished by their fractional electric charges:

charge in units e	quark type
2/3	u
- 1/3	d
- 2/3	$\bar{u}$
1/3	$\bar{d}$

With these charge assignments a proton is made of three quarks of type (uud), whereas a neutron contains three quarks of type (udd). Moreover, if quarks have spin $s = 1/2$ in units $\hbar$, it it is clear that the nucleons, the lowest-energy states made of three quarks, have spin 1/2 as well. Indeed, two u- or d-type quarks with opposite spin quantum numbers can be put in the same lowest-energy state, an S-state of angular momentum, together with a quark of different type, so as to have total spin angular momentum $s = 1/2$, without violating the Pauli principle.

In contrast to baryons, mesons are bound states of quarks and anti-quarks: $q\bar{q}$. For example, the charged pions have quark content

$$\pi^+ = (u\bar{d}), \qquad \pi^- = (d\bar{u}), \tag{165}$$

showing that indeed they are mutual anti-particles, whilst the neutral pion is a bound state of same-type quark/anti-quark pairs. It turns out that the wave function is constructed from an anti-symmetric combination:

$$\pi^0 = \frac{1}{\sqrt{2}} \left(u\bar{u} - d\bar{d} \right), \tag{166}$$

whereas the neutral η-meson represents the symmetric combination:

$$\eta = \frac{1}{\sqrt{2}} \left(u\bar{u} + d\bar{d} \right). \tag{167}$$

The baryon number of hadrons can now be reproduced by simply assigning a conserved fractional baryon number to quarks and anti-quarks:

$$B_q = \frac{1}{3}, \qquad B_{\bar{q}} = -\frac{1}{3}. \tag{168}$$

This automatically explains why mesons have no baryon number.

If quarks with fractional electric charges exist, it is natural to ask why particles with fractional charges have never been observed in experiments. The explanation of the absence of free particles with fractional charge must be found in the theory of quark interactions. Indeed pions, being bound states of quarks and anti-quarks, can not be themselves the quanta of a fundamental field of force; at best they can represent the effective mediators of interactions between hadrons, which must have a deeper origin in strong interactions between quarks.

The nature of fundamental quark interactions has been elucidated to a large extent in the past 35 years. The theoretical description of these interactions has considerable similarities with quantum electrodynamics (QED), the quantum version of Maxwell's theory. The theory of quark interactions is known as quantum chromodynamics (QCD), because it assigns to any quark a new kind of charge; this charge exists in three types, coded by colors usually chosen as red, green and blue (r, g, b). Thus every type of quark (like u or d) actually exists in three different varieties: with an r-, g- or b-charge. The anti-quarks then carry an anti-color charge $(\bar{r}, \bar{g}, \bar{b})$. There is a field of force acting on these color charges, simply known as the color field, just like the electromagnetic field acts between electric charges. The quanta of the color field are known as *gluons*, which play a similar role in QCD as photons in QED. QCD will be discussed in more detail later on. One aspect to be mentioned here is that in contrast to QED, in QCD the strength of the color interactions increases with distance: interquark forces behave at larger distances like a classical spring. This makes it difficult to separate color charges by large distances, and gives rise to the phenomenon of *color confinement*, the observation that no free particles with bare color charges exist, and therefore also no free particles with fractional electric charges.

The existence of color charges solves another riddle in the quark theory of hadrons, which we have not yet touched upon. This riddle is posed by the existence of certain baryon states like the Δ, which seem to defy the Pauli principle. Indeed, the Δ-baryons are the lowest-mass particles with spin 3/2, which one would normally associate with three quarks in an S-state of orbital angular momentum, with all spins are polarized in the same sense. Now the quark content of these particles is

$$\Delta = (\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}) = (uuu, uud, udd, ddd), \tag{169}$$

as is easy to verify from their electric charges. It would thus seem that there exist bound states of three u- or three d-quarks, all with the same spin and charge, in an S-state of angular momentum. As quarks are fermions, this contradicts the Pauli principle. However, the existence of color charges solves the riddle, by assigning to the three quarks all different color charges. Thus the detailed quark content of the Δ^{++} would be

$$\Delta^{++} = (u^r u^g u^b), \tag{170}$$

and therefore the three fermions are *not* in identical states. Although this may not seem obvious, such a state of three color charges actually corresponds to a colorless (a 'white') hadron, and is not in conflict with the principle of color confinement mentioned earlier.

Exercise 18.1
Explain why the asymmetric electric charges of the baryons, such as

$$N = (n^0, p^+), \qquad \Delta = (\Delta^-, \Delta^0, \Delta^+, \Delta^{++}),$$

together with their fermionic nature ($s = 1/2$), support the assignment of *different* fractional charges (2/3, -1/3) to the u- and d-quarks.

Summary
Tabel 18.1 below summarizes the properties of quarks, leptons and their antiparticles we have discussed so far.

particle	spin	electric charge	color multiplicity	baryon number	lepton number
u	1/2	2/3	3	1/3	0
d	1/2	-1/3	3	1/3	0
ν	1/2	0	1	0	1
e	1/2	-1	1	0	1
$\bar{u}$	1/2	-2/3	$\bar{3}$	-1/3	0
$\bar{d}$	1/2	1/3	$\bar{3}$	-1/3	0
$\bar{\nu}$	1/2	0	1	0	-1
$\bar{e}$	1/2	1	1	0	-1

Table 18.1: Quantum numbers of stable quarks and leptons

19. The muon

In the study of cosmics rays which produced the first evidence for the existence of strongly interacting pions, Powell ad his colleagues also found evidence for a weakly interacting charged particle with a mass slightly lower than the pion mass, which is since known as the *muon*; see fig. 19.1. The muon turned out to be a spin-1/2 charged lepton, just like the electron, but with a much larger mass;

$$m_\mu = 105.6\,\mathrm{MeV}/c^2. \tag{171}$$

Like the electron, the muon with negative charge is accompanied by an anti-particle of the same mass and positive charge.

The muons in cosmic ray air showers are produced predominantly by the decay of charged pions. In par. 17 we mentioned that charged pions decay into lepton/anti-lepton pairs; in almost all cases (99.99%) this is a muon and a neutrino:

$$\pi^+ \rightarrow \mu^+ + \nu_\mu, \qquad \pi^- \rightarrow \mu^- + \bar{\nu}_\mu, \tag{172}$$

The label μ has been attached to the symbol for the neutrino and anti-neutrino, because this neutrino turns out to be different from the neutrino produced in ordinary β-decay. For the same reason the neutrino produced in nuclear β-decay, accompanied by a positron, is usually labeled ν_e, whilst the anti-neutrino accompanied by an electron is labeled $\bar{\nu}_e$. For example, the β-decay process of the free neutron is more precisely

$$n \rightarrow p^+ + e^- + \bar{\nu}_e. \tag{173}$$

In very rare cases (about 1:10^4) charged pions can decay similarly by ordinary β-decay into electrons or positrons:

$$\pi^+ \rightarrow e^+ + \nu_e, \qquad \pi^- \rightarrow e^- + \bar{\nu}_e. \tag{174}$$

The experimental evidence for the difference between the two kinds of neutrinos comes from studying the inverse reactions

$$\nu_e + n \rightarrow e^- + p^+, \qquad \bar{\nu}_e + p^+ \rightarrow e^+ + n, \tag{175}$$

which were used by Reines and Cowan to establish the existence of the neutrino. The same reactions using muon-neutrinos $(\nu_\mu, \bar{\nu}_\mu)$ instead of electron neutrinos $(\nu_e, \bar{\nu}_e)$ –for example muon-neutrinos from pion-decay– have never been observed. Therefore muon-neutrinos must be different from electron-neutrinos. On the other hand, the reactions

$$\nu_\mu + n \rightarrow \mu^- + p^+, \qquad \bar{\nu}_\mu + p^+ \rightarrow \mu^+ + n, \tag{176}$$

have been observed, and are used in modern large-scale underground experiments to detect high-energy muon-neutrinos of astrophysical or cosmic origin.

All reactions discussed above are in agreement with interpreting the muon and muon-neutrino as a new pair of leptons, very similar to the electron and electron-neutrino; in particular we can assign the same lepton numbers to these particles:

$$L_\mu = L_{\nu_\mu} = +1, \qquad L_{\bar{\mu}} = L_{\bar{\nu}_\mu} = -1. \tag{177}$$

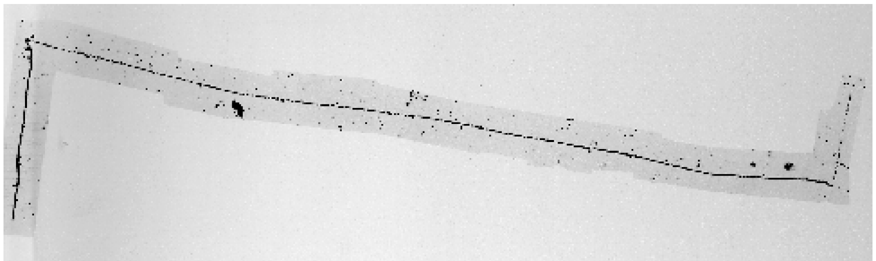

Fig. 19.1: Discovery of the pion and its muonic decay by Powell (1947)

Although the muon is considerably more stable than the pion, it decays after an average life-time of 2.2×10^{-6} seconds into an electron, a neutrino and an anti-neutrino:

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu, \qquad \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu. \tag{178}$$

Fig. 19.1 shows direct evidence of both pion and muon decay; in a photographic emulsion a pion from a cosmic-ray air shower is seen to enter on the lower left, decaying into a muon (moving to the right) and an unrecorded neutrino; the muon then decays into an electron (right upper track) and an unseen neutrino/anti-neutrino pair.

Observe that the decay process (178) is actually a form of β decay, like the neutron decay (173) and the rare pion decays (174), and thus a weak interaction process. That is why the muon life-time is orders of magnitude longer than the average hadronic life-time of the ρ or Δ, for example. Taking into account the assignments (177) the muon-decay process is consistent with the laws of conservation of charge, of angular momentum (spin) and of lepton number.

Finally, as the charged pions are actually bound states of a quark and anti-quark, we can interpret the pion-decay (172) at a more fundamental level as a quark-annihilation process:

$$u + \bar{d} \rightarrow \mu^+ + \nu_\mu, \qquad d + \bar{u} \rightarrow \mu^- + \bar{\nu}_\mu. \tag{179}$$

Again, it is easy to verify that charge, baryon and lepton number are all conserved in these transformation processes.

Exercise 19.1
a. A hydrogen-like atom can be made as a bound state of an electron e^- and an anti-muon μ^+. What are the binding energy and the Bohr radius in the ground state?
b. Explain why this atom is not absolutely stable, and by what kind of process it decays.
c. Argue that the anti-atom made of e^+ and μ^- has the same energy levels. How can one distinguish the atom and anti-atom from their decay products?
d. Another exotic type of atom is made as a bound state of a proton p^+ and a muon μ^-. What are the ground-state binding energy and Bohr radius of this atom?
e. Why is this atom ustable and how does it decay?

20. More quark and lepton families

After the muon was discovered, also new hadronic particles were identified in cosmic rays. Among these there was a particularly strange spin-0 meson doublet of particles now known as *kaons*: (K^0, K^+) and their anti-particles $(\bar{K}^0, \bar{K}^-)$. The masses of these particles are

$$m_{K^+} = 493.7\,\mathrm{MeV}/c^2, \qquad m_{K^0} = 497.7\,\mathrm{MeV}/c^2. \tag{180}$$

These mesons differ from the pions not only by their considerably larger mass, but also by the fact that the K^0 meson and the $\bar{K}^0$ anti-meson are different, whereas the π^0 is its own anti-particle. However, in some other respects the kaons actually resemble the pions; for example, the dominant decay mode (64%) of the charged kaons is

$$K^+ \to \mu^+ + \nu_\mu \qquad K^- \to \mu^- + \bar{\nu}_\mu, \tag{181}$$

which is a weak decay process, and the associated average life-time is also comparable to that of the charged pions:

$$\tau_{K^+} = 1.2 \times 10^{-8}\,\mathrm{s}. \tag{182}$$

Moreover, although the decay of charged kaons into pions: $K^+ \to \pi^+ + \pi^0$, is also possible, it takes about the same time as the weak decay (181), which is very unusual for a seemingly hadronic process. The decay of the neutral kaons is also rather unusual: sometimes a K^0 decays relatively fast (in 0.9×10^{-10} seconds) into *two* pions, and sometimes it decays more slowly (in about 5×10^{-8} seconds) into either *three* pions, or into a pion and a lepton/anti-lepton pair, e.g.:

$$K^0 \to \pi^- + e^+ + \nu_e, \qquad K^0 \to \pi^- + \mu^+ + \nu_\mu. \tag{183}$$

These processes involving a pion and a lepton/anti-lepton pair are called *semi-leptonic decay modes* of the neutral kaons.

In fact, all these decay processes are actually weak decays, with corresponding long life-times. This suggests that, like the pions, the kaons are *stable* under strong interactions. Such stability is natural if kaons possess a new quantum number which is conserved under strong interactions, but not under weak interactions. For obvious reasons this quantum number is called *strangeness* S, and the assignments are:

$$S_{K^0} = S_{K^+} = +1, \qquad S_{\bar{K}^0} = S_{\bar{K}^-} = -1, \tag{184}$$

whilst the hadrons we have encountered before, like pions and nucleons, all have strangeness $S = 0$. The only explanation for such a new quantum number is the existence of a new type of quark, appropriately called the strange quark s, with charge $-1/3$ and baryon number $+1/3$, such that the kaons are bound states

$$K^+ = (u\bar{s}), \quad K^0 = (d\bar{s}), \quad \bar{K}^0 = (\bar{d}s), \quad \bar{K}^- = (\bar{u}s). \tag{185}$$

Observe, that the definitions (184) then result in assigning negative strangeness to the s-quark:

$$S_s = -1, \qquad S_{\bar{s}} = +1, \tag{186}$$

which is a historical accident, much like the assignement of negative electric charge $-e$ to the electron.

The semi-leptonic decay modes of the neutral kaons can now be understood, at least at a qualitative level, if the s-quark is stable under strong interactions (acting on its color charge only), but is transformed by weak interactions into a u-quark:

$$s \rightarrow u + e^- + \bar{\nu}_e \quad \text{or} \quad s \rightarrow u + \mu^- + \bar{\nu}_\mu. \tag{187}$$

The first decay mode is actually the direct equivalent of the transformation of the d-quark into a u-quark responsible for the β-decay of the neutron, whilst the second, more common decay mode is its muonic counterpart. These weak decay processes obviously do *not* conserve the total strangeness quantum number S, hence there is no absolute conservation law for this quantity. The leptonic decay of the charged kaons is similarly the result of a quark annihilation process like (179):

$$u + \bar{s} \rightarrow \mu^+ + \nu_\mu, \qquad \bar{u} + s \rightarrow \mu^- + \bar{\nu}_\mu. \tag{188}$$

Now what about the decay of the charged kaons into pions? These decay modes look similar to the decay of the ρ-mesons, but they are really very different. In ρ-decay a pion is created by strong interactions, at a characteristic time-scale of 10^{-24} seconds. In K-decay a pion is created by transforming a strange quark into an up-quark:

$$s \rightarrow u + \bar{u} + d, \qquad \bar{s} \rightarrow \bar{u} + u + \bar{d}, \tag{189}$$

which is a weak interaction process like (187), in which the lepton/anti-lepton pair has been replaced by a quark/anti-quark pair. This allows reactions like

$$(u\bar{s}) \rightarrow (u\bar{u}) + (u\bar{d}) \quad \text{or} \quad K^+ \rightarrow \pi^0 + \pi^+. \tag{190}$$

As a result the decay of charged kaons into two pions is really a *weak* decay process, rather than a strong interaction process. This explains why the decay time of kaons into pions is similar to the decay time of kaons into leptons. It also reveals that there is a kind of hidden quark-lepton universality in weak interactions.

Exercise 20.1
Check that the decay modes (187) and (189), and the annihilation modes (188) of the strange quark all conserve charge, baryon number and lepton number and are consistent with the conservation of color charge and angular momentum.

The relation between the s- and the d-quark is seen to be very similar to the relation between the electron and the muon: one is just a more massive copy of the other, with the same charge, spin, baryon and lepton number number. Now the muon was accompanied by its own uncharged partner, the muon neutrino. To complete the parallel between quarks and leptons that is shown in table 18.1 there should also exist a heavy partner of the u-quark with charge $+2/3$; this quark has been given the name *charmed* quark c. Such a new quark was independently discovered in 1974 by Richter in Stanford, using collisions of electrons and positrons, and Ting in Brookhaven using high-energy protons colliding with nucleons in a beryllium target. Both experiments observed the existence of neutral $c\bar{c}$ bound state called the J/Ψ-meson[6]. This meson has a mass of

$$m_{J/\Psi} = 3.097\,\text{GeV}/c^2, \tag{191}$$

implying that the mass of a single c-quark is considerably larger than a proton.

