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PROPERTIES AND CALCULATIONS OF MULTI PARTON PROCESSES

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR AAN DE RIJKSUNIVERSITEIT TE LEIDEN, OP GEZAG VAN DE RECTOR MAGNIFICUS DR. J.J.M. BEENAKKER, HOOGLERAAR IN DE FACULTEIT DER WISKUNDE EN NATUURWETENSCHAPPEN, VOLGENS BESLUIT VAN HET COLLEGE VAN DEKANEN TE VERDEDIGEN OP DONDERDAG 5 OKTOBER 1989 TE KLOKKE 15.15 UUR

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Contents

Chapter I

Outline

The observation of jets at the $P\bar{P}$ collider in CERN by the UA1 and UA2 experiments [1] made it possible to interpret and organize the experimental data in a very clear and direct way. Jets are streams of hadrons moving more or less in the same direction. They can be understood as a result of large angle scattering of partons. These partons are the constituents of hadrons and are either quarks or gluons. Each outgoing parton which originates from such a large angle collision evolves into an observable jet through the mechanism of fragmentation. As long as one is interested in the global properties of jets, such as for example energy content or angular distributions, one can identify the outgoing parton with the jet. This viewpoint gives a very direct interpretation of the theoretical model in terms of experimental observables [2].

The model consists of two distinct parts. The first part describes the properties of the parton within the hadrons, such as momentum distributions of the partons. These distributions are the input parameters for the second part of the model. This describes the actual collision of two individual partons, each from one of the colliding hadrons. The interaction between these partons is described by Quantum Chromo Dynamics (QCD). The description of the partons within the hadron belongs to the realm of nonperturbative QCD, while the actual scattering process is described by perturbative QCD. As one can see the study of jets at the colliders is a direct test of the QCD description of the strong interactions. When one is also interested in the hadron contents of jets one adds to the above model another part, which describes the evolution of the outgoing partons into jets.

Another important reason to study jets is given by the fact that any new physics, such as the detection of the top quark or the Higgs particle, is done by looking at the decay products of these heavy particles. The decay products are leptons and partons, which evolve into jets. The resulting experimental signal can be mimicked by other, more conventional processes which give exactly the same final state [3]. An example is the case where the top quark is heavier than the W mass. It is possible that in a $P\bar{P}$ collision a top quark pair is produced. These quarks do not evolve into jets because of their large mass, instead they decay via electroweak interactions into a final state with jets and/or leptons. Other processes, the background processes, can easily reproduce the same final state. So an important issue is how to look for specific properties of the signal produced by the t[illegible] quarks. For instance, one considers the invariant mass distribution of the observed jets. One can then try to apply cuts on the signal which reduce the background in favour of the top quark signal [4]. Another important example is the search for the Higgs particle. Again the background processes threaten the ability to observe this particle, and all kinds of cuts must be applied to reduce this background with respect to that of the Higgs boson signal [5].

From the above it is clear that calculations of multi jet final states are crucial for the experiments at current and future colliders, especially with the increasing beam energies of the colliders. Any new and interesting physics is hidden in a background of multi jet final states, so it is important to be able to calculate the background processes. Unfortunately the calculations of such background processes are complicated. In this thesis we will study and develop methods to calculate these multi jet final states. In chap. 2 the parton model is described, the application of perturbative calculations is given and the shortcomings of the approach are discussed. Furthermore the difficulties in evaluating the processes are discussed following the historical development of the calculational methods. In chap. 3 we introduce our basic calculational tool, the Weyl-van der Waerden spinor calculus. This technique makes optimal use of the fact that in our calculations all partons are taken massless. This reduces a lot of algebra which is necessary in more conventional spinor calculus. The developed spinor calculus will be applied in chaps. 5, 6, 7 and 8, where the actual calculations of matrix elements are carried out. Chap. 4 introduces the recursion relations. These relations form a new technique, the basis for calculating multi parton matrix elements. In the next four chapters the recursive formulae will be applied to obtain relevant results. Firstly in chap. 5 the so called Parke-Taylor conjecture [6] and related conjectures are proven. The Parke-Taylor conjecture gives the matrix element for the scattering of two gluons to an arbitrary number of gluons for special helicity configurations of the gluons. The validity of this conjecture is important because approximate formulae for multi gluon processes are based on it [7]. The proof is extended to processes involving a quark pair with or without a vector boson. In chap. 6 we examine the soft gluon behaviour of multi parton processes. A number of factorization properties are proven for scattering amplitudes with an arbitrary number of gluons. These proofs make extensive use of the recursive techniques developed in the previous chapters. The subject of chaps. 7 and 8 is the explicit calculation of multi parton helicity amplitudes. Chap. 7 is concerned with n-gluon scattering, $n \leq 8$, while in chap. 8 the process involving a vector boson and up to 5 partons is calculated.

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Chapter II

Introduction

In this chapter we explain in sec. 1 the for this thesis relevant part of the theoretical model for jet phenomena. Also the necessity to apply phase space cuts on the outgoing particles is discussed. These cuts are closely related to the jet definitions and detector properties of the experiment. The second section gives a historical overview of the methods used to calculate multi parton processes. The problems encountered in such calculations will become clear from this survey.

1 The theoretical model

As already explained in the outline the model consists of two parts. The first part describes the parton inside the hadrons [1], while the second part contains the actual scattering of two partons into outgoing particles [2].

We will explain the model by giving the formula for the cross section of a $P\bar{P}$ collision and examine the details by looking at different aspects of this expression. The differential cross section for a $P\bar{P}$ collision at a beam energy $\sqrt{s}$ is given by

$$\frac{d\sigma(s)}{dPS} = \sum_{i,j} \int dx_1 \int dx_2 \; F_i^P(x_1, Q^2) F_j^{\bar{P}}(x_2, Q^2) \frac{d\hat{\sigma}_{ij}(x_1 x_2 s)}{dPS} \,. \qquad (1.1)$$

The measure for the probability of finding a parton of type i with momentum fraction x of the hadron H is given by the structure function $F_i^H(x, Q^2)$. Therefore formula (1.1) describes the collision of two partons, the active partons during a $P\bar{P}$ collision. As can be seen from the formula it is assumed that during the parton parton collision there is no interaction with the other partons, the spectator quarks. This is only justified if the collision takes place at a small time scale or equivalently at a high momentum transfer. This is exactly the type of collisions we want to study, because the jets are a result of high momentum transfer scattering. Only comparison with experiment can justify the above approximation. The summation in eq. (1.1) runs over all types of partons and an integration is performed over all possible momentum fractions.

The quantity $\hat{\sigma}_{ij}(x_1 x_2 s)$ is the parton cross section at the available energy $x_1 x_2 s$ for the partons. Up to a normalization constant, which contains statistical and phase space factors, the parton cross section is given by the matrix element squared denoted by $|\mathcal{M}|^2$. The calculation and properties of the matrix element $\mathcal{M}$ for multi parton final states is the subject of this thesis. The differential dPS stands for the phase space variables of the outgoing particles. The matrix element squared also contains the coupling constant $\alpha_S(Q^2)$. This coupling constant depends, just as the structure functions, on the scale Q^2 at which the parton collision takes place. Since QCD is an asymptotic free theory the coupling constant has the important property that $\alpha_S(Q^2) \longrightarrow 0$ as $Q^2 \longrightarrow \infty$ [3]. Because we are looking at large momentum transfer events, i.e. a large Q^2, we can use perturbative QCD to calculate the parton cross section. At the CERN collider, with $\sqrt{s} = 630$ GeV, it was found that the typical value of $\alpha_S \simeq 0.2$ for "jetty" final states.

To clarify the necessity of phase space cuts we will look at a specific process, the scattering of 2 gluons into $(n-2)$ gluons or in other words n-gluon scattering. The 4-gluon scattering at tree level, i.e. no loops included, is the lowest order process, being of order α_S^2 [4]. To calculate the next order in α_S of the cross section there are contributions coming from 4-gluon scattering, now including loop contributions, and from 5-gluon scattering at tree level approximation [5]. These two contributions are closely related as we will explain. The loop contribution to the 4-gluon scattering will contain divergencies. The ultraviolet divergencies can be absorbed in the coupling constant renormalization. Still there will be divergencies left. These divergencies are related to the collinear and infrared singularities of the 5-gluon matrix element. The divergencies occur when two gluons are collineair or the energy of one of the outgoing gluons becomes zero. In the language of the experiment this would mean two outgoing gluons are inside one jet, the jet is too close to the beam to be observed or a jet is too soft to be observed by the detector. The consequence is that not all 5-gluon scattering events will be observed as a 3-jet event, but rather as a 2-jet final state. Thus a part of the 5-gluon cross section must be added to the 4-gluon loop correction. The result is that the infrared singularities cancel [6] whereas the initial and the final state collineair divergencies remain. The latter type of singularities have to be absorbed in the initial parton distribution function or final state fragmentation function respectively. This procedure goes under the historical name of mass factorization [7]. Finally we are left with a finite answer.

As is clear from the above the resulting cross section will depend on how a jet is defined, for example what is the minimal separation angle between two outgoing gluons to evolve into two separate observable jets. These jet definitions have to be translated into phase space cuts which mimic the experimental situation as well as possible. For instance, one can choose a minimal separation angle between the gluons and a minimal transverse energy for an outgoing gluon. These cuts make a separation in the phase space of 5-gluon scattering. On the one hand we have events which do not fulfil the phase space cuts and are added to the higher order correction of the 2 jet cross section. On the other hand we have events which do fulfil the cuts, adding to the lowest order contribution of the 3 jet cross section. We will be interested in these lowest order tree level contributions to a specific process.

The fact that we limit ourselves to tree level calculations has important consequences for the results obtained from these calculations. In higher order calculations it is generally found that the corrections to tree level calculations only affect the overall normalization of the cross sections and not the shape of distributions. Furthermore the inclusion of higher order contributions reduces the dependence on the scale Q^2. This scale has to be chosen and depends on the process one is considering. It is not always clear what the correct scale choice is. It is found that for the n-gluon scattering the best choice for Q^2 is a constant times the average transverse momentum squared of the outgoing jets. The higher order terms reduce the effect of this scale choice and thus making the answer, especially the normalization, more reliable. When the number of outgoing jets is large this problem becomes acute, because the coupling constant is raised to a high power. For instance for the n-gluon scattering amplitude the coupling constant is given by α_S^{n-2}.

Because we limit ourselves to tree level calculations we must keep this in mind whenever we want to compare the results with experiment. We must always be aware of the fact that the overall normalization is not correct and that the result depends strongly on the choice of the scale Q^2. One can reduce these uncertainties by looking at ratios of cross sections or by comparing differential cross sections with the experimental data with the overall normalization as a free parameter. Let us look at an example of a ratio of cross sections and compare the tree level calculation with the experimental data. We define the quantity

$$f_n(W) = \frac{\sigma(W + n \text{ jets})}{\sum_{m=0}^{\infty} \sigma(W + m \text{ jets})} \tag{1.2}$$

for which the UA1 collaboration has presented experimental values [8]. The cross section $\sigma(W + n \text{ jets})$ stands for the production of a W vector boson in association with n jets. We can compare the experimental results with the theoretical prediction based on the tree level calculations of chap. 8 for n up to three. These results are also given in fig. 2.1 together with the theoretical uncertainty which results from the different scale choices and structure functions [9]. As can be seen from the figure there is a broad agreement between experiment and theory.

To give an idea of the normalization uncertainties of tree level calculations one has introduced the so called K-factor [10]. This factor gives the constant by which the tree level calculation must be multiplied to find the correct answer. The factor depends of course on the specific process under consideration, the chosen scale, beam energies, etc. For example in the case of four jet production at CERN the K-factor found from comparing experiment with the tree level calculation is given by $K = 1.35 \pm 0.08$. Multiplying the tree level predictions with this factor brings the theory in agreement with all kinds of experimental distributions and total event rate for the 4 jet process [11].

The calculation of the higher order terms is complicated, for instance for the n-gluon scattering only the first order correction is known to the 4-gluon process [5]. The calculation of tree level amplitudes can be seen as a first step towards complete calculation of a specific order in the coupling constant. Still the results of the tree level calculations are relevant for the experiment as long as one keeps in mind the above described uncertainties.

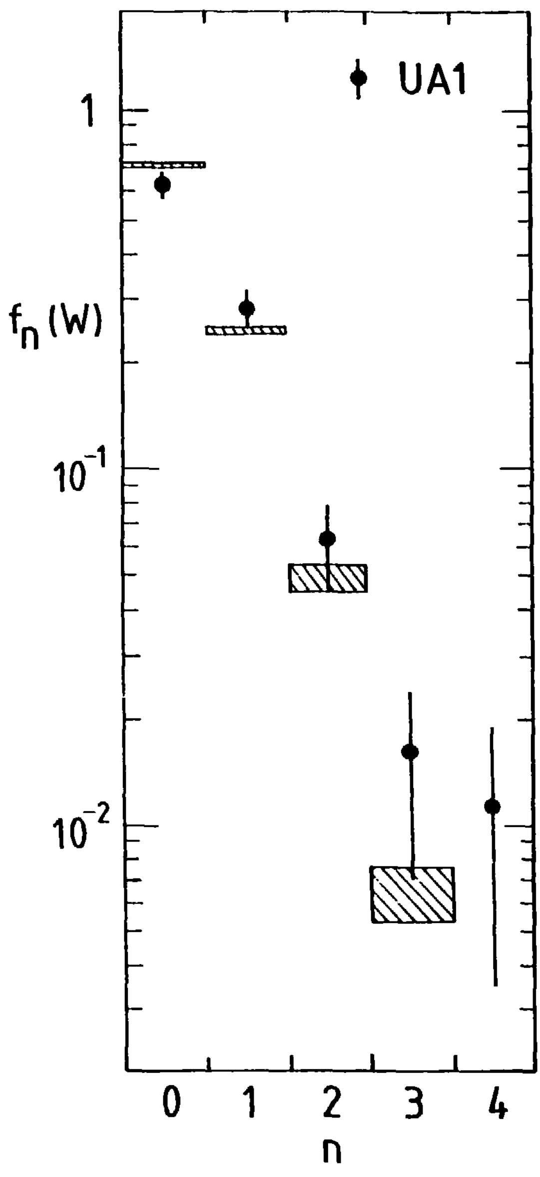


Fig. 2.1. Multiplicity of jets f_n as function of n. The shaded areas give the theoretical predictions (including the uncertainties).

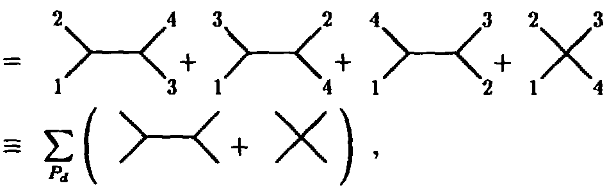
2 A historical review of the calculations

At first sight one could think that the calculation of tree level amplitudes is simply the combining of the vertices given by the Feynman rules into all possible contributing diagrams for a given process. To clarify the problems one encounters we will look at the historical development of the evaluation of n-gluon scattering. As we will see the adding of a parton to this process necessitates the development of new techniques in order to control the algebraic problems. The reasons for this are twofold, first by adding a gluon to a process the number of Feynman diagrams is increased by an order in magnitude [12]. On top of this each diagram in itself will be build up of more and more vertices. Each vertex in itself is a complicated object, for instance the 3-gluon vertex contains six terms. So putting these vertices together to construct a diagram gives an explosive growth in the number of terms contained in each diagram. Looking at these vertices another problem becomes clear. This is that apart from a space-time part, i.e. the momenta and metric tensors, the vertices also contain a colour part. These are a consequence of the non-abelian character of the theory. Controlling these colour structures, which increases in complexity with the adding of partons, is an important part of the problem.

We will now look at the history of the calculation of n-gluon scattering. For pure partonic processes the n-gluon matrix element is the process with the largest number of Feynman diagrams. Furthermore it consists only of 3- and 4-gluon vertices. As such it is a key process. If one succeeds to calculate the n-gluon process, the processes where one replaces gluons with quarks are easily obtained by applying the same techniques developed for the calculation of the pure gluonic amplitude.

The first calculation was 4-gluon scattering [4] as early as 1978. These calculations where motivated to explain high transverse momentum production of hadrons at the $P\bar{P}$ collider (one assumed the transverse momentum could be high enough to apply pertubative QCD). The 4-gluon matrix element consists of four diagrams

$\mathcal{M}(1,2,3,4) =$



(2.1)

where the sum runs over all different permutations of the external legs of the diagram. The calculation was done with the standard spin summation technique. This means one calculates $|\mathcal{M}|^2$ by summing over all the polarizations of the external states. So one has to consider $4 \times 4 = 16$ terms at one time. The spin and colour averaged matrix element squared turns out to be

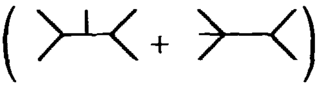
$$|\mathcal{M}(1,2,3,4)|^2 = \frac{9g^4}{2}\left(3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{st}{u^2}\right) \tag{2.2}$$

with s, t and u the Mandelstam variables of a $2 \to 2$ scattering process. The gauge group was choosen to be $SU(3)$ and g is given by the relation

$$\alpha_S = \frac{g^2}{4\pi} \ . \tag{2.3}$$

A complication in calculating the matrix element squared with the above method is the occurence of non-physical longitudinal components of the gluons. These components have to be removed. The usual method is the so called ghost substraction, i.e. by adding some new Feynman rules for ghosts the longitudinal components are removed.

In 1980 the 5-gluon scattering relevant for 3 jet production was calculated [13]. The finding of 3-jet events at PETRA [14] was a strong stimulus for these calculations. From the discovery of jets at the e^+e^- colliders one realized that at the $P\bar{P}$ colliders one should also see jets with the increased energy in the forthcoming experiments at CERN. The methods used in ref. [13] are the same as for the 4-gluon case, only now 25 diagrams are contributing

$$\mathcal{M}(1,2,3,4,5) = \sum_{P_d}$$

$. \qquad (2.4)$

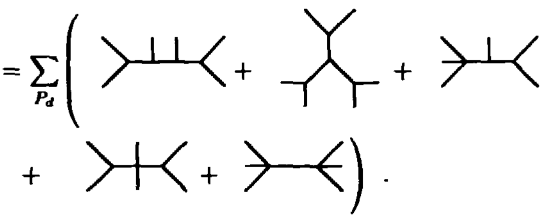
So for the evaluation of $|\mathcal{M}|^2$ one had to consider $25 \times 25 = 625$ complicated terms at the same time. Moreover a diagram with three 3-vertices contains $6^3 = 216$ terms, so the square of such a diagram contains $216^2 = 46656$ separate terms. Of course a lot of terms will be the same, but it is clear from the above numbers that one had to rely on algebraic manipulation programs to perform the calculation. As a result the final answer is cumbersome and long, but it can be fitted into a computer program in order to apply it phenomenologically. It is obvious one could not proceed this way for the calculation of 6-gluon scattering. One year later, in 1981, the CALCUL collaboration turned their attention to the 5-parton scattering amplitudes and found compact expressions [15]. These results can be most easily obtained by using helicity amplitudes and an appropriate choice of polarization vectors [16]. We will give the essential points of the method here, in chap. 3 we will come back to it in more detail because it is one of the basic tools to calculate multi gluon matrix elements. The helicity method consists of contracting all external gluon states in the Feynman diagrams with their corresponding helicity vector. The first consequence is that one can evaluate and simplify the matrix element before squaring (looking at 25 diagrams is easier than considering 25^2 squared diagrams). The second advantage is that no ghosts are needed for removal of longitudinal components of the gluon polarization since the helicity states are transverse. The last point is that one still has some freedom to choose the helicity vectors. By making clever choices, i.e. make a specific gauge choice, one can reduce the number of contributing diagrams (some diagrams will give a zero result in the specific chosen gauge) and the number of terms in each diagram can be reduced considerably. With the helicity method it is not necessary to use an algebraic manipulation program because of the enormous reduction in the number of terms one has to manipulate. By using the helicity formalism a very compact answer is readily obtained for the spin and colour averaged squared matrix element of 5-gluon scattering

$$|\mathcal{M}(1,2,3,4,5)|^2 = \frac{g^6}{4}\left(\frac{N^3}{N^2-1}\right)\left(\sum_{i=1}^{4}\sum_{j=i+1}^{5}(K_i \cdot K_j)^4\right) \times \left(\sum_{P(1234)} \frac{1}{(K_1\cdot K_2)(K_2\cdot K_3)(K_3\cdot K_4)(K_4\cdot K_5)(K_5\cdot K_1)}\right), \quad (2.5)$$

where N is the number of colours.

The finding of jets at the $P\bar{P}$ collider in CERN was a strong motivation to calculate the four jet final state. It is interesting both as a test for QCD and as a background process to new physics. The 6-parton matrix elements were calculated [17,18] using the improved CALCUL helicity formalism [19] (i.e. a specific phase for the polarization vectors is used) and supersymmetric relations between processes with different partons [17]. The number of diagrams for the 6-gluon matrix element is 220 and the matrix element is given by

$$\mathcal{M}(1,2,3,4,5,6) = \sum_{P_d}\Bigg(\quad\Bigg) . \quad (2.6)$$



The improved CALCUL method and supersymmetry were not sufficient to find a compact and presentable analytic expression. One could write computer programs fast enough for phenomenological applications. Considering that the next step, 7-gluon scattering, consists of 2485 diagrams it was clear this method could not be extended to more partons. Not even if one limites oneself to the writing of a computer program which calculates the cross section numerically.

The solution to obtain compact answers lies, somewhat surprisingly, in the colour structure of the process [20]. The 6-gluon amplitude calculated in this manner will be discussed in detail in chap. 7. The treatment of the colour structure in the matrix element can be done very systematical by projecting each colour structure on the so called trace base. The resulting form of the n-gluon amplitude is

$$\mathcal{M}(1,2,\ldots,n) = 2ig^{n-2}\sum_{P(12\cdots n-1)}^{(n-1)!} Tr(T^{a_1}\cdot T^{a_2}\cdot\cdots\cdot T^{a_n})\mathcal{C}(1,2,\ldots,n)\,, \quad (2.7)$$

as will be explained in chap. 4. The sum runs over all permutations of labels 1 through $(n-1)$. The generator of the $SU(N)$ gauge group in the fundamental representation is given by T^{a_i}. Its labels a_i represent the colour of the corresponding gluon. The function $\mathcal{C}(1,2,\ldots,n)$ is called the subamplitude and does not contain any colour factors. This subamplitude possesses a number of properties which makes it an interesting and fundamental object to study and calculate. First of all the subamplitude is cyclic invariant and has a reflective property, this in contrast to the amplitude which is invariant under all permutations of the gluons. The most important property is that the subamplitude is gauge invariant i.e. one is free to choose the polarization of the external gluons. Another nice feature of the colour decomposition in eq. (2.7) is that it is orthogonal up to leading order in the number of colours N, by which we mean that the amplitude squared, summed over the number of colours of the gluons is given by

$$|\mathcal{M}(1,2,\ldots,n)|^2 = \left(\frac{g^2 N}{2}\right)^{n-2} \left(N^2-1\right) \left(\sum_{P(12\cdots n-1)} |\mathcal{C}(1,2,\ldots,n)|^2 + \mathcal{O}\left(\frac{1}{N^2}\right)\right) . \tag{2.8}$$

Sc instead of calculating the full amplitude one removes the colours of the diagrams by projecting them on the trace base of eq. (2.7) and calculate the subamplitude by combining the contributing Feynman diagrams. This reduces the number of diagrams to be evaluated greatly. For instance for 6-gluon scattering we only have to evaluate $\mathcal{C}(1,2,3,4,5,6)$ which consists of 32 diagrams instead of the 220 diagrams of the full amplitude. Furthermore, as can be expected from the factorial summation in eq. (2.7), the number of contributing Feynman diagrams to a subamplitude will only grow slowly with increasing number of external gluons compared with the explosive growth of diagrams for the full amplitude.

From this point on progress was made in two directions. Firstly through recursion relations [21] which describes the exact amplitude by recursivity in the number of gluons, the subject of chap. 4. Secondly one has developed approximative formulae for n-gluon cross sections [22]. Though the results for the approximations are promising, they lack the ability to estimate the error and it is not clear how to obtain improvements in the approximation, i.e. it is not a truncated series for which the full series converges to the exact answer.

