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A short introduction to noncommutative geometry

This talk gives an elementary introduction to the basic ideas of noncommutative geometry *as a mathematical theory*, with some remarks on possible physical applications. Concepts will be emphasized and technical details avoided

Contents

1. 'Noncommutative manifolds'

Mathematicians define an n-dimensional manifold (or differentiable manifold) in an intrinsic way; here it is enough to think of something described in a 'smooth' way by systems of local coordinates $x^1, \dots, x^n$. Obvious examples are the 2-dimensional surface of a sphere, or of 4-dimensional space-time in general relativity.

There are various objects living on a manifold. The first and simplest of these are the infinitely differentiable functions. If $\mathcal{M}$ is a manifold, then the system of all such functions is called $C^\infty(\mathcal{M})$. Functions can be added and multiplied pointwise: $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. This means that $C^\infty(\mathcal{M})$ is an *algebra*, in fact a *commutative algebra*. (For remarks on the use of the word 'algebra' in mathematics and physics, see the Notes).

There are other useful geometrical objects on $\mathcal{M}$, and it is interesting and of great importance for what follows that these objects can all be defined in terms of the algebra $C^\infty(\mathcal{M})$.

The first such geometrical notion that comes to mind is that of a vector field. For us a vector field (a contravariant vector fields, as physicists say) is a system of functions $X^k(x)$, for $k = 1, \dots, n$, and $x = x^1, \dots, x^n$, which under change of local coordinates transform in a certain way. A vector field gives a differential operator acting on functions:

$$X^1 \frac{\partial}{\partial x^k} + \cdots + X^n \frac{\partial}{\partial x^n}.$$

This is a linear operator from $C^\infty(\mathcal{M})$ into itself – let me also call it X. It satifies, as an additional property, the Leibniz relation

$$X(fg) = X(f)\, g + f\, X(g),$$

for every pair f and g of functions.

Theorem: The vector fields, or rather the corresponding differential operators, are precisely the derivations of $C^\infty(\mathcal{M})$.

The situation is the same for other geometric objects on $\mathcal{M}$, such as differential forms, general tensor fields, etc.. They can all be defined purely algebraically from $C^\infty(\mathcal{M})$, first the vector fields, as indicated, then from these and $C^\infty(\mathcal{M})$ the general tensor fields. The additional structures that one may wish to put on the manifold, like for instance a Riemannian (metric) tensor with its curvature tensor, connections and covariant derivatives, as in general relativity and Yang-Mills field theory, can also be defined and studied in this algebraic context, without ever mentioning the manifold $\mathcal{M}$ itself or its points. Vector bundles, an important geometrical notion that I do not explain here, similarly have a purely algebraic definition.

That the algebra $C^\infty(\mathcal{M})$ indeed characterizes the manifold $\mathcal{M}$ completely, is shown by the following theorem:

Theorem: Two manifolds $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic (= the same as manifolds) if and only if the algebras of functions $C^\infty(\mathcal{M})$ and $C^\infty(\mathcal{N})$ are isomorphic (= the same as algebras).

Conclusion: *All the differential geometric properties of a manifold $\mathcal{M}$ are encoded in the algebra $C^\infty(\mathcal{M})$, the commutative algebra of the infinitely differentiable functions on $\mathcal{M}$. As soon a one has the algebra $C^\infty(\mathcal{M})$, the manifold $\mathcal{M}$ itself becomes superfluous. This is illustrated by the following diagram:*

manifold $\mathcal{M}$

$\downarrow$

algebra of functions $C^\infty(M)$

$\downarrow$

all geometric objects:
vector fields
differential forms
general tensor fields
various vector bundles
also:
Riemannian metric
connection, covariant derivative
curvature tensor

The diagram expresses the fact that differential geometry of a manifold $\mathcal{M}$ can be based on a commutative algebra, the algebra $C^\infty(\mathcal{M})$ of its infinitely differentiable functions. Let me therefore introduce here the terms ‘commutative differential geometry’ and ‘commutative manifold’. This leads to a suggestion for a natural generalization to ‘noncommutative differential geometry’ and ‘noncommutative manifold’:

Consider, instead of the commutative algebra $A = C^\infty(\mathcal{M})$, an algebra $\hat{A}$, *noncommutative*, but in some way or another similar A, for instance obtained from $C^\infty(\mathcal{M})$ by some deformation procedure. Use it to draw the following extended diagram, which illustrates the basic idea of noncommutative geometry:

$$\begin{array}{ccc}
\textbf{manifold } \mathcal{M} & & \text{`a noncommutative manifold' ?} \\
\downarrow & & \text{`}\downarrow\text{' ?} \\
\textbf{algebra } A = C^\infty(\mathcal{M}) & \longrightarrow & \textbf{a noncommutative algebra } \hat{A} \\
\downarrow & & \downarrow \\
\begin{array}{l}\textbf{all geometric}\\ \textbf{objects on } \mathcal{M}\textbf{,}\\ \textbf{obtained algebraically}\\ \textbf{from } A = C^\infty(\mathcal{M})\end{array} & \longrightarrow & \begin{array}{l}\textbf{objects,}\\ \textbf{in a similar way}\\ \textbf{obtained algebraically}\\ \textbf{from the algebra } \hat{A}\end{array}
\end{array}$$

In the right-hand column one mimicks the various definitions, based on $A = C^\infty(\mathcal{M})$ in the left-hand side. This turns out be a fruitful idea; many of the ordinary geometric definitions still make sense in this noncommutative algebraic context. The definition of vector field as derivations of $\hat{A}$ works quite well, as do the definitions of differential forms with an exterior derivative, with most of the usual properties. The same is true for general tensor fields. Vector bundles and covariant derivatives appear in this approach in a particularly transparent form. Etc., etc..

One gets in this manner 'noncommutative vector fields, differential forms, tensorfields, vector bundles. An obvious question is: "vector fields, etc., but on what space?" It would be nice if one could answer: "on a noncommutative manifold". However, such a space *does not exist.* There is no underlying manifold; strictly speaking, the vector fields, etc., only make sense in a purely algebraical world.

Conclusion: *Strictly speaking, 'noncommutative manifolds' do not exist; it is an imaginary but nevertheless very fruitful notion which gives intuitive guidance for interesting work on certain algebra structures which geralizes those from ordinary differential geometry.*

2. A fundamental theorem

This section will discuss a classical theorem in functional analysis (= 'infinite dimensional linear algebra'), which provides us in a rigorous way with what is the basic pedagogical example of a 'noncommutative space'. It was proved by Gelfand and Naimark, two Russian mathematicians, in the late 1940s, when 'noncommutative geometry' as such did not yet exist – the term was introduced much later by the French mathematician Alain Connes (more about him later on). It appears in each introductory exposition of noncommutative geometry; this talk will be no exception.

In order to state the theorem, I need two concepts:

1. Topological space: There is a precise mathematical definition as a set in which a system of subsets is specified which are by definition 'open', but here it is enough to think of a space for which the notions of convergence, limits and continuity are meaningful. A topological space may be deformed, stretched, but not torn. The surface of a sphere and that of a cube in three-dimensional space are the same as topological space. The obvious general example for our purpose is n-dimensional Euclidean space, or some part of if, with angles and distances irrelevant. Topological spaces may be *compact*, a property which always simplifies matters. I dont give the general definition; it is enough to know that in n-dimensional Eucliden space compact means closed (= containing all its limit points) and bounded. The continuous complex-valued functions on a topological space X form a commutative algebra, which I denote as $C(X)$.

2. C^*-algebra:

An (associative) algebra is a vector space in which there is in addition an associative multiplication between elements, with has some obvious properties. (Note that when physicists speak of an 'algebra' they usually mean a Lie-algebra, which is a different sort of object). I call an algbra 'abstract', when it consists of elements which are not specified further. Examples of 'concrete' algebras: the (commutative) algebra of functions, i.e. $C^\infty(\mathcal{M})$ or $C(X)$, and the (noncommutative) algebra of all n by n matrices. In a normed algebra elements a have a 'lenght', or *norm*, denoted as $||a||$, which has properties such as the triangle inequality. A norm defines a notion of convergence of sequence of elements; a normed algebra is called *complete* when all Cauchy sequences converge. A complete normed algebra is called a *Banach algebra*. In an *involutive* or $*$-*algebra* each elements a has a 'hermitian adjoint' a^*, with obvious properties as $(a^*)^* = a$ and $(ab)^* = b^*a^*$. Finally, an involutive Banach algebra with $||a^*a|| = ||a||^2$, for each element a, is called a C^*-*algebra*.

All this enables me to state the theorem:

Theorem (Gelfand-Naimark):

1. The algebra of continuous functions $C(X)$ of a compact topological space X is a commutative C^*-algebra, in which the maximum of a function f is its norm $||f||$, and the complex conjugate $\bar{f}$ is its hermitian adjoint f^*.