With the discovery of the c-quark the table of quarks and leptons had regained its balance; for every quark and lepton in table 18.1: (u, d, ν_e, e), there is a heavier counter part (c, s, ν_μ, μ) with exactly the same charges/quantum numbers. In the last quarter of the 20^{th} century a complete third family of quarks and leptons has been found, called top- and bottom quark, tau-neutrino and tau-lepton, denoted by (t, b, ν_τ, τ). Again, they are equal in all respects to their first- and second-family cousins, except for their masses which are still an order of magnitude larger:

$$m_t = 172\,\text{GeV}/c^2, \quad m_b = 4.2\,\text{GeV}/c^2, \quad m_\tau = 1.78\,\text{GeV}/c^2. \tag{192}$$

None of the neutrino masses are presently known, but there are measurements establishing their mass differences. Together with the bounds on the electron-neutrino mass, these indicate that neutrinos must be extremely light compared to all other particles including the electron, of the order of 1 eV/c^2 or below. Table 20.1 summarizes some properties of the three families of quarks and leptons.

electric charge Q	color multiplicity	baryon number B	lepton number L	fam. I	fam. II	fam. III
$2/3$	3	$1/3$	0	u	c	t
$-1/3$	3	$1/3$	0	d	s	b
0	1	0	1	ν_e	ν_μ	ν_τ
-1	1	0	1	e	μ	τ
$-2/3$	3	$-1/3$	0	$\bar{u}$	$\bar{c}$	$\bar{t}$
$1/3$	3	$-1/3$	0	$\bar{d}$	$\bar{s}$	$\bar{b}$
0	1	0	-1	$\bar{\nu}_e$	$\bar{\nu}_\mu$	$\bar{\nu}_\tau$
1	1	0	-1	$\bar{e}$	$\bar{\mu}$	$\bar{\tau}$

Table 20.1: Properties of known quarks and leptons.

[6] Its double name comes from the names it was given by its two discoverers, J by Ting and Ψ by Richter.

21. Weak interactions

Quarks carry a color charge, and interact through the exchange of gluons. Quarks and charged leptons also carry an electric charge, and they interact with each other by the exchange of photons. Neutrinos carry neither color charges nor electric charge, but are still known to interact with both leptons and quarks. These interactions are weak, in the sense that the probability for a neutrino to scatter with other particles, though non-vanishing, is very small. On the other hand, of the subatomic interactions the weak interactions are the most universal, as all quarks and leptons participate in weak interactions. For example, all β-decay type processes are mediated by the weak interactions.

Obviously, weak interactions do not act through electric or color charges, as neutrinos have neither. In fact, there is no kind of strictly conserved charge associated with weak interactions. However, there are quantum numbers one can associate with the weak interactions of particles which are *approximately* conserved and very useful in understanding the nature of the weak interactions. These quantum numbers characterizing the weak interactions of particles are called *isospin* and *hypercharge*. As we will see below, they would be well-defined and strictly conserved quantities in a world with only massless particles. It is only because weakly interacting particles have masses that the assignment of these quantum numbers is problematic, and their conservation in interactions, like scattering or decay, is not exact.

To gain an understanding of weak interactions, we consider a famous experiment by Chien-Shiung Wu (1957). She measured the spin of electrons emitted by a radioactive cobalt source polarized at low temperature in a magnetic field; see fig. 21.1.

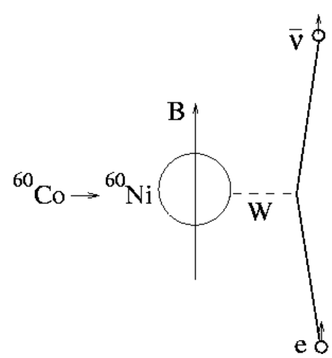

Fig. 21.1: Schematic set-up of the parity violation experiment of C.S. Wu.

In the experiment, ^{60}Co nuclei are aligned in a magnetic field. These nuclei are unstable and decay by β-decay into a stable ^{60}Ni nucleus, an electron and an anti-neutrino:

$$ {}^{60}_{27}\text{Co} \rightarrow {}^{60}_{28}\text{Ni} + e + \bar{\nu}. \tag{193}$$

The conservation of angular momentum requires, that in this process the electron and the anti-neutrino together carry way one unit of spin[7]. The remarkable result of the experiment

[7]The β-decay of ^{60}Co is actually followed by the emission of high-energy γ-rays; as this complication is not relevant to the rest of our argument, we do not account for the complete angular-momentum balance here.

was, that the electrons were preferentially emitted in a direction opposite to the magnetic field, with their spin parallel to the magnetic field. This implied, that the anti-neutrinos were emitted in the other hemisphere, in the direction of the magnetic field, with their spin also parallel to the direction of the magnetic field, i.e. parallel to the direction of motion.

The projection of a particle's spin on the direction of its momentum is called the *helicity*. For an electron it can be $\pm 1/2$ (in units $\hbar$), but this is an observer-dependent quantity. Indeed, if the electron moves with velocity v in the direction of its spin, then its helicity is $+1/2$. However, w.r.t. another observer moving with velocity $u > v$ in the same direction, the electron seems to move in the opposite direction, although the direction of its spin is not changed. Therefore in the frame of the second observer the helicity is $-1/2$.

The above argument becomes invalid for massless particles. As they move at the speed of light, there exist no inertial frames moving faster, with respect to which the velocity reverses sign. Hence for relativistic particles helicity is a well-defined and observer-independent concept. The invariant quantity characterizing the spin of a massless particle in its direction of motion is also called its handedness or *chirality*: a massless particle is called right-handed if its spin points in the same direction as its momentum, and left-handed if its spin is anti-parallel to its momentum. It is an invariant quantity in the sense that all inertial observers will always agree on the chirality of a massless particle.

Now neutrinos are so light, that even if their energy is very small they practically move at the velocity of light. This is certainly true for neutrinos emitted by ^{60}Co nuclei. At the time C.S. Wu performed her experiment it was in fact widely assumed that neutrinos were strictly massless. Therefore the handedness of the anti-neutrinos was considered a fundamental property of the neutrinos produced in β-decay. In contrast, the electron polarization (their apparent handedness in the rest frame of the decaying Co-nucleus) is not an intrinsic property, but dictated in this experiment by the conservation of angular momentum.

As a (massless) right-handed anti-neutrino is the anti-particle of a (massless) left-handed neutrino, the rule infered from this and later experiments was: neutrinos produced in weak decay processes (β-decay) are always left-handed, anti-neutrinos are right-handed. This rule says nothing about the existence of right-handed neutrinos or left-handed anti-neutrinos; it just says that if they exist, they are not produced in weak interactions. And since they also don't have electromagnetic or color interactions, if they exist they are for all practical purposes non-interacting and non-observable (barring any gravitational effects).

The fact that the interactions for left- and right-handed neutrinos are different (if the latter exist at all), implies that nature is not invariant under mirror symmetry, or *parity* transformations. Consider fig. 21.2; it shows a mirror in the x-y-plane, and a vector (which can be associated with linear motion) pointing along the z-axis. Of course, in the mirror the vector will point in the opposite direction. In contrast, a vector parallel to the plane of the mirror (in the x- or y-direction) will have the same direction in the mirror. Mathematically the mirror performs the operation $z \rightarrow -z$.

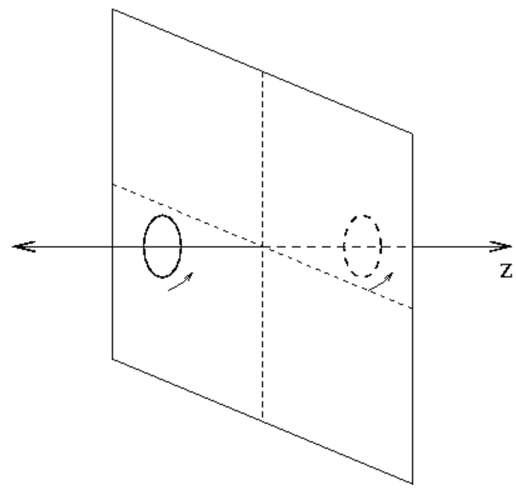

Fig. 21.2: Mirror reflection of linear and rotational motion.

Fig. 21.2 also shows a rotational motion around the z-axis, which we associate with an axial vector (or pseudo-vector) in the z-direction. Now in the mirror the rotational motion, being confined to the x-y-plane, looks exactly the same, no reversal of the direction of rotation takes place, and we would associate the same axial vector with the rotation on both sides of the mirror. Hence axial vectors in the z-direction do *not* change sign under reflection in the x-y-plane. In contrast, rotations in the x-z- or y-z-plane are reversed in the mirror, and axial vectors in the x-y-plane *do* change their orientation in the mirror. In all cases we therefore find: under reflections real vectors (as associated with linear motion) and axial vectors (as associated with rotational motion) behave under reflections in opposite ways. In particular after reflection a left-handed neutrino becomes a right-handed neutrino. But a left-handed neutrino has weak interactions, whilst a right-handed one does not; in particular, the mirror process of β-decay (in which a right-handed neutrino or a left-handed anti-neutrino is produced) does not occur in nature. Therefore the laws of nature are not invariant under reflection: parity (invariance under a mirror transformation like $z \to -z$) is violated.

Electro-weak unification

On the basis of this and other experiments, the theory of weak interactions is based on the assumption that in a world with massless quarks and leptons the interactions of left- and right-handed matter particles are different. In such a world there are actually two kinds of interactions, each associated with a conserved charge called isospin and hypercharge, respectively. More precisely, isospin and hypercharge are associated with fields that act on these charges, and that are created by these charges and their currents. Thus the weak charges are in the same class as electric charges and color charges, which couple a particle to the electromagnetic and color fields, respectively. However, the weak charges are assigned in such a way that only left-handed quarks and leptons carry an isospin charge, whilst left- and right-handed particles both carry hypercharge, but in different amounts. In this scheme parity is doubly violated: both by the isospin interactions and by the hypercharge interactions.

The hypercharge is rather simple to understand; it is a number y assigned to any particle to characterize how strongly it couples to the hypercharge field, just like the electric charge indicates the strength of the coupling of a particle to the electromagnetic field. The main difference is, that hypercharge is different for particles with different chirality, which is not the case for electric charge.

In contrast, isospin is a multi-valued charge, similar to color charge. Thus, a quark or lepton can appear in two different isospin states; indeed, this is precisely why there are two different left-handed quarks and leptons in each particle family. More generally, one can think of isospin as a vector in some abstract space (a subspace of the quantum-mechanical Hilbert space), which behaves like ordinary spin: particles can appear in singlets, doublets, triplets, etc., characterized by an isospin quantum number τ, with the particles in the same multiplet distinguished by the component in a fixed direction, usually called τ_3. This third isospin component can take $(2\tau + 1)$ values[8]:

$$\tau_3 = \tau, \tau - 1, ..., -\tau. \tag{194}$$

For a singlet, there is only one state: $\tau = \tau_3 = 0$; for a doublet the states are $\tau = 1/2$, $\tau_3 = (+1/2, -1/2)$; for a triplet: $\tau = 1$, $\tau_3 = (+1, 0, -1)$; etc. In this language, right-handed quarks and leptons are iso-singlets, whilst left-handed quarks and leptons form iso-doublets. In table 21.1 we present the detailed assignment of isospin and hypercharge to quarks and leptons.

particle	τ	τ_3	y	q
u_L	1/2	+1/2	1/6	2/3
d_L	1/2	−1/2	1/6	−1/3
u_R	0	0	2/3	2/3
d_R	0	0	−1/3	−1/3
$\nu_{e\,L}$	1/2	+1/2	−1/2	0
e_L	1/2	−1/2	−1/2	−1
$\nu_{e\,R}$	0	0	0	0
e_R	0	0	−1	−1

Table 21.1: Isospin and hypercharge assignments in a quark-lepton family.

Here the subscript (L, R) denotes the chirality of the particle. We have included the hypothetical right-handed neutrino, although it does not have any weak or other observable charges. The same isospin and hypercharge are assigned to the quarks and leptons in the other families: s-quarks and b-quarks the same as d-quarks, muons and tau-leptons the same as electrons, etc. We emphasize that the isospin and hypercharge assignments are strictly valid only in a world of massless particles, which have well-defined chirality.

In the real world we observe, quarks and leptons are not massless, and the helicity of a particle becomes observer-dependent. That implies that also the isospin and hypercharge become observer-dependent: a particle which changes its state of motion can also change

[8] A more detailed discussion of isospin is to be found in appendix E.

its weak charges. In this world these charges are therefore not conserved. However, there is one exception. The combination $y + \tau_3$ is the same for left- and right-handed particles; in fact this combination can be seen to equal the electric charge:

$$q = y + \tau_3. \tag{195}$$

Therefore this charge is conserved also in the real world for massive particles: the electric charge, being the same for both helicity states of the particle, does not depend on the state of motion of the particle. It turns out that not only the electric charge equals the combination of hypercharge and third isospin component, in a literal sense it *is* the electric charge: the electromagnetic field is effectively a combination of the hypercharge field and the third component of the isospin field. What we call the weak interactions are the remaining interactions which couple to non-conserved components of isospin and hypercharge. This can work in practice only, if these weak isospin and hypercharge field components do not have a long range like the electromagnetic field, but fall of exponentially fast over a finite distance away from the charges. In quantum language this implies, that the field quanta of the weak interactions are *massive* spin-1 particles; they are called intermediate vector bosons, of which there are three: (W^+, W^-, Z^0). The charged $W^\pm$-bosons are each others anti-particle and associated with isospin; therefore they only couple to left-handed quarks and leptons. The Z^0 is like a massive photon, associated with a mixture of isospin and hypercharge fields, but one which still couples differently to left- and right-handed particles. The photon itself is also a mixture of isospin and hypercharge states, but one that couples excusively to the electric charge, which is the same for left- and right-handed particles. As electric charge is strictly conserved, the photon can remain a massless particle in the real world of massive quarks and leptons.

The upshot of this discussion is, that there is a close connection between weak and electromagnetic interactions: they have a common origin in the combined isospin and hypercharge fields and charges. This common origin is hidden by the fact that quarks and leptons have a non-zero mass, but in an environment where all these particles are relativistic and effectively behave as massless particles, the isospin and hypercharge framework provides a better and more accurate description of the physics. Such an environment was presumably provided in the early universe, when temperatures far exceeded the rest-energy of intermediate vector bosons, quarks and leptons. Later in these lectures we return to the weak interactions to make these arguments more quantative.

22. Quantum field theory

We have become acquainted with two forms of matter: quarks and leptons, which together provide the ultimate building blocks for atoms. What breathes life into matter, and makes matter dynamical, are the various interactions between the building blocks. We have already encountered three kinds of fundamental interactions between quarks and leptons:
- colordynamics, acting on particles carrying color charges;
- electrodynamics, acting on particles carrying electric charge;
- weak interactions, acting on all quarks and leptons via isospin and hypercharge.

In principle we should also add gravity to this trio, as gravitational forces are not reducible to any of the three interactions above. However, gravity is so much weaker on the scale of quarks and leptons, that it is completely irrelevant for understanding their observed dynamics. Only on cosmological scales and in extreme astrophysical environments is gravity expected to play a role, in particular in explaining the evolution of matter in the early universe and the formation of nuclei in the interior of stars. In the following we will restrict ourselves to the interactions of quarks and leptons in Minkowski space-time, where non-trivial gravitational effects are absent.

The general framework at our disposal for the description of matter and its interactions is relativistic Quantum Field Theory (QFT). Relativistic QFT extends the matter-wave duality to the level where all particles are considered to be quanta of a corresponding field, similar to the way that photons are the quanta of the electromagnetic field. We briefly discussed this in sect. 15, based on the Planck-Einstein-De Broglie particle-wave duality relations

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}, \tag{196}$$

where ω and $\mathbf{k}$ are the angular frequency and wave vector of a monochromatic plane wave associated with the particle. The relativistic energy-momentum relation then translates into a dispersion relation for the frequency and wave vector of the plane waves:

$$E^2 = m^2c^4 + \mathbf{p}^2c^2 \quad \Leftrightarrow \quad \omega^2 = \frac{4\pi^2c^2}{\lambda_c^2} + \mathbf{k}^2c^2, \tag{197}$$

with λ_c the Compton wave length of the particle. The particular plane wave is of the form[9]

$$\Psi = A\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{198}$$

and satisfies the wave equation[10]

$$\left(-\hbar^2\,\Box + m^2c^2\right)\Psi \equiv \left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2\boldsymbol{\nabla}^2 + m^2c^2\right)\Psi = 0. \tag{199}$$

Via the plane-wave solutions (198), this relativistic wave equation is equivalent to the dispersion relation (197). Eq. (199) is generally known as the Klein-Gordon equation. It

[9] Here and in the following sections we will treat the fields as complex; for real fields in expressions like (198) one is supposed to take the real part.

[10] The box operator $\Box$ was introduced in appendix F, eq. (198).

follows, that any field describing free particles with an energy-momentum relation (197) must satisfy the Klein-Gordon equation, irrespective of what other equations are imposed on it.

23. Dirac fields

Free particles without spin are completely characterized by their mass, which fixes their energy-momentum relation (197). Since no other kinematical information is necessary to describe their motion in Minkoswki space-time, the Klein-Gordon equation provides a complete starting point for the quantum theory of spin-0 particles.