This concludes the historical overview for n-gluon scattering. As has become clear from the above it is far from trivial getting numbers out of the Feynman rules which can be compared with the experiment. For adding an additional gluon it is necessary to develop a new technique to control the algebraic problems.

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Chapter III

The helicity method

The helicity method for evaluating an amplitude will be discussed in this chapter. In the first section we will look at the historical development of the helicity formalism. The Weyl-van der Waerden method explained in sec. 2 is a next step along this line. The Weyl-van der Waerden implementation of the helicity method will be used extensively in chaps. 5, 6, 7 and 8 for the explicit calculation of helicity amplitudes.

1 Introduction

The most important aspect of the helicity method is the fact that it evaluates a matrix element on amplitude level, i.e. one acquires a numerical result for the amplitude before squaring. This is accomplished by introducing helicity states for the fermions as well as the vector bosons. In ref. [1] the helicity states for the fermions were introduced in order to reduce an amplitude with a fermion line, including its spinors and γ-matrices, to a simpler expression with momenta and polarization vectors. After this the squaring to obtain the cross section was easily performed and short expressions were found. The method was elaborated in ref. [2] and extended to amplitudes with more fermion lines. The extention to spin-$\frac{3}{2}$ fermions for the above method was given in ref. [3]. The polarization vectors for the vector bosons were introduced in ref. [4]. One made a special choice for these polarization vectors within the frame one was working. The authors of ref. [5] related the helicity vectors of massless vector bosons to other outgoing momenta in the amplitude under consideration. For instance, in the process

$$e^+(P_+) + e^-(P_-) \longrightarrow \mu^+(Q_+) + \mu^-(Q_-) + \gamma(K) \tag{1.1}$$

one could choose the helicity vectors of the photon γ to be equal to

$$\not{\epsilon}^{\pm} = \frac{1}{4} \frac{\left[\not{P}_+ \not{P}_- \not{K}\left(1 \mp \gamma_5\right) - \not{K} \not{P}_+ \not{P}_-\left(1 \pm \gamma_5\right)\right]}{\sqrt{(P_+ \cdot P_-)(P_+ \cdot K)(P_- \cdot K)}} . \tag{1.2}$$

This helicity vector has numerous advantages, one of which is that a number of Feynman diagrams will become zero which dramatically simplifies the calculations.

The result of this development is the powerful CALCUL method [6] for obtaining compact expressions of helicity amplitudes. This method was extended to massive spin-1 and spin-$\frac{3}{2}$ particles in ref. [7]. Later the CALCUL method was considerably improved by several authors [8]. The helicity vector in this improved CALCUL scheme depended only on one arbitrary light cone vector instead of the two external vectors of the original CALCUL method. The helicity vector is given by

$$\epsilon^{\mu}_{\lambda}(K) = \frac{1}{2}\frac{\bar{u}_{\lambda}(K)\gamma^{\mu}u_{\lambda}(P)}{\sqrt{P\cdot K}} \tag{1.3}$$

with K_μ the momentum of the spin-1 particle and P_μ an arbitrary light cone vector. This gauge four-vector can be choosen to be equal to one of the external vectors, which reduces the calculation enormously. Another line of development was the use of Weyl spinors [9,10,11,12]. The use of these Weyl spinors leads to considerable simplification in calculations involving a lot of γ-matrix algebra. In the next section we shall merge these two lines of development into one very powerful scheme, which uses both the advantages of the improved CALCUL method and the Weyl spinor method [12].

2 The Weyl-van der Waerden implementation

In this section we will use the Weyl-van der Waerden spinor calculus [13] to implement the helicity formalism. One advantage of the formalism is that the quantities concerning the space time part of an amplitude which appear in a calculation, i.e. momenta, spinors and helicity vectors, are expressed using only one object, the Weyl spinor. Another advantage is the absence of gamma matrices in the expressions. The often complicated relations between these matrices one has to use in order to simplify the expressions are replaced by simple rules for the Pauli matrices and the Weyl spinors. A possible disadvantage of the Weyl-van der Waerden formalism is that it is strictly confined to four dimensions. For the applications in this thesis this is no impediment.

This section is divided into four parts. The first subsection introduces the Weyl spinors and their calculational rules. In the second subsection the momenta are translated into the Weyl-van der Waerden formalism. Special attention is given to light cone vectors, being the momenta belonging to massless particles. The third and fourth subsection describes the spin-$\frac{1}{2}$ and spin-1 particles respectively in the language of the Weyl-van der Waerden formalism.

2.1 The formalism

The Weyl spinor, ψ_A, is a two dimensional complex vector. Its complex conjugate is denoted by $\psi_{\dot{A}}$. Define the spinorial inner product between two Weyl spinors as

$$\psi_{1A}\varepsilon^{BA}\psi_{2B} = \psi_{1A}\psi_2^A = \langle\psi_1\psi_2\rangle \tag{2.1}$$

and for its complex conjugates

$$\psi_{1\dot{A}}\varepsilon^{\dot{B}\dot{A}}\psi_{2\dot{B}} = \psi_{1\dot{A}}\psi_2^{\dot{A}} = \langle\psi_1\psi_2\rangle^* \,. \tag{2.2}$$

The matrix ε is defined as

$$\varepsilon_{AB} = \varepsilon^{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.3}$$

The only relations between the spinorial inner products which are used in the calculations are consequences of the following two identities for the ε-symbol

1. Antisymmetry : $\varepsilon^{AB} = -\varepsilon^{BA}$,
2. Schouten identity: $\varepsilon^{AB}\varepsilon^{CD} + \varepsilon^{AC}\varepsilon^{DB} + \varepsilon^{AD}\varepsilon^{BC} = 0$.

This last relation is a direct result from the fact that the spin space of the Weyl spinors is two dimensional. The two above identities give rise to two identities of the spinorial inner products. Contracting the first identity with $\psi_{1A}\psi_{2B}$ results in

$$\langle\psi_1\psi_2\rangle = -\langle\psi_2\psi_1\rangle , \tag{2.4}$$

in particular

$$\langle\psi_1\psi_1\rangle = 0 . \tag{2.5}$$

Similarly the Schouten identity leads to

$$\langle\psi_1\psi_2\rangle\langle\psi_3\psi_4\rangle + \langle\psi_1\psi_3\rangle\langle\psi_4\psi_2\rangle + \langle\psi_1\psi_4\rangle\langle\psi_2\psi_3\rangle = 0 . \tag{2.6}$$

These two simple rules are all the spinor calculus which is necessary to manipulate the Weyl spinors. Of course sometimes one needs relations in which not all labels are summed away. The Schouten identity then leads to equations like

$$\langle\psi_1\psi_2\rangle\psi_{3A} + \langle\psi_2\psi_3\rangle\psi_{1A} + \langle\psi_3\psi_1\rangle\psi_{2A} = 0 , \tag{2.7}$$

and

$$\psi_1^A\psi_2^B - \psi_1^B\psi_2^A = \varepsilon^{BA}\langle\psi_1\psi_2\rangle . \tag{2.8}$$

2.2 The momentum vectors

The first step is the translation of momentum vectors into the Weyl-van der Waerden formalism. To start the derivation take a general momentum vector with real components,

$$K^\mu = \left(K^0, K^1, K^2, K^3\right) = \left(K^0, \mathbf{K}\right) \tag{2.9}$$

with K^0 the energy component and (K^1, K^2, K^3) the components of the spatial momentum vector $\mathbf{K}$. This is translated in the spinor formalism by the relation

$$K_{\dot{A}B} \equiv \sigma^\mu_{\dot{A}B} K_\mu = \begin{pmatrix} K_0 + K_3 & K_1 + iK_2 \\ K_1 - iK_2 & K_0 - K_3 \end{pmatrix} , \tag{2.10}$$

where σ^0 is the unit matrix and σ^i are the Pauli matrices. From the definition of ε in eq. (2.3) we can raise the indices

$$K^{\dot{A}B} = \begin{pmatrix} K_{\dot{2}2} & -K_{\dot{2}1} \\ -K_{\dot{1}2} & K_{\dot{1}1} \end{pmatrix} = \begin{pmatrix} K_0 - K_3 & -K_1 + iK_2 \\ -K_1 - iK_2 & K_0 + K_3 \end{pmatrix} . \tag{2.11}$$

The complex conjugate of this matrix equals its transposed

$$(K_{\dot{A}B})^* = K_{\dot{B}A} \ .$$

A number of relations for Pauli matrices will turn out to be useful

$$\sigma^{\mu}_{\dot{A}B}\sigma_{\mu}^{\dot{C}D} = 2\delta_{\dot{A}}{}^{\dot{C}}\delta_B{}^D \ . \tag{2.12}$$

$$\sigma^{\mu}_{\dot{A}B}\sigma_{\nu}^{\dot{A}B} = 2\delta^{\mu}_{\nu} \ . \tag{2.13}$$

$$\sigma^{\mu}_{\dot{A}B}\sigma^{\nu\dot{A}C} + \sigma^{\nu}_{\dot{A}B}\sigma^{\mu\dot{A}C} = 2g^{\mu\nu}\delta_B{}^C \ . \tag{2.14}$$

From eq. (2.14) follows

$$\partial_{\dot{A}C}\partial^{\dot{A}B} = \partial_{\mu}\partial^{\mu}\delta^B{}_C \equiv \Box\delta^B{}_C \tag{2.15}$$

and from eq. (2.13)

$$K_{\dot{A}B}P^{\dot{A}B} \equiv \{K, P\} = 2K \cdot P \ . \tag{2.16}$$

Up to now we have used general four-vectors. At the energies of present day colliders the masses of the fermions can often be neglected. Therefore we will look in more detail at the properties of light cone vectors in the Weyl-van der Waerden formalism. A light cone momentum will be translated in a dyad of a Weyl spinor and its complex conjugate. In order to distinguish between a four-momentum and its related spinor the former is denoted by an upper case and the latter by a lower case letter. Eqs. (2.10) and (2.16) hold for any four-vector. For a null vector we have

$$K^2 = K \cdot K = \frac{1}{2}\{K, K\} = 0 \ , \tag{2.17}$$

which means that the determinant of the matrix $K_{\dot{A}B}$ vanishes, and that $K_{\dot{A}B}$ can be written as a dyad of its eigenvectors, which are Weyl spinors

$$K_{\dot{A}B} = k_{\dot{A}}k_B \ . \tag{2.18}$$

The spinor k_A is determined up to a phase factor. Suitable expressions are

$$k_A = e^{i\alpha}\begin{pmatrix} \sqrt{K_0 - K_3}e^{-i\omega/2} \\ \sqrt{K_0 + K_3}e^{+i\omega/2} \end{pmatrix} \tag{2.19}$$

with $e^{i\alpha}$ the arbitrary phase factor and ω given by the relations

$$\cos\omega = K_1/\sqrt{K_1^2 + K_2^2} \ , \ \sin\omega = K_2/\sqrt{K_1^2 + K_2^2} \ , \tag{2.20}$$

or a more for numerical applications useful form

$$k_A = e^{i\alpha}\begin{pmatrix} (K_1 - iK_2)/\sqrt{K_0 - K_3} \\ \sqrt{K_0 - K_3} \end{pmatrix} \tag{2.21}$$

with again an arbitrary phase factor. With this procedure all light cone vectors can be decomposed in corresponding momentum spinors. The inner product between two light cone vectors $K_{1\mu}$ and $K_{2\mu}$ can be written as

$$2K_{1\mu}K_2^{\mu} = 2K_1 \cdot K_2 = \{K_1, K_2\} = |\langle k_1 k_2\rangle|^2 \ . \tag{2.22}$$

Thus the spinor inner product can be considered as the (complexified) square root of the Minkowski inner product.

2.3 The spin-$\frac{1}{2}$ particles

One could start directly with a Lagrangian in terms of Weyl spinors, but to keep the derivations in a more familiar description, the usual Dirac Lagrangian is translated into the spinor notation. To do this take the γ-matrices in the Weyl representation, thus

$$(\gamma^{\mu})^{a}{}_{b} \;=\; \begin{pmatrix} 0 & -i\sigma^{\mu}_{\dot{B}A} \\ i\sigma^{\mu\dot{A}B} & 0 \end{pmatrix} , \tag{2.23}$$

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.24}$$

The Dirac spinor, its adjoint, and helicity projections in this representation are given by

$$\Psi = \begin{pmatrix} \psi_A \\ \phi^{\dot{A}} \end{pmatrix} , \tag{2.25}$$

$$\overline{\Psi} = \Psi^{*} \cdot \gamma_0 = i \begin{pmatrix} \phi^{A} & -\psi_{\dot{A}} \end{pmatrix} , \tag{2.26}$$

and

$$\Psi_{+} = \frac{1}{2}(1+\gamma_5)\Psi = \begin{pmatrix} \psi_A \\ 0 \end{pmatrix} ,$$

$$\Psi_{-} = \frac{1}{2}(1-\gamma_5)\Psi = \begin{pmatrix} 0 \\ \phi^{\dot{A}} \end{pmatrix} . \tag{2.27}$$

Using the translations

$$\overline{\Psi} \cdot \Psi = i \left(\psi_A \phi^A - \psi_{\dot{A}} \phi^{\dot{A}} \right) , \tag{2.28}$$

$$\overline{\Psi} \cdot \gamma^{\mu} \cdot \Psi \; A_{\mu} = \psi_{\dot{A}} A^{\dot{A}B} \psi_B + \phi_{\dot{A}} A^{\dot{A}B} \phi_B , \tag{2.29}$$

the Dirac Lagrangian

$$\mathcal{L}^{(\frac{1}{2})} = \frac{i}{2} \left(\overline{\Psi} \cdot \gamma^{\mu} \cdot (\partial_{\mu}\Psi) - (\partial_{\mu}\overline{\Psi}) \cdot \gamma^{\mu} \cdot \Psi \right) - m \, \overline{\Psi} \cdot \Psi \tag{2.30}$$

transforms into the Weyl Lagrangian

$$\begin{aligned} \mathcal{L}^{(\frac{1}{2})} \;=\; & \frac{i}{2} \left(\psi_{\dot{A}} \partial^{\dot{A}B} \psi_B - \psi_B \partial^{\dot{A}B} \psi_{\dot{A}} \right) \\ + \; & \frac{i}{2} \left(\phi_B \partial^{\dot{A}B} \phi_{\dot{A}} - \phi_{\dot{A}} \partial^{\dot{A}B} \phi_B \right) \\ - \; & im \left(\psi_A \phi^A - \psi_{\dot{A}} \phi^{\dot{A}} \right) . \end{aligned} \tag{2.31}$$

We will look at massless fermions, the equations of motion for massless spin-$\frac{1}{2}$ particles is given by

$$\partial^{\dot{A}B} \psi_B = \partial^{\dot{A}B} \phi_B = 0 . \tag{2.32}$$

In momentum space this leads to

$$K^{\dot{A}B} \tilde{\psi}_B = K^{\dot{A}B} \tilde{\phi}_B = 0 , \tag{2.33}$$

with $\tilde{\psi}_B$ and $\tilde{\phi}_B$ the Fourier transforms of ψ_B and ϕ_B respectively. After multiplying eq. (2.32) with $\partial_{\dot{A}C}$ and using eq. (2.15) one finds

$$\Box\psi_C = \Box\phi_C = 0 \Rightarrow K^2\tilde{\psi}_C = K^2\tilde{\phi}_C = 0 \; . \tag{2.34}$$

So, because the fermion is taken massless its corresponding momentum vector is a light cone vector. We can now use eq. (2.18) in eq. (2.33). This leads to

$$k^{\dot{A}}\langle\tilde{\psi}k\rangle = k^{\dot{A}}\langle\tilde{\phi}k\rangle = 0 \; . \tag{2.35}$$

Therefore $\tilde{\psi}_B$ and $\tilde{\phi}_B$ are proportional to k_B. For a proportionality factor of one we find as solutions of the massless Dirac equation (2.32)

$$\tilde{\psi}_A = \tilde{\phi}_A = k_A \; . \tag{2.36}$$

This normalization choice leads to

$$\tilde{\psi}_{\dot{A}}\tilde{\psi}_B = \tilde{\phi}_{\dot{A}}\tilde{\phi}_B = k_{\dot{A}}k_B = K_{\dot{A}B} \; , \tag{2.37}$$

and from eq. (2.27) follows

$$\begin{aligned} u_+(K) &= v_-(K) = \begin{pmatrix} k_B \\ 0 \end{pmatrix} \\ u_-(K) &= v_+(K) = \begin{pmatrix} 0 \\ k^{\dot{A}} \end{pmatrix} \; . \end{aligned} \tag{2.38}$$

The normalization choice is equivalent to the usual Dirac spinor normalization for massless fermions

$$\begin{aligned} \sum_\lambda u_\lambda(K)\bar{u}_\lambda(K) &= \begin{pmatrix} k_B \\ 0 \end{pmatrix}\begin{pmatrix} 0 & -ik_{\dot{A}} \end{pmatrix} + \begin{pmatrix} 0 \\ k^{\dot{B}} \end{pmatrix}\begin{pmatrix} ik^A & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -ik_{\dot{A}}k_B \\ ik^{\dot{B}}k^A & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -iK_{\dot{A}B} \\ iK^{\dot{B}A} & 0 \end{pmatrix} \\ &= \not{K} \; , \end{aligned} \tag{2.39}$$

where we have used eqs. (2.10), (2.18), (2.23) and (2.38). The Weyl spinor k_A is in a sense the square root of the the momentum K and is easy to use in calculations.

2.4 The spin-1 particles

The equation of motion for a free massless spin-1 particle is given by

$$\Box A_\mu - \partial_\mu(\partial \cdot A) = 0 \; . \tag{2.40}$$

In the Lorentz gauge the equation of motion reduces to

$$\begin{aligned} \Box A_\mu &= 0 \; , \\ \partial \cdot A &= 0 \; . \end{aligned} \tag{2.41}$$

This set of equations is easily translated in the spinor formalism by multiplying with $\sigma^{\mu}_{\dot{A}B}$,

$$\begin{aligned} \Box A_{\dot{A}B} &= 0 , \\ \{\partial, A\} &= 0 . \end{aligned} \tag{2.42}$$

Taking the Fourier transform of these equation one has

$$\begin{aligned} K^2 \tilde{A}_{\dot{A}B} &= 0 , \\ \{K, \tilde{A}\} &= 0 , \end{aligned} \tag{2.43}$$

from which follows that the momentum vector is a light cone vector and decomposes according to eq. (2.18) into Weyl spinors. The two degrees of freedom in the spin-1 field will be described in the helicity formalism. The right and left oriented helicity vectors $e_{\pm}$ have the following properties

$$\{e_{\pm}, e_{\pm}\} = 0 , \tag{2.44}$$

$$(e_{\pm})^{\dagger} = e_{\mp} , \tag{2.45}$$

$$\{K, e_{\pm}\} = 0 , \tag{2.46}$$

$$\{e_{\pm}, e_{\mp}\} = -2 , \tag{2.47}$$

where all helicity vectors are translated in the Weyl-van der Waerden formalism using eqs. (2.10) and (2.16). With these four properties one can construct the helicity vectors in terms of Weyl spinors. Property (2.44) prescribes the form

$$\begin{aligned} e_{+\dot{A}B} &= a_{\dot{A}} b_B , \\ e_{-\dot{A}B} &= c_{\dot{A}} d_B . \end{aligned} \tag{2.48}$$

where a_A, b_A, c_A and d_A are arbitrary spinors. Property (2.45) gives $a_A = d_A$ and $b_A = c_A$, thus the helicity vectors become

$$\begin{aligned} e_{+\dot{A}B} &= a_{\dot{A}} b_B , \\ e_{-\dot{A}B} &= b_{\dot{A}} a_B . \end{aligned} \tag{2.49}$$

To fulfil property (2.46) choose $a_A = k_A$. This choice is made (and not $b_A = k_A$) so that the positive helicity vector is given by e_+ . Now the helicity vectors take the form

$$\begin{aligned} e_{+\dot{A}B} &= k_{\dot{A}} b_B , \\ e_{-\dot{A}B} &= b_{\dot{A}} k_B . \end{aligned} \tag{2.50}$$

The normalization property (2.47) gives the condition

$$\langle kb \rangle = \sqrt{2} e^{i\alpha} , \tag{2.51}$$

where $e^{i\alpha}$ is an arbitrary phase factor. This condition is easily solved and gives

$$b_A = \sqrt{2} e^{i\alpha} \frac{g_A}{\langle kg \rangle} . \tag{2.52}$$

The new spinor g_A is called the gauge spinor and may be choosen freely except for k_A itself. Substituting this form of b_A in the helicity vectors results in

$$\begin{aligned} e_{+\dot{A}B}(g) &= \sqrt{2}e^{i\alpha}\frac{k_{\dot{A}}g_B}{\langle kg\rangle}\,, \\ e_{-\dot{A}B}(g) &= \sqrt{2}e^{-i\alpha}\frac{g_{\dot{A}}k_B}{\langle kg\rangle^*}\,. \end{aligned} \tag{2.53}$$

The argument of the helicity vector will often be omitted. Now an additional condition on the helicity vectors will be introduced, the so called phase condition

$$\{e_+(g_1), e_-(g_2)\} = -2\,. \tag{2.54}$$

This additional property is fulfilled when we set the phase factor equal to one, and the final form of the helicity vectors is

$$\begin{aligned} e_{+\dot{A}B}(g) &= \sqrt{2}\frac{k_{\dot{A}}g_B}{\langle kg\rangle}\,, \\ e_{-\dot{A}B}(g) &= \sqrt{2}\frac{g_{\dot{A}}k_B}{\langle kg\rangle^*}\,. \end{aligned} \tag{2.55}$$

The phase property is introduced because one is now free to make another choice of gauge spinor for each gauge invariant subset of diagrams without worrying about the possible phase differences between the subsets of diagrams when one is squaring the amplitude. Making another choice of the gauge spinor is a gauge transformation of the spin-1 field according to

$$e_{+\dot{A}B}(g_2) = e_{+\dot{A}B}(g_1) + \sqrt{2}\frac{\langle g_1 g_2\rangle}{\langle g_1 k\rangle\langle k g_2\rangle}K_{\dot{A}B}\,, \tag{2.56}$$

which is easily shown with the aid of the spinor relation (2.7).

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Chapter IV

The recursion relations

A method is presented in which multi gluon processes are calculated recursively. The technique is explicitly developed for processes where only gluons are produced and processes where in addition to the gluons also a quark-antiquark pair with or without a vector boson are present.

1 Introduction

Cross sections for processes involving a number of partons possibly together with a vector boson like W, Z or a virtual photon are of importance for present and future colliders. Although one may primarily be concerned with hadron colliders, multi jet events are also relevant for e^+e^- and e^-P collisions.

Amongst the parton processes the pure gluon processes play a special role. On the one hand, when one has techniques to calculate this process it is not too difficult to incorporate a quark-antiquark pair with or without a vector boson. On the other hand, in hadron collisions the gluons have the largest parton luminosity and have the largest parton cross sections. Thus one often focusses on the gluon processes [1]. A number of authors [2] derived involved expressions for 6-gluon scattering, which they could evaluate numerically. A more systematic answer to the problem of the six-gluon amplitude was subsequently found [3,4]. A major difference is the colour split-up of the amplitude, see also [5].

The amplitude is written as

$$\mathcal{M}(1,\ldots,n) \sim \sum_{P(1,\ldots,n-1)} Tr(T^{a_1}\cdots T^{a_n})C(1,\ldots,n), \tag{1.1}$$

where $a_1,\ldots,a_n$ denote the colours of the n gluons, T^{a_i} are the colour matrices in the fundamental representation. Moreover C is a subamplitude depending on the momenta and helicities of the gluons. Even with these simpler expressions it remains a formidable task to evaluate amplitudes with more than six gluons.

In this chapter we derive recursion relations as a technique to evaluate the exact parton amplitudes [6]. In fact, in first instance a matrix element for n gluons is calculated, one of which is off shell. From this current J the subamplitude $C(1,\ldots,n)$ is obtained. The advantage is that for the calculation of an $(n+1)$-gluon process one can use the calculation of the n-gluon process. Both for analytic and numerical evaluation this is an asset. A numerical evaluation of the seven [7,8] and eight [9] gluon process now becomes possible, although a straightforward use of the recursion relations without further tricks [8] remains time consuming. On the other hand, the recursion relation takes automatically into account all Feynman diagrams. Writing down those diagrams would be a problem in itself, which is now avoided.