2. Given a complex commutative C^*-algebra A, one can construct a unique compact topological space X, such that A can be identified with the function algebra $C(X)$.

This means that there is a one-to-one correspondence between compact topological spaces and complex commutative C^*-algebras. Part 1 is obvious; it is part 2 that is nontrivial. The theorem implies that all the information about a compact topological space is encoded in its algebra of functions. The space itself can be recovered in a certain way from the algebra. Note that this result is stronger than that for manifolds: there one does not yet know precisely what kind of 'abstract' algebra gives a manifold.

Conclusion: *Studying commutative C^* algebras amounts to studying compact topological spaces, and vice versa.*

From the general point of view of noncommutative geometry, explained in the preceding section, it is natural to suggest:

Studying noncommutative C^*-algebras amounts to studying 'noncommutative compact topological spaces'.

One can illustrate this by drawing a scheme similar to the second diaram in the preceding section:

a compact **topological space** X		'a noncommutative topological space' ?
$\downarrow\uparrow$ **(GN-theorem)**		'$\downarrow$' **?**
a commutative C^***-algebra** $A = C(X)$	$\longrightarrow$	**a noncommutative** C^***-algebra** $\hat{A}$
$\downarrow$		$\downarrow$
various topological **objects on** X**,** **obtained algebraically** **from** $A = C(X)$ **(e. g.** K**-theory)**	$\longrightarrow$	**objects,** **in a similar way** **obtained algebraically** **from the algebra** $\hat{A}$ **(e. g.** K**-theory)**

Again, stricly speaking, like in the case of manifolds, 'noncommutative topological spaces do not exist, but the idea is they nevertheless a useful intuitive guide in the study of general noncommutative C^*-algebras. This happens, for instance, in parts of algebraic topology where one has something called 'K-theory', which becomes nicer when considered in a general noncommutative context. (See the section Notes for a general remark on the use of the term 'topology')

3. Various approaches

The term 'noncommutative geometry' is now generally associated with the work of Alain Connes, and rightly so. However, before discussing this, I want to look briefly at other approaches, or realizations of the same basic idea.

We all are familiar with an important physical theory that has been with us for a long time, and which is a perfect example of noncommutative geometry in the sense explained in the previous sections, namely **quantum mechanics**.

Classical mechanics has a *phase space*. For a system of N nonrelativistic particles this is just $6N$-dimensional Euclidean space, but for other situations, for example, particles moving on a surface, or under certain constraints, it can be a nontrivial differential manifold. This is a special type of manifold, it is *symplectic*, a notion that I do no explain here, but which leads to special coordinates, the canonical positions and momenta q^j and p_j. The physical observables, energy, angular momentum, etc., are functions of these coordinates. So we have a manifold $\mathcal{M}$, the phase space, and an algebra $A = C^\infty(\mathcal{M})$, the observables.

The corresponding quantum theory is obtained by defining operators in a Hilbert space $\mathcal{H}$, first operators $\hat{Q}^j$, $\hat{P}_j$, satisfying Heisenberg's commutation relations, then the other observables, as operator expressions in these basic observables by the prescription $f(p,q) \to \hat{f}(\hat{q},\hat{p})$ – as much as this is possible. This means that we have instead of the commutative algebra of classical observables $A = C^\infty(\mathcal{M})$ a noncommutative algebra of quantum observables $\hat{A}$, consisting of operators in $\mathcal{H}$. With these data the same type of diagram as before can be drawn:

phase space (a symplectic manifold $\mathcal{M}$ with canonical coordinates p_j, q^j)		'noncommutative phase space ?
$\downarrow$		'$\downarrow$' ?
physical observables (the algebra $A = C^\infty(\mathcal{M})$ of functions $f(p,q)$ on $\mathcal{M}$)	$\longrightarrow$	**physical observables** (the algebra $\hat{A}$ of operators in a Hilbert space $\mathcal{H}$)
$\downarrow$		$\downarrow$
classical mechanics (with time evolution and symmetries)	$\longrightarrow$	**quantum mechanics** (with time evolution and symmetries)

Note that the physical observables in quantum mechanics are represented by *hermitian operators – selfadjoint* is the more precise mathematical term. These do not form an algebra; the product of two hermitian operators is not hermitian, unless they commute. The algebra $\hat{A}$ in the right-hand column of the diagram is therefore the complex algebra of all operators, in which the hermitian operators are embedded. For the consistency of the diagram there also has to be a complex algebra in the left-hand column; the algebra $A = C^\infty(\mathcal{M})$ consists in this scheme of all infinitely differentiable complex-valued functions on $\mathcal{M}$, with the real-valued functions as the subalgebra of physical observables. Note also that I glossed over the distinction between bounded and unbouded operators.