This is not true for particle with spin, including all quarks and leptons. Their wave functions, and the fields from which they are constructed, must have several components describing states of different polarization. The relativistic equation for spin-1/2 particles was developed by Dirac. The starting point for its derivation is the fact that spin-1/2 particles have two polarization states with spin eigenvalues $\pm\hbar/2$. The spin operators for such particles can be represented by the 2×2 Pauli matrices; in this representation the spin operators $\mathbf{\Sigma}$ are defined by

$$\mathbf{\Sigma} = \frac{\hbar}{2}\,\boldsymbol{\sigma}, \quad \Rightarrow \quad [\Sigma_i, \Sigma_j] = i\hbar\,\varepsilon_{ijk}\,\Sigma_k. \tag{200}$$

Therefore we incorporate spin in the field-description of fermions by introducing a 2-component field

$$\psi = \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right], \tag{201}$$

of which the components represent the polarization states. Now the Pauli-matrices satisfy the anti-commutation relations

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\,\delta_{ij}\,\mathbf{1}, \tag{202}$$

where $\mathbf{1}$ is the 2×2 unit matrix. This allows us to construct a square root of the d'Alembert operator, which is the main idea behind the Dirac construction. First note, that the operator $\boldsymbol{\sigma}\cdot\mathbf{p}$ satisfies the identity

$$(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \frac{1}{2}\sum_{i,j}(\sigma_i\sigma_j + \sigma_j\sigma_i)\,p_ip_j = \mathbf{p}^2\,\mathbf{1}. \tag{203}$$

Now consider a massless fermion. For such a particle the relativistic energy-momentum relation can be written as

$$\mathbf{p}^2 = p_0^2 = \frac{E^2}{c^2}. \tag{204}$$

Then the 2-component representation of the fermion field (201) can be taken to satisfy the wave equation

$$-i\hbar\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,\psi = \pm i\hbar\,\partial_0\psi. \tag{205}$$

Indeed, if the two components are of the plane-wave form (198):

$$\psi(\mathbf{r},t) = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left[\begin{array}{c} A_1 \\ A_2 \end{array} \right], \tag{206}$$

then the wave equation (205) is equivalent to the eigenvalue equation

$$c\mathbf{p} \cdot \boldsymbol{\sigma}\, \psi = \pm E\, \psi. \tag{207}$$

Re-applying the same operator on the left-hand side this gives

$$c^2 (\mathbf{p} \cdot \boldsymbol{\sigma})^2 \psi = \pm cE\, \boldsymbol{\sigma} \cdot \mathbf{p} \psi = E^2 \psi. \tag{208}$$

Using eq. (203) this is seen to reduce to the equation

$$\left(\mathbf{p}^2 c^2 - E^2 \right) \psi = 0 \quad \Leftrightarrow \quad -\hbar^2 \Box\, \psi = 0. \tag{209}$$

This can be established directly by squaring the differential operators on the left- and right-hand side of eq. (205); first,

$$\left(-i\hbar \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right)^2 \psi = \left(\pm i\hbar\, \partial_0 \right)^2 \psi = -\frac{\hbar^2}{c^2} \frac{\partial^2 \psi}{\partial t^2}. \tag{210}$$

But also, by the anti-commutation property of the Pauli matrices:

$$\left(-i\hbar \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right)^2 \psi = -\hbar^2 \boldsymbol{\nabla}^2 \psi. \tag{211}$$

Combining these results reproduces eq. (209). Thus eq. (205) is seen to imply the massless Klein-Gordon equation, and therefore is acceptable as a relativistic wave equation for massless spin-1/2 fermions. After its discoverer, it is called the massless Dirac equation.

In addition to implying the correct energy-momentum relation for free massless particles, the massless Dirac equation also contains information about the spin-polarization of the particle. This information is encoded in the *helicity operator*

$$\frac{\mathbf{p} \cdot \boldsymbol{\Sigma}}{\hbar |\mathbf{p}|} = \frac{c\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}, \tag{212}$$

with the eigenvalues $\pm 1/2$, as follows from eq. (207):

$$\frac{c\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}\, \psi = \pm \frac{1}{2}\, \psi. \tag{213}$$

The dimensionless helicity eigenvalues are just the eigenvalues of the spin component in the direction of motion, in units of $\hbar$. In general, the eigenstates of the helicity operator are said to describe *right-handed* fermions if the eigenvalue is positive, and *left-handed* fermions if the eigenvalue is negative. As discussed before in sect. 21, the handedness or *chirality* of a particle with spin is well-defined and observer-independent only for massless particles; indeed, a massive particle can always be described in its rest frame, in which there is no direction of motion. The sign choice in eq. (213) is determined by the sign choice in the massless Dirac equation (205), hence positive and negative helicity particles are described by two different Dirac equations. This confirms that for massless particles the helicity eigenstates are independent and chirality is a Lorentz-invariant quantity.

Next we extend the construction to massive spin-1/2 fermions. From eq. (203) it follows that for massive particles

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \left(p_0^2 - m^2 c^2 \right) \quad \Leftrightarrow \quad (p_0 - \mathbf{p} \cdot \boldsymbol{\sigma})\,(p_0 + \mathbf{p} \cdot \boldsymbol{\sigma}) = m^2 c^2 \neq 0, \tag{214}$$

where in line with general custom we have suppressed the explicit writing of unit matrices; we will follow that custom from now on. Eq. (214) implies that for $m \neq 0$ the 2-component field ψ can not satisfy the massless Dirac equation, and we must introduce two 2-component fields (ψ, χ) related by

$$mc\,\chi \equiv i\hbar\,(\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\,\psi \quad \Rightarrow \quad mc^2\,\chi = (E - c\mathbf{p} \cdot \boldsymbol{\sigma})\,\psi, \tag{215}$$

where in the second equation we have substituted a plane wave solution (206); and by an argument similar to that presented in eq. (210), (211), there also must be a second relation

$$mc\,\psi = i\hbar\,(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\,\chi \quad \Rightarrow \quad (E + c\mathbf{p} \cdot \boldsymbol{\sigma})\,\chi = mc^2 \psi. \tag{216}$$

Together these relations imply the Klein-Gordon equation with non-zero mass:

$$\left(-\hbar^2 \Box + m^2 c^2 \right) \psi = \left(-\hbar^2 \Box + m^2 c^2 \right) \chi = 0. \tag{217}$$

The important point is, that to describe a massive fermion we need *four* field components (ψ, χ), rather than two, as in the massless case. This is because *both* sign-choices for the helicity components are present in the massive Dirac equation. Indeed, from eqs. (215) and (216) it follows that in the limit $mc \rightarrow 0$,

$$\frac{c\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}\,\psi = \frac{1}{2}\,\psi, \qquad \frac{c\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}\,\chi = -\frac{1}{2}\,\chi. \tag{218}$$

In other words, in the relativistic high-momentum limit ψ describes the positive-helicity components, and χ the negative-helicity components. The occurrence and mixing of both helicity spinor fields (ψ, χ) in eqs. (215) and (216) for massive particles was then to be expected, as for massive particles helicity is an observer-dependent quantity.

Exercise 23.1
The Pauli matrices are

$$\sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

a. Show that they satisfy the commutation relations

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_3 = 2i\,\varepsilon_{ijk} \sigma_k,$$

with ε_{ijk} the completely anti-symmetric symbol

$$\varepsilon_{ijk} = \left\{ \begin{array}{rl} +1, & (ijk) = \text{ even permutation of}(123); \\ -1, & (ijk) = \text{ odd permutation of}(123); \\ 0, & \text{all other cases.} \end{array} \right. .$$

b. Establish the anti-commutation relations

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\,\mathbf{1},$$

and from this derive that

$$\boldsymbol{\Sigma}^2 = \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = s(s+1)\hbar^2\,\mathbf{1}, \quad \text{with } s = \frac{1}{2}.$$

c. Prove that if $(\mathbf{a}, \mathbf{b})$ are two vectors, then

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}.$$

Covariant Dirac equation
The Dirac theory of a free massive spin-1/2 particle is defined by equations (215), (216):

$$i\hbar\left(\partial_0 + \boldsymbol{\sigma}\cdot\nabla\right)\psi(x) = mc\,\chi(x), \qquad i\hbar\left(\partial_0 - \boldsymbol{\sigma}\cdot\nabla\right)\chi(x) = mc\,\psi(x). \tag{219}$$

In the massless case, these equations decouple and one can use either one of them, depending on the handedness (chirality) of the particle. By construction, the solutions of the Dirac equations describe relativistic particles.

It is often convenient to rewrite the two equations (219) for two 2-component spinors as a single equation for a 4-component spinor, as follows. Define a 4-component spinor by a direct sum of the left- and right-handed ones

$$\Psi = \left[\begin{array}{c} \psi \\ \chi \end{array}\right] = \left[\begin{array}{c} \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right) \\ --- \\ \left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right) \end{array}\right]. \tag{220}$$

Furthermore, introduce a set of 4×4 Dirac matrices

$$\gamma^\mu = \left(\gamma^0, \boldsymbol{\gamma}\right); \qquad \gamma^0 = \left(\begin{array}{cc} 0 & 1_2 \\ 1_2 & 0 \end{array}\right), \quad \boldsymbol{\gamma} = \left(\begin{array}{cc} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{array}\right), \tag{221}$$

where 1_d stands for the d-dimensional unit matrix[11]. Then the pair of Dirac equations (219) can be written as a single equation

$$\left(-i\hbar\gamma^\mu\partial_\mu + mc\right)\Psi = 0 \quad \Leftrightarrow \quad \left(\begin{array}{c|c} mc & -i\hbar(\partial_0 - \boldsymbol{\sigma}\cdot\nabla) \\ \hline -i\hbar(\partial_0 + \boldsymbol{\sigma}\cdot\nabla) & mc \end{array}\right)\left[\begin{array}{c} \psi \\ \chi \end{array}\right] = 0. \tag{222}$$

Now the Dirac matrices γ^μ satisfy the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\,\eta^{\mu\nu}\,\mathbf{1}. \tag{223}$$

[11] As remarked before, often the unit matrices are not written explicitly.

It is then easy to establish, that the Dirac equation implies the Klein-Gordon equation:

$$\left(-\Box + \mu^2\right) \Psi = (i\gamma \cdot \partial + \mu)(-i\gamma \cdot \partial + \mu) \Psi = 0, \tag{224}$$

where

$$\mu = \frac{mc}{\hbar} = \frac{2\pi}{\lambda_c} \tag{225}$$

is the Compton wave number.

Exercise 23.2
a. The (4×4) Dirac matrices are defined in eq. (221):

$$\gamma^\mu = \left(\gamma^0, \boldsymbol{\gamma}\right); \qquad \gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}.$$

Prove that these matrices satisfy the relations (223)

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu}\, 1_4,$$

where the Minkowski metric has the standard components diag$(-1, +1, +1, +1)$.
b. Show that for a plane wave solution

$$\Psi = A\, e^{ik_\mu x^\mu}, \quad \text{with} \quad k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right),$$

the covariant Dirac equation takes the form

$$(\gamma^\mu k_\mu + \mu)\, A = 0.$$

c. Use this result to derive eq. (197) for plane waves

$$(\gamma^\mu k_\mu - \mu)(\gamma^\nu k_\nu + \mu)\, A = \left(\frac{\omega^2}{c^2} - \mathbf{k}^2 - \mu^2\right) A = 0.$$

d. Define

$$\Sigma_i \equiv \frac{i\hbar}{4} \sum_{j,k} \varepsilon_{ijk}\, \gamma_j \gamma_k.$$

Prove that

$$\Sigma_i = \frac{\hbar}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},$$

and that the operators Σ_i also satisfy the angular momentum commutation relations:

$$[\Sigma_i, \Sigma_j] = i\hbar\, \varepsilon_{ijk}\, \Sigma_k.$$

24. Vector fields

Massive spin-1 particles have three polarization states, characterized by by the value of the spin in a fixed direction, e.g. the z-direction: $s_z = (+\hbar, 0, -\hbar)$. The spin-operators for such particles are described by (3×3)-matrices satisfying the commutations relations of angular momentum, like the Pauli matrices for spin-1/2 particles. In a convenient basis these spin-operators read

$$\Sigma_1 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \Sigma_2 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (226)$$

with the property

$$[\Sigma_i, \Sigma_j] = i\hbar \, \varepsilon_{ijk} \, \Sigma_k. \qquad (227)$$

These matrices act in a three-dimensional space, therefore we expect that spin-1 particles are described by a field $\mathbf{A}$ with three components. However, in a relativistic framework there are no three-component objects transforming in a simple (linear) way under Lorentz transformations. To guarantee that the field equations take the same form in all inertial frames, the best we can do is embed a three-component vector in a Minkowski four-vector with one time- and three space-components: $A_\mu = (A_0, \mathbf{A})$. We can then eliminate the time-component as an independent degree of freedom by imposing an additional constraint on the four-vector. Of course, for the description to be valid in all inertial frames, this relation between the components of A_μ should itself be Lorentz invariant. The required constraint is

$$\partial^\mu A_\mu = 0 \quad \Leftrightarrow \quad \partial_0 A_0 = \boldsymbol{\nabla} \cdot \mathbf{A}. \qquad (228)$$

Clearly the time-component A_0 can be expressed in terms of the three space-components $\mathbf{A}$, and the four-vector A_μ really depends on only three independent degrees of freedom.

Now we look for a field equation for A_μ which implies both the constraint (228) and the Klein-Gordon equation (199). To achieve our aim, we first define an anti-symmetric four-tensor

$$F_{\mu\nu} = -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (229)$$

Mathematically, the components of $F_{\mu\nu}$ represent a four-dimensional generalization of the components of the curl of a vector. In three dimensions the curl of vector has three components:

$$\boldsymbol{\nabla} \times \mathbf{A} = (\nabla_2 A_3 - \nabla_3 A_2, \nabla_3 A_1 - \nabla_1 A_3, \nabla_1 A_2 - \nabla_2 A_1), \qquad (230)$$

whereas in four dimension the generalized curl has six components: three which involve mixed time and space components:

$$F_{0j} = \partial_0 A_j - \nabla_j A_0, \qquad j = (1, 2, 3); \qquad (231)$$

and three which only involve space components as in (230):

$$F_{ij} = \nabla_i A_j - \nabla_j A_i, \qquad i, j = (1, 2, 3). \qquad (232)$$

In terms of $F_{\mu\nu}$ the field equation we need is

$$\partial^\mu F_{\mu\nu} = \frac{m^2c^2}{\hbar^2}\, A_\nu. \tag{233}$$

This equation is known as the Proca equation; it describes fields of which the quanta are massive spin-1 particles. To show this, we first note that

$$\partial^\mu \partial^\nu F_{\mu\nu} = 0. \tag{234}$$

The reason is, that partial derivatives commute: $\partial^\mu\partial^\nu = \partial^\nu\partial^\mu$, hence the product of derivatives is *symmetric* under interchange of μ and ν; however, $F_{\mu\nu} = -F_{\nu\mu}$ is *antisymmetric* under this interchange, and therefore the contracted form (234) must vanish. Now taking the four-dimensional divergence of the left- and right-hand side of eq. (233), and using the result (234), it follows that

$$\frac{m^2c^2}{\hbar^2}\, \partial^\mu A_\mu = 0. \tag{235}$$

Hence for massive spin-1 particles ($m \neq 0$), we indeed reobtain the constraint (228). Next, write out eq. (233), and bring all terms to the same side, to get

$$-\Box A_\nu + \partial_\nu\, \partial^\mu A_\mu + \frac{m^2c^2}{\hbar^2}\, A_\nu = 0. \tag{236}$$

Using the constraint (228) to eliminate the middle term, we then finally get the Klein-Gordon equation

$$\left(-\hbar^2\,\Box + m^2c^2\right) A_\nu = 0. \tag{237}$$

For plane-wave fields

$$A_\nu(t, \mathbf{r}) = a_\nu\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{238}$$

we then again find the correct energy-momentum relation

$$E^2 = \mathbf{p}^2c^2 + m^2c^4,$$

and in addition a constraint on the amplitude

$$\frac{\omega}{c}\, a_0 + \mathbf{k}\cdot\mathbf{a} = 0 \quad \Rightarrow \quad a_0 = -\frac{c\mathbf{p}\cdot\mathbf{a}}{E}. \tag{239}$$

This shows explicitly how the time component of the amplitude is expressed in terms of the space components. Clearly for a massive spin-1 particle at rest: $\mathbf{p} = 0$, $E = mc^2$, the time component of the amplitude vanishes: $a_0 = 0$. It follows that the plane-wave representation of a particle at rest reduces to a vector with only three components:

$$A_\mu = (0, \mathbf{A}), \qquad \mathbf{A} = \mathbf{a}\, e^{-imc^2t/\hbar}. \tag{240}$$

In this frame the spin-operators $\mathbf{\Sigma}$ defined in eq. (226) determine the polarization states of the field: the vector amplitude $\mathbf{a}$ can be an eigenvector of Σ_3 corresponding to any of the eigenvalues $(+1, 0, -1)$ (in units $\hbar$), or any linear combination thereof.