Once one knows the gluon currents, they can be used as building blocks for those reactions, where besides n gluons, a quark-antiquark pair with or without a vector boson is produced. Thus in this chapter we introduce a recursive calculational technique for parton processes, which is suitable for analytical and numerical evaluation.

The actual outline of this chapter is as follows. The recursion relation relevant for the pure gluonic processes is derived in sec. 2, whereas the extension to processes with a quark-antiquark pair is given in sec. 3. Sec. 4 shows how to obtain the amplitudes and what the expressions for their squares in leading order in the colour are.

2 The gluon recursion relation

In this section an expression will be derived for a matrix element of $(n+1)$ outgoing gluons, where one gluon is off shell. This quantity will be called a n-gluon current $\hat{J}^x_\xi(1,2,\ldots,n)$, where x and ξ denote the colour and vector index of the off shell gluon. The $(n+1)$ particle amplitude can be obtained from this current by a suitable contraction with a polarization vector of the last gluon.

The current will first be introduced for 1, 2 and 3 gluons. It turns out that the current $\hat{J}^x_\xi$ can be decomposed in a colour part and a space-time part J_ξ. The latter has a number of symmetry properties, related to permutations of the gluons and is moreover a conserved current. For n gluons $\hat{J}^x_\xi(1,2,\ldots,n)$ is again related to $J_\xi(1,2,\ldots,n)$. For the latter a recursion relation holds, which relates it to all $J_\xi(1,2,\ldots,m)$ with $m < n$. Again, this current is conserved and obeys certain symmetry properties.

For one gluon we define

$$\hat{J}^x_\xi(1) = \delta^{a_1 x} e_\xi = \delta^{a_1 x} J_\xi(1) = 2Tr(T^{a_1} T^x) J_\xi(1) = 2(a_1 x) J_\xi(1) \ , \tag{2.1}$$

where e_ξ is the polarization vector of the gluon, depending on the helicity and momentum K_1 of the particle. The colour of the gluon is a_1, the indices x and ξ are summation indices. The $SU(N)$ matrices in the fundamental representation are denoted by T^{a_i}. The normalization is such that

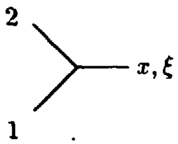
$$\begin{aligned} Tr(T^{a_1} T^{a_2}) &= (a_1 a_2) = \frac{1}{2}\delta^{a_1 a_2} \ , && (2.2) \\ [T^{a_1}, T^{a_2}] &= i f^{a_1 a_2 x} T^x \ , && (2.3) \\ (T^x)_{ij}(T^x)_{kl} &= \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl} \ . && (2.4) \end{aligned}$$

Obviously we have

$$K_1 \cdot J(1) = 0 \ . \tag{2.5}$$

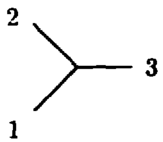
For two gluons we use the 3-vertex and introduce a propagator

$$\hat{J}^{x}_{\xi}(1,2) \quad = $$



$$= \quad \frac{ig}{(K_1+K_2)^2} f^{a_1 a_2 x} V^{\alpha_1\alpha_2}{}_{\xi}(K_1, K_2, -(K_1+K_2)) J_{\alpha_1}(1) J_{\alpha_2}(2) \;, \tag{2.6}$$

with $V_{\alpha_1\alpha_2\alpha_3}(K_1, K_2, K_3)$ given by the 3-vertex



$$\begin{aligned} &= -g f^{a_1 a_2 a_3} V_{\alpha_1\alpha_2\alpha_3}(K_1, K_2, K_3) \\ &= -g f^{a_1 a_2 a_3} \Big[(K_1 - K_2)_{\alpha_3}\, g_{\alpha_1\alpha_2} + (K_2 - K_3)_{\alpha_1}\, g_{\alpha_2\alpha_3} \\ &\qquad + (K_3 - K_1)_{\alpha_2}\, g_{\alpha_3\alpha_1} \Big] \;. \end{aligned} \tag{2.7}$$

It is convenient to introduce

$$\begin{aligned} J_{\xi}(1,2) &= \frac{1}{(K_1+K_2)^2} V^{\alpha_1\alpha_2}{}_{\xi}(K_1, K_2, -(K_1+K_2)) J_{\alpha_1}(1) J_{\alpha_2}(2) \\ &= \frac{1}{(K_1+K_2)^2} [2K_2 \cdot J(1) J_{\xi}(2) - 2K_1 \cdot J(2) J_{\xi}(1) \\ &\qquad + (K_1 - K_2)_{\xi} J(1) \cdot J(2)] \;, \end{aligned} \tag{2.8}$$

which obeys

$$(K_1 + K_2) \cdot J(1,2) = 0 \;, \tag{2.9}$$

$$J_{\xi}(1,2) = -J_{\xi}(2,1) \;. \tag{2.10}$$

Because of the antisymmetry property we introduce the suggestive notation

$$J_{\xi}(1,2) = \frac{1}{(K_1+K_2)^2} [J(1), J(2)]_{\xi} \;, \tag{2.11}$$

the bracket is however not a commutator. With this definition and using the relation

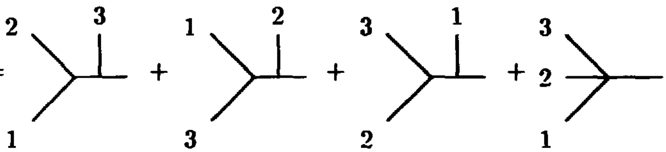
$$\frac{i}{2} f^{a_1 a_2 a_3} = (a_1 a_2 a_3) - (a_3 a_2 a_1) \tag{2.12}$$

we now have

$$\begin{aligned} \hat{J}^{x}_{\xi}(1,2) &= 2g \sum_{P(1,2)} Tr(T^{a_1} T^{a_2} T^{x}) J_{\xi}(1,2) \\ &= 2g \sum_{P(1,2)} (a_1 a_2 x) J_{\xi}(1,2) \;, \end{aligned} \tag{2.13}$$

with a sum over the permutations of $(1,2)$.

In the case of three gluons the following four diagrams should be considered

$\hat{J}^x_\xi(1,2,3) =$ 

. (2.14)

The first three diagrams give

$$
\begin{aligned}
\hat{D}^x_\xi(1,2,3) &= \frac{ig}{(K_1+K_2+K_3)^2}\frac{1}{2!}\sum_{P(1,2,3)} f^{ya_3x}\hat{J}^y_\eta(1,2)J_{\alpha_3}(3) \\
&\quad \times V^{\eta\alpha_3}{}_\xi(K_1+K_2,K_3,-(K_1+K_2+K_3)) \\
&= \frac{(2g)^2}{[\kappa(1,3)]^2}\frac{1}{2!}\sum_{P(1,2,3)}[(ya_3x)-(a_3yx)]\sum_{P(1,2)}(a_1a_2y)[J(1,2),J(3)]_\xi \\
&= 2g^2\sum_{P(1,2,3)}(a_1a_2a_3x)D_\xi(1,2,3)\ , \qquad (2.15)
\end{aligned}
$$

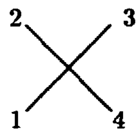
where $P(1,2,3)$ denotes a summation over the permutations of $(1,2,3)$ and a factor $\frac{1}{2!}$ is necessary to avoid multiple counting. Moreover

$$
\kappa(m,n) = K_m + K_{m+1} + \ldots + K_n\ , \qquad (2.16)
$$

and

$$
D_\xi(1,2,3) = \frac{1}{[\kappa(1,3)]^2}([J(1),J(2,3)]_\xi + [J(1,2),J(3)]_\xi)\ . \qquad (2.17)
$$

The fourth diagram of eq. (2.14) contains the 4-vertex. This 4 gluon vertex is given by



$= -ig^2 \sum_{C(1,2,3)} f^{a_1a_2y}f^{ya_3a_4}K(\alpha_1,\alpha_2;\alpha_3,\alpha_4)\ ,$ (2.18)

with

$$
K(\alpha_1,\alpha_2;\alpha_3,\alpha_4) = g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4} - g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_3} \qquad (2.19)
$$

and $C(1,2,3)$ denotes a cyclic sum. With this vertex the last diagram of eq. (2.14) becomes

$$
\begin{aligned}
\hat{F}^x_\xi(1,2,3) &= \frac{2g^2}{[\kappa(1,3)]^2}\sum_{C(1,2,3)}\{(a_1a_2a_3x)-(a_2a_1a_3x)+(a_3a_2a_1x)-(a_3a_1a_2x)\} \\
&\quad \times K(\alpha_1,\alpha_2;\alpha_3,\xi)J^{\alpha_1}(1)J^{\alpha_2}(2)J^{\alpha_3}(3) \\
&= \frac{2g^2}{[\kappa(1,3)]^2}\sum_{P(1,2,3)}(a_1a_2a_3x)\{J(1),J(2),J(3)\}_\xi \qquad (2.20)
\end{aligned}
$$

where we have used the relation

$$
\begin{aligned}
i^2 f^{yx_1x_2}f^{yx_3x}(\Omega_1x_1)(\Omega_2x_2)(\Omega_3x_3) = \frac{1}{4}\ \ \{ &\ (\Omega_1\Omega_2\Omega_3x)-(\Omega_2\Omega_1\Omega_3x) \\
+ &\ (\Omega_3\Omega_2\Omega_1x)-(\Omega_3\Omega_1\Omega_2x)\}\ . \qquad (2.21)
\end{aligned}
$$

The summation has been carried out over x_1, x_2, x_3 and y. The quantities Ω_i are strings of T matrices. The definition

$$\{J(1), J(2), J(3)\}_\xi = 2J(1)\cdot J(3)J_\xi(2) - J(1)\cdot J(2)J_\xi(3) - J(2)\cdot J(3)J_\xi(1) \quad (2.22)$$

shows the same symmetry properties as eq. (2.17). The full three-gluon current of eq. (2.14) now is

$$\begin{aligned} \hat{J}^x(1,2,3) &= \hat{D}^x(1,2,3) + \hat{F}^x(1,2,3) \\ &= 2g^2 \sum_{P(1,2,3)} (a_1 a_2 a_3 x) J(1,2,3) \end{aligned} \quad (2.23)$$

with

$$J(1,2,3) = \frac{1}{[\kappa(1,3)]^2}\Big([J(1), J(2,3)] + [J(1,2), J(3)] + \{J(1), J(2), J(3)\}\Big)\ , \quad (2.24)$$

where we have suppressed the index ξ. The current has the following properties

$$J(3,2,1) = J(1,2,3)\ , \quad (2.25)$$
$$J(1,2,3) + J(2,1,3) + J(2,3,1) = 0\ , \quad (2.26)$$
$$\kappa(1,3)\cdot J(1,2,3) = 0\ . \quad (2.27)$$

Properties (2.25) and (2.26) easily follow from the symmetry properties manifest in eqs. (2.17) and (2.22). Current conservation (2.27) follows by writing out the current.

The n-gluon current is a generalization of eqs. (2.1), (2.13) and (2.23)

$$\hat{J}^x_\xi(1,2,\ldots,n) = 2g^{n-1} \sum_{P(1,2,\ldots,n)} (a_1 a_2 \cdots a_n x) J_\xi(1,2,\ldots,n)\ , \quad (2.28)$$

where J_ξ is a generalization of eqs. (2.11) and (2.24)

$$\begin{aligned} J(1,2,\ldots,n) = \frac{1}{\kappa(1,n)^2}\Bigg(&\sum_{m=1}^{n-1}[J(1,\ldots,m), J(m+1,\ldots,n)] \\ &+ \sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\{J(1,\ldots,m), J(m+1,\ldots,k), J(k+1,\ldots,n)\}\Bigg)\ . \end{aligned} \quad (2.29)$$

The notation is a generalization of eqs. (2.8) and (2.22)

$$\begin{aligned} [J(1,\ldots,m), J(m+1,\ldots,n)] = \\ & 2\kappa(m+1,n)\cdot J(1,\ldots,m)\, J(m+1,\ldots,n) \\ & -2\kappa(1,m)\cdot J(m+1,\ldots,n)\, J(1,\ldots,m) \\ & +J(1,\ldots,m)\cdot J(m+1,\ldots,n)\, (\kappa(1,m) - \kappa(m+1,n))\ , \end{aligned} \quad (2.30)$$

$$\begin{aligned} \{J(1,\ldots,m), J(m+1,\ldots,k), J(k+1,\ldots,n)\} = \\ & 2J(1,\ldots,m)\cdot J(k+1,\ldots,n)\, J(m+1,\ldots,k) \\ & -J(1,\ldots,m)\cdot J(m+1,\ldots,k)\, J(k+1,\ldots,n) \\ & -J(m+1,\ldots,k)\cdot J(k+1,\ldots,n)\, J(1,\ldots,m)\ . \end{aligned} \quad (2.31)$$

The current $J(1,2,\ldots,n)$ has the properties

$$J(n,n-1,\ldots,1) = (-1)^{n-1}J(1,2,\ldots,n) \; , \tag{2.32}$$

$$\kappa(1,n)\cdot J(1,2,\ldots,n) = 0 \; , \tag{2.33}$$

$$\sum_{\mathrm{Perm}(i,j)} J(i_1,\ldots,i_m,j_1,\ldots,j_k) = 0 \; , \; 1 \le m \le n-1 \; , \; m+k=n \; . \tag{2.34}$$

Relation (2.34) needs some explanation. The permutations over which the summation is performed are those permutations of the set $(i_1,\ldots,i_m,j_1,\ldots,j_k)$ where the order within each set $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_k)$ is preserved. For example taking $m=2$ and $k=3$ leads to

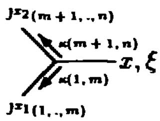
$$\begin{aligned} & J(1,2,3,4,5)+J(1,3,2,4,5)+J(1,3,4,2,5)+J(1,3,4,5,2) \\ + \; & J(3,1,2,4,5)+J(3,1,4,2,5)+J(3,1,4,5,2)+J(3,4,1,2,5) \\ + \; & J(3,4,1,5,2)+J(3,4,5,1,2)=0 \; . \end{aligned} \tag{2.35}$$

A special case arises when $m=1$

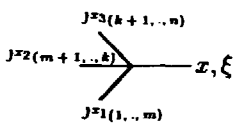
$$J(1,2,3,\ldots,n)+J(2,1,3,\ldots,n)+\cdots+J(2,3,\ldots,n,1)=0 \; , \tag{2.36}$$

which is a generalization of eqs. (2.10) and (2.26). We will now prove recursively in the number of gluons the validity of the above relations. The correctness for $n=2, 3$ is already shown. The correctness of eqs. (2.28) and (2.29) follows by induction. So assume the validity for $m<n$, then the n-gluon current is obtained from considering a 3-vertex and a 4-vertex with all possible currents attached

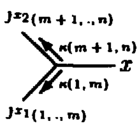
$$\hat{J}^x_\xi(1,\ldots,n) = \frac{1}{[\kappa(1,n)]^2}\sum_{P(1,\ldots,n)}\left(\sum_{m=1}^{n-1}\frac{1}{2!}\frac{1}{m!}\frac{1}{(n-m)!}\right.$$



$$\left. + \sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\frac{1}{3!}\frac{1}{m!}\frac{1}{(k-m)!}\frac{1}{(n-k)!}\right) \; . \tag{2.37}$$



In eq. (2.37) a summation over all permutations of the n gluons is performed. In order to avoid multiple counting factors like $\frac{1}{m!}$ are introduced, since $\hat{J}(1,\ldots,m)$ contains all $m!$ permutations of the particles. Using the expression (2.28) one rewrites the terms in (2.37)

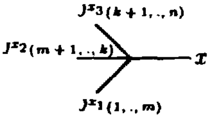


$$= ig\frac{1}{\kappa(1,n)^2}f^{x_1x_2x}V^{\alpha_1\alpha_2}{}_\xi\left(\kappa(1,m),\kappa(m+1,n),-\kappa(1,n)\right)$$

$$\times \hat{J}_{\alpha_1}(1,\ldots,m)\hat{J}_{\alpha_2}(m+1,\ldots,n)$$

$$
\begin{aligned}
&= 8g^{n-1}\frac{1}{\kappa(1,n)^2}\sum_{P(1,\ldots,m)}\sum_{P(m+1,\ldots,n)}\Big([x_1,x_2]x\Big)\Big(\Omega_1x_1\Big)\Big(\Omega_2x_2\Big)\\
&\quad\times V^{\alpha_1\alpha_2}{}_{\xi}\left(\kappa(1,m),\kappa(m+1,n),-\kappa(1,n)\right)\\
&\quad\times J_{\alpha_1}(1,\ldots,m)J_{\alpha_2}(m+1,\ldots,n)\\
&= g^{n-1}\sum_{P(1,\ldots,m)}\sum_{P(m+1,\ldots,n)}\Big([x_1,x_2]x\Big)\Big(\Omega_1x_1\Big)\Big(\Omega_2x_2\Big)\\
&\quad\times\Big[J(1,\ldots,m),J(m+1,\ldots,n)\Big]\ ,
\end{aligned}
\tag{2.38}
$$

where Ω_1 and Ω_2 are the strings of T matrices with labels $a_1\cdots a_m$ and $a_{m+1}\cdots a_n$. Carrying out the x_1,x_2 summation the first term of eq. (2.29) is obtained. The 4-vertex term becomes



$$
= g^{n-1}\sum_{P(1,\ldots,m)}\sum_{P(m+1,\ldots,k)}\sum_{P(k+1,\ldots,n)}
$$

$$
\begin{aligned}
&\left\{\sum_{C(1,2,3)}2[(\Omega_1\Omega_2\Omega_3x)-(\Omega_2\Omega_1\Omega_3x)+(\Omega_3\Omega_2\Omega_1x)-(\Omega_3\Omega_1\Omega_2x)]K(\alpha_1,\alpha_2;\alpha_3,\xi)\right\}\\
&\times J^{\alpha_1}(1,\ldots,m)J^{\alpha_2}(m+1,\ldots,k)J^{\alpha_3}(k+1,\ldots,n),
\end{aligned}
\tag{2.39}
$$

where Ω_1, Ω_2 and Ω_3 are the strings of T matrices with labels $a_1\cdots a_m$, $a_{m+1}\cdots a_k$ and $a_{k+1}\cdots a_n$. The 4-vertex term contains a cyclic permutation over (Ω_i,α_i) inside the brackets. This particular expression follows from eqs. (2.18) and (2.21) and can be rewritten as

$$
\{\cdots\}=2\sum_{P(1,2,3)}(\Omega_1\Omega_2\Omega_3x)[K(\alpha_1,\alpha_2;\alpha_3,\xi)+K(\alpha_3,\alpha_2;\alpha_1,\xi)].
\tag{2.40}
$$

From this, the second term in (2.29) easily follows. This completes the proof of eqs. (2.28) and (2.29).

The three properties (2.32) - (2.34) of the n-gluon current will now be proven. The reflective property (2.32) is proven recursivly, so assume eq. (2.32) holds for $J(1,\ldots,m)$ with $m<n$ then

$$
\begin{aligned}
J(1,\ldots,n) &= \frac{1}{[\kappa(1,n)]^2}\left(\sum_{m=1}^{n-1}[\,J(1,\ldots,m),J(m+1,\ldots,n)\,]\right.\\
&\quad\left.+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\{\,J(1,\ldots,m),J(m+1,\ldots,k),J(k+1,\ldots,n)\,\}\right)
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
&= \frac{1}{[\kappa(1,n)]^2}\left(\sum_{m=1}^{n-1}-(-1)^{(m-1)+(n-(m+1))}\right.\\
&\qquad\qquad\times[\,J(n,\ldots,m+1),J(m,\ldots,1)\,]\\
&\quad+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}(-1)^{(m-1)+(k-m-1)+(n-k-1)}
\end{aligned}
$$

$$\times \{ J(n,\ldots,k+1), J(k,\ldots,m+1), J(m,\ldots,1) \} \Big)$$

$$= (-1)^{n-1} J(n,n-1,\ldots,1) .$$

To prove current conservation, eq. (2.33), we contract (2.29) with $\kappa(1,n)$ to get terms of the following type

$$\kappa(1,n) \cdot [\, J(1,\ldots,m), J(m+1,\ldots,n)\,] = \left([\kappa(1,m)]^2 - [\kappa(m+1,n)]^2 \right) J(1,\ldots,m) \cdot J(m+1,\ldots,n) , \tag{2.42}$$

$$\begin{aligned} \kappa(1,n) \cdot \{ J(1,\ldots,m), J(m+1,\ldots,k), J(k+1,\ldots,n) \} = \\ J(1,\ldots,m) \cdot [\, J(m+1,\ldots,k), J(k+1,\ldots,n)] \\ - \; J(k+1,\ldots,n) \cdot [\, J(1,\ldots,m), J(m+1,\ldots,k)] \; . \end{aligned} \tag{2.43}$$

Expression (2.43), inserted in the summation of eq. (2.29) leads to

$$\begin{aligned} & \sum_{m=1}^{n-2} \sum_{k=m+1}^{n-1} J(1,\ldots,m) \cdot [\, J(m+1,\ldots,k), J(k+1,\ldots,n)] \\ & \quad - \sum_{m=1}^{n-2} \sum_{k=m+1}^{n-1} J(k+1,\ldots,n) \cdot [\, J(1,\ldots,m), J(m+1,\ldots,k)] \\ & = \sum_{m=1}^{n-1} J(1,\ldots,m) \cdot J(m+1,\ldots,n) \, [\kappa(m+1,n)]^2 \\ & \quad - \sum_{k=1}^{n-1} J(1,\ldots,k) \cdot J(k+1,\ldots,n) \, [\kappa(1,k)]^2 \; . \end{aligned} \tag{2.44}$$

This sum cancels the sum of the terms of type (2.42) in eq. (2.29), thus proving current conservation.

The proof of eq. (2.34) can be easily given when we consider the current $\hat{J}_\xi^x$ of eq. (2.28). The colour factor in eq. (2.28) originates from the colour factors of the three- and four-vertices

$$i f^{a_1 a_2 x} = 2([a_1, a_2] x) , \tag{2.45}$$

$$\sum_y f^{a_1 a_2 y} f^{y a_3 a_4} = -4 \sum_y ([a_1, a_2] y)([a_3, a_4] y) . \tag{2.46}$$

The replacement of the structure constants by eqs. (2.45) and (2.46) leads to terms like $(a_1 \cdots a_n x)$ irrespective of the precise properties of the T^{a_i}. Suppose that we take for T^{a_1} a special matrix T^{b_1} which equals the identity matrix. This would mean that the coupling of a gauge boson with colour b_1 to the other gauge bosons is zero since the structure constants are zero or in physical terms a photon does not couple to a gluon. Therefore $\hat{J}_\xi^x(1,\ldots,n)$ vanishes, which implies that for each ordering of the colour structure $(a_2 \cdots a_n)$ the term in eq. (2.28) should vanish. This gives the condition

$$\begin{aligned} &(b_1 a_2 a_3 \cdots a_n x) J(1,2,3,\ldots,n) + (a_2 b_1 a_3 \cdots a_n x) J(2,1,3,\ldots,n) + \cdots \\ + \;& (a_2 a_3 \cdots a_n b_1 x) J(2,3,\ldots,n,1) = 0 \;. \end{aligned} \tag{2.47}$$

Since all the above colour factors are the same we have proved eq. (2.36). Property (2.34) is obtained in a similar way. Replace $T^{a_1},\ldots,T^{a_m}$ by matrices $T^{b_1},\ldots,T^{b_m}$ such that

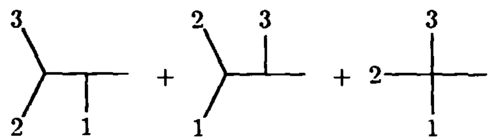
$$[T^{b_i}, T^{a_j}] = 0 \;,\; i = 1,\ldots,m \;,\; j = m+1,\ldots,n \;. \tag{2.48}$$

It means that we have two kinds of gluons. Within each kind the usual non-abelian interaction is present but between the two kinds there is no interaction. Thus the current with gluons with colours $b_1,\ldots,b_m,a_{m+1},\ldots,a_n$ vanishes. Since all traces are the same for which the order in the set $\{b_i\}$ and in the set $\{a_i\}$ are fixed we get the vanishing of the coefficient of that type of trace. This coefficient is given by eq. (2.34). One could also take the colour x of the off shell gluon equal to b_1. This results in the equation

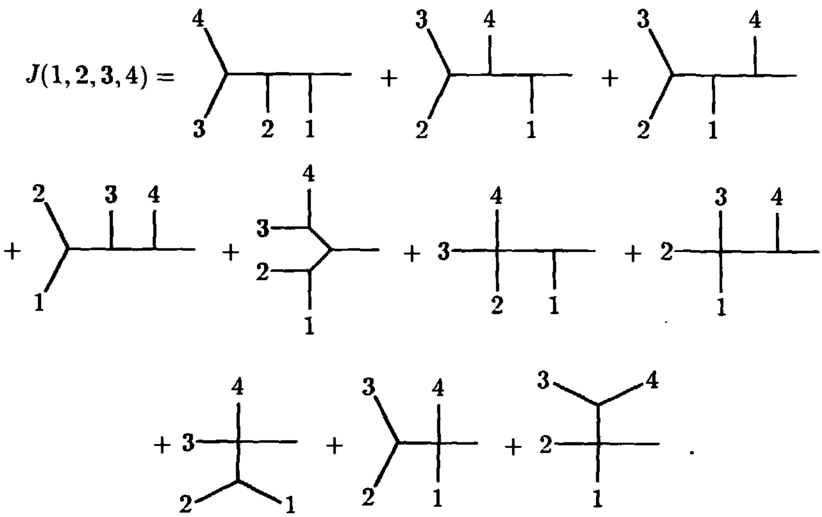
$$\sum_{C(1,\ldots,n)} J(1,2,\ldots,n) = 0 \;. \tag{2.49}$$

Sofar we have restricted ourselves to two kinds of gluons in eq. (2.28). When this is generalized to more kinds of gluons eq. (2.34) can be generalized accordingly.