The naive idea of 'quantization', the construction of a quantum mechanical theory from a given classical theory, is simple: classical expressions $f(p_j, q^j)$ lead in quantum mechanics to analogous operator expressions $\hat{f}(\hat{P}_j, \hat{Q}^j)$. Moreover, Poisson brackets $\{f, g\}$ become commutators $-i\hbar\,[\hat{f}, \hat{g}]$. However this simple prescription is ambiguous. The operators $\hat{P}_j$ and $\hat{Q}^k$ do not all commute, so the prescription which assigns an operator expression to a function of the p_j, q^k, say a polynomial, is not well-defined. Various choices are possible, which lead to different quantizations of a given classical theory. Note that the relation between the classical Poisson bracket and the quantum commutator is more complicated than the naive one-to-one relation mentioned above, which is only the lowest order term in a power series in $\hbar$. e.g. by symmetrization (Weyl quantization), or Moyal quantization, a notion that I do not explain here. These two quantizations make sense only for simple linear phase spaces. Since the beginning of quantum theory people have looked for a unambigous quantization procedure, and for a classification of these different procedures.

From the 1970s onwards, the 'Dijon school' of Flato, Sternheimer, and others, together with the mathematician Lichnerowicz (Paris) developed an approach to this problem, called ***deformation quantization***. They start from a phase space, a general symplectic (or Poisson) manifold, together with the commutative algebra of classical observables $C^\infty(\mathcal{M})$ and try to construct a noncommutative algebra of quantum observables, by defining in $C^\infty(\mathcal{M})$ *as a vector space* a new noncommutative product, a so-called *star-product*, not surprisingly denoted as $*$. This construction uses power series in $\hbar$. The $*$ is defined as a power series, with the zeroth order term the 'classical' commutative product and the first order term containing the classical Poisson bracket. Flato and many others have worked on this; have shown that star products exists in various cases and have given explicit constructions and also classifications in term of cohomology theory. (See a brief explanation of the notion of cohomology in the Section Notes). The result is an algebra $\hat{A}$ which is the same as $A = C^\infty(\mathcal{M})$ as a vector space, but different as an algebra. The quantum algebra $\hat{A}$ still has to be represented as an algebra of operators in some Hilbert space $\mathcal{H}$. This aspect has sofar has not been discussed in a fully satisfactory manner.

In the last few years have seen a broad development of deformation quantization as a purely mathematical theory, separated however completely from the questions originally posed by physics.

There have been approaches to noncommutative geometry by **John Madore** and **Giovanni Landi**, who both have written books on noncommutative geometry in general and their own work more in particular. (See the Section References for this). **Julius Wess**, one of the inventors of supersymmetry and supergravity, has developed an approach in which the coordinates x^μ of space-time are deformed, with $[x^\mu, x^\nu] = 0$ being transformed in $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$.

Finally a well-known early realization of the idea of noncommutative geometry, which is noncommutative in a very minimal sense, should be mentioned: **supermanifolds**, in supersymmetry and supergravity. The algebra $\hat{A}$ of 'functions' on a supermanifold is a so-called *supercommutative algebra* or *superalgebra*, which differs from a commutative algebra only by the appearance of minus signs. Whether in this case the underlying supermanifold exists as a set of points, like in commutative geometry, is a matter of discussion.

4. Connes' version of noncommutative geometry

Alain Connes (1947):

- Professor at the Collège de France, Paris
- Permanent staff member of the IHES
  (Institut des Hautes Études Scientifiques, Bûres-sur-Yvette, near Paris)
- Fields Medal in 1982, for his work on the theory of operator algebras

Alain Connes is an outstanding, extremely productive mathematician, with a deep interest and a considerable knowledge of modern physics. He is also an enthousiastic and stimulating lecturer.

The theory of operators in Hilbert space was developed in the late 1920s and early 1930s by John von Neumann, in order to understand the mathematical basis of quantum mechanics. It has since then grown into an ever expanding and lively field of mathematics, which has far outgrown its physical origin.