Exercise 24.1
a. Show that the spin operators (226) satisfy the commutation relations (227).
b. Prove that spin operators satisfy the relation

$$\mathbf{\Sigma}^2 = \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = s(s+1)\hbar^2\,\mathbf{1}, \quad \text{with } s = 1.$$

c. An equivalent representation of the spin operators is

$$\Sigma_1' = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_2' = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Sigma_3' = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Prove that these matrices obey the same commutation relations (227), and find the unitary transformation U relating the two sets of spin operators:

$$\Sigma_i' = U \Sigma_i U^{\dagger}.$$

Exercise 24.2
Prove the identity (234) by writing out the implicit summation over the indices (μ, ν).

For massless spin-1 particles, like photons and gluons, this analysis has to be modified. Taking $m = 0$ in eq. (233), the equation simplifies to

$$\partial^{\mu} F_{\mu\nu} = 0. \tag{241}$$

In the absence of the mass term on the right-hand side, we can no longer derive the constraint (228). However, this is compensated by the appearance of a new property of the field equation: *gauge invariance*. Gauge invariance is a direct consequence of the fact that $F_{\mu\nu}$ is a four-dimensional curl[12]: the components of $F_{\mu\nu}$ are invariant under transformations

$$A_{\mu} \;\rightarrow\; A_{\mu}' = A_{\mu} + \partial_{\mu}\Lambda, \tag{242}$$

where Λ is an arbitrary scalar function. These transformations are refered to as *local gauge transformations*. The gauge invariance is easy to verify:

$$\partial_{\mu}\partial_{\nu}\Lambda - \partial_{\nu}\partial_{\mu}\Lambda = 0 \quad \Rightarrow \quad F_{\mu\nu}' = \partial_{\mu}A_{\nu}' - \partial_{\nu}A_{\mu}' = F_{\mu\nu}. \tag{243}$$

The arbitrariness in the vector field A_{μ} can be used to impose an additional constraint on A_{μ}, which can be chosen to be the Lorentz invariant condition (228):

$$\partial^{\mu} A_{\mu} = 0. \tag{244}$$

Then we are back in the previous situation, with the field equation (241) and the above gauge condition (244) together implying the massless Klein-Gordon equation:

$$-\Box A_{\nu} = 0. \tag{245}$$

[12]In mathematical terminology F is a two form: $F = dA = dx^{\mu} \wedge dx^{\nu}\, \partial_{\mu} A_{\nu} = \frac{1}{2}\, dx^{\mu} \wedge dx^{\nu}\, F_{\mu\nu}$.

Exercise 24.3

a. Check that for plane waves

$$A_\mu = a_\mu e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

the gauge condition (244) reduces to the constraint (239) on the amplitudes:

$$\frac{\omega}{c}\, a_0 + \mathbf{k}\cdot\mathbf{a} = 0.$$

b. Show that even after imposing the gauge condition (244) it is still possible to make further gauge transformations (242) leaving $F_{\mu\nu}$ invariant, provided Λ satisfies the equation

$$\Box\Lambda = 0,$$

with the solution

$$\Lambda = \varepsilon e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.$$

c. Such a special gauge transformation changes the amplitude by

$$a'_\mu = a_\mu + ik_\mu\varepsilon.$$

or in components:

$$a'_0 = a_0 - \frac{i\omega}{c}\,\varepsilon, \qquad \mathbf{a}' = \mathbf{a} + i\mathbf{k}\,\varepsilon.$$

Show, that ε can be chosen such that

$$a'_0 = 0, \qquad \mathbf{k}\cdot\mathbf{a}' = 0,$$

and that for the full plane wave solution this is equivalent to

$$A'_0 = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{A}' = 0.$$

From this, conclude that the amplitude of the wave field A_μ is purely transverse and possesses only *two* independent components.

25. Quantum electrodynamics

The description of massless spin-1 particles can be applied directly to the quantum theory of photons. This becomes obvious by noting that eq. (241) provides just the manifestly relativistic form of the free Maxwell's equations. To show this, identify the scalar and vector potential of electrodynamics with the components of the vector field A_μ:

$$A_0 = \frac{\phi}{c}, \quad \Rightarrow \quad A_\mu = \left(\frac{\phi}{c}, \mathbf{A} \right). \tag{246}$$

Then the components of $F_{\mu\nu}$ can be split in two sets: the electric components (231):

$$-cF_{0j} = \partial_j \phi - \partial_t A_j \equiv E_j \quad \Leftrightarrow \quad \mathbf{E} = \boldsymbol{\nabla} \phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{247}$$

and the magnetic components

$$F_{ij} = \partial_i A_j - \partial_j A_i \equiv \varepsilon_{ijk} B_k \quad \Leftrightarrow \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{248}$$

In terms of these electric and magnetic field components, the field equations (241) become

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0, \tag{249}$$

which are indeed Maxwell's equations in empty space. Hence interpreted in the framework of quantum theory, the plane-wave solutions of the free Maxwell equations represent massless spin-1 particles of well-defined energy and momentum: photons.

Exercise 25.1
Prove that **E** and **B** also satisfy the homogeneous Maxwell equations

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0.$$

So far we have only discussed free electromagnetic fields. The interactions of photons with charged particles is described by the inclusion of charges and currents in the theory. Let ρ and $\mathbf{j}$ represent the charge and current density of charged particles; then the Maxwell equations (249) are modified to read

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0 c^2} \mathbf{j}. \tag{250}$$

These two equations are consistent only if the charge and current density satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{251}$$

This equation is the local version of the equation of charge conservation. To establish this, compute the total charge in a fixed volume V:

$$Q = \int_V d^3x \, \rho. \tag{252}$$

By the equation of continuity and Gauss' law, the change in the total charge is

$$\frac{dQ}{dt} = \int_V d^3x \, \frac{\partial \rho}{\partial t} = - \int_V d^3x \, \boldsymbol{\nabla} \cdot \mathbf{j} = - \oint_\Sigma d^2\sigma \, j_n, \tag{253}$$

where Σ is the surface of the volume V, $d^2\sigma$ is a surface integration elements and j_n is the normal component of the current density across the surface. In words this equation states that the only change in the charge in the volume V comes from currents flowing in or out of the volume across the surface. In particular, if there are no currents across the surface the total charge in V is constant.

The derivation of the equation of continuity is particularly simple in the relativistically covariant notation. The derivation starts by defining the current four-vector

$$j^\mu = (j^0, \mathbf{j}) = (\rho c, \mathbf{j}). \tag{254}$$

The inhomogeneous Maxwell equations (250) then take the covariant form

$$\partial^\mu F_{\mu\nu} = - \frac{1}{\epsilon_0 c^2} \, j_\nu. \tag{255}$$

The identity (234) then implies

$$\partial^\mu j_\mu = - \epsilon_0 c^2 \, \partial^\mu \partial^\nu F_{\mu\nu} = 0. \tag{256}$$

Writing out this equation in components (254) gives back the equation of continuity in the form (251).

At the microscopic level, charges and currents are composed of charged particles like electrons, quarks, protons or heavier nuclei. These particles act as the source of photons. Following Feynman, the interaction between photons and charged Dirac particles can be represented graphically by a diagram of the type shown in fig. 25.1. In the diagram a charged particle emits, absorbs or exchanges a single photon, depending on the direction in which time runs in the figure. A more detailed discussion of Feynman diagrams is presented in appendix F.

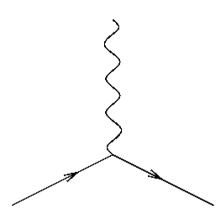

Fig. 25.1: Interaction of charged particle and photon.

A field-theoretical description of the electromagnetic interactions of charged particles involves, in addition to the Maxwell equations (255), an equation for the dynamics of the charged particles and an expression for the components of the four-current on the r.h.s. of the Maxwell equations. We now show how to do this for charged spin-0 particles.

The key to our construction is the concept of gauge invariance. Gauge invariance is important to have a consistent theory of massless vector particles, like photons. Therefore we wish to extend this invariance to the full theory of electro-magnetic fields interacting with charged matter. Now any linear equation for a complex matter field Ψ, like the Klein-Gordon or the Dirac equation, is invariant under phase transformations

$$\Psi \,\rightarrow\, \Psi' = e^{i\alpha}\Psi, \tag{257}$$

where α is a real constant. Consider the Klein-Gordon equation, and define $\mu = mc/\hbar$; if α is a constant it then follows that

$$\left(-\Box + \mu^2\right)\Psi' = e^{i\alpha}\left(-\Box + \mu^2\right)\Psi = 0. \tag{258}$$

Hence if Ψ is a solution of the equation, then Ψ' is a solution of the same equation. However, if the phase transformation depends on the point in space-time, this is no longer true. The reason is simple: the equation involves partial derivatives w.r.t. space-time and therefore

$$\partial_\mu\Psi' = e^{i\alpha}\left(\partial_\mu + i\partial_\mu\alpha\right)\Psi. \tag{259}$$

The situation can be repared by replacing the partial derivative with a linear differential operator that includes the vector field A_μ:

$$D_\mu\Psi \equiv \left(\partial_\mu - ieA_\mu\right)\Psi. \tag{260}$$

This object is known as the gauge-covariant derivative of the field Ψ. It has the property that a local phase transformation of Ψ can be compensated by a gauge transformation of A_μ with gauge parameter $\Lambda = \alpha/e$:

$$A'_\mu = A_\mu + \frac{1}{e}\,\partial_\mu\alpha. \tag{261}$$

Applying this transformation and the phase transformation (259) simultaneously, we get

$$\begin{aligned}(D_\mu\Psi)' &= \left(\partial_\mu - ieA'_\mu\right)\Psi' = e^{i\alpha}\left(\partial_\mu + i\partial_\mu\alpha - ie\left(A_\mu + \frac{1}{e}\partial_\mu\alpha\right)\right)\Psi \\ &= e^{i\alpha}\left(\partial_\mu - ieA_\mu\right)\Psi = e^{i\alpha}\,D_\mu\Psi.\end{aligned} \tag{262}$$

In the same way we deduce that

$$(D^\mu D_\mu\Psi)' = e^{i\alpha}D^\mu D_\mu\Psi, \tag{263}$$

This suggests that we modify the free Klein-Gordon equation to include the vector field A_μ through the gauge-covariant derivative so as to become

$$\left(-D^\mu D_\mu + \mu^2\right)\Psi = 0. \tag{264}$$

It follows, that

$$\left[\left(-D^\mu D_\mu + \mu^2\right)\Psi\right]' = e^{i\alpha}\left(-D^\mu D_\mu + \mu^2\right)\Psi = 0. \tag{265}$$

Hence if Ψ is a solution of eq. (264) for given electromagnetic vector potential A_μ, then Ψ' is a solution for the gauge-transformed potential A'_μ. Now if the charged particles are described by the field Ψ, the four-current j_μ must be constructed in terms of this field. As a gauge-transformation of A_μ does not change the electromagnetic field strength tensor $F_{\mu\nu}$ in the Maxwell equations (255), we must require this four-current to be unchanged by the combined phase and gauge transformations (259) and (261) as well. It should also satisfy the continuity equation (256). To construct such a current, first note that the complex conjugate field transforms with the opposite phase, and that

$$\left(D_\mu \Psi\right)^* = \left(\partial_\mu + ieA_\mu\right)\Psi^*. \tag{266}$$

Indeed, it follows in a straightforward way that

$$\Psi^{*\prime} = e^{-i\alpha}\Psi^*, \qquad \left(D_\mu \Psi^*\right)' = e^{-i\alpha} D_\mu \Psi^*. \tag{267}$$

Then the following expression for the current fits all our requirements:

$$j_\mu = -ie\left(\Psi^* D_\mu \Psi - \Psi D_\mu \Psi^*\right). \tag{268}$$

To prove this, use eqs. (262) and (267) to infer the invariance of the current:

$$j'_\mu = -ie\left(\Psi^* D_\mu \Psi - \Psi D_\mu \Psi^*\right)' = j_\mu. \tag{269}$$

Next, the equation of continuity applies as well:

$$\partial^\mu j_\mu = -ie\left(\Psi^* D^2 \Psi - \Psi D^2 \Psi^*\right) = 0, \tag{270}$$

with $D^2 = D^\mu D_\mu$ and

$$D^2\Psi = \mu^2\Psi, \qquad D^2\Psi^* = \mu^2\Psi^*. \tag{271}$$

Hence we have a complete theory of interacting charged scalar and electromagnetic fields, defined by eqs. (255), (268) and (264), which are all invariant under the combined gauge and phase transformations (257) and (261).

Exercise 25.2

a. Complete the steps in the proof of eq. (270).

b. Write out the expression for the gauge-covariant derivatives to prove that

$$j_\mu = -ie\left(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*\right) - 2e^2 \Psi^* \Psi\, A_\mu.$$

26. Spontaneously broken gauge invariance and dynamical mass generation

The Maxwell equations provide an example of how to couple massless vector fields and charged particle fields in a consistent way. The field equations for other vector fields, like gluons and weak vector bosons, follow a similar pattern.

In this section we consider a general massless vector field A_μ coupled to a complex scalar field Ψ, described by the equations

$$\partial^\mu F_{\mu\nu} = -J_\nu, \qquad J_\nu = -ig\left(\Psi^* D_\mu \Psi - \Psi D_\mu \Psi^*\right). \tag{272}$$

Here the electric charge e has been replaced by a general coupling constant g, which could also represent for example a weak charge like isospin or hypercharge, and the field A_μ has been rescaled to absorb factors like $\varepsilon_0 c^2$. In particular, the gauge-covariant derivatives take the form

$$D_\mu \Psi = \left(\partial_\mu - igA_\mu\right)\Psi, \qquad D_\mu \Psi^* = \left(\partial_\mu + igA_\mu\right)\Psi^*. \tag{273}$$

As in exercise (25.2) the current J_μ and the field strength tensor $F_{\mu\nu}$ are invariant under combined gauge transformations

$$\Psi' = e^{i\alpha}\Psi, \qquad A'_\mu = A_\mu + \frac{1}{g}\,\partial_\mu \alpha. \tag{274}$$

Moreover, the current is conserved:

$$\partial^\mu J_\mu = 0, \tag{275}$$

provided Ψ satisfies a Klein-Gordon equation of the form

$$D^2 \Psi = M^2 \Psi. \tag{276}$$

Here M^2 is a real quantity, which we identified before as the reduced mass (i.e., the Compton wave number) $\mu^2 = m^2c^2/\hbar^2$; however, in general it doesn't have to be a constant: it can be some space-time dependent quantity $M^2(x)$, as long as it is real.

Exercise 26.1
Check that the equation of continuity

$$\partial^\mu J_\mu = 0$$

holds for general $M^2(x)$.

The explicit expression for the current J_μ takes the same form as in exercise (25.2):

$$J_\mu = -ig\left(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*\right) - 2g^2|\Psi|^2 A_\mu. \tag{277}$$

This result has the very interesting consequence, that for a non-vanishing constant field $\Psi = v =$ constant the current takes the form

$$J_\mu = -2g^2|v|^2 A_\mu, \tag{278}$$

and the field equation for the vector field becomes effectively

$$\partial^\mu F_{\mu\nu} = \kappa^2 A_\mu, \qquad \kappa^2 = 2g^2|v|^2. \tag{279}$$

Comparing with eq. (233) we observe, that A_μ now satisfies the Proca equation for a *massive* vector field, with

$$m^2 = \frac{2g^2\hbar^2|v|^2}{c^2}. \tag{280}$$

Thus the constant scalar field $\Psi = v$ behaves as a medium in which massless vector particles effectively become massive, and slow down to velocities less than that of light. At the same time we observe, that in this situation the gauge invariance (274) has disappeared; this is manifest as the solution for the scalar field changes under a gauge transformation:

$$\Psi = v \; \rightarrow \; \Psi' = e^{i\alpha} v, \tag{281}$$

hence our solution picks out a particular value of Ψ which is not unique. Moreover, the Proca equation (279) is itself not gauge invariant, because A_μ on the right-hand side changes under a gauge transformation, whilst $F_{\mu\nu}$ on the left-hand side does not. We conlude that, even though our fundamental theory represented by eqs. (272) and (276) is gauge invariant, the solutions of the equations are not necessarily gauge invariant. In this situation one speaks of *spontaneously broken* gauge invariance, and we observe that it leads to dynamical mass generation for the vector particles, an effect described for the first time in a relativistic context by Robert Brout and François Englert.

It remains to discuss how a non-vanishing constant value of Ψ can be a solution of the generalized Klein-Gordon equation (276). Now a constant value $v \neq 0$ has vanishing gradients[13]: $\partial_\mu v = 0$. Therefore, in the vacuum with $A_\mu = 0$ the scalar equation effectively becomes

$$M^2\big|_{\Psi=v}\, v = 0 \quad \Rightarrow \quad M^2\big|_{\Psi=v} = 0. \tag{282}$$

The way to achieve this is to take M^2 to be a polynomial in $|\Psi|^2$ with a zero at $|\Psi|^2 = |v|^2$. In the simplest case we can take

$$M^2[\Psi] = \lambda \left(|\Psi|^2 - |v|^2 \right) \quad \Rightarrow \quad M^2[v] = 0. \tag{283}$$

With this choice, the modified Klein-Gordon equation becomes

$$\left(D^2 + \lambda |v|^2 \right) \Psi = \lambda |\Psi|^2 \Psi. \tag{284}$$

This is a non-linear extension of the scalar field equation, which implies that the scalar field has self-interactions, represented by the cubic term on the right-hand side.