It is sometimes useful to describe the successive terms in eq. (2.29) by means of diagrams. For instance,

$J(1,2,3) =$ 

and

$J(1,2,3,4) =$ 

These diagrams have a clock-wise orientation for the labels $1, 2, 3, \ldots, n$. This is necessary since $J(1, 2, \ldots, m)$ is not symmetric in the indices.

3 The spinorial recursion relation

In this section an expression will be derived for a matrix element, where a quark-antiquark pair and n gluons are outgoing. The antiquark is off shell such that we have a spinorial current $\hat{J}^j_c(Q; 1, 2, \ldots, n)$. In this notation Q stands for the quark momentum. Moreover the numbers $1, 2, \ldots, n$ denote the n gluons and their momentum. The colour j and the spinor index c of the off shell antiquark are written explicitly, but will often be suppressed. The other colour indices will be manifest in the explicit formulae. As in sec. 2 the current $\hat{J}$ will be expressed in a sum of terms consisting of a colour factor and a current $J_c(Q; 1, \ldots, n)$ which is independent of colour and for which a recursion relation holds.

For a single quark and no gluon we have

$$\hat{J}^j_c(Q) = \delta_{ij} \bar{u}_c(Q) \tag{3.1}$$

or

$$\hat{J}(Q) = \delta_{ij} \bar{u}(Q) = \delta_{ij} J(Q) \ . \tag{3.2}$$

The one-gluon spinorial current is

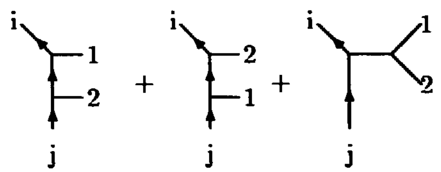
$$\begin{aligned} \hat{J}(Q; 1) &= igT^{a_1}_{ij} \bar{u}(Q) \not{\epsilon}_1 \frac{i}{\not{Q} + \not{K}_1 - m} \\ &= gT^{a_1}_{ij} J(Q; 1) \\ &= g(a_1)_{ij} J(Q; 1) \ , \end{aligned} \tag{3.3}$$

with

$$J(Q; 1) = -J(Q) \not{J}(1) \frac{1}{\not{Q} + \not{K}_1 - m} \ . \tag{3.4}$$

For two gluons the following diagrams should be considered.

$$\hat{J}(Q; 1, 2) \ =$$



$$\begin{aligned} = \ & ig^2 \{ (a_1 a_2)_{ij} J(Q; 1) \not{J}(2) + (a_2 a_1)_{ij} J(Q; 2) \not{J}(1) \\ & + 2(x)_{ij} [(a_1 a_2 x) J(Q) \not{J}(1, 2) + (a_2 a_1 x) J(Q) \not{J}(2, 1)] \} \\ & \times \frac{i}{\not{Q} + \not{K}_1 + \not{K}_2 - m} \ . \end{aligned} \tag{3.5}$$

The one- and two-gluon currents eqs. (2.1) and (2.13) have been used in eq. (3.5). The summation over x is carried out with the help of eq. (2.4), giving

$$\hat{J}(Q; 1, 2) = g^2 \sum_{P(1,2)} (a_1 a_2)_{ij} J(Q; 1, 2) \ , \tag{3.6}$$

with

$$J(Q;1,2) = -\big(J(Q;1)\not{J}(2) + J(Q)\not{J}(1,2)\big)\frac{1}{\not{Q} + \not{K}_1 + \not{K}_2 - m} . \qquad (3.7)$$

We generalize the above derived expressions for the 0-, 1- and 2-gluon spinorial currents to arbitrary number of gluons, giving

$$\hat{J}(Q;1,2,\ldots,n) = g^n \sum_{P(1,\ldots,n)} (a_1 a_2 \cdots a_n)_{ij} J(Q;1,2,\ldots,n) , \qquad (3.8)$$

where

$$J(Q;1,2,\ldots,n) = -\sum_{m=0}^{n-1} J(Q;1,2,\ldots,m)\not{J}(m+1,\ldots,n)\frac{1}{\not{Q} + \not{q}(1,n) - m} . \qquad (3.9)$$

This is proven by induction

$$\begin{aligned}\hat{J}(Q;1,\ldots,n) \;&=\; 2ig^n \sum_{P(1,\ldots,n)} \sum_{m=0}^{n-1} \frac{1}{m!}\frac{1}{(n-m)!} \sum_{P(1,\ldots,m)} \sum_{P(m+1,\ldots,n)} \\ &\quad (a_1 \cdots a_m)_{il}(x)_{lj}(a_{m+1} \cdots a_n x) J(Q;1,\ldots,m)\not{J}(m+1,\ldots,n) \\ &\quad \times \frac{i}{\not{Q} + \not{q}(1,n) - m} . \qquad (3.10)\end{aligned}$$

The colour sum over x give terms $\frac{1}{2}(a_1 \cdots a_n)_{ij}$ and $-\frac{1}{2N}(a_1 \cdots a_m)_{ij}(a_{m+1} \cdots a_n)$. The second term does not contribute since for each choice of the labels $1,\ldots,m$ we perform a summation over all permutations of the remaining $m+1,\ldots,n$ labels which implies sums over cyclic permutations of the pure gluon current. Due to eq. (2.49) the result for these terms is zero. In the special case where we have $-\frac{1}{2N}(a_1 \cdots a_{n-1})_{ij}(a_n)$ this contribution also vanishes because T^{a_i} is traceless. Thus we are left with the $(a_1 \cdots a_n)_{ij}$ terms which lead to the eqs. (3.8) and (3.9).

In the following it will be useful to have a spinorial current with the outgoing quark instead of the antiquark off shell. In exactly the same fashion we derive

$$\hat{J}^j(1,\ldots,n;P) = g^n \sum_{P(1,\ldots,n)} (a_1 a_2 \cdots a_n)_{ji} J(1,\ldots,n;P) , \qquad (3.11)$$

where j in the colour index of the off shell outgoing quark and P the momentum of the outgoing antiquark. The current J is given by

$$J(P) = v(P) \qquad (3.12)$$

and in general

$$J(1,\ldots,n;P) = \frac{1}{\not{P} + \not{q}(1,n) + m} \sum_{m=1}^{n} \not{J}(1,\ldots,m) J(m+1,\ldots,n;P) . \qquad (3.13)$$

The notation for the currents (3.8) and (3.11) is such that the position of the momentum Q or P determines whether one deals with an adjoint spinor or a spinor.

By means of the charge conjugation matrix C for which

$$C\, v_{\pm} = -\bar{u}_{\pm}^T , \; \bar{u}_{\pm} C^{-1} = v_{\pm}^T \qquad (3.14)$$

where

$$v_{\pm} = \frac{1}{2}(1 \mp \gamma_5)\, v\ , \ u_{\pm} = \frac{1}{2}(1 \pm \gamma_5)\, u\ , \tag{3.15}$$

one has

$$C\, J(1,\ldots,n;P\pm) = (-1)^{n-1} J^T(P\pm;n,\ldots,1) \tag{3.16}$$

or

$$J(P\pm;1,\ldots,n)\, C^{-1} = -(-1)^{n-1} J^T(n,\ldots,1;P\pm)\ . \tag{3.17}$$

The $\pm$ sign in eqs. (3.16) and (3.17) denotes the helicity of the outgoing quark or antiquark.

4 From currents to amplitudes and cross sections

From the pure n-gluon current and the spinorial currents the amplitudes and cross sections for the parton processes with a large number of gluons can be obtained. We discuss in this section the most relevant processes i.e. those with only gluons and processes where in addition to n gluons a quark-antiquark pair is produced, possibly together with a W, Z or a virtual photon.

4.1 Scattering of n gluons

The amplitude for n-gluon scattering is obtained from the $(n-1)$ gluon current $\hat{J}^x_\xi(1,\ldots,n-1)$ by removing the propagator of the off shell gluon, contracting the current with the polarization vector of the n^{th} gluon i.e. $\hat{J}^x_\xi(n)$ and demanding overall momentum conservation $\kappa(1,n)=0$,

$$\begin{aligned}
\mathcal{M}(1,\ldots,n) &= \hat{J}^x(1,\ldots,n-1)\cdot \hat{J}_x(n)\, i[\kappa(1,n-1)]^2\Big|_{\kappa(1,n)=0} \\
&= 2ig^{n-2}\sum_{P(1,\ldots,n-1)} (a_1\cdots a_n) J(1,\ldots,n-1)\cdot J(n)[\kappa(1,n-1)]^2\Big|_{\kappa(1,n)=0} \\
&= 2ig^{n-2}\sum_{P(1,\ldots,n-1)} (a_1\cdots a_n)\, \mathcal{C}(1,2,\ldots,n)\ ,
\end{aligned} \tag{4.1}$$

with

$$\mathcal{C}(1,2,\ldots,n) = [\kappa(1,n-1)]^2 J(1,\ldots,n-1)\cdot J(n)\Big|_{\kappa(1,n)=0}\ . \tag{4.2}$$

From the definitions (4.2) and (2.29) of the gluon current together with the properties (2.32)-(2.34) it is clear that the function $\mathcal{C}(1,\ldots,n)$ has the following four properties :

1. The subamplitude $\mathcal{C}$ is invariant under cyclic permutations,

$$\mathcal{C}(1,\ldots,n) = \mathcal{C}(m+1,\ldots,n,1,\ldots,m)\ . \tag{4.3}$$

2. The subamplitude $\mathcal{C}$ has a reflective property,

$$\mathcal{C}(1,\ldots,n) = (-1)^n \mathcal{C}(n,\ldots,1)\ . \tag{4.4}$$

3. There are linear relations between the subamplitudes C

$$\sum_{\mathrm{Perm}(i,j)} C(i_1,\ldots,i_m,j_1,\ldots,j_k,n+1)=0\;,\;1\le m\le n-1\;,\;m+k=n\;. \quad (4.5)$$

The relations limit the number of independent n-gluon subamplitudes to $(n-2)!$ [7]. For $m=1$ we have the special case

$$\sum_{C(1,\ldots,n-1)} C(1,\ldots,n)=0\;. \quad (4.6)$$

4. The quantity C is gauge invariant.

The first property stems from the fact, that one can obtain $C(1,\ldots,n)$ from any $(n-1)$ gluon current $\hat{J}(m+1,\ldots,n,1,\ldots,m-1)$ by contraction with $\hat{J}(m)$. In eq. (4.6) we keep n fixed, but one could also fix another label m, using eq. (4.3). The property of gauge invariance here means that another gauge choice for a gluon leads to the same subamplitude.

For the cross section one must square the amplitude (4.1) and sum over all colours. In general this becomes complicated [4,6,7,8]. However the terms in the cross section of leading order in N can be obtained for any number of gluons. An arbitrary term in the cross section contains the colour term

$$\begin{aligned}\sum_{a_n}(a_1\cdots a_n)(a_n a_{m_{n-1}}\cdots a_{m_1}) &= \frac{1}{2}(a_1\cdots a_{n-1}a_{m_{n-1}}\cdots a_{m_1})\\ &- \frac{1}{2N}(a_1\cdots a_{n-1})(a_{m_{n-1}}\cdots a_{m_1})\;. \quad (4.7)\end{aligned}$$

In leading order in N the second term can be omitted. The remaining term is now summed over a_{n-1}. Two different type of structures can occur

$$\sum_{a_{n-1}}(\Omega_1 a_{n-1}a_{n-1}\Omega_2)=\frac{1}{2}N(\Omega_1\Omega_2)-\frac{1}{2N}(\Omega_1)(\Omega_2)\;, \quad (4.8)$$

and

$$\sum_{a_{n-1}}(\Omega_1 a_{n-1}\Omega_2 a_{n-1})=\frac{1}{2}(\Omega_1)(\Omega_2)-\frac{1}{2N}(\Omega_1\Omega_2)\;, \quad (4.9)$$

So in leading order in N only the first structure is of relevance leading to $\frac{1}{2}N(\Omega_1\Omega_2)$. When summing over a_{n-2} the terms with neighbouring matrices a_{n-2} are again leading. Repeating this process we find

$$\begin{aligned}&\sum_{a_1,\ldots a_n}|\mathcal{M}(1,\ldots,n)|^2\\ &= 4g^{2n-4}\left(\sum_{P(1,\ldots n-1)}(a_1\cdots a_n)C(1,\ldots,n)\right)\left(\sum_{P(1,\ldots n-1)}(a_n a_{n-1}\cdots a_1)C^*(1,\ldots,n)\right)\\ &= \left(\frac{g^2N}{2}\right)^{n-2}(N^2-1)\left(\sum_{P(1,\ldots,n-1)}|C(1,\ldots,n)|^2+\mathcal{O}(\frac{1}{N^2})\right)\;. \quad (4.10)\end{aligned}$$

The last summation concerns (a_1a_1) which gives with the normalization of eq. (2.2) $\frac{1}{2}(N^2-1)$. Up to 5 gluons the $\mathcal{O}(N^{-2})$ term is zero. From 6 gluons onward interference terms between different C functions are present in the cross section, but are suppressed by colour [3,4,7,8,9].

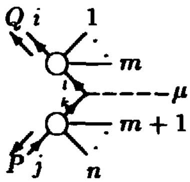


Fig. 4.1. The process producing $q\bar{q}$, n gluons and a vector boson.

4.2 The process producing $q\bar{q}$ and n gluons

The amplitude for a process with n outgoing gluons, a quark with momentum Q, colour i and antiquark with momentum P, colour j can be obtained from either one of the spinorial currents.

When we use eq. (3.8) we find

$$\begin{aligned} \mathcal{M}(Q;1,2,\ldots,n;P) &= -i\hat{J}(Q;1,\ldots,n)[\not{Q}+\not{\kappa}(1,n)-m]v(P)\Big|_{P+Q+\kappa(1,n)=0} \\ &= -ig^n \sum_{P(1,\ldots,n)} (a_1\cdots a_n)_{ij}\mathcal{D}(Q;1,\ldots,n;P)\ , \end{aligned} \tag{4.11}$$

with

$$\mathcal{D}(Q;1\ldots,n;P) = J(Q;1,\ldots,n)[\not{Q}+\not{\kappa}(1,n)-m]v(P)|_{P+Q+\kappa(1,n)=0}\,. \tag{4.12}$$

For the matrix element squared, summed over the colours of the partons we have

$$\begin{aligned} &\sum_{i,j,\{a_m\}} |\mathcal{M}(Q;1,\ldots,n;P)|^2 \\ &= g^{2n}\left(\sum_{P(1,\ldots,n)}(a_1\cdots a_n)_{ij}\mathcal{D}(Q;1,\ldots,n;P)\right)\left(\sum_{P(1,\ldots,n)}(a_n\cdots a_1)_{ji}\mathcal{D}^*(Q;1,\ldots,n;P)\right) \end{aligned} \tag{4.13}$$

$$= \left(\frac{g^2N}{2}\right)^n \frac{(N^2-1)}{N}\left(\sum_{P(1,\ldots,n)} |\mathcal{D}(Q;1,\ldots,n;P)|^2 + \mathcal{O}(N^{-2})\right)\ , \tag{4.14}$$

where the last expression shows the leading N behaviour.

4.3 The process producing $q\bar{q}$, n gluons and a vector boson

The typical structure for this process is depicted in fig. 4.1. The index μ denotes a vertex $ie\delta_{lk}\Gamma_\mu^{V,f_lf_k}$ to which a vector boson V_μ can be attached. The colour labels are l,k and flavour labels f_1, f_2 are indicated in fig. 4.1. The vertex contains left- and right-handed couplings

$$\Gamma_\mu^{V,f_1f_2} = L^V_{f_1f_2}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right) + R^V_{f_1f_2}\gamma_\mu\left(\frac{1+\gamma_5}{2}\right)\ , \tag{4.15}$$

where for a photon

$$L^{\gamma}_{f_1 f_2} = R^{\gamma}_{f_1 f_2} = -Q_{f_1}\delta_{f_1 f_2}, \tag{4.16}$$

and for a Z boson

$$L^{Z}_{f_1 f_2} = \frac{I_3^{f_1} - s_W^2 Q_{f_1}}{s_W\, c_W}\delta_{f_1 f_2}\ ,\ R^{Z}_{f_1 f_2} = \frac{-s_W Q_{f_1}}{c_W}\delta_{f_1 f_2}\ . \tag{4.17}$$

In eqs. (4.16) and (4.17) the charge Q_{f_1} and the weak isospin component $I_3^{f_1}$ of the fermions occur i.e. -1, $-\frac{1}{2}$ for an electron, $\frac{2}{3}$, $\frac{1}{2}$ for an u-quark and $-\frac{1}{3}$, $-\frac{1}{2}$ for a d-quark. The sine of the weak angle θ_W is given by s_W and the cosine by c_W. For a W boson one has

$$R^{W}_{f_1 f_2} = 0\ ,\ L^{W}_{f_1 f_2} = \frac{1}{\sqrt{2}s_W}\delta_{\tilde{f}_1 f_2}\ . \tag{4.18}$$

In the last formula $\tilde{f}_1$ is the partner of f_1 in the doublet, the Kobayashi-Maskawa mixing matrix has been set to unity. Often we will denote the vertex of eq. (4.15) by the shorthand notation Γ_μ.

The matrix element takes the form

$$\begin{aligned} &\mathcal{M}(Q,P;V;1,\ldots,n) \\ &= ie \sum_{P(1,\ldots,n)} \sum_{m=0}^{n} \frac{1}{m!}\frac{1}{(n-m)!}\hat{J}(Q;1,\ldots,m)\Gamma_\mu V^\mu \delta_{kl}\hat{J}(m+1,\ldots,n;P) \\ &= ieg^n \sum_{P(1,\ldots,n)} (a_1\cdots a_n)_{ij}\mathcal{S}_\mu(Q;1,\ldots,n;P)V^\mu, \end{aligned} \tag{4.19}$$

with

$$\mathcal{S}_\mu(Q;1,\ldots,n;P) = \sum_{m=0}^{n} J(Q;1,\ldots,m)\Gamma_\mu J(m+1,\ldots,n;P) \tag{4.20}$$

and V^μ the polarization vector of the boson.

Using the C matrix of eq. (3.14) for which

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T\ , \tag{4.21}$$

we have for $\Gamma_\mu = \gamma_\mu$

$$\mathcal{S}_\mu(Q;1,\ldots,n;P) = (-1)^n \mathcal{S}_\mu(P;n,\ldots,1;Q)\ . \tag{4.22}$$

The matrix element squared, summed over all colours is again obtained in leading order in N

$$\begin{aligned} |\mathcal{M}(Q,P;V;1,\ldots,n)|^2 &= e^2\left(\frac{g^2N}{2}\right)^n \frac{(N^2-1)}{N} \\ &\times \left(\sum_{P(1,\ldots,n)} |\mathcal{S}_\mu V^\mu|^2 + \mathcal{O}(N^{-2})\right)\ . \end{aligned} \tag{4.23}$$

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Chapter V

Solutions of the recursion relations

The recursion relations of chap. 4 are used to derive rigorously amplitudes for certain helicity configurations, where most of the gluons have the same helicities. This proves a number of conjectures in the literature.

1 Introduction

In chap. 4 recursion relations for several currents were derived. They can be used to calculate step by step any current with a certain number of gluons having a specific helicity configuration. For some special helicity configurations the answers become so simple and systematic that a generalization to an arbitrary number of gluons presents itself [1]. They are the QCD counterpart of proven QED relations [2]. They also exist in quantum gravity as was shown in ref. [3].

The generalizations to arbitrary number of gluons for these special helicity amplitudes were conjectured in the literature. The first conjecture was made for the n-gluon amplitude in ref. [4]. After that similar conjectures were proposed for special helicity amplitudes of $q\bar{q} + n$ gluons [5], $q\bar{q}q\bar{q} + n$ gluons [6] and $V + q\bar{q} + n$ gluons [6] where V stands for a vector boson. Actually the proof [1] already existed when the last conjecture was made.

The special helicity amplitudes form the basis of approximative methods for calculating multi gluon processes. In ref. [7] approximations are proposed to describe the pure gluon scattering, while in ref. [8] also a quark pair is included. The approximation of the $e^+e^- \to q\bar{q} + n$ gluon scattering is given in ref. [9]. Because all these approximations are based on the special helicity amplitudes it is very important to prove the conjectures of refs. [4,5,6] for arbitrary number of gluons.

The currents related to these special helicity amplitudes can be shown to satisfy the recursion relation, so that we have the explicit solution of the recursion relation for an arbitrary number of gluons. The helicity configurations for which this is possible are those currents in which all gluon helicities are the same or all but one are the same. With these gluon currents we can prove the conjecture of ref. [4]. With a simple algebraic relation between the gluon current and the spinorial current the latter can also be derived. With these currents we are able to prove the proposed helicity amplitudes of refs. [5,6]. It should be noted that the relation between gluonic and spinorial currents is proven without the help of an imbedding of QCD in a $N = 1$ supersymmetric theory [10].

The outline of this chapter is as follows. Sec. 2 explains the steps necessary for the recursive proof by looking at the simpler case of QED. A number of helicity amplitudes with an arbitraty number of photons, which were proven in ref. [2], will be derived with the aid of the QED recursion relation. In sec. 3 the gluon recursion relation is solved for a number of helicity combinations. Whereas in sec. 4 the spinorial recursion relation is examined. In sec. 5 the solutions of the gluonic and spinorial recursion relations are used to construct specific helicity amplitudes for various processes and thus proving the conjectures of refs. [4,5,6].

2 QED examples for n photon emission

Before deriving the explicit expressions for the above currents, it is useful to explain the various steps in the proof for a simpler case. That is the QED case for the emission of n photons with the same helicity in a number of processes [2]. Firstly a recursion relation for a QED spinorial current is given. Then it will be solved for the equal helicity configuration of the photons.