Connes' involvement in Hilbert space operator theory, together with a long-standing interest in quantum mechanics, has led him, over a period of more than twenty years, to develop a formulation of differential geometry in terms of commutative algebras, which lends itself to a noncommutative generalization. There have been earlier versions of the general idea of noncommutative geometry, as I indicated in the preceding section, but Connes coined the term and gave a version which by now dominates all others by its originality and mathematical richness.

Two introductory remarks:

1. Geometry is for Connes 'metric geometry', i.e. the study of manifolds with a Riemannian structure given by a metric tensor $g_{\mu\nu}$. (Think of space-time in general relativity).

2. The probabilist and mathematical physicist Mark Kac, Lorentz-professor at this Institute in 1963, used to give a talk called: “Can one hear the shape of a drum?”. In this he explained that important properties of the shape of a bounded two-dimensional surface could be derived from the assymptotic behaviour of the discrete eigenvalues of the Laplace operator, assuming certain boundary conditions.

In a certain way Connes has adopted this idea, found an interesting analogue and carried it further. He considers manifolds of arbitrary dimension, with a Riemannian structure, which give rise to a first order differential operator, the Dirac oprator. He shows that the manifold, including the metric tensor, can be completely reconstructed from the discrete eigenvalues of this operator. Note that the manifold has to be compact, a serious restriction to which I shall return later. He encodes the properties of the spectrum in a mathematical object, called by him a *spectral triple*, which contains a number of algebraic data, and which completely describes the Riemannian manifold. This means a formulation of ordinary Riemannian geometry as a ‘commutative Riemannian geometry’.

Commutative spectral triples

Consider an n-dimensional compact manifold $\mathcal{M}$ with Riemannian metric, given by a metric tensor g, or $g_{\mu\nu}$ in local coordinates. Under very weak additional condition there is a so-called *spin structure*. This means that there is a space of ‘spinor fields’ $\psi(x)$ on $\mathcal{M}$ (mathematicians would say ‘sections of a spinor bundle’), on which the *Dirac operator* acts, in local coordinates the differential operator $D = i\gamma_\mu \partial x^\mu$, with the γ_μ the Dirac γ-matrices, which satisfy the well-known condition $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$. Note that in the general case both the $g_{\mu\nu}$ and the γ_μ are x-dependent. There is a natural inner product for spinor fields, which makes the space of this fields into a Hilbert space $\mathcal{H}$, in which D is a selfadjoint (= ‘hermitian’) operator. Spinor fields in $\mathcal{H}$ can be muliplied by functions on $\mathcal{M}$. In this manner $C^\infty(\mathcal{M})$ is represented by, or can be seen as an algebra A of operators in $\mathcal{H}$.

Connes calls the system consisting of $\mathcal{H}$, the algebra $A = C^\infty(\mathcal{M})$ and the Dirac operator D the *spectral triple* $(\mathcal{H}, A, D)$ *associated with the Riemannian manifold* $(\mathcal{M}, g)$. He exhibits a number of algebraic properties of this system, characterizing the various characteristics of the manifold and the metric and puts them in a list of seven properties. He then proves a theorem, that might be called the ‘*Gelfand-Naimark theorem for compact Riemannian manifolds and spectral triples*’:

Theorem:

1. For every compact Riemannian manifold $(\mathcal{M}, g)$ there is an associated spectral triple $(\mathcal{H}, A, D)$ as defined above.

2. For every spectral triple $(\mathcal{H}, A, D)$, with $\mathcal{H}$ a Hilbert space, A a commutative algebra of operators in $\mathcal{H}$, and D a linear operator in $\mathcal{H}$, satisfying the seven properties, there exist a unique compact manifold $(\mathcal{M}, g)$, such that $(\mathcal{H}, A, D)$

is the spectral triple associated with $(\mathcal{M}, g)$. Moreover both the manifold $\mathcal{M}$ and the metric tensor can be constructed in an explicit way from $(\mathcal{H}, A, D)$.

It will be clear that 2 is the nontrivial part of the theorem. I shall not say anything on the proof of this, except that it is hard and very complicated.

The notion of spectral triple, as an ‘abstract’ algebraic notion, can be generalized to a version in which one has a noncommutative algebra $\hat{A}$.

Noncommutative spectral triples

Definition: A ‘noncommutative’ spectral triple is a system $(\mathcal{H}, \hat{A}, D)$ consisting of the following objects:

- a Hilbert space $\mathcal{H}$,
- a noncommutative algebra $\hat{A}$ of operators in $\mathcal{H}$,
- a selfadjoint operator D in $\mathcal{H}$,

satisfying a list of seven axioms, the properties mentioned earlier.