Finally, let us consider a slightly more general solution, in which Ψ has tiny ripples around the constant value v:

$$\Psi = e^{i\alpha}(v + h(x)), \qquad A_\mu = \frac{1}{g}\, \partial_\mu \alpha, \tag{285}$$

[13] More generally, for $\Psi = e^{i\alpha} v$ and $gA_\mu = \partial_\mu \alpha$ we have $D_\mu v = 0$.

where the local phase $\alpha(x)$ is chosen such that $h(x)$ is real everywhere. Then

$$D_\mu \Psi = e^{i\alpha} \partial_\mu h, \tag{286}$$

and the scalar equation (283) becomes

$$\left(-\Box + \mu^2\right) h = \mathcal{O}(h^2), \qquad \mu^2 = 2\lambda |v|^2. \tag{287}$$

which in the limit $|h| \ll |v|$ becomes the free Klein-Gordon equation for spin-0 particles with mass

$$m_h^2 = \frac{\hbar^2 \mu^2}{c^2} = \frac{2\lambda \hbar^2 |v|^2}{c^2}. \tag{288}$$

Therefore, as first observed by Peter Higgs, vector particles which get their mass through dynamical mass generation must be accompanied by a massive scalar particle: the Higgs particle. The observation of such a Higgs particle signifies spontaneously broken gauge invariance as the mechanism behind vector-particle masses.

27. Vector boson mixing

As we have seen in sect. 21, the weak interactions look much simpler in the limit in which all particles can be considered massless. This was argued by noting that the weak charges isospin and hypercharge, which depend on the particle chirality, are well-defined only in this approximation. We also found in sect. 24 that there is a fundamental difference in the description of massless and massive vector bosons: they differ in the number of polarization states, which is 2 for massless vector particles and 3 for massive ones. In the mathematical equations this becomes manifest, as the theory of massless vector fields (Maxwell-type) is gauge invariant, whereas the theory of massive vector fields (Proca-type) is not.

In this section we show how spontaneously broken gauge invariance can explain the weak interactions, and how electromagnetism emerges from this theory as the only remaining long-range interaction at low energy. In order to simplify the discussion, we only consider the vector bosons coupling to hypercharge and the third component of isospin. At the end of the section we briefly indicate how things change when we take into account all components of isospin.

We introduce massless vector fields W_μ and B_μ coupling to the weak charges τ_3 and y. Table 21.1 lists the values of hypercharge and isospin of quarks and leptons. If the unit of isospin is denoted by g and that of hypercharge by g', we can modify the Dirac equation by replacing ordinary partial derivatives by gauge-covariant derivatives

$$\partial_\mu \;\rightarrow\; D_\mu = \partial_\mu - i\tau_3 g\, W_\mu - iyg' B_\mu. \tag{289}$$

For example, for a massless right-handed electron described by the field ψ_e, and a massless left-handed electron represented by the field χ_e, the Dirac equation becomes

$$i\hbar \left(D_0 + \boldsymbol{\sigma}\cdot\mathbf{D}\right) \psi_e = 0, \qquad i\hbar \left(D_0 - \boldsymbol{\sigma}\cdot\mathbf{D}\right) \chi_e = 0, \tag{290}$$

where

$$D_\mu \psi_e = \left(\partial_\mu + ig' B_\mu\right) \psi_e, \qquad D_\mu \chi_e = \left(\partial_\mu + \frac{ig}{2}\, W_\mu + \frac{ig'}{2}\, B_\mu\right) \chi_e, \tag{291}$$

and for the neutrinos:

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu, \qquad D_\mu \chi_\nu = \left(\partial_\mu - \frac{ig}{2}\, W_\mu + \frac{ig'}{2}\, B_\mu\right) \chi_\nu. \tag{292}$$

These gauge-covariant derivatives render the massless Dirac equations (290) invariant under two sets of gauge transformations: the isospin transformations

$$W'_\mu = W_\mu + \frac{1}{g}\, \partial_\mu \alpha, \qquad \psi'_e = \psi_e, \qquad \chi'_e = e^{-i\alpha/2} \chi_e; \tag{293}$$

and the hypercharge transformations

$$B'_\mu = B_\mu + \frac{1}{g'}\, \partial_\mu \beta, \qquad \psi'_e = e^{-i\beta} \psi_e, \qquad \chi'_e = e^{-i\beta/2} \chi_e. \tag{294}$$

Similarly, the Dirac equations for massless neutrinos are invariant if the neutrino fields transform as

$$\psi'_\nu = \psi_\nu, \qquad \chi'_\nu = e^{i(\alpha-\beta)/2} \chi_\nu. \tag{295}$$

Next suppose we have a complex scalar field φ with isospin $\tau_3 = 1/2$ and hypercharge $y = -1/2$, similar to the left-handed neutrino field χ_ν. Therefore this field transforms under the gauge transformations (293) and (294) as

$$\varphi' = e^{i(\alpha-\beta)/2} \varphi, \qquad \varphi^{*\prime} = e^{-i(\alpha-\beta)/2} \varphi^*. \tag{296}$$

Now the massless Dirac equations (290) can be extended in a gauge-invariant way to include the weakly charged scalar field, as follows

$$i\hbar\, (D_0 + \boldsymbol{\sigma} \cdot \mathbf{D})\, \psi_e = f_e \varphi\, \chi_e, \qquad i\hbar\, (D_0 - \boldsymbol{\sigma} \cdot \mathbf{D})\, \chi_e = f_e \varphi^* \psi_e, \tag{297}$$

where f_e is a constant. It is now easy to check that both sides of the first equation are multiplied by a factor $e^{-i\beta}$ under the full set of gauge transformations, whilst both sides of the second equation are multiplied by $e^{-i(\alpha+\beta)/2}$. This shows that these Dirac equations are consistent with both isospin and hypercharge gauge transformations. It also shows that if the scalar field φ takes a constant value $v \neq 0$, then the Dirac equations (297) become those of a *massive* electron, with mass

$$m_e c = f_e v. \tag{298}$$

This is yet another example of dynamical mass generation by spontaneously broken gauge invariance. The same scalar field can also generate a mass for the neutrinos, if their Dirac equation is similarly modified:

$$i\hbar\, (D_0 + \boldsymbol{\sigma} \cdot \mathbf{D})\, \psi_\nu = f_\nu \varphi^* \chi_\nu, \qquad i\hbar\, (D_0 - \boldsymbol{\sigma} \cdot \mathbf{D})\, \chi_\nu = f_\nu \varphi\, \psi_\nu, \tag{299}$$

such that $m_\nu c = f_\nu v$. The constants $f_{e,\nu}$ are known as *Yukawa coupling constants*; their role is to determine the relative size of particle masses.

Now the scalar field φ itself then satisfies a Klein-Gordon equation of the type (276):

$$\left(-D^2 + M^2\right) \varphi = 0, \tag{300}$$

with

$$D_\mu \varphi = \left(\partial_\mu - \frac{ig}{2} W_\mu + \frac{ig'}{2} B_\mu\right) \varphi. \tag{301}$$

This suggests we define the following orthogonal combinations of vector fields:

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g W_\mu - g' B_\mu\right), \qquad A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g' W_\mu + g B_\mu\right). \tag{302}$$

It is convenient to parametrize this mixing of the vector fields by an angle θ_W defined by

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{303}$$

which is equivalent to

$$g = g_W \cos\theta_W, \qquad g' = g_W \sin\theta_W, \qquad g_W = \sqrt{g^2 + g'^2}. \tag{304}$$

Then the covariant derivative of the scalar field becomes

$$D_\mu \varphi = \left(\partial_\mu - \frac{i g_W}{2} Z_\mu\right) \varphi. \tag{305}$$

Furthermore, defining

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}} = g_W \sin\theta_W \cos\theta_W, \tag{306}$$

the covariant derivatives of the electron and neutrino fields take the form

$$\begin{aligned}
D_\mu \psi_e &= \left(\partial_\mu + ie A_\mu - \frac{i g_W}{2} \left(1 - \cos 2\theta_W\right) Z_\mu\right) \psi_e, \\
D_\mu \chi_e &= \left(\partial_\mu + ie A_\mu + \frac{i g_W}{2} \cos 2\theta_W Z_\mu\right) \chi_e, \\
D_\mu \psi_\nu &= \partial_\nu \psi_\nu, \\
D_\mu \chi_\nu &= \left(\partial_\mu - \frac{i g_W}{2} Z_\mu\right) \chi_\nu.
\end{aligned} \tag{307}$$

From the results (305) and (307) we can draw many interesting conclusions. First, the scalar field couples only to Z_μ, not to A_μ. Therefore, a constant value $\varphi = v$ in the vacuum leads to a mass for the vector field Z_μ, but not for A_μ which remains massless. Second, left- and right-handed electrons couple to the massless field A_μ with the same strength $-e$, whilst neutrinos do not couple to A_μ at all. Therefore we can identify A_μ with the Maxwell field, i.e. photons, whereas Z_μ represents a massive vector boson acting differently on left- and righthanded leptons as expected for weak intereactions. The electric charge proportional to e then remains conserved, while the weak charge proportional to g_W is not.

Exercise 27.1
a. Using the isospin and hypercharge values listed in table 21.1, construct the gauge-covariant derivatives

$$D_\mu = \partial_\mu - ig\tau_3 W_\mu - ig' y B_\mu,$$

for the left- and righthanded quarks (ψ_u, χ_u) and (ψ_d, χ_d).
b. Rewrite the general gauge-covariant derivative above in terms of the fields A_μ and Z_μ and prove that the coefficient of A_μ is given by the electric charge $q = y + \tau_3$ in units e.

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28. Charged vector bosons

We have discussed in the previous section, how the photon and the Z-boson are represented by fields acting on hypercharge and isospin eigenstates. In these interactions the electric charge $q = y + \tau_3$ of particles is strictly conserved. This implies that the photon and Z-boson are electrically neutral themselves: emission or absorption of a photon or Z-boson can not change the electric charge of a quark or lepton.

Now in quantum theory the isospin charge τ_3 is the eigenvalue of an isospin operator T_3, which is one of a triplet of isospin operators operators $\mathbf{T} = (T_1, T_2, T_3)$. Acting on the quark and lepton states in table 21.1, the isospin components $T_{1,2}$ are not diagonal and mix the eigenstates of T_3. More conveniently, one can define conjugate linear combinations

$$T_\pm = \frac{1}{2}\left(T_1 \pm iT_2\right), \tag{308}$$

which change the isospin τ_3 precisely by one unit while leaving the hypercharge the same, as explained in appendix E. Therefore these operators also raise or lower the electric charge by one unit.

In quantum field theory the two vector bosons associated with these non-diagonal isospin operators are represented by fields $W^\pm$. Interactions with these fields change the isospin eigenvalue τ_3 of particles, but not the hypercharge y. Therefore they change the electric charge q by one unit as well. As the total electric charge is strictly conserved in all interactions, this implies that the $W^\pm$-bosons carry one unit of charge themselves. Thus emission of a $W^\pm$-boson changes a d-quark into a u-quark, and an electron into a neutrino, or vice versa. Similarly, the decay of W-bosons into other particles must produce a net charge of ± 1 unit.

The existence of the charged $W^\pm$-bosons explains β-decay processes and their relatives, such as

$$\begin{aligned} &\mu^- \rightarrow \nu_\mu + W^- \rightarrow \nu_\mu + \bar{\nu}_e + e^-, \\ &d \rightarrow u + W^- \rightarrow u + \bar{\nu}_e + e^-, \\ &c \rightarrow s + W^+ \rightarrow s + \bar{d} + u. \end{aligned} \tag{309}$$

Like the Z-boson, the W-bosons are massive due to spontaneously broken isospin gauge-invariance. The masses of these particles are actually very large:

$$m_W = 80.4\,\mathrm{GeV}/c^2, \qquad m_Z = 91.2\,\mathrm{GeV}/c^2, \tag{310}$$

comparable to the mass of a heavy nucleus like rubidium. This implies that relatively light particles like a muon or d-quark can never produce a real W-boson. The W-boson here exists only as a virtual particle in an intermediate state, and splits immediately into lighter fragments like (e, ν_e) or $(u, \bar{d})$. That is why we included in eqs. (309) also the second stage of the process. A graphical representation in terms of a Feynman diagram is shown in appendix F, fig. f.9.

29. Neutrino oscillations

Let us now return to the Dirac equations (299) for neutrinos. As all neutrinos have the same hypercharge and isospin, the most general gauge-invariant form of these equations for three neutrino families ν_i, $i = (e, \mu, \tau)$, reads

$$i\hbar \left(D_0 + \boldsymbol{\sigma} \cdot \mathbf{D} \right) \psi_\nu^i = \sum_{j=e,\mu,\tau} f_\nu^{ij}\, \varphi^* \chi_\nu^j, \qquad i\hbar \left(D_0 - \boldsymbol{\sigma} \cdot \mathbf{D} \right) \chi_\nu^i = \sum_{j=e,\mu,\tau} f_\nu^{ij}\, \varphi\, \psi_\nu^j. \tag{311}$$

Indeed, under local hypercharge and isospin transformations both sides of the first equation are still invariant, whilst both sides of the second equation are multiplied by the same factor $e^{i(\alpha-\beta)/2}$. The only thing we have done is to promote the constant f_ν for a single neutrino to a matrix f_ν^{ij} in the case of several neutrinos. This simple generalization can however have dramatic consequences.

By definition, we can take the neutrinos labeled $(\nu_e, \nu_\mu, \nu_\tau)$ to be those produced in W^+-decay in association with the anti-leptons (e^+, μ^+, τ^+). For example, if in a β-decay process we detect a positron, we know that the associated neutrino produced is ν_e. Now if the matrix of Yukawa-couplings f_ν^{ij} is not diagonal, then these neutrinos do not have a well defined mass: the Higgs field φ couples the right-handed electron neutrino represented by ψ_ν^e to a mixture of left-handed neutrinos $(\chi_\nu^e, \chi_\nu^\mu, \chi_\nu^\tau)$, and a vacuum-expectation value $v = \langle \varphi \rangle$ leads to a non-diagonal mass matrix

$$m_\nu^{ij} c = f_\nu^{ij}\, v. \tag{312}$$

We can of course diagonalize this mass matrix by some unitary transformation U:

$$UmU^{-1} = \left(\begin{array}{ccc} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{array} \right). \tag{313}$$

The corresponding eigenstates $(\tilde{\psi}_\nu, \tilde{\chi}_\nu) = (U\psi_\nu, U\chi_\nu)$ then represent particles of well-defined mass. However, now the coupling of these fields to the W-bosons are no longer diagonal, implying that the particle produced in association with a positron is a superposition of neutrino mass-eigenstates, and similarly for the neutrinos produced in association with the other charged anti-leptons.

In quantum-mechanical Hilbert-space language we represent the neutrino interaction states by $|\nu_i\rangle$, $i = (e, \mu, \tau)$, and the mass eigenstates by $|\nu_a\rangle$, $a = (1, 2, 3)$. Now in the absence of external fields, the energy E_a of a neutrino mass eigenstate with momentum $\mathbf{p}_a$ is

$$E_a = \sqrt{m_a^2 c^4 + \mathbf{p}_a^2 c^2}, \tag{314}$$

and the free neutrino mass-eigenstate depends on space and time as a plane wave

$$|\Psi_a(t, \mathbf{r})\rangle = e^{i(\mathbf{p}_a \cdot \mathbf{r} - E_a t)/\hbar} |\nu_a\rangle. \tag{315}$$

Now a neutrino starting as a particular interaction eigenstate at the origin, is at time $t = 0$ given by the state

$$|\Psi_i(0)\rangle = |\nu_i\rangle = \sum_{a=1,2,3} U_{ia} |\nu_a\rangle. \tag{316}$$

At time t this state takes the form

$$|\Psi_i(t, \mathbf{r})\rangle = \sum_{a=1,2,3} e^{i(\mathbf{p}_a \cdot \mathbf{r} - E_a t)/\hbar} U_{ia} |\nu_a\rangle. \tag{317}$$

The probability amplitude that this state is observed as a neutrino interaction state $|\nu_j\rangle$ is

$$\langle \nu_j | \Psi_i(t, \mathbf{r})\rangle \quad = \quad \sum_{a=1,2,3,} e^{i(\mathbf{p}_a \cdot \mathbf{r} - E_a t)/\hbar} U_{ia} \langle \nu_j | \nu_a \rangle = \sum_{a=1,2,3,} e^{i(\mathbf{p}_a \cdot \mathbf{r} - E_a t)/\hbar} U_{ia} U_{aj}^\dagger. \tag{318}$$

Hence a neutrino starting at time $t = 0$ in the origin as a well-defined interaction state $|\nu_i\rangle$, can at a later time t at position $\mathbf{r}$ be observed as a neutrino interaction state $|\nu_j\rangle$ with probability

$$P_{i \to j}(t, \mathbf{r}) = |\langle \nu_j | \Psi_i(t, \mathbf{r})\rangle|^2 = \left| \sum_a e^{i(\mathbf{p}_a \cdot \mathbf{r} - E_a t)/\hbar} U_{ia} U_{aj}^\dagger \right|^2. \tag{319}$$

This process of a space-time dependent probability for changing identity is refered to as *neutrino oscillations*. Such oscillations have been observed in the neutrinos emitted by the sun: solar neutrinos are produced in the sun by inverse β-decay: $p + e \to n + \nu_e$, but neutrino experiments on earth detect equal numbers of ν_e, ν_μ and ν_τ.