Analogously to the n-gluon spinorial current (IV.3.9) we introduce an n-photon spinorial current with an off shell positron

$$J(Q;1,\ldots,n) = -e\sum_{m=1}^{n} J(Q;1,\ldots,m-1,m+1,\ldots,n)\,\not{\epsilon}(m)\frac{1}{\not{Q}+\not{K}(1,n)}\ , \quad (2.1)$$

where the electron mass is taken to be zero. The no-photon and 1-photon currents are given by

$$J(Q) = \bar{u}(Q)\ , \quad (2.2)$$

$$J(Q;1) = -eJ(Q)\,\not{\epsilon}(1)\frac{1}{\not{Q}+\not{K}_1}\ . \quad (2.3)$$

Specifying the electron helicity to ± 1 gives a simple modification of the current

$$J(Q\pm;1,\ldots,n) = J(Q;1,\ldots,n)\left(\frac{1\mp\gamma_5}{2}\right)\ . \quad (2.4)$$

For the current where the electron is off shell and the positron on shell we have

$$J(1,\ldots,n;P) = e\frac{1}{\not{P}+\not{K}(1,n)}\sum_{m=1}^{n}\not{\epsilon}(m)\,J(1,\ldots,m-1,m+1,\ldots,n;P) \quad (2.5)$$

with $J(P) = v(P)$. Again the helicity of the outgoing positron can be easily specified

$$J(1,\ldots,n;P\pm) = \left(\frac{1\mp\gamma_5}{2}\right)J(1,\ldots,n;P)\ . \quad (2.6)$$

To incorporate the photon helicities it is convenient to translate the currents into the Weyl-van der Waerden spinor formalism [11] of chap. 3. For electron helicity $+\frac{1}{2}$ and positron helicity $-\frac{1}{2}$ one has

$$\begin{aligned}
J_{\dot{A}}(Q+) &= -iq_{\dot{A}} \,, && (2.7)\\
J_{\dot{A}}(Q+;1) &= ie\, q_{\dot{C}}\, e^{\dot{C}B}(1)\, \frac{(Q+K_1)_{\dot{A}B}}{(Q+K_1)^2} \,, && (2.8)\\
J_{\dot{A}}(Q+;1,\ldots,n) &= -e\, \frac{(Q+\kappa(1,n))_{\dot{A}B}}{(Q+\kappa(1,n))^2} \\
&\quad \times \sum_{m=1}^{n} J_{\dot{C}}(Q+;1,\ldots,m-1,m+1,\ldots,n)\, e^{\dot{C}B}(m) \,, && (2.9)\\
J_B(P-) &= p_B \,, && (2.10)\\
J_B(1,\ldots,n;P-) &= e\, \frac{(P+\kappa(1,n))_{\dot{C}B}}{(P+\kappa(1,n))^2} \\
&\quad \times \sum_{m=1}^{n} J_D(1,\ldots,m-1,m+1,\ldots,n;P-)\, e^{\dot{C}D}(m) \,. && (2.11)
\end{aligned}$$

In these expressions the q and p denote the Weyl-van der Waerden spinors related to the null four vectors Q and P. The polarization spinors of the photon are given by eq. (III.2.55)

$$e_+^{\dot{A}B} = \sqrt{2}\frac{k^{\dot{A}} g^B}{\langle kg \rangle} \,, \qquad (2.12)$$

$$e_-^{\dot{A}B} = \sqrt{2}\frac{g^{\dot{A}} k^B}{\langle kg \rangle^*} \,. \qquad (2.13)$$

A current like (IV.4.20) can be introduced as well, where we take for the vector boson an off shell photon

$$\mathcal{S}_\mu(Q+;1,2,\ldots,n;P-) = ie \sum_{m=0}^{n} J(Q+;1,\ldots,m)\, \gamma_\mu\, J(m+1,\ldots,n;P-) \,, \qquad (2.14)$$

or in spinor language

$$\begin{aligned}
\mathcal{S}_{\dot{A}B}(Q+;1,2,\ldots,n;P-) &= \sigma^\mu_{\dot{A}B} \mathcal{S}_\mu(Q+;1,\ldots,n;P-) \\
&= -2e \sum_{m=0}^{n} J_{\dot{A}}(Q+;1,\ldots,m)\, J_B(m+1,\ldots,n;P-) \,. && (2.15)
\end{aligned}$$

The helicity amplitude for the production of n photons and an electron-positron pair is given by

$$\mathcal{M}(P_-+;1,\ldots,n;P_+-) = \frac{1}{2}\, \mathcal{S}_{\dot{A}B}(P_-+;1,\ldots,n-1;P_+-)\, e^{\dot{A}B}(n)\Big|_{P_++P_-+\kappa(1,n)=0} \,. \qquad (2.16)$$

For the reaction

$$e^+(P_+) + e^-(P_-) \longrightarrow \mu^+(Q_+) + \mu^-(Q_-) + \gamma(K_1) + \cdots + \gamma(K_n) \tag{2.17}$$

with the photon emission off the muons the amplitude reads

$$\mathcal{M}(P_+-;P_-+;Q_+-;Q_-+;1,\ldots,n) = e\,\frac{p_{+\dot{A}}p_{-B}}{(P_+ + P_-)^2}\,\mathcal{S}^{\dot{A}B}(Q_-+;1,\ldots,n;Q_+-)\,, \tag{2.18}$$

where

$$P_+ + P_- = Q_+ + Q_- + \kappa(1,n)\,. \tag{2.19}$$

In order to evaluate (2.16) and (2.18) for certain helicity combinations, the currents $J(P_-+;1+,\ldots,n+)$, $J(1+,\ldots,n+;P_+-)$ and $\mathcal{S}_\mu(P_-+;1+,\ldots,n+;P_+-)$ are required. The actual calculation will be simplified by choosing as gauge spinor p_+, such that

$$e_+^{\dot{A}B}(m) = \sqrt{2}\,\frac{k_m^{\dot{A}}p_+^B}{\langle k_m p_+\rangle}\,. \tag{2.20}$$

The one-photon currents for these specific helicities are

$$\begin{aligned}
J_{\dot{A}}(P_-+;1+) &= i\sqrt{2}\,e\,\frac{(P_- + K_1)_{\dot{A}B}}{(P_- + K_1)^2}\,\frac{k_1^{\dot{C}}\,p_+^B}{\langle k_1 p_+\rangle}\,p_{-\dot{C}} \\
&= i\sqrt{2}\,e\,\frac{(P_- + K_1)_{\dot{A}B}\,p_+^B}{\langle p_+ k_1\rangle\langle k_1 p_-\rangle}\,, &(2.21)\\
J_B(1+;P_+-) &= 0\,, &(2.22)\\
\mathcal{S}_{\dot{A}B}(P_-+;1+;P_+-) &= -2e\,J_{\dot{A}}(P_-+;1+)\,p_{+B} &(2.23)\\
&= -2i\,\sqrt{2}e^2\frac{(P_- + K_1)_{\dot{A}C}\,p_+^C\,p_{+B}}{\langle p_+ k_1\rangle\langle k_1 p_-\rangle}\,, &(2.24)
\end{aligned}$$

Inserting (2.21) in (2.9) gives the two photon current

$$\begin{aligned}
J_{\dot{A}}(P_-+;1+,2+) &= -e\,\frac{(P_- + K_1 + K_2)_{\dot{A}B}}{(P_- + K_1 + K_2)^2} \\
&\quad \left[J_{\dot{C}}(P_-+;1+)\,e^{\dot{C}B}(2) + J_{\dot{C}}(P_-+;2+)\,e^{\dot{C}B}(1)\right] \\
&= -i(\sqrt{2}\,e)^2\,\frac{(P_- + K_1 + K_2)_{\dot{A}B}}{(P_- + K_1 + K_2)^2} \\
&\quad \times \left\{\frac{(P_- + K_1)_{\dot{C}D}\,p_+^D\,k_2^{\dot{C}}\,p_+^B\,\langle k_2 p_-\rangle}{\langle p_+ k_1\rangle\langle k_1 p_-\rangle\langle k_2 p_+\rangle\langle k_2 p_-\rangle}\right. \\
&\quad \left.+\frac{(P_- + K_2)_{\dot{C}D}\,p_+^D\,k_1^{\dot{C}}\,p_+^B\,\langle k_1 p_-\rangle}{\langle p_+ k_2\rangle\langle k_2 p_-\rangle\langle k_1 p_+\rangle\langle k_1 p_-\rangle}\right\}\,. &(2.25)
\end{aligned}$$

To simplify the current we use the relations

$$(P_- + K_i)_{\dot{C}D}\,k_j^{\dot{C}} = (P_- + K_i + K_j)_{\dot{C}D}\,k_j^{\dot{C}}\,, \tag{2.26}$$

and

$$
\begin{aligned}
&(P_- + K_1 + K_2)_{\dot{C}D}\, p_+^D\, (K_1 + K_2)^{\dot{C}E}\, p_{-E} \\
&\quad = (P_- + K_1 + K_2)_{\dot{C}D}\, (P_- + K_1 + K_2)^{\dot{C}E}\, p_+^D\, p_{-E} && (2.27) \\
&\quad = -(P_- + K_1 + K_2)^2\, \langle p_+ p_- \rangle\ , && (2.28)
\end{aligned}
$$

where we have used the relation (III.2.14). Applying these relations in eq. (2.25) results in

$$
J_{\dot{A}}(P_-+;1+,2+) = i(\sqrt{2}e)^2\, (P_- + K_1 + K_2)_{\dot{A}B}\, p_+^B\, \frac{\langle p_+ p_- \rangle}{\prod_{i=1}^2 \langle p_+ k_i \rangle \langle k_i p_- \rangle}\ . \tag{2.29}
$$

For the other two-photon spinorial current we find

$$
J_B(1+,2+;P_+-) = 0\ , \tag{2.30}
$$

and consequently, through eq. (2.15), for the vector current

$$
\begin{aligned}
&S_{\dot{A}B}(P_-+;1+,2+;P_+-) = \\
&\quad -2i(\sqrt{2})^2 e^3\, (P_- + K_1 + K_2)_{\dot{A}C}\, p_+^C\, p_{+B}\, \frac{\langle p_+ p_- \rangle}{\prod_{i=1}^2 \langle p_+ k_i \rangle \langle k_i p_- \rangle}\ . && (2.31)
\end{aligned}
$$

A generalization of these results for arbitrary n presents itself in the form

$$
\begin{aligned}
&J_{\dot{A}}(P_-+;1+,2+,\ldots,n+) = \\
&\quad i(\sqrt{2}e)^n\, (P_- + \kappa(1,n))_{\dot{A}B}\, p_+^B\, \frac{\langle p_+ p_- \rangle^{n-1}}{\prod_{i=1}^n \langle p_+ k_i \rangle \langle k_i p_- \rangle}\ , && (2.32) \\
&J_B(1+,2+,\ldots,n+;P_+-) = 0\ , && (2.33) \\
&S_{\dot{A}B}(P_-+;1+,\ldots,n+;P_+-) = \\
&\quad -2i(\sqrt{2})^n e^{n+1}\, (P_- + \kappa(1,n))_{\dot{A}C}\, p_+^C\, p_{+B}\, \frac{\langle p_+ p_- \rangle^{n-1}}{\prod_{i=1}^n \langle p_+ k_i \rangle \langle k_i p_- \rangle}\ . && (2.34)
\end{aligned}
$$

The conjecture (2.32) requires a proof, eq. (2.33) is obvious and eq. (2.34) follows directly from inserting eqs. (2.32) and (2.33) in eq. (2.15).

Assume eq. (2.32) to be valid for $l < n$, then for n photons we have from eq. (2.9) :

$$
\begin{aligned}
&J_{\dot{A}}(P_-+;1+,\ldots,n+) \\
&\quad = -e\, \frac{(P_- + \kappa(1,n))_{\dot{A}B}}{(P_- + \kappa(1,n))^2} \sum_{m=1}^{n} J_{\dot{C}}(P_-;1,\ldots,m-1,m+1,\ldots,n)\, e^{\dot{C}B}(m) \\
&\quad = i(\sqrt{2}e)^n\, \frac{(P_- + \kappa(1,n))_{\dot{A}B}}{(P_- + \kappa(1,n))^2}\, p_+^B (P_- + \kappa(1,n))_{\dot{C}D} \sum_{m=1}^{n} p_+^D\, k_m^{\dot{C}} \langle k_m p_- \rangle \\
&\quad \times \frac{\langle p_+ p_- \rangle^{n-2}}{\prod_{i=1}^n \langle p_+ k_i \rangle \langle k_i p_- \rangle} && (2.35) \\
&\quad = i(\sqrt{2}e)^n\, (P_- + \kappa(1,n))_{\dot{A}B}\, p_+^B \frac{\langle p_+ p_- \rangle^{n-1}}{\prod_{i=1}^n \langle p_+ k_i \rangle \langle k_i p_- \rangle}\ . && (2.36)
\end{aligned}
$$

To obtain eq. (2.35) use has been made of eq. (2.26). For eq. (2.36) steps (2.27) and (2.28) have been used for the case of $\kappa(1,n)$ instead of $\kappa(1,2)$.

Inserting the vector current (2.34) in the amplitudes (2.16) and (2.18) gives

$$\mathcal{M}(P_-+;1+,\ldots,n+;P_+-)=0\ , \tag{2.37}$$

$$\begin{aligned}
&\mathcal{M}(P_-+;1+,\ldots,(n-1)+,n-;P_+-)\\
&=-i(\sqrt{2})^n e^n\,(-P_+-K_n)_{\dot{A}C}\,\frac{p_+^C\,p_{+B}\,p_+^{\dot{A}}\,k_n^B}{\langle k_np_+\rangle^*}\,\frac{\langle p_+p_-\rangle^{n-2}}{\prod_{i=1}^{n-1}\langle p_+k_i\rangle\langle k_ip_-\rangle}\\
&=-i(\sqrt{2}e)^n\,\frac{\langle p_+k_n\rangle^2\langle p_+p_-\rangle^{n-2}}{\prod_{i=1}^{n-1}\langle p_+k_i\rangle\langle k_ip_-\rangle}\ ,
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
&\mathcal{M}(P_+-;P_-+;Q_+-;Q_-+;1+,\ldots,n+)\\
&=-2i(\sqrt{2})^n e^{n+2}(-Q_++P_++P_-)_{\dot{A}C}\,\frac{q_+^C\,q_{+B}\,p_+^{\dot{A}}\,p_-^B}{|\langle p_+p_-\rangle|^2}\,\frac{\langle q_+q_-\rangle^{n-1}}{\prod_{i=1}^{n}\langle q_+k_i\rangle\langle k_iq_-\rangle}\\
&=i(\sqrt{2}e)^{n+2}\,\frac{\langle p_-q_+\rangle^2}{\langle p_-p_+\rangle}\,\frac{\langle q_+q_-\rangle^{n-1}}{\prod_{i=1}^{n}\langle q_+k_i\rangle\langle k_iq_-\rangle}\ .
\end{aligned} \tag{2.39}$$

Changing the fermion helicities in (2.38) and (2.39) to opposite values amounts to the replacement $\langle p_+k_n\rangle \to \langle p_-k_n\rangle$ and $\langle p_-q_+\rangle \to \langle p_+q_-\rangle$. The results are in agreement with those of ref. [2]. For the cross section of $e^+e^- \to n$ photons or $e^+e^- \to \mu^+\mu^- + n$ photons, we can just square the expressions (2.38) and (2.39). In the former the momenta P_+, P_- should be changed into $-P_+$, $-P_-$ and the helicities of the incoming electron and positron are -1 and $+1$ respectively.

3 The gluon recursion relation

The gluon recursion relation (IV.2.29) will be solved for two special helicity configurations. The configurations are those where all helicities are the same (here +1) or all but one are the same. One can in these cases choose a gauge for the helicity spinors such that for all polarization vectors $(e_i \cdot e_j) = 0$. Through the recursion relation (IV.2.29) the currents keep this orthogonality property, thus the 4-vertex contributions vanish. As in the previous section we use the Weyl-van der Waerden spinor calculus. We have

$$J_{\dot{A}B}(1,\ldots,n)=\frac{1}{2\kappa(1,n)^2}\sum_{m=1}^{n-1}[\,J(1,\ldots,m),J(m+1,\ldots,n)\,]_{\dot{A}B}\ , \tag{3.1}$$

where in eq. (IV.2.30) the inner product between 4-vectors has been replaced by spinor contractions e.g.

$$\begin{aligned}
\kappa(m+1,n)\cdot J(1,\ldots,m) \;&\to\; \{\kappa(m+1,n),J(1,\ldots,m)\}\\
&\equiv \kappa_{\dot{C}D}(m+1,n)\,J^{\dot{C}D}(1,\ldots,m)\ ,
\end{aligned} \tag{3.2}$$

which necessitates the factor $\frac{1}{2}$ in eq. (3.1). A choice of gauge spinors which gives $\{e_i, e_j\} = 0$ is for the current $J(1+, 2+, \ldots, n+)$

$$e^{+}_{\dot{A}B}(i) = -\sqrt{2}\frac{k_{i\dot{A}} b_{+B}}{\langle i+\rangle} \ , \ 1 \le i \le n \ , \tag{3.3}$$

and for $J(1-, 2+, \ldots, n+)$

$$\begin{aligned} e^{-}_{\dot{A}B}(1) &= -\sqrt{2}\frac{k_{2\dot{A}} k_{1B}}{\langle 12\rangle^*} \ , & (3.4) \\ e^{+}_{\dot{A}B}(i) &= -\sqrt{2}\frac{k_{i\dot{A}} k_{1B}}{\langle i1\rangle} \ , \ 2 \le i \le n \ , & (3.5) \end{aligned}$$

where b_+ is at the moment an arbitrary spinor and

$$\langle i+\rangle = \langle k_i b_+\rangle \ , \langle ij\rangle = \langle k_i k_j\rangle \ . \tag{3.6}$$

Note that we have chosen a different phase convention as in eq. (III.2.55). One obtains eqs. (3.4) and (3.5) by taking $\alpha = \pi$ in eq. (III.2.53). The property $J_{\dot{A}B}(i) = X_{\dot{A}}(i) b_B$ with $b = b_+$ or $b = k_1$ extends through the recursion relation (3.1) to the n-gluon current

$$J_{\dot{A}B}(1, \ldots, n) = X_{\dot{A}}(1, \ldots, n) b_B \ . \tag{3.7}$$

Consequently, the recursion relation takes the simple form

$$\begin{aligned} J_{\dot{A}B}(1, \ldots, n) = \frac{1}{\kappa(1,n)^2} \sum_{m=1}^{n-1} & \Big(\{\kappa(m+1, n), J(1, \ldots, m)\} J_{\dot{A}B}(m+1, \ldots, n) \\ & - \{\kappa(1, m), J(m+1, \ldots, n)\} J_{\dot{A}B}(1, \ldots, m) \Big) \ . \end{aligned} \tag{3.8}$$

Consider the equal helicity case in more detail

$$\begin{aligned} J_{\dot{A}B}(1+, 2+) &= \frac{1}{\kappa(1,2)^2} \Big(\{K_2, e(1)\} e_{\dot{A}B}(2) - \{K_1, e(2)\} e_{\dot{A}B}(1) \Big) \\ &= 2 b_{+B} \frac{\langle 21\rangle^* \langle 2+\rangle k_{2\dot{A}} - \langle 12\rangle^* \langle 1+\rangle k_{1\dot{A}}}{\langle 1+\rangle\langle 2+\rangle\langle 12\rangle\langle 12\rangle^*} \\ &= 2 \frac{\kappa_{\dot{A}C}(1,2)\, b_+^C\, b_{+B}}{\langle +1\rangle\langle 12\rangle\langle 2+\rangle} \ . \end{aligned} \tag{3.9}$$

This leads to the conjecture

$$J_{\dot{A}B}(1+, 2+, \ldots, m+) = (\sqrt{2})^m \frac{\kappa_{\dot{A}C}(1,m)\, b_+^C\, b_{+B}}{\langle\langle +1, m+\rangle\rangle} \ , \tag{3.10}$$

where $\langle\langle +1, m+\rangle\rangle = \langle +1\rangle\langle 12\rangle \cdots \langle m-1, m\rangle\langle m+\rangle$. For one gluon eq. (3.10) reduces to eq. (3.3), which explains the introduction of a minus sign in the definition of the polarization vectors, eqs. (3.4) and (3.5). The conjecture will be proven by induction. Suppose (3.10) to be valid for $m < n$, then using (3.8) we find

$$
\begin{aligned}
J_{\dot{A}B}(1+,2+,\ldots,n+) &= \frac{1}{\kappa(1,n)^2}\sum_{m=1}^{n-1} \\
&\left(\kappa_{\dot{C}D}(m+1,n)\,(\sqrt{2})^m\,\frac{\kappa^{\dot{C}}{}_E(1,m)\,b_+^E\,b_+^D}{\langle\langle +1,m+\rangle\rangle}J_{\dot{A}B}(m+1,\ldots,n)\right. \\
&\left.-\kappa_{\dot{C}D}(1,m)\,(\sqrt{2})^{n-m}\,\frac{\kappa^{\dot{C}}{}_E(m+1,n)\,b_+^E\,b_+^D}{\langle\langle +(m+1),n+\rangle\rangle}J_{\dot{A}B}(1,\ldots,m)\right) \\
&= (\sqrt{2})^n\,\frac{1}{\kappa(1,n)^2}\kappa_{\dot{A}C}(1,n)\,b_+^C\,b_{+B}\,P_1^{n-1}\ ,
\end{aligned}
\tag{3.11}
$$

where

$$
P_1^{n-1} = -\sum_{m=1}^{n-1}\frac{\kappa_{\dot{C}D}(1,m)\,b_+^D\,\kappa^{\dot{C}}{}_E(m+1,n)\,b_+^E}{\langle\langle +1,m+\rangle\rangle\langle\langle +(m+1),n+\rangle\rangle}\ . \tag{3.12}
$$

When it is shown that

$$
P_1^{n-1} = \frac{\kappa(1,n)^2}{\langle\langle +1,n+\rangle\rangle}\ , \tag{3.13}
$$

we have proven the conjecture (3.10). The validity of (3.13) can again be shown by induction. For $n = 2$ it is easily verified to be correct. Suppose P_1^{n-2} is valid, then

$$
\begin{aligned}
P_1^{n-1} &= -\sum_{m=1}^{n-2}\frac{\kappa_{\dot{C}D}(1,m)\,b_+^D\,\kappa^{\dot{C}}{}_E(m+1,n-1)\,b_+^E}{\langle\langle +1,m+\rangle\rangle\langle\langle +(m+1),(n-1)+\rangle\rangle}\frac{\langle n-1,+\rangle}{\langle n-1,n\rangle\langle n+\rangle} \\
&\quad -\sum_{m=1}^{n-1}\frac{\kappa_{\dot{C}D}(1,m)\,b_+^D\,K_n^{\dot{C}}{}_E\,b_+^E}{\langle\langle +1,n+\rangle\rangle}\frac{\langle m,m+1\rangle}{\langle m+\rangle\langle +,m+1\rangle} \\
&= \frac{\langle n-1,+\rangle}{\langle n-1,n\rangle\langle n+\rangle}P_1^{n-2} - \sum_{m=1}^{n-1}\frac{\kappa_{\dot{C}D}(1,m)\,b_+^D\,K_n^{\dot{C}}{}_E\,b_+^E}{\langle\langle +1,n+\rangle\rangle}\frac{\langle m,m+1\rangle}{\langle m+\rangle\langle +,m+1\rangle} \\
&= \frac{1}{\langle\langle +1,n+\rangle\rangle} \\
&\quad\times\left(\kappa(1,n-1)^2 - \sum_{m=1}^{n-1}\kappa_{\dot{C}D}(1,m)\,b_+^D\,K_n^{\dot{C}}{}_E\,b_+^E\frac{\langle m,m+1\rangle}{\langle m+\rangle\langle +,m+1\rangle}\right) \\
&= \frac{1}{\langle\langle +1,n+\rangle\rangle} \\
&\quad\times\left(\kappa(1,n-1)^2 - \sum_{i=1}^{n-1}\left(\sum_{m=i}^{n-1}K_{i\dot{C}D}\,b_+^D\,K_n^{\dot{C}}{}_E\,b_+^E\frac{\langle m,m+1\rangle}{\langle m+\rangle\langle +,m+1\rangle}\right)\right)\ .
\end{aligned}
\tag{3.14}
$$

Since (cf. eq. (III.2.6))

$$
\frac{\langle ab\rangle}{\langle a+\rangle\langle +b\rangle} + \frac{\langle bc\rangle}{\langle b+\rangle\langle +c\rangle} = \frac{\langle ac\rangle}{\langle a+\rangle\langle +c\rangle} \tag{3.15}
$$

we have

$$\sum_{m=i}^{n-1} \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle} = \frac{\langle in\rangle}{\langle i+\rangle\langle +n\rangle} \tag{3.16}$$

and

$$\sum_{m=i}^{n-1} K_{i\dot{C}D}\, b_+^D\, K_n^{\dot{C}}{}_E\, b_+^E \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle} = -2K_i \cdot K_n \tag{3.17}$$

such that eq. (3.14) indeed reduces to (3.13).