Such a system, according to Connes, is a ‘noncommutative’ geometry. The intuitive idea behind this point of view can be illustrated by a diagram of the type that I have drawn before:

Riemannian manifold $(\mathcal{M}, g)$		‘noncommutative Riemannian manifold’ ?
$\downarrow \uparrow$ (‘GN-theorem’)		‘$\downarrow$’ ?
spectral triple $(\mathcal{H}, A, D)$ (commutative algebra A)	$\longrightarrow$	**spectral triple** $(\mathcal{H}, \hat{A}, D)$ (noncommutative algebra $\hat{A}$)

Connes, in joint work with Giovanni Landi, Michel Dubois-Violette and others, has constructed several examples of noncommutative geometries in this sense, for instance so-called spherical manifolds, noncommutative deformations of S^3 and S^4, the spheres in 3- and 4-dimensional space.

5. Physical applications. Further outlook

The main attempts at physical application of noncommutative geometry, by Connes and others, have been in **relativistic quantum theory** and **particle physics**. The idea behind this is that noncommutative geometry might give a better understanding of space-time at the micro-scale.

The most serious problem of quantum field theory, singular behaviour and divergencies, manifests itself at very short distances. This suggests that there is something basically wrong with our picture of space-time, as a 4-dimensional Riemannian manifold. Intuitively clear as this is, it could nevertheless be wrong

Noncommutative geometry, in the version of Connes, but also in those of Madore, Landi, Wess and others, has tried to give a more adequate description. However, it has not been easy to turn this natural and appealing idea into concrete and realistic theoretical proposals.

Together with John Lott, Connes has developed an ingenious formulation of the **Standard Model** of elementary particle physics in the framework of noncommutative geometry, later developed further by the 'Marseille school', Thomas Schücker, Daniel Kastler and others.

All this is interesting, but one has to admit that up till now it has not given enough new insights to make a real impact. The same is true for the general algebraic formalism for perturbative quantum field theory developed by Connes together with Dirk Kreimer.

In 1997 Connes wrote, together with the mathematician Albert Schwarz and the string theorist Micheal Douglas, a paper on the relation between noncommutative geometry and M theory, a branch of **string theory** (Alain Connes, Michael R. Douglas and Albert Schwarz: Noncommutative Geometry and Matrix Theory: Compactification on Tori, hep-th/9711162). This introduced noncommutative geometry into the string world and stimulated quite a few papers there. Connes himself did not follow this up; probably because he has no great interest in string theory.

Connes' version of noncommutative geometry has also been applied on a very different subject, the **quantum Hall effect**, independently by Jürg Fröhlich and Jean Bellissard. More recently, Leonard Susskind has also applied noncommutative geometry, in a somewhat different spririt, to the quantum Hall effect.

Further outlook

Noncommutative geometry, based on Connes' ideas, is a very lively mathematical field of interest, with many connections to other parts of mathematics. In its implications for physics it remains a tantalizing idea, which definitely will not go away.

6. What I am trying to do myself

What I am trying to do in noncommutative geometry is mathematical work that is motivated by physics.

For applications in relativistic field theory and elementary particle physics, Connes' noncommutative geometry has at this moment two serious drawbacks:

1. It only makes sense for *compact* spaces.

2. Connes only considers *Euclidean* Riemannian differential geometry, i.e. with a positive-definite metric tensor. This reflects the interests of mathematicians in general. In most mathematics books there is very little on what is often called pseudo-Riemannian manifolds. Of course, compact is always easier than noncompact, and Riemannian easier than pseudo-Riemannian. However physics, in particular relativity, requires noncompact manifolds with an indefinite metric tensor. (There are compact manifolds with an indefinite metric. The 2-dimensional torus, which is of course compact, can easily be provided with a Minkowskian metric. However there would not be acceptable notion of causality; time would go in circles. This happens for all compact pseudo-Riemannian manifolds).

In the beginning Connes considered this a technical problem that eventually would solve itself; by now it is generally seen as a serious question. Almost nothing has been done on it.

My aim is tackle this problem. I hope to make a few very small steps in the direction of a solution.

A Riemannian (or pseudo-Riemannian) manifold $(\mathcal{M}, g)$ is a composite notion: there is the underlying differentiable manifold $\mathcal{M}$, and added to that one has the Riemannian (or pseudo-Riemannian) structure given by a metric tensor g.