Exercise 29.1
Consider two kinds of neutrino with interaction eigenstates $|\nu_{e,\mu}\rangle$, and with mass eigenstates $|\nu_{1,2}\rangle$, related by

$$|\nu_i\rangle = \sum_a U_{ia}|\nu_a\rangle, \qquad U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The parameter θ is called the mixing angle.
a. Show that

$$P_{\nu_e\to\nu_\mu}(t,\mathbf{r}) = 2\cos^2\theta\sin^2\theta\left(1-\cos\left(\Delta E\,t - \Delta\mathbf{p}\cdot\mathbf{r}\right)/\hbar\right),$$

with $\Delta E = E_2 - E_1$, and $\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$.
b. In the limit of relativistic neutrinos moving in a fixed direction over a distance $L = ct$

$$\frac{1}{\hbar}\left(\Delta E\,t - \Delta pL\right) = \frac{L}{\hbar c}\left(\Delta E - c\Delta p\right).$$

Show that with $p_i \gg m_i c$ one can to good approximation write

$$\frac{1}{\hbar}\left(\Delta E\,t - \Delta pL\right) = \frac{L}{2\hbar}\left(\frac{m_2^2c^2}{p_2} - \frac{m_1^2c^2}{p_1}\right) \approx \frac{\Delta m^2\,c^3L}{2\hbar E},$$

where $E \approx p_1 c \approx p_2 c$.
c. Show that the final transition probability is

$$P_{\nu_e\to\nu_\mu}(t,\mathbf{r}) = \frac{1}{2}\,\sin^2 2\theta\,\sin^2\left(\frac{\Delta m^2\,c^3L}{4\hbar E}\right) = \frac{1}{2}\,\sin^2 2\theta\,\sin^2\left(1.27\,\frac{\Delta m^2\,L}{E}\right),$$

with the neutrino energy E expressed in GeV, the mass difference $\Delta m^2 = m_2^2 - m_1^2$ expressed in $(\mathrm{eV}/c^2)^2$, and L expressed in km.

Appendix

A. Special relativity

Special relativity is based on two important empirical observations:
1. the existence of a special class of co-ordinate systems, in which all free particles are in rest or move uniformly in a straight line;
2. the universality and invariance of the speed of light, which is the same for observers in all inertial systems.

Newton's first law states, that free particles are at rest or move with constant velocity on straight lines. This is only true for observers who are not accelerated by external forces themselves. The co-ordinate systems associated with such obervers are called *inertial frames*.

If a particle moves with constant velocity on a straight line in one such frame, it will also move on a straight line in any shifted, rotated or moving frame, provided the translation or change in orientation is constant, or the velocity of one system w.r.t. the others is constant in a fixed direction. Therefore frames connected to an inertial frame by a constant translation, constant rotation or constant linear motion are also inertial frames. Examples:
a. *translation:*

$$x' = x + a, \qquad y' = y, \qquad z' = z; \tag{320}$$

b. *rotation:*

$$x' = x \cos\alpha - y \sin\alpha, \qquad y' = x \sin\alpha + y \cos\alpha, \qquad z' = z; \tag{321}$$

c. *linear motion:*

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \tag{322}$$

with v the relative velocity and γ a proportionality constant to be determined.

According to the special theory of relativity Minkowski space-time intervals are the same in all inertial frames:

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2, \tag{323}$$

where the light velocity c has the same value on both sides. Indeed, for an observer in the frame (x, y, z, t) a light ray moves with velocity $\mathbf{v}$ given by

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0 \quad \Rightarrow \quad \mathbf{v}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2. \tag{324}$$

By eq. (323) this is then also true in any other inertial frame:

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = 0 \quad \Rightarrow \quad \mathbf{v}'^2 = c^2. \tag{325}$$

Therefore the velocity of light is a universal constant, taking the same value in all inertial frames.

It is easy to see that the space-time interval (323) is invariant under the translations (320)

$$dx' = d(x + a) = dx, \qquad dy' = dy, \qquad dz' = dz, \tag{326}$$

and under the rotations (321):

$$dx' = dx \cos\alpha - dy \sin\alpha, \qquad dy' = dx \sin\alpha + dy \cos\alpha, \qquad dz' = dz. \tag{327}$$

It is also invariant for frames connected by linear motion, provided the constant γ takes the special value:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{328}$$

and the clock time is adjusted in a similar way:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \qquad ct' = \frac{ct - vx/c}{\sqrt{1 - v^2/c^2}}, \tag{329}$$

with the result that

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}, \qquad cdt' = \frac{cdt - vdx/c}{\sqrt{1 - v^2/c^2}}. \tag{330}$$

Indeed, it is straightforward to establish that

$$dx'^2 - c^2 dt'^2 = dx^2 - c^2 dt^2. \tag{331}$$

The transformations (329) are called *special Lorentz transformations.*

The Minkoswki interval (323) can be obtained by combining the *contravariant* components of the space-time interval

$$dx^\mu = (dx^0, dx^1, dx^2, dx^3) = (cdt, dx, dy, dz) \tag{332}$$

with the *covariant* components of the space-time interval

$$dx_\mu = (dx_0, dx_1, dx_2, dx_3) = (-cdt, dx, dy, dz). \tag{333}$$

Then

$$\sum_{\mu=0}^{3} dx^\mu dx_\mu = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{334}$$

We have seen above, that if x^μ and x'^μ are co-ordinates refering to two inertial frames, related by any combinaton of translations, rotations or special Lorentz transformations, they measure the same space-time intervals:

$$\sum_{\mu=0}^{3} dx^\mu dx_\mu = \sum_{\mu=0}^{3} dx'^\mu dx'_\mu \tag{335}$$

Next we introduce the general concept of four-vectors, which are defined by their properties under space-time transformations. Any set of quantities $a^\mu = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3)$, of which the components in different inertial frames are related exactly as the contravariant intervals dx^μ, is called a *contravariant four-vector.* For example, for two frames moving with relative velocity v in the x-direction the components of a' must be related to those of a by (330)

$$a'^1 = \frac{a^1 - va^0/c}{\sqrt{1 - v^2/c^2}}, \qquad a'^0 = \frac{a^0 - va^1/c}{\sqrt{1 - v^2/c^2}}. \tag{336}$$

Similarly, a set of quantities $a_\mu = (a_0, a_1, a_2, a_3)$ transforming between inertial frames as the covariant intervals dx_μ is called a *covariant four-vector.* They must be related to the contravariant components like the covariant and contravariant differentials:

$$(a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3). \tag{337}$$

Therefore

$$\sum_{\mu=0}^{3} a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2. \tag{338}$$

It is standard practice in such multipication of covariant and contravariant vectors to omit the summation sign; this is known as the Einstein summation convention:

$$a^\mu a_\mu \equiv \sum_{\mu=0}^{3} a^\mu a_\mu. \tag{339}$$

Because the transformation rules of a^μ and a_μ are the same as those of dx^μ and dx_μ, and as these transformation rules guarantee that $dx^\mu dx_\mu$ is invariant (i.e., it has the same value in any interial frame), it follows that the number $a^\mu a_\mu$ is an invariant as well. Hence if we compute this number in any inertial frame it is guaranteed to have the same value in any other inertial frame.

The switch from contravariant to covariant components is achieved by changing the sign of the time component; formally this can be written as multiplication with the Minkowski metric or its inverse:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{340}$$

Indeed, it follow that

$$a_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} a^\nu \equiv \eta_{\mu\nu} a^\nu, \quad a^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} a_\nu \equiv \eta^{\mu\nu} a_\nu, \tag{341}$$

whilst similar matrix multiplication gives

$$\eta^{\mu\lambda} \eta_{\lambda\nu} = \delta^\mu_\nu. \tag{342}$$

In all these products we always sum over a common upper and lower index. We can also use the Minkowski metric to define the Minkowski space-time interval

$$\begin{aligned} \eta_{\mu\nu}\, dx^\mu dx^\nu &= \eta_{00}(dx^0)^2 + \eta_{11}(dx^1)^2 + \eta_{22}(dx^2)^2 + \eta_{33}(dx^3)^2 \\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2. \end{aligned} \tag{343}$$

One can distinguish three kinds of space-time intervals: space-like, light-like and time-like. The distinction depends on the sign of the expression (343):

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = \begin{cases} ds^2 > 0 & \Rightarrow \quad \text{space-like;} \\ 0 & \Rightarrow \quad \text{light-like;} \\ -c^2d\tau^2 < 0 & \Rightarrow \quad \text{time-like.} \end{cases} \tag{344}$$

Note that intervals between two points on the worldline of a single particle are time-like; in particular

$$d\tau = dt\, \sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{r}}{dt}\right)^2} = dt\, \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}, \tag{345}$$

where $\mathbf{v}$ is the velocity of the particle. Therefore $d\tau$ is the time interval measured on a clock in the rest frame of the particle (in which $\mathbf{v} = 0$); this is known as the *proper time*.

Similar to the Lorentz-invariant line element (343), we can define the Lorentz-invariant Laplace operator, also called *d'alembertian*:

$$\Box = \eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{346}$$

It is customary to use the abbreviated notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial^\mu = \eta^{\mu\nu}\partial_\nu = \frac{\partial}{\partial x_\mu}, \qquad \Box = \partial^\mu \partial_\mu. \tag{347}$$

An example of another 4-vector is the 4-velocity of a particle, obtained as the derivative of the space-time position w.r.t. proper time:

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{348}$$

Using relation (345) between proper time and observer time, it follows that

$$u^\mu = (\gamma c, \gamma \mathbf{v}), \qquad \mathbf{v} = (v^1, v^2, v^3) = (v_x, v_y, v_z). \tag{349}$$

Here γ is the relativistic time-dilation factor, and $\mathbf{v}$ is the ordinary velocity in observer co-ordinates:

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{350}$$

As proper time is observer-invariant, it is easy to check that u^μ transforms under Lorents transformations as the co-ordinate differentials themselves, cf. eq. (336). Also note, that we can construct an invariant

$$u^\mu u_\mu = -c^2. \tag{351}$$

Similarly we define the 4-momentum as the product of mass and 4-velocity:

$$p^\mu = m u^\mu = \left(\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z \right), \tag{352}$$

which is a 4-vector because u^μ is, whilst m is invariant[14]. The 4-momentum satisfies

$$p^\mu p_\mu = -m^2 c^2. \tag{353}$$

Explicitly, the space- and time-components read

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}, \qquad p^0 = \frac{mc}{\sqrt{1 - \mathbf{v}^2/c^2}} = \frac{E}{c}, \tag{354}$$

whilst eq. (353) can be written in the more familiar form

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4. \tag{355}$$

In the limit $\mathbf{v}^2/c^2 \rightarrow 0$ the space components $\mathbf{p}$ reduce to the standard newtonian momentum vector $m\mathbf{v}$. On the other hand, in this limit the time component is given by

$$E \rightarrow mc^2 + \frac{1}{2} m\mathbf{v}^2, \tag{356}$$

which is the sum of the rest energy (a constant) and the newtonian kinetic energy. An important property of the momentum 4-vector is, that all 4-momentum components are conserved in elastic collisions:

$$P_f^\mu = P_i^\mu, \qquad P_{i,f}^\mu = \sum p_{i,f}^\mu, \tag{357}$$

where the labels (i, f) refer to the initial and final states of motion.

[14] In some texts this quantity is refered to as the *rest mass*. In our terminology the mass is simply a number, the same for all observers, with the property that mc^2 is the *rest energy*.

Exercise A.1

An inertial system Σ' moves w.r.t. an inertial system Σ with (constant) velocity $\mathbf{v} = (v, 0, 0)$. The co-ordinates in Σ' are related to those in Σ by the special Lorentz transformations

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}} \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}.$$

a. Show that the Minkowski space-time interval is invariant under these transformations:

$$dx'^2 - c^2 dt'^2 = dx^2 - c^2 dt^2.$$

b. What is the inverse transformation for (ct, x) in terms of (ct', x')?
c. A light-ray moves in system Σ with velocity $dx/dt = c$. Show that in Σ' its velocity dx'/dt' is the same.
d. Suppose that at $t = 0$ the origins of the systems (Σ, Σ') coincide. Which point at distance x from the origin in the system Σ coincides at $t = 0$ with a point at distance x' from the origin in Σ'? How does one interpret this in terms of lengths measured in the two systems?
e. O' measures a time interval $\Delta t'$ on a clock at rest in his system; show that in Σ the corresponding interval is

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \mathbf{v}^2/c^2}}.$$

Exercise A.2

Show that a Lorentz transformation of the form (336) leaves the quadratic expression

$$a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$$

invariant.

Exercise A.3

a. A muon is a particle with a mass m_μ such that

$$m_\mu c^2 = 105 \text{ MeV}.$$

The muon is instable: in its restframe it decays on average after a time

$$\tau_\mu = 2.2\ \mu\text{s}.$$

If the muon has a total energy $E = 1$ GeV, how far can it travel on average before it decays?

Exercise A.4

We consider the collision of two (relativistic) particles with masses m_1 and m_2. In the lab-frame particle 1 is at rest: $\mathbf{p}_1 = 0$, whilst particle 2 moves along the x-axis with momentum $\mathbf{p}_2 = (p, 0, 0)$.

a. Give the expressions for the energies (E_1, E_2) of the particles.
b. Give the components of the momentum 4-vectors p_1^μ and p_2^μ.
c. The center of mass (CM) frame is the frame in which the total 3-momentum vanishes:

$$\mathbf{p}_1' + \mathbf{p}_2' = 0.$$

Construct the Lorentz transformation from the lab frame to the CM frame, by performing the following steps:
(i) explain that the CM-frame moves with a velocity $\mathbf{v} = (v, 0, 0)$, where v is to be determined.
(ii) Using the Lorentz transformation of 4-vectors, show that the energy and momentum for each particle in the CM-frame takes the form

$$E' = \frac{E - vp_x}{\sqrt{1 - v^2/c^2}}, \qquad p_x' = \frac{p_x - vE/c^2}{\sqrt{1 - v^2/c^2}}.$$

(iii) Show that in the CM-frame the momenta are of the form

$$\mathbf{p}_1' = (-p', 0, 0), \qquad \mathbf{p}_2' = (p', 0, 0),$$

where

$$p' = \frac{m_1 v}{\sqrt{1 - v^2/c^2}} = \frac{E_1 v/c^2}{\sqrt{1 - v^2/c^2}},$$

and v is given by

$$\frac{v}{c} = \frac{pc}{E_1 + E_2}.$$

(iv) From this derive the result

$$p' = \frac{pE_1}{\sqrt{(E_1 + E_2)^2 - p^2c^2}}.$$

d. Define

$$s = -\left(p_1 + p_2\right)^2 c^2 \equiv -\left(p_1^\mu + p_2^\mu\right)\left(p_{1\mu} + p_{2\mu}\right) c^2.$$

(i) Explain why s has the same value in the lab frame and in the CM frame.
(ii) Show that $\sqrt{s}$ represents the total energy in the CM frame.
(iii) Show that the equation for p' can be written in the form

$$p' = \frac{m_1 c^2}{\sqrt{s}}\, p.$$

B. The Bohr model

The first quantitative understanding of the energy levels of the hydrogen atom was provided by Niels Bohr in 1911, many years before the Schrödinger equation was invented. In his model of the hydrogen atom Bohr treated the dynamics of electrons in a classical way, reserving the role of quantum theory to apply only to the emission and absorption of electromagnetic radiation (light). The results are in full agreement with the observations and confirmed by the modern treatment sketched above. It is of interest as an example of how a correspondence principle can guide one's way to understanding in a new domain of physics where the mathematics is still to be developed, but some of the underlying physical principles are already at hand. It also gives a feeling for the magnitude of quantities in atomic physics, in particular the size of electron orbits and the spectrum of energy levels.

The empirical starting point for Bohr's treatment of the hydrogen atom was its well-known set of spectral lines. They form series, such as the Balmer series, the Paschen series, and several others; it was found that the frequencies and wavelengths in vacuum of these series satisfy the simple formula

$$\nu_n = \frac{c}{\lambda_n} = cR_H \left(\frac{1}{n^2} - \frac{1}{m^2} \right), \tag{358}$$

where (n, m) are integers: $n = 1, 2, 3, ...$ and $m = n+1, n+2, ...$; R_H is the Rydberg constant:

$$R_H = 1.097 \times 10^{-2}\,\text{nm}^{-1}. \tag{359}$$

Bohr constructed a relation between these frequencies and the energy absorbed or emitted by an electron changing its orbit. In his model the electron energy in the n-th orbit is

$$E_n = -\frac{hcR_H}{n^2}, \qquad n = 1, 2, ..., \tag{360}$$

where the zero point of energy has been set to correspond to the ionization energy: $E_\infty = 0$. This rule was motivated by the quantum hypothesis of Planck and Einstein, which implies that the frequency of a photon emitted during a transition between adjacent orbits is

$$h\nu_n = \Delta E_n = E_{n+1} - E_n; \tag{361}$$

with Bohr's assumption (360) this equation returns the frequencies eq. (358) with $m = n+1$.