In the case of $J(1-2+\cdots n+)$ we use as starting point the currents

$$J_{\dot{A}B}(1-) = e^-_{\dot{A}B}(1) = -\sqrt{2}\frac{k_{2\dot{A}}\, k_{1B}}{\langle 12\rangle^*} , \tag{3.18}$$

$$J_{\dot{A}B}(i+\cdots m+) = (\sqrt{2})^{m-i+1}\, \frac{\kappa_{\dot{A}C}(i,m)\, k_{1B}\, k_1^C}{\langle\langle 1i, m1\rangle\rangle} . \tag{3.19}$$

Using these currents and eq. (3.8) we find

$$J(1-2+) = \frac{1}{2\kappa(1,2)^2}[J(1-), J(2+)] = 0 , \tag{3.20}$$

$$\begin{aligned} J(1-2+3+) &= \frac{1}{\kappa(1,3)^2}\{\kappa(2,3), J(1-)\}\, J(2+3+) \\ &= -\sqrt{2}J(2+3+)\frac{\langle 31\rangle\, k_{3\dot{C}}\, k_{1D}\, K_2^{\dot{C}D}}{\kappa(1,2)^2\kappa(1,3)^2} . \end{aligned} \tag{3.21}$$

The induction conjecture for $l \geq 3$ is given by

$$J(1-2+\cdots l+) = -\sqrt{2}\, J(2+\cdots l+)\; c_l \tag{3.22}$$

where

$$c_l = \sum_{m=3}^{l} \lambda_m , \tag{3.23}$$

$$\lambda_m = \frac{\langle m1\rangle\, k_{m\dot{C}}\, k_{1D}\, \kappa^{\dot{C}D}(2,m)}{\kappa(1,m-1)^2\kappa(1,m)^2} \tag{3.24}$$

With eq. (3.22) for $l < n$ in the recursion relation (3.8) we have

$$\begin{aligned} J(1-2+\cdots n+) = \frac{1}{\kappa(1,n)^2}\Big(& \{\kappa(2,n), J(1-)\}\, J(2+\cdots n+) \\ & + \sum_{m=3}^{n-1}[J(1-2+\cdots m+), J((m+1)+\cdots n+)]\Big) \end{aligned} \tag{3.25}$$

$$\begin{aligned} = \; & -\sqrt{2}\frac{J(2+\cdots n+)}{\kappa(1,n)^2}\frac{\kappa_{\dot{C}D}(3,n)\, k_2^{\dot{C}}\, k_1^D}{\langle 12\rangle^*} \\ & -\sqrt{2}\sum_{m=3}^{n-1} \frac{c_m}{\kappa(1,n)^2}(\{\kappa(m+1,n), J(2+\cdots m+)\}\; J((m+1)+\cdots n+) \end{aligned}$$

$$
\begin{aligned}
&-\{\kappa(1,m), J((m+1)+\cdots n+)\}\, J(2+\cdots m+)) && (3.26)\\
= \;& -\sqrt{2}\frac{J(2+\cdots n+)}{\kappa(1,n)^2}\left(\frac{\kappa_{\dot{C}D}(3,n)\, k_2^{\dot{C}}\, k_1^D}{\langle 12\rangle^*}\right.\\
&\left. -\sum_{m=3}^{n-1} c_m\, \kappa_{\dot{C}D}(2,m)\, k_1^D\, \kappa^{\dot{C}}{}_E(m+1,n)\, k_1^E\, \frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}\right) . && (3.27)
\end{aligned}
$$

The way in which eq. (3.27) is obtained is similar to that of arriving at eq. (3.11). The proportionality factor in eq. (3.27) is c_{n-1} for $J(2+\cdots(n-1)+)$ from the induction hypothesis and it should be proven that it is c_n for $J(2+\cdots n+)$. The difference of these proportionality factors is from eq. (3.23) known to be

$$
\kappa(1,n)^2 c_n - \kappa(1,n-1)^2 c_{n-1} = \kappa(1,n)^2\lambda_n + 2K_n\cdot\kappa(1,n-1)\; c_{n-1} \;, \tag{3.28}
$$

whereas from eq. (3.27) it follows

$$
\begin{aligned}
\kappa(1,n)^2 c_n - \kappa(1,n-1)^2 c_{n-1} = \;& \frac{K_{n\dot{C}D}\, k_2^{\dot{C}}\, k_1^D}{\langle 12\rangle^*}\\
& -\sum_{m=3}^{n-1} c_m\, \kappa_{\dot{C}D}(2,m)\, k_1^D K_n^{\dot{C}}{}_E\, k_1^E\, \frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle} \,. && (3.29)
\end{aligned}
$$

In order to prove that eq. (3.29) reduces to eq. (3.28) we carry out the summation in the second term of eq. (3.29), using eqs. (III.2.7), (III.2.14), (3.16), (3.17), (3.23) and (3.24)

$$
\begin{aligned}
&\sum_{m=3}^{n-1}\left(\sum_{i=3}^{m}\lambda_i\right) \kappa_{\dot{C}D}(2,m)\, k_1^D K_n^{\dot{C}}{}_E\, k_1^E\, \frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}\\
&= \sum_{i=3}^{n-1}\lambda_i\left(\sum_{m=i}^{n-1}\kappa_{\dot{C}D}(2,m)\, k_1^D K_n^{\dot{C}}{}_E\, k_1^E\, \frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}\right)\\
&= \sum_{m=3}^{n-1}\lambda_m\left(-\kappa_{\dot{A}B}(1,m-1)\, k_n^{\dot{A}}\, k_1^B\frac{\langle mn\rangle}{\langle m1\rangle} - 2\,\kappa(m,n-1)\cdot K_n\right)\\
&= \sum_{m=3}^{n-1}\lambda_m\left(-2\,\kappa(1,n-1)\cdot K_n + \kappa_{\dot{A}B}(1,m-1)\, k_n^{\dot{A}}\, k_m^B\, \frac{\langle n1\rangle}{\langle m1\rangle}\right)\\
&= -2\,K_n\cdot\kappa(1,n-1)\; c_{n-1} + \langle n1\rangle \sum_{m=3}^{n-1}\frac{k_{m\dot{C}}\, k_{1D}\, \kappa^{\dot{C}D}(1,m)\kappa_{\dot{A}B}(1,m)\, k_n^{\dot{A}}\, k_m^B}{\kappa(1,m-1)^2\kappa(1,m)^2}\\
&= -2\,K_n\cdot\kappa(1,n-1)\; c_{n-1} + X \,. && (3.30)
\end{aligned}
$$

At this point the first term in eq. (3.30) gives indeed the second term in eq. (3.28). The rest R should give the term $\kappa(1,n)^2\lambda_n$. We find

$$
\begin{aligned}
R \;&=\; \frac{K_{n\dot{C}D} k_2^{\dot{C}}\, k_1^D}{\langle 12\rangle^*} - X\\
&=\; -\langle n1\rangle\left\{-\frac{\langle n2\rangle^*}{\langle 12\rangle^*} + k_{1D}\, k_{n\dot{A}}\right.
\end{aligned}
$$

$$\times \sum_{m=3}^{n-1} \frac{2\kappa(1,m-1)\cdot K_m\, \kappa^{\dot{A}D}(1,m-1) - K_m^{\cdot\dot{C}D}\kappa_{\dot{C}B}(1,m-1)\kappa^{\dot{A}B}(1,m-1)}{\kappa(1,m-1)^2\kappa(1,m)^2} \Bigg\} \tag{3.31}$$

$$= -\langle n1\rangle \left\{ -\frac{\langle n2\rangle^*}{\langle 12\rangle^*} + k_{1D}\, k_{n\dot{A}} \right.$$

$$\left. \times \sum_{m=3}^{n-1} \left[\kappa^{\dot{A}D}(1,m-1)\left(\frac{1}{\kappa(1,m-1)^2} - \frac{1}{\kappa(1,m)^2}\right) - \frac{K_m^{\cdot\dot{A}D}}{\kappa(1,m)^2} \right] \right\}$$

$$= -\langle n1\rangle k_{1D}\, k_{n\dot{A}} \left\{ -\frac{\kappa^{\dot{A}D}(1,2)}{\kappa(1,2)^2} + \sum_{m=3}^{n-1} \left[\frac{\kappa^{\dot{A}D}(1,m-1)}{\kappa(1,m-1)^2} - \frac{\kappa^{\dot{A}D}(1,m)}{\kappa(1,m)^2} \right] \right\} \tag{3.32}$$

$$= \langle n1\rangle \frac{k_{1D}\, k_{n\dot{A}}\, \kappa^{\dot{A}D}(1,n)}{\kappa(1,n-1)^2}\,, \tag{3.33}$$

where use has been made of eq. (III.2.14) for obtaining eqs. (3.31) and (3.32). The final result (3.33) is indeed $\kappa(1,n)^2\lambda_n$, as can been seen from eq. (3.24). Thus the general form (3.22) for $J(1-2+\cdots l+)$ is valid. Complex conjugation of the currents (3.10) and (3.22) gives $J(1-,2-,\ldots,n-)$ and $J(1+,2-,\ldots,n-)$.

4 The quark current and its recursion relation

Firstly we rewrite the recursion relation (3.8), which is valid for the special gauge choice (3.3)-(3.5). Since it follows in general from eq. (III.2.7) that

$$\begin{aligned} \kappa_{\dot{A}C}(1,n)\, J_{\dot{B}D}(1,\ldots,m)\, J^{\dot{B}C}(m+1,\ldots,n) = \\ \{\kappa(1,n), J(m+1,\ldots,n)\}\, J_{\dot{A}D}(1,\ldots,m) \\ -\{\kappa(m+1,n), J(1,\ldots,m)\}\, J_{\dot{A}D}(m+1,\ldots,n) \\ +J_{\dot{A}C}(m+1,\ldots,n)\, J^{\dot{B}C}(1,\ldots,m)\kappa_{\dot{B}D}(1,n) \end{aligned} \tag{4.1}$$

we obtain from eq. (3.8) another form of the recursion relation

$$J_{\dot{A}D}(1,\ldots,n) = -\frac{\kappa_{\dot{A}C}(1,n)}{\kappa(1,n)^2} \sum_{m=1}^{n-1} J_{\dot{B}D}(1,\ldots,m)\, J^{\dot{B}C}(m+1,\ldots,n)\,. \tag{4.2}$$

For this, use has been made of current conservation (IV.2.33) and the special form (3.7), which leads to the vanishing of the last term in eq. (4.1). In terms of $X_{\dot{A}}(1,\ldots,n)$ one has

$$X_{\dot{A}}(1,\ldots,n) = -\frac{\kappa_{\dot{A}C}(1,n)}{\kappa(1,n)^2} \sum_{m=1}^{n-1} X_{\dot{B}}(1,\ldots,m)\, J^{\dot{B}C}(m+1,\ldots,n)\,. \tag{4.3}$$

For a quark with positive helicity the recursion relation (IV.3.9) is translated into the Weyl-van der Waerden spinor formalism in the same way as was done for the photon current in sec. 2:

$$J_{\dot{A}}(Q+;1,\ldots,n) = -\frac{[Q+\kappa(1,n)]_{\dot{A}C}}{[Q+\kappa(1,n)]^2} \sum_{m=0}^{n-1} J_{\dot{B}}(Q+;1,\ldots,m)\, J^{\dot{B}C}(m+1,\ldots,n) \tag{4.4}$$

with

$$J_{\dot{B}}(Q+) = -iq_{\dot{B}} \ . \tag{4.5}$$

We now see that the recursion relations for $X_{\dot{A}}(1,\dots,n)$ and $J_{\dot{A}}(1;2,\dots,n)$ are the same. The starting point for $X_{\dot{B}}(1+,2+,\dots,(n+1)+)$ is

$$X_{\dot{B}}(1+) = -\sqrt{2}\frac{k_{1\dot{B}}}{\langle 1+\rangle} \ . \tag{4.6}$$

Thus we find

$$\begin{aligned} J_{\dot{A}}(Q+;1+,2+,\dots,n+) &= \frac{i}{\sqrt{2}}\langle q+\rangle X_{\dot{A}}(Q+,1+,2+,\dots,n+) \qquad (4.7)\\ &= i(\sqrt{2})^n\langle q+\rangle\frac{(Q+\kappa(1,n))_{\dot{A}C}b_+^C}{\langle +q\rangle\langle q1\rangle\langle 12\rangle\cdots\langle n+\rangle} \\ &= -i(\sqrt{2})^n\frac{(Q+\kappa(1,n))_{\dot{A}C}b_+^C}{\langle q1\rangle\langle 12\rangle\cdots\langle n+\rangle} \ . \qquad (4.8)\end{aligned}$$

From the expression for $X_{\dot{A}}(Q+,2+,\dots,m+,1-,(m+1)+,\dots,n+)$, which one could evaluate in the gauge (3.4)-(3.5), we have similarly

$$\begin{aligned} &J_{\dot{A}}(Q+;2+,\dots,m+,1-,(m+1)+,\dots,n+) = \\ &\qquad \frac{i}{\sqrt{2}}\langle q1\rangle\, X_{\dot{A}}(Q+,2+,\dots,m+,1-,(m+1)+,\dots,n+) \ . \end{aligned} \tag{4.9}$$

5 The amplitudes for specific helicity configurations

Since we have solved the recursion relation for currents in cases of specific helicity configurations we can calculate the amplitudes for these situations as well. We do this for n-gluon scattering with and without the production of other particles. The additional particles are a quark-antiquark pair alone or in combination with a vector boson. The results prove certain conjectures in the literature [4,5,6] to be correct.

5.1 Scattering of n gluons

From the currents we make subamplitudes and from them the helicity amplitudes according to eqs. (IV.4.1) and (IV.4.2). With the explicit expression for the current $J^{\dot{A}B}(2+,\dots,n+)$ in eq. (3.10) we have

$$\begin{aligned} C(1\pm,2+,\dots,n+) &= \frac{1}{2}\kappa(2,n)^2 e^{\pm}_{1\dot{A}B}\, J^{\dot{A}B}(2+,\dots,n+)\Big|_{\kappa(1,n)=0} \\ &= \frac{1}{2}(\sqrt{2})^{n-1}\,\kappa(2,n)^2 e^{\pm}_{1\dot{A}B}\,\frac{\kappa^{\dot{A}}{}_C(2,n)\,b_+^C\,b_+^B}{\langle(+2,n+)\rangle}\Bigg|_{\kappa(1,n)=0} \\ &= 0 \ . \end{aligned} \tag{5.1}$$

The vanishing of this C-function is due to the overall momentum conservation, which leads to a vanishing $\kappa(2,n)^2$. With the cyclic symmetry of the C-function, also the $\mathcal{C}$-function with one negative helicity in an arbitrary position vanishes. The helicity amplitude then vanishes as well

$$\mathcal{M}(1\pm,2+,\ldots,n+) = 0 \ . \tag{5.2}$$

The first non-trivial helicity amplitude is $\mathcal{M}(1-,2-,3+,\ldots,n+)$, for which we have to know the subamplitude $\mathcal{C}(1-,3+,\ldots,m+,2-,(m+1)+,\ldots,n+)$. With the current $J(2-,3+,\ldots,n+)$ we only obtain a $\mathcal{C}$-function with adjacent negative helicity gluons, arising from the $\kappa(2,n)^{-2}$ term in eqs. (3.22)-(3.24):

$$\begin{aligned}
\mathcal{C}(1-,2-,3+,\ldots,n+) &= \frac{1}{2}\kappa(2,n)^2 e^-_{1\dot{A}B}\, J^{\dot{A}B}(2-,3+,\ldots,n+)\,\Big|_{\kappa(1,n)=0} \\
&= \frac{g_{\dot{A}}\, k_{1B}}{\langle 1g\rangle^*} J^{\dot{A}B}(3+,\ldots,n+)\ \frac{\langle n2\rangle\, k_{2C}\, k_{n\dot{D}}\, \kappa^{\dot{D}C}(2,n)}{\kappa(2,n-1)^2}\Bigg|_{\kappa(1,n)=0} \\
&= (\sqrt{2})^{n-2}\frac{g_{\dot{A}}\, k_{1B}\, K_1^{\cdot\dot{A}}{}_E\, k_2^B\, k_2^E\, \langle n2\rangle\, k_{2C}\, k_{n\dot{D}} K_1^{\cdot\dot{D}C}}{\langle 1g\rangle^* \langle\langle 23,n2\rangle\rangle \langle n1\rangle\langle n1\rangle^*}\ \Bigg|_{\kappa(1,n)=0} \\
&= \frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle\langle 12,n1\rangle\rangle}\ .
\end{aligned} \tag{5.3}$$

For gluon 1 an arbitrary gauge spinor $g_{\dot{A}}$ is choosen. Due to gauge invariance this spinor should drop out, as is explicitly shown in eq. (5.3). From the subcyclic (IV.4.5,IV.4.6) and cyclic (IV.4.3) identities we find

$$\begin{aligned}
&\mathcal{C}(1-,3+,2-,4+,\ldots,n+) = -\mathcal{C}(1-,2-,3+,4+,\ldots,n+) \\
&\quad -\mathcal{C}(1-,2-,4+,3+,\ldots,n+) - \cdots - \mathcal{C}(1-,2-,4+,\ldots,n+,3+) \\
&= -\mathcal{C}(2-,3+,4+,\ldots,n+,1-) - \mathcal{C}(2-,4+,3+,\ldots,n+,1-) \\
&\quad - \cdots - \mathcal{C}(2-,3+,4+,\ldots,n+,3+,1-)\ .
\end{aligned} \tag{5.4}$$

From eq. (III.2.6) one derives

$$[\langle ab\rangle\langle bc\rangle\langle cd\rangle]^{-1} + [\langle ac\rangle\langle cb\rangle\langle bd\rangle]^{-1} = \frac{\langle ad\rangle}{\langle ab\rangle\langle bd\rangle}\ [\langle ac\rangle\langle cd\rangle]^{-1}\ , \tag{5.5}$$

and in general

$$\begin{aligned}
&[\langle ab\rangle\langle bc\rangle\langle cd\rangle\cdots\langle yz\rangle]^{-1} + [\langle ac\rangle\langle cb\rangle\langle bd\rangle\cdots\langle yz\rangle]^{-1} + \cdots + [\langle ac\rangle\langle cd\rangle\cdots\langle yb\rangle\langle bz\rangle]^{-1} \\
&\qquad = \frac{\langle az\rangle}{\langle ab\rangle\langle bz\rangle}\ [\langle ac\rangle\langle cd\rangle\cdots\langle yz\rangle]^{-1}\ .
\end{aligned} \tag{5.6}$$

With these relations we obtain from eq. (5.4)

$$\mathcal{C}(1-,3+,2-,4+,\ldots,n+) = \frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 45\rangle\cdots\langle n1\rangle}\ . \tag{5.7}$$

Using again the cyclic and subcyclic properties we arrive at

$$\begin{aligned}
&\mathcal{C}(1-,3+,4+,2-,5+,\ldots,n+)+\mathcal{C}(1-,4+,3+,2-,5+,\ldots,n+) \\
&\quad = \frac{(\sqrt{2})^n}{2}\left(\frac{\langle 12\rangle}{\langle 24\rangle\langle 41\rangle}\right)\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 25\rangle\langle 56\rangle\cdots\langle n1\rangle} \\
&\quad = \frac{(\sqrt{2})^n}{2}\left(\frac{1}{\langle 13\rangle\langle 34\rangle\langle 42\rangle}+\frac{1}{\langle 14\rangle\langle 43\rangle\langle 32\rangle}\right)\frac{\langle 12\rangle^4}{\langle 25\rangle\langle 56\rangle\cdots\langle n1\rangle}\ .
\end{aligned} \tag{5.8}$$

Since we know from the recursion relations what kind of pole terms belong to a $\mathcal{C}$-function we see that the first and second term on the left hand side correspond to the first and second term on the right hand side. Repeating these arguments we find in the general case

$$\begin{aligned}
&\mathcal{C}(1-,3+,4+,\ldots,m+,2-,(m+1)+,\ldots,n+) \\
&\qquad = \frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle\cdots\langle m2\rangle\langle 2(m+1)\rangle\cdots\langle n1\rangle}\ ,
\end{aligned} \tag{5.9}$$

which gives the amplitude

$$\mathcal{M}(1-,2-,3+,\ldots,n+) = i(\sqrt{2})^n g^{n-2}\langle 12\rangle^4 \sum_{P(1,\ldots,n-1)}\frac{(a_1\cdots a_n)}{\langle\langle 12,n1\rangle\rangle}\ . \tag{5.10}$$

The contribution to the cross section arises from squaring the amplitude. Using the leading order approximation (IV.4.10) we find

$$\begin{aligned}
&|\mathcal{M}(1-,2-,3+,\ldots,n+)|^2 = g^{2n-4}\,2^{4-n}\,N^{n-2}\,(N^2-1) \\
&\quad\times(K_1\cdot K_2)^4\left[\sum_{P(1,\ldots,n-1)}\frac{1}{(K_1\cdot K_2)(K_2\cdot K_3)\cdots(K_n\cdot K_1)}+\mathcal{O}(\frac{1}{N^2})\right]\ .
\end{aligned} \tag{5.11}$$

This now proves the conjecture of ref. [4].

5.2 The process producing $q\bar{q}$ and n gluons

In the amplitude for the production of $q\bar{q}$ and k gluons we need eq. (IV.4.12), which in the Weyl-van der Waerden notation reads

$$\mathcal{D}(Q+;1,\ldots,k;P-) = i\,J_{\dot{A}}(Q+;1,\ldots,k)\,\left[Q+\kappa(1,k)\right]^{\dot{A}B}p_B\ , \tag{5.12}$$

with

$$Q+P+\kappa(1,k)=0\ . \tag{5.13}$$

For the special case, where all gluons have the same helicity we insert eq. (4.8) for $J_{\dot{A}}$ and find a vanishing $\mathcal{D}$ and thus a vanishing amplitude. Consider the case where one of the gluons has a negative helicity, we use again the polarization vectors

$$\epsilon^{+}_{\dot{A}B}(i) = -\sqrt{2}\,\frac{k_{i\dot{A}}b_{+B}}{\langle i+\rangle}\ , \tag{5.14}$$

$$\epsilon^{-}_{\dot{A}B}(j) = -\sqrt{2}\,\frac{b_{-\dot{A}}k_{jB}}{\langle j-\rangle^*}\ . \tag{5.15}$$

Inserting the recursion relation (4.4), using eq. (3.7) and

$$J_{\dot{A}}(Q+;1,\ldots,m) = c X_{\dot{A}}(Q+,1,\ldots,m) \tag{5.16}$$

with some factor c we find

$$\begin{aligned}\mathcal{D}(Q+;1,\ldots,k;P-) &= -i\sum_{m=0}^{k-1} J_{\dot{B}}(Q+;1,\ldots,m)\, J^{\dot{B}C}(m+1,\ldots,k)\, p_C \\ &= -i\, c\,\langle p+)\sum_{m=0}^{k-1} X_{\dot{B}}(Q+,1,\ldots,m)\, X^{\dot{B}}(m+1,\ldots,k)\ .\end{aligned} \tag{5.17}$$

On the other hand, the n-gluon $\mathcal{C}$-function is also related to the quantities $X_{\dot{A}}$

$$\mathcal{C}(1,2,\ldots,n-) = \frac{1}{2}\,[\kappa(1,n-1)]^2\, J_{\dot{A}D}(1,2,\ldots,n-1)\, e_-^{\dot{A}D}(n) \tag{5.18}$$

with

$$\kappa(1,n) = 0\ . \tag{5.19}$$

Using eqs. (4.2) or (4.3) and (5.15) we have

$$\begin{aligned}\mathcal{C}(1,2,\ldots,n-) &= \frac{1}{2} K_{n\dot{A}C}\, b_{+D}\, e_-^{\dot{A}D}(n)\, b_+^C \sum_{m=1}^{n-2} X_{\dot{B}}(1,\ldots,m)\, X^{\dot{B}}(m+1,\ldots,n-1) \\ &= -\frac{1}{\sqrt{2}}\langle n+)(+n\rangle \sum_{m=1}^{n-2} X_{\dot{B}}(1,\ldots,m)\, X^{\dot{B}}(m+1,\ldots,n-1)\ .\end{aligned} \tag{5.20}$$

Since both eqs. (5.17) and (5.20) contain a similar sum of terms, we find

$$\begin{aligned}&\mathcal{D}(Q+;2+,\ldots,m+,1-,(m+1)+,\ldots,n;P-) \\ &= i\sqrt{2}\,\frac{c\,\langle p+)}{\langle p+)(+p\rangle}\,\mathcal{C}(Q+,2+,\ldots,m+,1-,(m+1)+,\ldots,n,P-) \\ &= \frac{\langle q1\rangle}{\langle p1\rangle}\,\mathcal{C}(Q+,2+,\ldots,m+,1-,(m+1)+,\ldots,n,P-)\ .\end{aligned} \tag{5.21}$$

where the factor c is taken from eq. (4.9) in which case $b_+ = k_1$. With the help of the explicit form of the $\mathcal{C}$-function (5.9) we now have

$$\begin{aligned}&\mathcal{D}(Q+;2+,\ldots,m+,1-,(m+1)+,\ldots,n;P-) \\ &\qquad = (\sqrt{2})^n\,\frac{\langle p1\rangle^3\langle q1\rangle}{\langle pq\rangle\langle q2\rangle\langle 23\rangle\cdots\langle m1\rangle\langle 1,m+1\rangle\cdots\langle np\rangle}\ ,\end{aligned} \tag{5.22}$$

and therefore

$$\begin{aligned}&\mathcal{M}(Q+;1-,2+,\ldots,n+;P-) \\ &\qquad = -i\,(\sqrt{2})^n g^n\,\frac{\langle p1\rangle^3\langle q1\rangle}{\langle pq\rangle}\sum_{P(1,\ldots,n)}(a_1\cdots a_n)_{ij}\,\frac{1}{\langle q1\rangle\langle 12\rangle\cdots\langle np\rangle}\ .\end{aligned} \tag{5.23}$$

This proves the conjecture of ref. [5]. For the square of this helicity amplitude we find after summing over the colours (IV.4.14)

$$|\mathcal{M}(Q+;1-,2+,\ldots,n+;P-)|^2$$
$$= 2^{2-n}g^{2n}N^{n-1}(N^2-1)\frac{(P\cdot K_1)^3(Q\cdot K_1)}{(P\cdot Q)}\sum_{P(1,\ldots,n)}\frac{1}{(Q\cdot K_1)(K_1\cdot K_2)\cdots(K_n\cdot P)}\ . \tag{5.24}$$

The case, where the helicities of the quarks are opposite from those in eq. (5.23) is obtained by an interchange of p and q (modulo a phase factor), as can be seen by C-conjugation, eqs. (IV.3.14)-(IV.3.17). When one wants the helicities of all particles in (5.23) to have the opposite value one should take the complex conjugate of all spinorial inner products $\langle ij\rangle$.