In order to understand noncompactness, in the first place a property of the manifold, and the difference between Riemannian and pseudo-Riemannian, a property of the metric structure superimposed on the manifold, one has to make a distinction between manifold and metric properties. Connes' description of $(\mathcal{M}, g)$ is a 'black box', in which this distinction is very hard to make: it is difficult to see which of the the seven properties have to do with the manifold as such and which with the metric.

My program is therefore as follows:

• Break up Connes' spectral triple.

• Study first topological and then differentiable manifolds; try, as a first step, to prove a 'Gelfand-Naimark theorem for (possibly noncompact) manifolds. To my knowledge such a theorem does not exist, probably because people have not tried very hard, so the use of the proper books and papers and above all some low-brow but hard work in standard point set topology and functional analysis might do the job.

• Try to find a substitute for the Dirac operator. This is the really difficult part, for which new ideas are needed. Connes' orginal, very intuitive idea was that a noncommutative Riemannian manifold should have a 'quantized' length element ds. From quantum mechanics he took the suggestion – in a way that I do not understand – that this quantized distance should be the 'inverse' of the Dirac operator, i.e. its propagator. On a pseudo-Riemannian manifold ds is not a true distance. Something else should take its place. What this should be is unclear at this moment. So indeed, if I ever reach this point I shall need really new ideas.

7. Notes

1. The use of the term 'algebra'

In *mathematics* the term 'algebra' has two different meanings:

1. Algebra as a subject, like analysis, group theory, etc.

2. Algebra as a mathematical object. In this sense an algebra – the simplest definition – is a vector space in which the the elements not only can be added and multiplied by scalars, but can in addition be multiplied with each other. When this multiplication is associative, i.e. $(ab)c = a(bc)$, the algebra is called associative. The algebras of $(n \times n)$ matrices and algebras of functions are associative. The most important example of a nonassociative algebra is a Lie algebra. In that case the product is written as a bracket of two elements.

In *physics*, especially in particle physics / quantum field theory, 'algebra' means very often Lie algebra, or more particularly, a system of commutation relation for a Lie algebra, with respect to a certain system of basis vectors. Note that a statement like 'Such and such field satisfies such and such algebra' would be incomprehensible to a mathematician.

2. The use of the term 'topology'

In *mathematics* the term 'topology, like 'algebra', has two meanings:

1. Topology as a subject. In this *general topology* or *point set topology* is concerned with the basic notions of the subject: definition and general properties of topological spaces, the notions of continuity, convergence, limits, etc.. In Section 2 the word 'topology' is used in this sense. A special part of the subject is *algebraic topology*, which discusses properties of spaces such as connectedness, the number of holes and handles, etc.; *differential topology* investigates such properties in differentiable manifolds.

2. Topological as a mathematical object: The system $\mathcal{T}$ of subsets of a given set X, specification of which makes X into a topological space. The members of $\mathcal{T}$ are by definition 'open'.

In *physics* one uses the term almost always in statements like "the topology of such and such space", meaning the properties of that space according to algebraic topology, i.e. holes, handles, etc.. *Topological field theory* is field theory, in which one does not have, or does not use, a (Riemannian) metric on the underlying manifold, only its differential topological properties.

3. Cohomology

When a speaker in a physics talk uses the term 'cohomology', there is, not without reason, a suspicion that he is trying to impress his audience. To avoid this I give here a very short explanation of this important mathematical notion, even though it is not needed in this talk.

The 'Ur' cohomology is the *de Rham cohomology.* Let $\mathcal{M}$ be an n-dimensional manifold, with (local) coordinates $x^1, \ldots, x^n$. On $\mathcal{M}$ there is a system of differential forms. A 0-form is just a function $f(x)$, a 1-form is an expression of the form

$$\alpha_1 = \sum_{j=1}^{n} A_j(x)\, dx^j,$$

a 2-form an expression

$$\alpha_2 = \frac{1}{2} \sum_{j_1=1, j_2=1}^{n} A_{j_1 j_2}(x)\, dx^{j_1} \wedge dx^{j_2},$$

and generally a k-form

$$\alpha_k = \frac{1}{k!} \sum_{j_1=1,\ldots,j_k=1}^{n} A_{j_1,\ldots,j_k}(x)\, dx^{j_1} \wedge \ldots \wedge dx^{j_k}.$$

The component functions $A_{j_1 \ldots j_k}$ are antisymmetric in the indices $j_1 \ldots j_k$; they form what is called by physicists an antisymmetric covariant tensor field. For $k > n$ the k-forms are identically 0, because of this antisymmetry.