Now consider an electron in a Coulomb potential. Classically, the electron could be in any Kepler-like orbit, but we consider only circular orbits here. The potential energy in such an orbit is

$$V_n = -\frac{e^2}{4\pi\varepsilon_0 r_n} = -\frac{\alpha\hbar c}{r_n}. \tag{362}$$

According to classical mechanics, the radial acceleration in this orbit is

$$\frac{m_e v_n^2}{r_n} = \frac{e^2}{4\pi\varepsilon_o r_n^2} \quad \Rightarrow \quad \frac{v_n^2}{c^2} = \frac{\alpha\hbar}{m_e c}\frac{1}{r_n}. \tag{363}$$

It follows, that the classical kinetic energy is

$$T_n = \frac{1}{2} m_e v_n^2 = \frac{\alpha \hbar c}{2 r_n}, \tag{364}$$

and the total energy is given by

$$E_n = T_n + V_n = -\frac{\alpha \hbar c}{2 r_n}. \tag{365}$$

Again, the zero point of energy here is taken to correspond to the fully ionized state $r_n = \infty$. By comparison with the expression (360) we then find for the radius r_n

$$r_n = a_0\, n^2, \tag{366}$$

with a_0 the Bohr radius:

$$a_0 = r_1 = \frac{\alpha}{4\pi R_H} \approx 0.053\,\mathrm{nm}. \tag{367}$$

Of course, according to modern quantum mechanics one can not speak about the precise position of the electron, and the above value of r_n represents only a statistical average. But the energy levels (360) are in agreement with the full quantum mechanical treatment; moreover, in the limit of large n quantum mechanics should go over into classical mechanics. As a result (367) is still a good estimate for the size of the hydrogen atom: its diameter is about 0.1 nm.

One can even use the Bohr model to derive the value of the Rydberg constant. In the classical limit $n \to \infty$ we can equate the spectral frequency ν_n:

$$\nu_n = \frac{\Delta E_n}{h} \approx \frac{2 c R_H}{n^3}, \tag{368}$$

with the orbital frequency:

$$\nu_n = \frac{v_n}{2\pi r_n} = \frac{2c}{\alpha n^3} \sqrt{\frac{2h}{m_e c}}\, R_H^{3/2}. \tag{369}$$

It then follows that

$$R_H = \frac{\alpha^2 m_e c}{2h} \approx 1.097 \times 10^{-2}\,\mathrm{nm}^{-1}, \tag{370}$$

in agreement with the experimental value (359).

C. The Thomson cross section

In this appendix we derive the scattering of electromagnetic radiation —such as light— by charged particles. This scattering process was first analysed in classical theory by Thomson as discussed in par. 4.

In the classical context, the charges are treated as non-relativistic particles. Let $\boldsymbol{\xi}(t)$ be the position of a point charge q at time t; the scalar and vector potential of the charge moving at velocity $\mathbf{v} = d\boldsymbol{\xi}/dt$ are

$$\phi(\mathbf{x}, t) = \frac{q}{R - \mathbf{u}\cdot\mathbf{R}}, \qquad \mathbf{A}(\mathbf{x}, t) = \frac{q\mathbf{u}/c}{R - \mathbf{u}\cdot\mathbf{R}}, \tag{371}$$

where

$$q = \frac{e}{4\pi\epsilon_0}, \qquad \mathbf{u} = \frac{\mathbf{v}}{c}, \qquad \mathbf{R} = \mathbf{x} - \boldsymbol{\xi}(t'), \qquad R = |\mathbf{R}|\,, \tag{372}$$

with t' the retarded time

$$t' = t - \frac{R}{c}. \tag{373}$$

Observe that these expressions reduce to the Coulomb potential in the limit $\mathbf{u} \to 0$. The electric and magnetic field strength at point $\mathbf{x}$ at time t are computed from

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{374}$$

with the result

$$\begin{aligned} \mathbf{E} &= \frac{q}{(R - \mathbf{u}\cdot\mathbf{R})^3}\left[(\mathbf{R} - R\mathbf{u})\left(1 - \mathbf{u}^2\right) + \mathbf{R} \times ((\mathbf{R} - R\mathbf{u}) \times \dot{\mathbf{u}}/c)\right]. \\ c\mathbf{B} &= \hat{\mathbf{R}} \times \mathbf{E}, \qquad \hat{\mathbf{R}} = \mathbf{R}/R. \end{aligned} \tag{375}$$

Here $\dot{\mathbf{u}} = d^2\boldsymbol{\xi}/dt'^2$ is the retarded acceleration. Observe that

$$c^2\mathbf{B}^2 = \mathbf{E}^2 = \frac{1}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) = \frac{\mathcal{H}}{\epsilon_0}, \tag{376}$$

with $\mathcal{H}$ the energy density in the field.

If we go to a very large distance from the point charge, the first term in the expression for $\mathbf{E}$ becomes negligeable w.r.t. the second one, as it contains one less power of R. Then if the velocity is small ($u \ll 1$) the expression simplifies to

$$\mathbf{E} = \frac{q}{cR^3}\,\mathbf{R} \times (\mathbf{R} \times \dot{\mathbf{u}})\,, \qquad c\mathbf{B} = \frac{q}{cR^2}\,\dot{\mathbf{u}} \times \mathbf{R}. \tag{377}$$

The Poynting vector $\mathbf{N}$ is given by

$$c\mathbf{N} = \epsilon_0 c\,\mathbf{E} \times \mathbf{B} = -\epsilon_0\,\mathbf{E} \times \left(\mathbf{E} \times \hat{\mathbf{R}}\right) = \epsilon_0\mathbf{E}^2\,\hat{\mathbf{R}} = \mathcal{H}\,\hat{\mathbf{R}}, \tag{378}$$

and represents the flux of field energy, directed perpendicular the electric and magnetic field strength. To see this, recall that Maxwell's equations in free space imply

$$H = \frac{\epsilon_0}{2} \int d^3x \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right), \qquad \frac{dH}{dt} = -c \int d^3x \, \nabla \cdot \mathbf{N}. \tag{379}$$

This shows directly that $\mathbf{N}$ represents the momentum density in the field, i.e. the energy flux across the surface bounding the volume of integration. For a large sphere centered at distance R from the position of the point charge this becomes

$$\Phi = -\frac{dH}{dAdt} = c^2 \left| \mathbf{N} \right| = c\mathcal{H}, \tag{380}$$

where we have taken into account that the Poynting vector is directly centrally, i.e. outward from and perpendicular to the spherical surface. To evaluate this flux it is now sufficient to calculate the magnetic field $\mathbf{B}$, eq. (377).

We now calculate $\mathbf{B}$ for motion of the charge under influence of a plane electromagnetic wave incident along the z-axis. We consider a charge moving under the influence of an electromagnetic wave moving in the z-direction:

$$\mathbf{E}_{inc} = E_0 \cos \left(\omega t - kz \right) \mathbf{i}, \qquad c\mathbf{B}_{inc} = \mathbf{k} \times \mathbf{E} = E_0 \cos \left(\omega t - kz \right) \mathbf{j}. \tag{381}$$

Then the Lorentz force on the point charge located at $z = 0$ is

$$\mathbf{F} = e \left(\mathbf{E}_{inc} + \mathbf{v} \times \mathbf{B}_{inc} \right) \approx eE_0 \cos \omega t \, \mathbf{i}. \tag{382}$$

In the last expression we have neglected the magnetic force, which is smaller by a factor $u = v/c \ll 1$. The magnetic field emitted by the oscillating point charge is given by (377):

$$c\mathbf{B} = \frac{q}{c^2 R} \dot{\mathbf{v}} \times \hat{\mathbf{R}} = \frac{q}{mc^2 R} \mathbf{F} \times \hat{\mathbf{R}}. \tag{383}$$

Now $\mathbf{F}$ points in the x-direction, and

$$\left| \mathbf{i} \times \hat{\mathbf{R}} \right| = \sin \psi, \tag{384}$$

where ψ is the angle between the centrally directed vector $\hat{\mathbf{R}}$ and the direction of polarization $\mathbf{i}$ of the electric field of the incident waves. Thus the outward energy flux induced by this polarized radiation is

$$\Phi_{pol} = \frac{\epsilon_0 q^2}{m^2 c^3 R^2} e^2 E_0^2 \cos^2 \omega t \, \sin^2 \psi. \tag{385}$$

Taking the time average:

$$\overline{\cos^2 \omega t} = \frac{1}{2}, \tag{386}$$

the time averaged energy flux becomes

$$\overline{\Phi}_{pol} = \frac{e^4}{4\pi\epsilon_0} \frac{E_0^2 \sin^2 \psi}{8\pi m^2 c^3 R^2}. \tag{387}$$

Now the flux of incident radiation is

$$\Phi_{inc} = \frac{c\epsilon_0}{2} \left(\mathbf{E}_{inc}^2 + c^2 \mathbf{B}_{inc}^2 \right) = c\, \epsilon_0 E_0^2 \cos^2 \omega t \quad \Rightarrow \quad \overline{\Phi}_{inc} = \frac{c\epsilon_0}{2}\, E_0^2. \tag{388}$$

The fraction of incident radiation scattered into the area element $dA = R^2 \sin\theta d\theta d\varphi$ then is

$$d^2\sigma_{pol} = \frac{\overline{\Phi}_{pol}}{\overline{\Phi}_{inc}}\, dA = \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \sin^2 \psi\, d\Omega. \tag{389}$$

The classical radius of the electron is defined such that

$$\frac{e^2}{4\pi\epsilon_0\, r_e} = mc^2 \quad \Leftrightarrow \quad r_e = \frac{e^2}{4\pi\epsilon_0\, mc^2}; \tag{390}$$

with this definition

$$d^2\sigma_{pol} = r_e^2\, \sin^2 \psi\, d\Omega. \tag{391}$$

The above computation holds for incident radiation with the electric field polarized along the x-axis. To average over all directions of polarization in the x-y-plane, we must re-express the angle ψ in terms of new polar angles θ and ϕ, defined by the direction of $\mathbf{E}_{inc}$ rather than that of the x-axis. This is done as follows; if $\mathbf{n}$ represents the unit vector in the direction of the electric field $\mathbf{E}_{inc}$, then

$$\sin^2 \psi = \left| \mathbf{n} \times \hat{\mathbf{R}} \right|^2 = \mathbf{n}^2 \hat{\mathbf{R}}^2 - \left(\mathbf{n} \cdot \hat{\mathbf{R}} \right)^2 = 1 - \sin^2 \theta\, \cos^2 \phi, \tag{392}$$

where θ still is the angle of outgoing radiation with the z-axis and can therefore be identified with the earlier θ appearing in $d\Omega$. Then the average over all polarizations of the electric field strength in the x-y-plane is equivalent to an average over all directions ϕ (which in general is not necessarily the same as the angle φ of the outgoing radiation w.r.t. the x-axis):

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi\, \sin^2 \psi = \frac{1}{2} \left(1 + \cos^2 \theta \right). \tag{393}$$

It follows that the fraction of incident flux of unpolarized radiation scattered into the spherial angle $d\Omega$ is:

$$d^2\sigma = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, d^2\sigma_{pol} = \frac{r_e^2}{2} \left(1 + \cos^2 \theta \right) d\Omega. \tag{394}$$

The quantity $d\sigma/d\Omega$ is the *differential scattering cross section* for scattering of electromagnetic waves by a free electron in the non-relativistic limit ($u = v/c \ll 1$).

Finally, the total cross section for this process is obtained by integrating over all directions represented by $d\Omega$, and is called the Thomson cross section:

$$\sigma_T = \int_{unit\ sphere} d^2\sigma = \frac{8\pi}{3}\, r_e^2. \tag{395}$$

D. The scattering Green's function

In this appendix we prove: the Green's function

$$G_k(\mathbf{r}) = \frac{1}{4\pi}\,\frac{e^{ikr}}{r} \tag{396}$$

satisfies the inhomogeneous linear partial differential equation

$$-\left(\Delta + k^2\right) G_k(\mathbf{r}) = \delta^3(\mathbf{r}). \tag{397}$$

First

$$\boldsymbol{\nabla}\,\frac{e^{ikr}}{r} = (ikr-1)\,\frac{e^{ikr}}{r^2}\,\hat{\mathbf{r}}, \tag{398}$$

where $\hat{\mathbf{r}}$ is the radial unit vector. Then, taking the divergence of this result, it immediately follows that for $\mathbf{r} \neq 0$

$$\left(\Delta + k^2\right) G_k(\mathbf{r}) = 0. \tag{399}$$

Hence we concentrate on the region near the origin $\mathbf{r} = 0$. We prove: for a spherical volume V_R with radius R

$$\lim_{R\to 0} \int_{V_R} \left(\Delta + k^2\right) G_k(\mathbf{r})\, d^3V = -1, \tag{400}$$

showing that the integral remains finite even if the sphere is contracted to a point. First, we prove that the second term does not contribute:

$$\int_{V_R} \frac{e^{ikr}}{r}\, d^3V = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\theta \int_0^R dr\, r^2\, \frac{e^{ikr}}{r} = \frac{4\pi}{k^2}\,\left(e^{ikR} - 1\right), \tag{401}$$

and therefore

$$\lim_{R\to 0}\, k^2 \int_{V_R} \frac{e^{ikr}}{r}\, d^3V = \lim_{R\to 0}\, 4\pi\left(e^{ikR}-1\right) = 0. \tag{402}$$

Finally we consider the first term; we use Gauss' theorem to get

$$\int_{V_R} \Delta\,\frac{e^{ikr}}{r}\, d^3V = \int_{V_R} \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\,\frac{e^{ikr}}{r}\right) d^3V = \int_{\Sigma_R} \left(\nabla\,\frac{e^{ikr}}{r}\right)_n d^2\Sigma, \tag{403}$$

where Σ_R is the surface of the sphere V_R, n denotes the normal (radial) component of the gradient on the surface, and $d^2\Sigma$ is the two-dimensional integration element. Now using the result (398) evaluated at $r = R$ one finds

$$\left.\left(\nabla\,\frac{e^{ikr}}{r}\right)_n\right|_{r=R} = (ikR-1)\,\frac{e^{ikR}}{R^2}. \tag{404}$$

The spherical surface element is

$$d^2\Sigma = R^2 \sin\theta\, d\theta d\varphi. \tag{405}$$

Substitution in eq. (403) gives

$$\int_{V_R} \Delta \frac{e^{ikr}}{r} \, d^3V = 4\pi \, (ikR - 1) \, e^{ikR}. \tag{406}$$

Combining the results (400), (402) and (406) we finally get

$$\lim_{R \to 0} \int_{V_R} \left(\Delta + k^2\right) G_k(\mathbf{r}) \, d^3V = \lim_{R \to 0} \, (ikR - 1) \, e^{ikR} = -1. \tag{407}$$

This proves the result (400).

E. Isospin

Table 21.1 shows, that the left-handed u- and d-quarks belong to the same isospin doublet, differing in the value of the third component τ_3. Similarly, also the left-handed neutrino and electron form an isospin doublet. In contrast, the right-handed quarks and leptons are iso-singlets: they carry no isospin, only hypercharge.

In quantum mechanics hypercharge and isospin are represented by operators $(Y, \mathbf{T})$, such that the values y of the hypercharges and τ_3 of isospin are eigenvalues of the operators (Y, T_3). The quantum states representing massless quarks and leptons can then be associated with vectors in Hilbert space which are classified by these eigenvalues:

$$Y|y, \tau_3\rangle = y|y, \tau_3\rangle, \qquad T_3|y, \tau_3\rangle = \tau_3|y, \tau_3\rangle. \tag{408}$$

For example, the left-handed u and d-quarks are represented by states

$$|1/6, 1/2\rangle = |u_L\rangle, \qquad |1/6, -1/2\rangle = |d_L\rangle, \tag{409}$$

whilst the right-handed quarks are represented by states

$$|2/3, 0\rangle = |u_R\rangle, \qquad |-1/3, 0\rangle = |d_R\rangle. \tag{410}$$

Similarly, for the leptons

$$\begin{array}{ll} |-1/2, 1/2\rangle = |\nu_L\rangle, & |-1/2, -1/2\rangle = |e_L\rangle, \\ |0, 0\rangle = |\nu_R\rangle, & |-1, 0\rangle = |e_R\rangle. \end{array} \tag{411}$$

The various isospin multiplets can be characterized by the *total isospin* quantum number τ with values $\tau = 0$ for iso-singlets, $\tau = 1/2$ for iso-doublets, $\tau = 1$ for iso-triplets, etc., such that there are $(2\tau + 1)$ possible eigenvalues of τ_3 within a given multiplet:

$$\tau_3 = \tau, \tau - 1, ..., -\tau. \tag{412}$$

Thus we should actually characterize the Hilbert-space vectors corresponding to the various quarks and leptons by three numbers; $|y; (\tau, \tau_3)\rangle$, with for example

$$|1/6; (1/2, 1/2)\rangle = |u_L\rangle, \qquad |1/6; (1/2, -1/2)\rangle = |d_L\rangle, \qquad \text{etc.} \tag{413}$$

Whilst quarks and leptons are members of iso-singlets ($\tau = 0$) or iso-doublets ($\tau = 1/2$), the vector bosons of the weak interactions (in the massless limit) form an iso-singlet B_μ coupling to hypercharge, and an iso-triplet W_μ coupling to isospin.