5.3 The process producing $q\bar{q}$, n gluons and a vector boson

In this subsection we show how processes with vector boson production can be incorporated, giving again explicit expressions for definite helicity combinations. Specifically we consider

$$e^+(P_+)+e^-(P_-)\to\gamma^*\to\bar{q}(Q_+)+q(Q_-)+g(K_1)+\cdots+g(K_n) \tag{5.25}$$

and

$$Z(P)\to\bar{q}(Q_+)+q(Q_-)+g(K_1)+\cdots+g(K_n)\ . \tag{5.26}$$

Similar results hold when a W instead of a Z is participating. Denoting the outgoing quark and antiquark flavours by f_- and f_+ we need the expression $\mathcal{S}_\mu$ of eq. (IV.4.20) with the general vertex

$$\Gamma_\mu^{V,f_-f_+}=L^V_{f_-f_+}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)+R^V_{f_-f_+}\gamma_\mu\left(\frac{1+\gamma_5}{2}\right)\ , \tag{5.27}$$

as given in eq. (IV.4.15).

First we consider the helicity combination, where only the right-handed part of Γ_μ, which is proportional to $R^V_{f_-f_+}$, can contribute. We find as in eqs. (2.14) and (2.15)

$$\mathcal{S}_\mu(Q_-+;1,2,\ldots,n;Q_+-)$$
$$=R^V_{f_-f_+}\sum_{m=0}^{n}J(Q_-+;1,\ldots,m)\,\gamma_\mu\,\frac{1+\gamma_5}{2}\,J(m+1,\ldots,n;Q_+-) \tag{5.28}$$
$$=iR^V_{f_-f_+}\sigma_\mu^{\dot{A}B}\sum_{m=0}^{n}J_{\dot{A}}(Q_-+;1,\ldots,m)\,J_B(m+1,\ldots,n;Q_+-)\ , \tag{5.29}$$

where we have used in the last step eqs. (III.2.23) and (III.2.24). For the gluons we choose the positive helicities with the polarization as in eq. (3.3) with $b_+=q_+$ such that we have (cf. eqs. (2.33) and (4.8)) only one term

$$\begin{aligned}\mathcal{S}_\mu(Q_-+;1+,\ldots,n+;Q_+-)&=iR^V_{f_-f_+}\sigma_\mu^{\dot{A}B}\,J_{\dot{A}}(Q_-+;1+,\ldots,n+)\,q_{+B}\\&=R^V_{f_-f_+}(\sqrt{2})^n\sigma_\mu^{\dot{A}B}\frac{[Q_-+\kappa(1,n)]_{\dot{A}C}\,q_+^C\,q_{+B}}{\langle q_-1\rangle\langle 12\rangle\cdots\langle nq_+\rangle}\ .\end{aligned} \tag{5.30}$$

In a similar fashion one obtains

$$S_\mu(Q_- -; 1+, \ldots, n+; Q_+ +) = -L^V_{f_- f_+} (\sqrt{2})^n \sigma_\mu^{\dot{A}B} \frac{[Q_+ + \kappa(1,n)]_{\dot{A}C}\, q_-^C\, q_{-B}}{\langle q_- 1\rangle\langle 12\rangle \cdots \langle n q_+\rangle} \,. \quad (5.31)$$

To this helicity combination only the left-handed part, proportional to the coefficient $L^V_{f_- f_+}$ in eq. (5.27), contributes. The quantity S_μ has to be contracted with V^μ which for reaction (5.25) assumes the forms

$$V_\mu(P_+ +, P_- -) = e\, \frac{p_{+D}\, \sigma_\mu^{\dot{E}D}\, p_{-\dot{E}}}{\langle p_+ p_-\rangle\langle p_+ p_-\rangle^*} \,, \quad (5.32)$$

$$V_\mu(P_+ -, P_- +) = e\, \frac{p_{+\dot{E}}\, \sigma_\mu^{\dot{E}D}\, p_{-D}}{\langle p_+ p_-\rangle\langle p_+ p_-\rangle^*} \,. \quad (5.33)$$

The matrix element of process (5.25) for a specific helicity combination now becomes

$$\begin{aligned}
&\mathcal{M}(P_+ +, P_- -, Q_- +, Q_+ -; 1+, \cdots, n+) \\
&= ieg^n \sum_{P(1,\ldots,n)} (a_1 \cdots a_n)_{ij} V^\mu(P_+ +, P_- -) S_\mu(Q_- +; 1+, \ldots, n+; Q_+ -) \\
&= ie^2 R^V_{f_- f_+} (\sqrt{2})^n g^n \frac{p_{+D}\, p_{-\dot{E}}\, \sigma^{\mu \dot{E} D}\, \sigma_\mu^{\dot{A}B} (P_+ + P_-)_{\dot{A}C}\, q_+^C\, q_{+B}}{\langle p_+ p_-\rangle\langle p_+ p_-\rangle^*} \\
&\quad \times \sum_{P(1,\ldots,n)} (a_1 \cdots a_n)_{ij} \frac{1}{\langle q_- 1\rangle\langle 12\rangle \cdots \langle n q_+\rangle} \\
&= -2ie^2 R^V_{f_- f_+} (\sqrt{2})^n g^n \frac{\langle p_+ q_+\rangle^2}{\langle p_+ p_-\rangle} \sum_{P(1,\ldots,n)} \frac{(a_1 \cdots a_n)_{ij}}{\langle q_- 1\rangle\langle 12\rangle \cdots \langle n q_+\rangle}
\end{aligned} \quad (5.34)$$

and

$$\begin{aligned}
&|\mathcal{M}(P_+ +, P_- -, Q_- +, Q_+ -; 1+, \ldots, n+)|^2 = 2^{2-n} g^{2n} e^4 (R^V_{f_- f_+})^2 N^{n-1} (N^2 - 1) \\
&\times \frac{(P_+ \cdot Q_+)^2}{(P_+ \cdot P_-)} \left(\sum_{P(1,\ldots,n)} \frac{1}{(Q_- \cdot K_1)(K_1 \cdot K_2) \cdots (K_n \cdot Q_+)} + \mathcal{O}(\frac{1}{N^2}) \right) \,.
\end{aligned} \quad (5.35)$$

Similar expressions arise for other helicity combinations of the fermions. This formula also applies to

$$e^-(P_- -) + q(Q_+ +) \to e^-(P_+ -) + q(Q_- +) + g(K_1 +) + \cdots + g(K_n +) \,. \quad (5.36)$$

In a similar fashion one could consider the process where a Z or W is produced by a lepton pair. Expressions like (5.32) and (5.33) should be used (see chap. 8 for further details). Instead of this we sum over all polarization states of the Z resulting in process (5.26). The polarization sum is given by

$$\sum_{pol.} V_\mu V_\nu^* = -g_{\mu\nu} + \frac{P_\mu P_\nu}{m_Z^2} \,. \quad (5.37)$$

Since $P^\mu S_\mu = 0$, we have in the case of eq. (5.30)

$$\sum_{pol.} |S_\mu V^\mu|^2 = -g_{\mu\nu} S^\mu S^{\nu *} = 2^2 e^2 (R^Z_{f_- f_+})^2 \frac{(P \cdot Q_+)^2}{(Q_- \cdot K_1)(K_1 \cdot K_2) \cdots (K_n \cdot Q_+)} . \quad (5.38)$$

For reaction (5.26) we have

$$\sum_{pol.Z} |\mathcal{M}(Q_-+, Q_+-; P; 1+, \ldots, n+)|^2 = e^2 (R^Z_{f_- f_+})^2 g^{2n} 2^{2-n} N^{n-1} (N^2 - 1)$$
$$\times (P \cdot Q_+)^2 \sum_{P(1,\ldots,n)} \frac{1}{(Q_- \cdot K_1)(K_1 \cdot K_2) \cdots (K_n \cdot Q_+)} , \quad (5.39)$$

$$\sum_{pol.Z} |\mathcal{M}(Q_--, Q_++; P; 1+, \ldots, n+)|^2 = e^2 (L^Z_{f_- f_+})^2 g^{2n} 2^{2-n} N^{n-1} (N^2 - 1)$$
$$\times (P \cdot Q_-)^2 \sum_{P(1,\ldots,n)} \frac{1}{(Q_- \cdot K_1)(K_1 \cdot K_2) \cdots (K_n \cdot Q_+)} . \quad (5.40)$$

For Q_- and Q_+ we can take incoming momenta in eqs. (5.39) and (5.40) and for P an outgoing momentum. Then (5.39) refers to a $-,+$ helicity combination for Q_-, Q_+ and (5.40) has a $+,-$ helicity combination for the incoming quarks. In this way we describe quarkpair annihilation into a Z and n gluons. When the Z is replaced by a W eq. (5.39) vanishes and in eq. (5.40) one should use the parameter $L^W_{f_- f_+}$. These relations proof the conjecture of ref. [6].

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Chapter VI

Multiple soft gluon radiation

The recursive calculational method [1] of chap. 4 is used to analyse the systematics of multiple soft gluon emission in multi parton processes [2]. Factorization properties of the subamplitudes are discussed. The case of soft gluons with equal helicities leads to simple expressions. Explicit formulae for double soft gluon emission are given. The hard partons in the processes are gluons, a $q\bar{q}$ pair with or without a vector boson. The special assumption of strong ordering of soft gluon momenta transforms the general expression into the ones known from the literature.

1 Introduction

It is well known that soft gluon emission is much more complicated than soft photon emission. In QED one has the factorization property of the soft bremsstrahlung amplitude

$$\mathcal{M}(P_1,\ldots,P_l;K_1,\ldots,K_m) = S(P_1,\ldots,P_l;K_1,\ldots,K_m)\mathcal{M}_H(P_1,\ldots,P_l)\ . \quad (1.1)$$

The momenta of the soft photons are denoted by $K_1,\ldots,K_m$ whereas $P_1,\ldots,P_l$ are the momenta of the hard process, which is described by the amplitude $\mathcal{M}_H$. In addition to the factorization of $\mathcal{M}$ into a bremsstrahlung part S and a hard part $\mathcal{M}_H$, the factor S itself factorizes

$$S(P_1,\ldots,P_l;K_1,\ldots,K_m) = \prod_{i=1}^{m} S(P_1,\ldots,P_l;K_i)\ , \quad (1.2)$$

which gives rise to independent photon emission. In QCD the amplitudes don't obey the simple factorization property (1.1). When one decides however to consider a decomposition of $\mathcal{M}$ into subamplitudes according to eq. (IV.4.1), one finds for those subamplitudes factorization like in eq. (1.1). Although eq. (1.2) is then still not valid, there are some special kinematical situations for which another kind of factorization for S holds. These are the situations in which all soft gluon helicities are the same and the case of a strong ordering of the soft gluon energies i.e. the first one being much softer than the second one, which in turn is much softer than the third etc.

In the literature soft gluon emission cross sections have been studied [3,4,5]. In ref. [3,4] the emphasis is on special momenta configurations for which a property like (1.2) holds, in ref. [5] single soft gluon emission in specific four parton processes is considered. In this chapter we studie the systematics of multiple soft gluon emission [2]. When we have m soft gluons with momenta

$$K_i = (\omega_i, \mathbf{K}_i) \ , \ i = 1, \ldots, m \ , \tag{1.3}$$

we study the most singular terms in the amplitudes which behave like

$$\mathcal{M} \sim (\omega_1 \omega_2 \cdots \omega_m)^{-1} \widetilde{\mathcal{M}} \ , \tag{1.4}$$

where

$$\omega_i \ll |\mathbf{P}_j| \, , \ i = 1, \ldots, m \ , j = 1, \ldots, l \ . \tag{1.5}$$

We shall analyse processes where only gluons participate and processes where besides gluons also a quark pair with or without a vector boson is present. It will be shown that the bremsstrahlung factor S does not depend on the underlying hard process.

The advantage of introducing the subamplitudes $\mathcal{C}$ of eq. (IV.4.2) here is that the soft gluon emission for the subamplitude can be shown to factorize as in (1.1). Then it can be investigated whether some factorization equivalent to eq. (1.2) occurs in special situations. Of course, even when the bremsstrahlung factors S are known for the $\mathcal{C}$-functions the expression for $\mathcal{M}$ becomes involved. This complicated structure propagates into the cross section. Nevertheless it is worthwhile to have a systematic procedure to find the soft gluon matrix elements. This could be of importance for higher order calculations which require infrared cancellations. The usual strong ordering assumption can be compared to the exact soft gluon emission cross section. The simplest test case is the emission of two soft gluons, for which we will give explicit expressions. The procedure to evaluate multiple soft gluon emission will be presented and applied to some special cases.

The actual outline of this chapter is as follows. In sec. 2, 3, 4 and 5 we consider pure gluonic processes. In sec. 2 the behaviour of the gluon currents under gauge transformations is examined. This behaviour has to be known in the next sections. Single bremsstrahlung is evaluated in sec. 3. Sec. 4 treats double bremsstrahlung in great detail, whereas sec. 5 is concerned with multiple soft gluon emission. The equal helicity case is shown to be simple and the strong ordering limit is applied. Sec. 6 introduces a $q\bar{q}$ pair into the previous discussions, whereas sec. 7 adds an electroweak vector boson.

2 The gauge behaviour of the gluonic current

We will need to know the gauge dependence of the gluon current (IV.2.29) in the following sections. By this we mean we must know the current when replacing a polarization vector by its corresponding momentum vector. Replacing polarization vector $J(m)$ by its momentum vector K_m in the gluonic current (IV.2.29) gives the result

$$J_\mu(1,\ldots,K_m,\ldots,n) = \frac{\kappa(1,n)^\nu}{\kappa(1,n)^2}[\kappa_\nu(1,n)Y_\mu(1,\ldots,K_m,\ldots,n)-\kappa_\mu(1,n)Y_\nu(1,\ldots,K_m,\ldots,n)] , \tag{2.1}$$

where

$$\begin{aligned} Y_\mu(1,\ldots,K_m,\ldots,n) &= \sum_{l=1}^{m-1} J_\mu(1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,K_m,\ldots,n)}{\kappa(l+1,n)^2} \\ &- \sum_{l=m}^{n-1} \frac{\kappa(1,l)\cdot Y(1,\ldots,K_m,\ldots,l)}{\kappa(1,l)^2} J_\mu(l+1,\ldots,n) , \\ & \qquad 1 < m < n , \end{aligned} \tag{2.2}$$

$$Y_\mu(K_1,2,\ldots,n) = J_\mu(2,\ldots,n) - \sum_{l=2}^{n-1} \frac{\kappa(1,l)\cdot Y(K_1,2,\ldots,l)}{\kappa(1,l)^2} J_\mu(l+1,\ldots,n) , \tag{2.3}$$

$$\begin{aligned} Y_\mu(1,\ldots,n-1,K_n) &= -J_\mu(1,\ldots,n-1) \\ &+ \sum_{l=1}^{n-2} J_\mu(1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,n-1,K_n)}{\kappa(l+1,n)^2} . \end{aligned} \tag{2.4}$$

We will now prove eqs. (2.1)-(2.4). The gluon current consists of two parts. One part, expressed in a quantity $Y_\mu(1,\ldots,n)$, depends on the specific gauge choice for the helicity vectors and will not contribute to the physical amplitude. Another part is gauge independent and is denoted by $G_\mu(1,\ldots,n)\kappa(1,n)^{-2}$. The physical subamplitude (IV.4.2) is determined by this gauge invariant part in the following way

$$\mathcal{C}(1,\ldots,n,n+1) = G_\mu(1,\ldots,n)J^\mu(n+1)\Big|_{\kappa(1,n+1)=0} . \tag{2.5}$$

Because of the gauge independence of the subamplitude one must have

$$G_\mu(1,\ldots,K_m,\ldots,n) = 0 , \tag{2.6}$$

$$\kappa_\mu(1,n)G^\mu(1,\ldots,n) = 0 . \tag{2.7}$$

Using eqs. (2.5) and (2.7) and the current conservation (IV.2.33) of the current J_μ we arrive at the general form

$$\begin{aligned} J_\mu(1,\ldots,n) = \frac{1}{\kappa(1,n)^2}\Big\{&G_\mu(1,\ldots,n) \\ &+\kappa^\nu(1,n)\Big[\kappa_\nu(1,n)Y_\mu(1,\ldots,n) - \kappa_\mu(1,n)Y_\nu(1,\ldots,n)\Big]\Big\} . \end{aligned} \tag{2.8}$$

Substituting for $J(m)$ the momentum K_m gives the gauge terms originating from the gluon current of eq. (2.1) where the explicit expressions for $Y_\mu(1,\ldots,K_m,\ldots,n)$

are given in eqs. (2.2)-(2.4). That these specific forms hold will be proven with the help of the recursion relation (IV.2.29) and the generalized subcyclic identities (IV.2.34).

The first step is to prove eq. (2.3) from which eq. (2.4) follows from the reflective property (IV.2.32). When we replace in the two-gluon current $J(1)$ by K_1 we find from eq. (IV.2.8)

$$J_\mu(K_1,2) = \frac{1}{\kappa(1,2)^2}\Big[2K_1\cdot K_2 J_\mu(2) - K_1\cdot J(2)(K_1+K_2)_\mu\Big]\ , \tag{2.9}$$

from which follows

$$Y_\mu(K_1,2) = J_\mu(2)\ . \tag{2.10}$$

Again, explicit calculation gives

$$Y_\mu(K_1,2,3) = J_\mu(2,3) - \frac{K_1\cdot Y(K_1,2)}{\kappa(1,2)^2}J_\mu(3)\ . \tag{2.11}$$

By means of induction and using the recursion relation it is established that

$$Y_\mu(K_1,2,\ldots,n) = J_\mu(2\ldots,n) - \sum_{l=2}^{n-1}\frac{\kappa(1,l)\cdot Y(K_1,2,\ldots,l)}{\kappa(1,l)^2}J_\mu(l+1,\ldots,n)\ . \tag{2.12}$$

In order to prove eq. (2.2) the generalized subcyclic relations (IV.2.34) are used for which we use the notation

$$\begin{aligned}\sum_{\mathrm{Perm}(i,j)} J(i_1,\ldots,i_m,j_1,\ldots,j_k) &= J(\underbrace{i_1,\ldots,i_m},\underbrace{j_1,\ldots,j_k}) \\ &= 0\ ,\Big(1\le m\le n-1\ ,m+k=n\Big)\ . \end{aligned}\tag{2.13}$$

A special case is

$$J(\underbrace{1},\underbrace{2,\ldots,n}) = 0\ , \tag{2.14}$$

which implies

$$Y(\underbrace{1},\underbrace{K_2,3,\ldots,n}) = 0\ . \tag{2.15}$$

Thus we can write

$$Y(1,K_2,3,\ldots,n) = -\sum_{i=2}^{n-1}Y(K_2,3,\ldots,i,1,i+1,\ldots,n)\ , \tag{2.16}$$

which upon insertion of eq. (2.12) proves eq. (2.2) for $m=2$. Suppose that eq. (2.2) holds for $1<l<m$. Then we prove it to hold for $l=m$. Indeed the relation

$$Y(\underbrace{1,\ldots,m-1},\underbrace{K_m,m+1,\ldots,n}) = 0 \tag{2.17}$$

expresses the $l=m$ case into the known $l<m$ cases, which can be seen to add to eq. (2.2) with $l=m$.

3 Single soft gluon emission in an n-gluon process

As a warming up for the next sections we will look at the single soft gluon behaviour of the gluonic current. Consider the limit in which gluon 2 becomes soft which will be denoted by underlining label 2. We then have from eqs. (IV.2.8), (IV.2.24), (2.1), (2.10) and (2.11)

$$J(1,\underline{2}) = s_{1\underline{2}}J(1) + t_{1\underline{2}}K_1 \ , \tag{3.1}$$

$$J(\underline{2},3) = s_{\underline{2}3}J(3) + t_{\underline{2}3}K_3 \ , \tag{3.2}$$

$$J(1,\underline{2},3) = (s_{1\underline{2}} + s_{\underline{2}3})J(1,3) + t_{1\underline{2}}J(K_1,3) + t_{\underline{2}3}J(1,K_3) \ , \tag{3.3}$$

$$s_{i\underline{j}} = -s_{\underline{j}i} = \frac{K_j \cdot J(i)}{K_j \cdot K_i} \ . \tag{3.4}$$

$$t_{i\underline{j}} = -t_{\underline{j}i} = -\frac{J(i)\cdot J(j)}{2K_j \cdot K_i} \ . \tag{3.5}$$

From the recursion relation (IV.2.29), eq. (3.2) is easily generalized to

$$J(\underline{2},3,4,\ldots,n) = s_{\underline{2}3}J(3,4,\ldots,n) + t_{\underline{2}3}J(K_3,4,\ldots,n) \tag{3.6}$$

and similarly eq. (3.1), such that eq. (3.3) now becomes

$$\begin{aligned} J(1,\ldots,m-1,\underline{m},m+1,\ldots,n) &= (s_{m-1\,\underline{m}} + s_{\underline{m}\,m+1})J(1,\ldots,m-1,m+1,\ldots,n) \\ &+ t_{m-1\,\underline{m}}J(1,\ldots,m-2,K_{m-1},m+1,\ldots,n) \\ &+ t_{\underline{m}\,m+1}J(1,\ldots,m-1,K_{m+1},m+2,\ldots,n) . \end{aligned} \tag{3.7}$$

Since we are interested in $\mathcal{C}$-functions and amplitudes only the first term in eq. (3.7) is relevant, as can be seen from eqs. (2.1) and (IV.4.2). One has

$$\mathcal{C}(1,\ldots,m-1,\underline{m},m+1,\ldots,n) = s_{m-1\,\underline{m}\,m+1}\mathcal{C}(1,\ldots,m-1,m+1,\ldots n) \ , \tag{3.8}$$

where

$$s_{m-1\,\underline{m}\,m+1} = s_{m-1\,\underline{m}} + s_{\underline{m}\,m+1} \tag{3.9}$$

$$= \frac{K_{m-1}\cdot F_m \cdot K_{m+1}}{K_{m-1}\cdot K_m\, K_m \cdot K_{m+1}} \ , \tag{3.10}$$

with $F_m^{\mu\nu}$ the abelian part of the gluon field strength

$$F_m^{\mu\nu} = K_m^{\mu}\, J^{\nu}(m) - K_m^{\nu}\, J^{\mu}(m) \ , \tag{3.11}$$

which shows explicitly the gauge invariance of the soft gluon factor.