There is an *exterior derivative* d which maps k-forms into $(k+1)$-forms. It is nilpotent, i.e. $d^2 = 0$. In terms of components this derivative acts on a 0-form as a gradient, $(df)_j = \partial_j f$, on a 2-form as $(d\alpha_1)_{j_1 j_2} = \partial_{j_1} A_{j_2} - \partial_{j_2} A_{j_1}$, etc.. (There is an elegant formulation of Maxwell equations in terms of differential forms, with the relativistic 4-potential $A_j(x)$ a 1-form and the field tensor $F_{jk}(x)$ a 2-form.)

A form α is called *closed* if $d\alpha = 0$ and *exact* if there is a form β such that $\alpha = d\beta$. Because of the nilpotency of d, an exact form is necessarily closed. The converse is in general not true: a closed form need not to be exact. The space of exact forms is a subspace of the space of closed forms. Whether it is equal or stricly smaller, and 'how much smaller' depends on the properties of the manifold, is in fact characteristic for those properties. The study of this is the aim of cohomology theory, in this case de Rham cohomology theory.

From this example in differential geometry, the idea of cohomology theory, essentially an algebraic notion, has been generalized to a wide range of other mathematical situations and has become a central mathematical concept. The idea of any cohomology theory is always the same: one has a space with a linear operator d with $d^2 = 0$ (nilpotency). The space of elements α with $d\alpha = 0$ contains the space of elements with $\alpha = d\beta$, for some β. Cohomology theory then studies the relation between this spaces. The results usually give characteristics of some other mathematical object. In this way one has for instance cohology of Lie groups and Lie algebras. *Cyclic cohomology*, which characterizes properties of general (associative) algebras, was developed by Connes for noncommutative geometry. It is one of the byproducts of noncommutative geometry that have been widely used in other areas of mathematics.

8. References

BOOKS

The two principal books on Connes' version of noncommutative geometry are the book by Alain Connes himself, and a more recent book by Gracia-Bondia et al.:

Alain Connes
Noncommutative Geometry
Academic Press 1994
ISBN 0-12-185860-X, 667 pages.

This is the 'Principia' of noncommutative geometry, a book for the serious student with a thorough education in mathematics. Even then it is does not make easy reading, for instance, because many proofs are incomplete. To try to read it all is madness. By now it is ten years old, which means that it does not contain Connes' more recent work. Nevertheless, it remains a great book, enormously interesting and stimulating.

José M. Gracia-Bondía, Joseph C. Várilly and Hector Figueroa
Elements of Noncommutative Geometry
Birkhauser 2001
ISBN 0-8176-4124-6, 685 pages.

A complete and rigorous textbook, in a certain way a companion volume to Connes' book. An excellent book, that is more recent, and that in addition seems to have Connes' official blessing. As a textbook it is precise and complete. It also requires a good mathematical background.

Fortunately, there are two introductory books, by John Madore and by Giovanni Landi. Both autors discuss noncommutative geometry in general, and more in particular their own approach. They are readable for physicts familiar with relativistic field theory and explain all the mathematics that they use:

John Madore
An Introduction to Noncommutative Differenial Geometry and its Physical Applications. Second Edition
Cambridge University Press, Cambridge 1999
ISBN 0-521-65991-4 paperback. 321 pages.

Giovanni Landi
Noncommutative Spaces and their Geometry
Lecture Notes in physics
Springer 2002
ISBN 0940-7677.

(The complete book, or rather an earlier version, is available on the preprint archive 'arXiv' as hep-th/9701078)

As far as I know there is no book on deformation quantization, at least not on deformation quantization in the physical context that I discussed in Section 3.

ARTICLES, LECTURE NOTES, ETC.

Joseph C. Várilly: An Introduction to Noncommutative Geometry
gr-qc/9909059

John Madore: Noncommutative Geometry for Pedestrians
gr-qc/9906059

There are various review talks on deformation quantization. In the following two typical examples one finds on the first 5-10 pages useful information on physical and mathematical aspects, together with something on the historical background, but after that they descend into rather inaccessible technicalities:

Daniel Sternheimer
Deformation Quantization: Twenty Years After
math/9809056

S. Gutt
Variations on Deformation Quantization
math/0003107