The isospin operator T_3 can now be complemented by conjugate operators $T_\pm = T_\mp^\dagger$, which change the isospin eigenvalue τ_3 within the same isospin-multiplet: they can turn a u_L-quark into a d_L-quark and vice-versa, and similarly a neutrino ν_L into a charged lepton e_L, without changing the value of the hypercharge or the total isospin:

$$T_+|y, (\tau, \tau_3)\rangle = c_+(\tau, \tau_3)|y, (\tau, \tau_3 + 1)\rangle, \qquad T_-|y, (\tau, \tau_3)\rangle = c_-(\tau, \tau_3)|y, (\tau, \tau_3 - 1)\rangle. \tag{414}$$

Here $c_\pm(\tau,\tau_3)$ are co-efficients determined by the normalization of the state vectors:

$$c_+(\tau,\tau_3) = \frac{1}{2}\sqrt{(\tau(\tau+1) - \tau_3(\tau_3+1)}, \qquad c_-(\tau,\tau_3) = \frac{1}{2}\sqrt{(\tau(\tau+1) - \tau_3(\tau_3-1)}. \quad (415)$$

We now explain these rules. First, because T_3 is diagonal simultaneously with Y, these operators commute:

$$[T_3, Y] = 0. \quad (416)$$

Also, the raising and lowering operators $T_\pm$ do not change the hypercharge; in mathematical language:

$$[T_\pm, Y] = 0. \quad (417)$$

Thus we can consider a subspace of states with fixed eigenvalue y of the hypercharge, which we will assume from now on. Next, $T_\pm$ raise or lower the eigenvalue τ_3 of T_3 by one unit, which is equivalent to the commutation relations

$$[T_3, T_\pm] = \pm T_\pm. \quad (418)$$

To see this, note that (ignoring the hypercharge)

$$\begin{aligned} T_3\left(T_\pm|\tau,\tau_3\rangle\right) &= \left([T_3,T_\pm] + T_\pm T_3\right)|\tau,\tau_3\rangle \\ &= T_\pm\left(\pm 1 + T_3\right)|\tau,\tau_3\rangle = (\tau_3 \pm 1)\left(T_\pm|\tau,\tau_3\rangle\right). \end{aligned} \quad (419)$$

Therefore the states $T_\pm|\tau,\tau_3\rangle$ are proportional to the eigenstates $|\tau,\tau_3\pm 1\rangle$, as stated by eqs. (414). The co-efficients (415) have been chosen such that the remaining commutation relation reads

$$[T_+, T_-] = \frac{1}{2}\,T_3, \quad (420)$$

whilst the states $|\tau,\tau_3\rangle$ are normalized:

$$\langle\tau,\tau_3|\tau,\tau_3\rangle = 1. \quad (421)$$

First observe, that if we first raise and then lower the eigenvalue of T_3 by one unit, or the other way around, we get back the original eigenstate; therefore

$$\begin{aligned} \left(T_+T_- - T_-T_+\right)|\tau,\tau_3\rangle &= \left(c_-(\tau,\tau_3)c_+(\tau,\tau_3-1) - c_+(\tau,\tau_3)c_-(\tau,\tau_3+1)\right)|\tau,\tau_3\rangle \\ &= \tfrac{1}{2}\,\tau_3|\tau,\tau_3\rangle, \end{aligned} \quad (422)$$

in agreement with (420). Finally

$$\langle\tau,\tau_3|T_\pm T_\mp|\tau,\tau_3\rangle = c_\mp(\tau,\tau_3)c_\pm(\tau,\tau_3\mp 1)\,\langle\tau,\tau_3|\tau,\tau_3\rangle = |c_\mp(\tau,\tau_3)|^2, \quad (423)$$

provided all states are normalized as in eq. (421).

Exercise E.1
Define the operators

$$T_1 = T_+ + T_-, \qquad T_2 = -i\,(T_+ - T_-)\,.$$

a. Show that these operators are hermitean.
b. Prove they satisfy the commutation relations

$$[T_i, T_j] = i\varepsilon_{ijk}\,T_k, \qquad i,j,k = (1,2,3).$$

Now consider a state with maximal τ_3, i.e. a state $|\tau, \tau\rangle$; such a state is special as

$$c_+(\tau,\tau) = 0 \quad \Rightarrow \quad T_+|\tau,\tau\rangle = 0. \tag{424}$$

Therefore, as anticipated, we can not raise the value of τ_3 above the value τ, and states $|\tau,\tau\rangle$ are called *highest weight* states. If $\tau \neq 0$, we can however lower the τ_3-value of the state by applying T_- :

$$c_-(\tau,\tau) = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad T_-|\tau,\tau\rangle = \frac{1}{\sqrt{2}}\,|\tau,\tau-1\rangle. \tag{425}$$

This procedure can be continued until $\tau_3 = -\tau$; then

$$c_-(\tau,-\tau) = 0 \quad \Rightarrow \quad T_-|\tau,-\tau\rangle = 0. \tag{426}$$

For any given (integer or half-integer) value τ we can thus construct precisely $2\tau+1$ states in which τ_3 takes all values $(\tau, ..., -\tau)$.

Exercise E.2
Define

$$\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2 = 2T_+T_- + 2T_-T_+ + T_3^2.$$

a. Prove:

$$\left[T_\pm, \mathbf{T}^2\right] = \left[T_3, \mathbf{T}^2\right] = 0.$$

Conclude, that $\mathbf{T}^2$ and T_3 can be diagonalized simultaneously.
b. Show that

$$\mathbf{T}^2|\tau,\tau_3\rangle = \tau(\tau+1)|\tau,\tau_3\rangle,$$

for any value of τ_3; it follows that the eigenvalues of $\mathbf{T}^2$ are fully determined by the total isospin τ.
c. From the result in (a), show that raising or lowering τ_3 by application of $T_\pm$ does not change the total isospin τ.

A trivial example of multiplets of isospin states is the singlet $|0,0\rangle$. As both coefficients $c_\pm(0,0) = 0$, there are no other states in the multiplet. All right-handed quarks and leptons are represented by singlet states.

The simplest non-trivial example is the isospin-doublet with $\tau = 1/2$, consisting of the states

$$|1/2, 1/2\rangle, \quad |1/2, -1/2\rangle, \tag{427}$$

examples of which are provided by the left-handed quarks and leptons. If the Dirac fields of left-handed quarks are represented by a 2-component vector of quark-spinors:

$$\Psi(x) = \left(\begin{array}{c} \chi_u(x) \\ \chi_d(x) \end{array} \right), \tag{428}$$

then the isospin operators act on these components as

$$T_+ = \frac{1}{2} \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad T_- = \frac{1}{2} \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad T_3 = \frac{1}{2} \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \tag{429}$$

Finally, a triplet such as the W-bosons in the massless (non-broken) phase of the weak isospin interactions form a three-dimensional isospin-vector

$$W_\mu = \left(\begin{array}{c} W_\mu^+ \\ W_\mu^0 \\ W_\mu^- \end{array} \right), \tag{430}$$

with the isospin operators represented in matrix notation by

$$T_+ = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad T_- = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \quad T_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right). \tag{431}$$

F. Feynman diagrams

Relativistic quantum field theory (QFT) describes both particles and their interactions, in a single unified formalism. In QFT the particles of matter —such as electrons, quarks or neutrinos— are quanta of Dirac fields, with specific energy and momentum proportional to the frequency and wave-vector of the plane-wave solutions, as in exercise 22.2. The interactions between these particles can be described by the exchange of bosons, such as photons, gluons and massive weak vector bosons. In addition, bosons can also interact with each other; for example, some weak vector bosons carry electric charge and interact with photons, and gluons carry color charge and interact among themselves. In this formalism the scattering of decay of particles can become quite complicated processes, involving the exchange of many intermediate particles.

A major contribution to QFT was made by Richard Feynman. He considered the probability amplitude for a system starting in a quantum state $|i\rangle$ to finish in a quantum state $|f\rangle$, such that the associated transition probability is of the form

$$P_{i\to f} = |\langle f|S|i\rangle|^2 \,. \tag{432}$$

Feynman showed that the transition (or scattering) matrix element $\langle f|S|i\rangle$ can be computed by adding the contribution of all possible ways that the process can take place; this procedure is known as the *sum over histories.* A way to visualize the various contributions to a certain process is to draw the various particle-exchange schemes that can lead from the initial state to the final state.

To illustrate this idea, let us first consider the theory of quantum electrodynamics (QED). The lightest and most common charged particles in nature are electrons and their anti-particles, the positrons. The scattering of two electrons can be represented schematically by fig. f.1, where time runs from left to right.

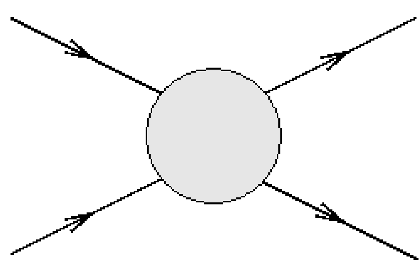

Fig. f.1: Scattering of two electrons; time runs from left to right.

Charged particles like electrons or positrons are represented by solid lines carrying an arrow which indicates the direction of flow of (negative) charge. Representing photons by wavy lines, we can draw explicit contributions to the scattering process; the simplest ones are shown in fig. f.2:

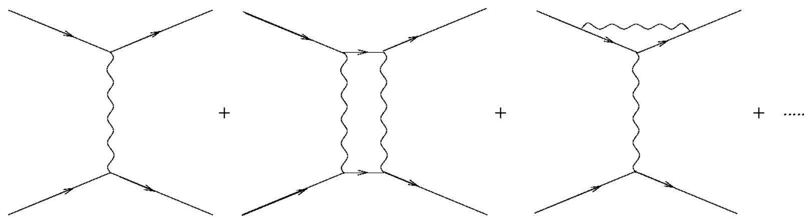

Fig. f.2: Scattering of two electrons by virtual photon exchange.

A characteristic property of QED is, that charged particles can emit, absorb or exchange only one photon at a time. Thus at any interaction point —or *vertex*— in the diagrams one incoming charged particle, one outgoing charged particle and one photon meet. No interaction points involving two or more photons are possible. This holds also for photons by themselves: photons carry no electric charge, so they don't couple directly to other photons.

Each of the diagrams in fig. f.2 represents a specific mathematical expression describing a contribution to the matrix element. Every vertex where a photon couples to a charged particle contributes a corresponding factor of the electron charge e to the term; indeed in the limit that the electric charge vanishes, there is no longer an interaction of the particle with photons, and the expression corresponding to the diagram must vanish. By simple book-keeping of these factors, we observe that the first diagram in fig. f.2 is proportional to e^2, whereas the second and third diagram are proportional to e^4. There are also diagrams of order e^{2n} for any natural number n. Actually, the factors e^2 are accompanied by sufficiently many powers of $c, \hbar, ...$, etc, to turn the diagrams into an expansion in the dimensionless parameter α, the fine-structure constant, with a numerical value close to $1/137$. This implies, that the successive terms in powers of α tend to become less and less significant. In many cases it is sufficient to consider only the contribution of the simplest diagrams, proportional to low powers of the fine-structure constant α.

The diagrams in fig. f.2 can also represent the scattering of other charged particles, e.g. the scattering of an electron with a muon or a quark. Of course the corresponding expressions in the matrix element will be changed, e.g. by replacing whenever appropriate the mass of the electron by the mass of the muon, or the electron charge $-e$ by the quark charge $2e/3$ or $-e/3$.

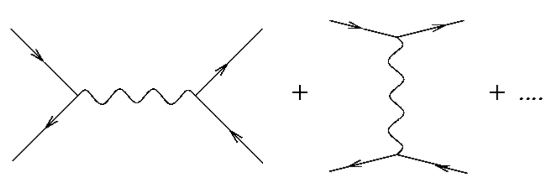

Fig. f.3: Scattering of an electron and a positron by virtual photon exchange.

A new effect arises in the scattering of an electron with a positron, illustrated in fig. f.3. The positron is represented by a solid line with a reversed arrow, indicating that the charge flows in the oposite direction compared to the electron. Only the contributions of order e^2 are shown; these include one more diagram than in the case of electron-electron scattering.

The reason is, that an electron and a positron can annihilate, creating an intermediate state with one or more virtual photons, which then produce an electron-positron pair again in the final state. Such an annihilation diagram is not possible in electron-electron scattering.

In some processes the first diagram f.3 can also represent the *only* contribution of order e^2. For example, an electron and a positron may annihilate, producing an intermediate state with one or more photons, and then produce a μ^+-μ^--pair. Fig. f.4 shows the e^2- and some of the e^4-contributions.

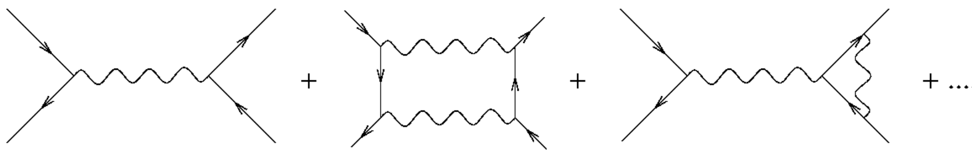

Fig. f.4: Electron-positron annihilation creating a muon and an anti-muon.

Finally, an electron and a positron can also annihilate to create two or more photons; the lowest-order contribution to photon pair-creation is shown in fig. f.5.

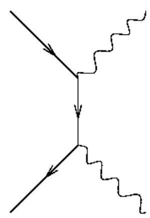

Fig. f.5: Electron-positron annihilation creating a photon pair.

The particles which appear in the intermediate state are not observed; they only exist as a possible intermediate state in the sum over histories. But no history is the *real* history in any classical sense: in the evolution of a quantum state *all* histories contribute. Therefore the particles in the intermediate states are refered to as *virtual* particles.

In quantum chromodynamics (QCD) the rules are quite similar to those of QED, but with an important difference: gluons carry a color and an anti-color charge, and therefore the interaction of quarks with gluons changes the color charge of the quark.

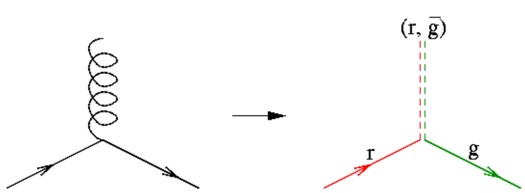

Fig. f.6: Quark-gluon vertex.

Fig. f.6. shows the general quark-gluon vertex, and a specific instance where a red quark interacts with a (red, anti-green)-gluon to become a green quark. The same vertex can also represent other color-charge combinations. As they carry color and anti-color charges, the gluons can also interact among themselves. In fact there are two different gluon-gluon vertices, with three or four gluons interacting at a point, as illustrated in fig. f.7:

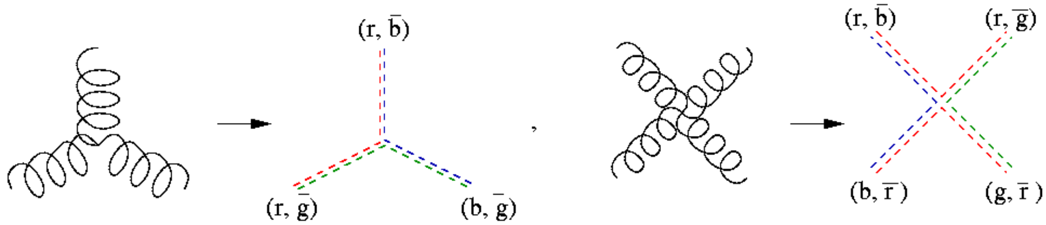

Fig. f.7: Gluon-gluon vertices.

In each case we have drawn one particular combination of color charges flowing into or out of the vertex, but this combination represents only one of several possible ones. As a result of the gluon-gluon interactions, there are contributions to quark-quark scattering by gluons which have no equivalent in the case of QED, such as the one drawn in fig. f.8:

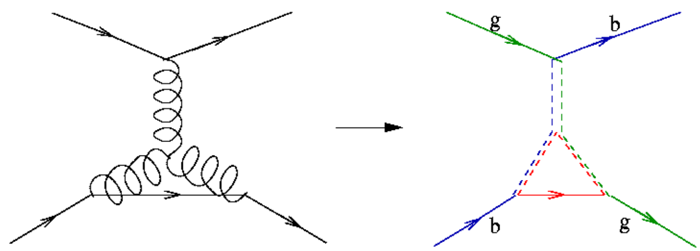

Fig. f.8: Contribution to quark-quark scattering from QCD.

Similar effects exist in the theory of weak interactions, although in that case it is not the color charge of the particles which changes, but the flavor, i.e. the particle type. For example, a muon can change into a muon neutrino while emitting a charged W-boson; in turn this boson can produce an electron and anti-neutrino, thus explaining β-decay. The lowest-order contribution to this process is the one shown in fig. f.9.

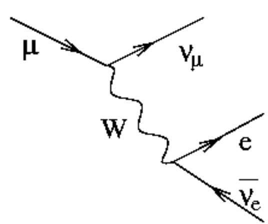

Fig. f.9: β-decay of the muon.