In the following it is often useful to present results for specific helicities. As has been shown in chap. 3, Weyl-van der Waerden spinor calculus [6] is very convenient for this purpose. For the polarization vectors we choose

$$J^{+}_{\dot{A}B}(i) = e^{+}_{\dot{A}B}(i) = -\sqrt{2}\frac{k_{i\dot{A}}\, b_B}{\langle ib\rangle} \ , \tag{3.12}$$

$$J^-_{\dot{A}B}(i) = e^-_{\dot{A}B}(i) = -\sqrt{2}\frac{b_{\dot{A}}\, k_{iB}}{\langle ib\rangle^*} \,, \tag{3.13}$$

leading to

$$s^+_{\underline{m}\, m+1} = \sqrt{2}\frac{\langle m{+}1\, b\rangle}{\langle mb\rangle}\,\frac{1}{\langle m\, m{+}1\rangle} \,, \tag{3.14}$$

$$s^+_{m-1\,\underline{m}\, m+1} = (s^-_{m-1\,\underline{m}\, m+1})^* = \sqrt{2}\frac{\langle m{-}1\, m{+}1\rangle}{\langle m{-}1\, m\rangle\langle m\, m{+}1\rangle} \,. \tag{3.15}$$

A similar result was reported in ref. [7]. The spinorial inner products $\langle ij\rangle$ have as usual the property

$$|\langle ij\rangle|^2 = 2K_i \cdot K_j \,. \tag{3.16}$$

Although the soft gluon factorization is simple when considering subamplitudes like in eq. (3.8) it becomes more involved for the amplitude (IV.4.1) and the squared matrix element. In leading order in the colour one has, after performing the colour sums (IV.4.10)

$$|\mathcal{M}(1,\ldots,n)|^2 = \alpha^{n-2}(N^2-1) \sum_{P(1,\ldots,n-1)} |\mathcal{C}(1,\ldots,n-1,n)|^2 \,, \tag{3.17}$$

with

$$\alpha = \frac{g^2 N}{2} \,. \tag{3.18}$$

For $n = 4,5$ eq. (3.17) is exact, whereas for higher values of n interference terms between different $\mathcal{C}$-functions arise, which are however suppressed being of order N^{-2} with respect to the terms in eq. (3.17). Consider gluon n as soft, apply eqs. (IV.4.3) and (3.8) and rearrange the summation such that we obtain

$$\begin{aligned} |\mathcal{M}(1,\ldots,\underline{n})|^2 &= \alpha^{n-2}(N^2-1) \sum_{P(1,\ldots,n-2)} \sum_{C(1,\ldots,n-1)} \left|s_{n-1\,\underline{n}\,1}\right|^2 |\mathcal{C}(1,2,\ldots,n-1)|^2 \\ &= \alpha^{n-3}(N^2-1) \sum_{P(1,\ldots,n-2)} S(1,2,\ldots,n-1;n)\,|\mathcal{C}(1,\ldots,n-1)|^2 \,, \end{aligned} \tag{3.19}$$

with

$$S(1,2,\ldots,n-1;n) = \alpha \sum_{C(1,2,\ldots,n-1)} \left|s_{n-1\,\underline{n}\,1}\right|^2 \,. \tag{3.20}$$

Thus the cross section does not factorize when gluon n becomes soft, instead one has a weighted sum of squares of $\mathcal{C}$-functions belonging to the $(n-1)$ gluon process. For the cases $n = 5$ and $n = 6$ we explicitly evaluate eq. (3.19), where we also perform the sum over helicities. We find from eq. (3.15)

$$\begin{aligned} \sum_{\lambda_5} S(1,2,3,4;5) &= \alpha \sum_{\lambda_5} \sum_{C(1234)} \left|s_{4\underline{5}1}\right|^2 \\ &= 2\alpha\left(\frac{(1\cdot 2)}{(1\cdot 5)(5\cdot 2)} + \frac{(2\cdot 3)}{(2\cdot 5)(5\cdot 3)} + \frac{(3\cdot 4)}{(3\cdot 5)(5\cdot 4)} + \frac{(4\cdot 1)}{(4\cdot 5)(5\cdot 1)}\right) , \end{aligned} \tag{3.21}$$

with

$$(i \cdot j) = K_i \cdot K_j \,. \tag{3.22}$$

For $n > 5$ we get similar expressions.The explicit $\mathcal{C}$-functions for 4 or 5 hard gluons are obtained from the general formula (V.5.9)

$$\begin{aligned}\mathcal{C}(1-,3+,\ldots,m+,2-,(m+1)+,\ldots,n+)\\ = \frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle\cdots\langle m2\rangle\langle 2\,m{+}1\rangle\cdots\langle n1\rangle}\,,\end{aligned} \tag{3.23}$$

which gives

$$\sum_{\lambda_i}|\mathcal{C}(1\lambda_1,2\lambda_2,3\lambda_3,4\lambda_4)|^2 = 4\,\frac{\sum_{i=1}^{3}\sum_{j=i+1}^{4}(i\cdot j)^4}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 1)}\,, \tag{3.24}$$

and

$$\sum_{\lambda_i}|\mathcal{C}(1\lambda_1,\ldots,5\lambda_5|^2 = 8\,\frac{\sum_{i=1}^{4}\sum_{j=i+1}^{5}(i\cdot j)^4}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 5)(5\cdot 1)}\,. \tag{3.25}$$

Here use has been made of the fact that for all helicities equal, or all but one equal the $\mathcal{C}$-functions vanish. Combining the above results, one finds

$$\begin{aligned}\sum_{\lambda_i}|\mathcal{M}(1\lambda_1,\ldots,4\lambda_4,\underline{5}\lambda_5)|^2 &= 8\alpha^3(N^2-1)\left[\sum_{i=1}^{3}\sum_{j=i+1}^{4}(i\cdot j)^4\right]\\ &\times\sum_{P(1234)}\frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 5)(5\cdot 1)}\,,\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\sum_{\lambda_i}|\mathcal{M}(1\lambda_1,\ldots,5\lambda_5,\underline{6}\lambda_6)|^2 &= 16\alpha^4(N^2-1)\left[\sum_{i=1}^{4}\sum_{j=i+1}^{5}(i\cdot j)^4\right]\\ &\times\sum_{P(12345)}\frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 5)(5\cdot 6)(6\cdot 1)}\,.\end{aligned} \tag{3.27}$$

We have verified numerically that eq. (3.26) agrees with the soft gluon expression in ref. [5]. It should be noted that although eq. (3.17) for $n = 6$ is leading order in N, eq. (3.27) turns out to be numerically a very good approximation to the exact soft gluon expression. The interference terms of order N^{-2} undergo sizable cancellations in the soft limit. For $n > 6$ this is not anymore the case [8].

4 Double soft gluon emission

Let gluons 2 and 3 be soft and consider $J(\underline{2},\underline{3},4)$. The most singular terms can be read off from the recursion relation (IV.2.24), giving

$$\begin{aligned}J(\underline{2},\underline{3},4) &= \frac{1}{2K_4\cdot\kappa(2,3)}\Big([J(\underline{2}),J(\underline{3},4)] + [J(\underline{2},\underline{3}),J(4)]\Big)\\ &= s_{\underline{23}4}J(4) + t_{\underline{23}4}K_4\end{aligned} \tag{4.1}$$

with

$$s_{\underline{23}4} = \frac{K_4 \cdot (J(\underline{2},\underline{3}) + s_{\underline{3}4}J(\underline{2}))}{K_4 \cdot (K_2 + K_3)} , \tag{4.2}$$

$$t_{\underline{23}4} = \frac{-J(4) \cdot (J(\underline{2},\underline{3}) + s_{\underline{3}4}J(\underline{2})) + t_{\underline{3}4}K_4 \cdot J(\underline{2})}{2K_4 \cdot (K_2 + K_3)} . \tag{4.3}$$

Use of eq. (IV.2.25) gives

$$J(4,\underline{3},\underline{2}) = s_{4\underline{32}}J(4) + t_{4\underline{32}}K_4 \tag{4.4}$$

with

$$s_{4\underline{32}} = s_{\underline{23}4} \ , \ t_{4\underline{32}} = t_{\underline{23}4} \ . \tag{4.5}$$

The recursion relation, eqs. (4.1) and (4.4) lead to

$$\begin{aligned} J(1,\underline{2},\underline{3},4) \ &= \ s_{1\underline{23}4}J(1,4) + (t_{1\underline{23}} + t_{1\underline{2}}s_{\underline{3}4})J(K_1,4) \\ &\quad +(t_{\underline{23}4} + s_{1\underline{2}}t_{\underline{3}4})J(1,K_4) + t_{1\underline{2}}t_{\underline{3}4}J(K_1,K_4) \ , \end{aligned} \tag{4.6}$$

with

$$s_{1\underline{23}4} = s_{1\underline{23}} + s_{1\underline{2}}s_{\underline{3}4} + s_{\underline{23}4} \ . \tag{4.7}$$

The general n particle case takes the form

$$\begin{aligned} &J(1,2,\ldots,m-1,\underline{m},\underline{m+1},m+2,\ldots,n) = \\ &\quad s_{m-1\,\underline{m}\,\underline{m+1}\,m+2}J(1,\ldots,m-1,m+2,\ldots,n) \\ &\quad +(t_{m-1\,\underline{m}\,\underline{m+1}} + t_{m-1\,\underline{m}}s_{\underline{m+1}\,m+2})J(1,\ldots,K_{m-1},m+2,\ldots,n) \\ &\quad +(t_{\underline{m}\,\underline{m+1}\,m+2} + s_{m-1\,\underline{m}}t_{\underline{m+1}\,m+2})J(1,\ldots,m-1,K_{m+2},\ldots,n) \\ &\quad +t_{m-1\,\underline{m}}t_{\underline{m+1}\,m+2}J(1,\ldots,K_{m-1},K_{m+2},\ldots,n) \ , \end{aligned} \tag{4.8}$$

with

$$s_{m-1\,\underline{m}\,\underline{m+1}\,m+2} = s_{m-1\,\underline{m}\,\underline{m+1}} + s_{m-1\,\underline{m}}s_{\underline{m+1}\,m+2} + s_{\underline{m}\,\underline{m+1}\,m+2} \tag{4.9}$$

and

$$s_{\underline{m}\,\underline{m+1}\,m+2} = s_{m+2\underline{m+1}\,\underline{m}} = \frac{K_{m+2} \cdot (J(\underline{m},\underline{m+1}) + s_{\underline{m+1}\,m+2}J(\underline{m}))}{K_{m+2} \cdot \kappa(m,m+1)} . \tag{4.10}$$

Again, the relevant two soft gluon factor can be expressed in the gauge invariant abelian field strengths F of eq. (3.11):

$$\begin{aligned} s_{m-1\,\underline{m}\,\underline{m+1}\,m+2} \ &= \ \frac{K_{m-1} \cdot F_m \cdot F_{m+1} \cdot K_{m+2}}{(K_{m-1} \cdot K_m)(K_m \cdot K_{m+1})(K_{m+1} \cdot K_{m+2})} \\ &\quad - \frac{K_{m-1} \cdot F_m \cdot F_{m+1} \cdot K_{m-1}}{(K_{m-1} \cdot K_m)(K_m \cdot K_{m+1})(K_{m-1} \cdot \kappa(m,m+1))} \\ &\quad - \frac{K_{m+2} \cdot F_m \cdot F_{m+1} \cdot K_{m+2}}{(K_m \cdot K_{m+1})(K_{m+1} \cdot K_{m+2})(K_{m+2} \cdot \kappa(m,m+1))} \ . \end{aligned} \tag{4.11}$$

We note two properties of the soft gluon factor. The first one concerns its symmetric sum

$$s_{m-1\,\underline{m}\,\underline{m+1}\,m+2} + s_{m-1\,\underline{m+1}\,\underline{m}\,m+2} = s_{m-1\,\underline{m}\,m+2}s_{m-1\,\underline{m+1}\,m+2} \ . \tag{4.12}$$

This relation follows from

$$s_{\underline{m}\,\underline{m+1}\,m+2} + s_{\underline{m+1}\,\underline{m}\,m+2} = s_{\underline{m}\,m+2}s_{\underline{m+1}\,m+2} \;, \tag{4.13}$$

a similar relation for $s_{m-1\,\underline{m}\,\underline{m+1}}$, eqs. (3.4), (3.9) and (4.9). Notice that with this symmetrization we have independent gluon emission as in eq. (1.2).

The second relation concerns the strong ordering of the soft gluons i.e.

$$\omega_m \ll \omega_{m+1} \;. \tag{4.14}$$

In that case eq. (4.10) gives

$$s_{\underline{m}\,\underline{m+1}\,m+2} = s_{\underline{m}\,m+1}s_{\underline{m+1}\,m+2} + t_{\underline{m}\,m+1} \;, \tag{4.15}$$

$$s_{m-1\,\underline{m}\,\underline{m+1}} = s_{m-1\,\underline{m+1}}(s_{m-1\,\underline{m}} + s_{\underline{m}\,m+1}) + t_{m\,\underline{m+1}} \;, \tag{4.16}$$

such that

$$s_{m-1\,\underline{m}\,\underline{m+1}\,m+2} = s_{m-1\,\underline{m}\,m+1}s_{m-1\,\underline{m+1}\,m+2} \;. \tag{4.17}$$

The terms $t_{\underline{m}\,m+1}$ and $t_{m\,\underline{m+1}}$ in eqs. (4.15) and (4.16) can be considered as gauge terms arising in $s_{m-1\,\underline{m}\,\underline{m+1}\,m+2}$ by the replacement of $J(m+1)$ by K_{m+1}. Since the factor $s_{m-1\,\underline{m}\,\underline{m+1}\,m+2}$ is gauge invariant these gauge terms should cancel. Here one can also see this cancellation explicitly upon using eq. (3.5).

As anticipated in sec. 1 we have a factorization of the $\mathcal{C}$-function

$$\mathcal{C}(1,\dots m-1,\underline{m},\underline{m+1},m+2,\dots,n) = s_{m-1\,\underline{m}\,\underline{m+1}\,m+2}\mathcal{C}(1,\dots,m-1,m+2\dots,n) \;, \tag{4.18}$$

where in the special case of strong ordering the soft gluon factor itself factorizes, but in a way different from eq. (1.2). When one takes the symmetric sum of the soft gluon factors the result factorizes as in eq. (1.2).

When we consider the helicity dependence of the soft gluon factor we see in the equal helicity case a split-up like in eq. (4.17) without the requirement of strong ordering (4.14). Use of eqs. (3.12) and (3.13) in eqs. (4.2) and (4.5) gives

$$s^{++}_{\underline{m}\,\underline{m+1}\,m+2} = 2\frac{\langle m{+}2\;b\rangle}{\langle m b\rangle}\,\frac{1}{\langle m\;m{+}1\rangle\langle m{+}1\;m{+}2\rangle} \;, \tag{4.19}$$

$$s^{++}_{m-1\,\underline{m}\,\underline{m+1}} = 2\frac{\langle m{-}1\;b\rangle}{\langle m{+}1\;b\rangle}\,\frac{1}{\langle m{-}1\;m\rangle\langle m\;m{+}1\rangle} \;, \tag{4.20}$$

and with eqs. (3.14) and (4.9)

$$s^{++}_{m-1\,\underline{m}\,\underline{m+1}\,m+2} = 2\frac{\langle m{-}1\;m{+}2\rangle}{\langle m{-}1\;m\rangle\langle m\;m{+}1\rangle\langle m{+}1\;m{+}2\rangle} \;, \tag{4.21}$$

which can be written as eq. (4.17) and thus factorizes. Similarly, or directly from eq. (4.11), one obtains

$$\begin{aligned} s^{-+}_{m-1\,\underline{m}\,\underline{m+1}\,m+2} \;=\; & \frac{1}{K_m\cdot K_{m+1}}\left[\frac{\langle m{-}1\;m{+}1\rangle^*\langle m{+}2\;m\rangle}{\langle m{-}1\;m\rangle^*\langle m{+}2\;m{+}1\rangle}\right.\\ & -\frac{\langle m{-}1\;m{+}1\rangle^{*2}}{2K_{m-1}\cdot(K_m+K_{m+1})}\,\frac{\langle m{-}1\;m\rangle}{\langle m{-}1\;m\rangle^*}\\ & \left.-\frac{\langle m{+}2\;m\rangle^2}{2K_{m+2}\cdot(K_m+K_{m+1})}\,\frac{\langle m{+}2\;m{+}1\rangle^*}{\langle m{+}2\;m{+}1\rangle}\right] \;, \end{aligned} \tag{4.22}$$

and

$$s^{--}_{m-1\,\underline{m}\,\underline{m+1}\,m+2} = \left(s^{++}_{m-1\,\underline{m}\,\underline{m+1}\,m+2}\right)^* , \tag{4.23}$$

$$s^{+-}_{m-1\,\underline{m}\,\underline{m+1}\,m+2} = \left(s^{-+}_{m-1\,\underline{m}\,\underline{m+1}\,m+2}\right)^* . \tag{4.24}$$

We see from eq. (4.22) that in the unequal helicity case no simple factorization occurs as in the equal helicity case. Only after assuming strong ordering one obtains (4.17).

Next we turn to the double soft gluon limit in the matrix element squared. In eq. (3.17) gluons $(n-1)$ and n become soft. Depending on the position of gluon $(n-1)$ we get a different factorization

$$\mathcal{C}(1,2,\ldots,n-2,\underline{n-1},\underline{n}) = s_{n-2\,\underline{n-1}\,\underline{n}\,1}\,\mathcal{C}(1,2,\ldots,n-2) , \tag{4.25}$$

$$\mathcal{C}(1,2,\ldots,m-1,\underline{n-1},m,\ldots,n-2,\underline{n}) = s_{m-1\,\underline{n-1}\,m}\,s_{n-2\,\underline{n}\,1}\,\mathcal{C}(1,2,\ldots,n-2) . \tag{4.26}$$

We introduce

$$\begin{aligned}
&S(1,2,\ldots,n-2;n-1,n) \\
&= \alpha^2 \sum_{C(12\cdots n-2)} \left[\left|s_{n-2\,\underline{n-1}\,\underline{n}\,1}\right|^2 + \left|s_{n-2\,\underline{n}\,\underline{n-1}\,1}\right|^2 + \left(\sum_{m=2}^{n-2}\left|s_{m-1\,\underline{n-1}\,m}\right|^2\right)\left|s_{n-2\,\underline{n}\,1}\right|^2\right]
\end{aligned} \tag{4.27}$$

$$= \alpha^2 \sum_{C(12\cdots n-2)} \left[\left|s_{n-2\,\underline{n}\,1}\right|^2 \sum_{C(12\cdots n-2)} \left|s_{n-2\,\underline{n-1}\,1}\right|^2 - 2Re\left(s^*_{n-2\,\underline{n-1}\,\underline{n}\,1}\,s_{n-2\,\underline{n}\,\underline{n-1}\,1}\right)\right] \tag{4.28}$$

$$\begin{aligned}
&= S(1,2,\ldots,n-2;n-1)S(1,2,\ldots,n-2;n) \\
&\quad - 2\alpha^2 \sum_{C(12\cdots n-2)} Re\left[s^*_{n-2\,\underline{n-1}\,\underline{n}\,1}\,s_{n-2\,\underline{n}\,\underline{n-1}\,1}\right] .
\end{aligned} \tag{4.29}$$

Use has been made of eq. (4.12) to obtain eq. (4.28) whereas the definition (3.20) is introduced in eq. (4.29). Note that the soft factor S for two soft gluons has a part which factorizes into independent single gluon emission functions and a part in which the soft gluons depend on each other. The latter part is typical for the non-abelian theory.

In the limit that $\omega_{n-1} \ll \omega_n$ we use eq. (4.17) in eq. (4.27) such that we find

$$S(1,2,\ldots,n-2;\underline{n-1}\,n) = S(1,2,\ldots,n-2,n;n-1)\,S(1,2,\ldots,n-2;n) , \tag{4.30}$$

or for $\omega_n \ll \omega_{n-1}$

$$S(1,2,\ldots,n-2;n-1\,\underline{n}) = S(1,2,\ldots,n-2;n-1)\,S(1,2,\ldots,n-1;n) . \tag{4.31}$$

For equal helicity gluons n and $n-1$ the same factorization holds, e.g.

$$\begin{aligned}
&S(1,2,\ldots,n-2;(n-1)+n+) \\
&\qquad = S(1,2,\ldots,n-2,n+;(n-1)+)\,S(1,2,\ldots,n-2;n+) \\
&\qquad = S(1,2,\ldots,n-2;(n-1)+)\,S(1,2,\ldots,(n-1)+;n+) .
\end{aligned} \tag{4.32}$$

For the colour summed matrix element squared in the large N limit we have

$$
\begin{aligned}
&|\mathcal{M}(1,2,\ldots,\underline{n-1},\underline{n})|^2 \\
&\quad = \alpha^{n-4}(N^2-1) \sum_{P(12\cdots n-3)} S(1,2,\ldots,n-2;n-1,n)\,|\mathcal{C}(1,2,\ldots,n-2)|^2 \,.
\end{aligned}
\tag{4.33}
$$

When we also sum over the helicities we get

$$
\begin{aligned}
&\sum_{\lambda_i} |\mathcal{M}(1,2,\ldots,\underline{n-1},\underline{n})|^2 \\
&\quad = \alpha^{n-4}(N^2-1) \sum_{P(12\cdots n-3)} \left(\sum_{\lambda_{n-1}\lambda_n} S(1,2,\ldots,n-2;n-1,n) \right) \\
&\qquad \times \sum_{\lambda_1\cdots\lambda_{n-2}} |\mathcal{C}(1,2,\ldots,n-2)|^2 \,.
\end{aligned}
\tag{4.34}
$$

The quantity $\sum S$ can be evaluated explicitly for any n when use is made of eqs. (3.15), (3.20), (4.21)-(4.24) and (4.29). The $\mathcal{C}$-functions are known in an analytic form up to seven gluons [8], such that $n = 9$ is at present the maximum value in eq. (4.34).

5 Multiple soft gluon emission

The multiple soft gluon case can be derived in a way similar to the double soft gluon case. When we consider $(m-1)$ soft gluons we should know all the soft gluon factors for a smaller number of gluons.

We give a few steps leading to the $(m-1)$ soft gluon factor

$$
J(\underline{2},\underline{3},\ldots,\underline{m},m{+}1) = s_{\underline{23\cdots m}\,m+1} J(m{+}1) + t_{\underline{23\cdots m}\,m+1} K_{m+1} \,, \tag{5.1}
$$

with

$$
\begin{aligned}
s_{\underline{23\cdots m}\,m+1} &= \frac{1}{K_{m+1}\cdot\kappa(2,m)} K_{m+1}\cdot \Big(J(\underline{2},\underline{3},\ldots,\underline{m}) + s_{\underline{m}\,m+1} J(\underline{2},\underline{3},\ldots,\underline{m-1}) \\
&\quad + s_{\underline{m-1}\,\underline{m}\,m+1} J(\underline{2},\underline{3},\ldots,\underline{m-2}) + \cdots + s_{\underline{34\cdots m}\,m+1} J(\underline{2}) \Big) \,,
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
t_{\underline{23\cdots m}\,m+1} &= \frac{1}{2\,K_{m+1}\cdot\kappa(2,m)} \Big[-J(m{+}1)\cdot \Big(J(\underline{2},\underline{3},\ldots,\underline{m}) \\
&\quad + s_{\underline{m}\,m+1} J(\underline{2},\underline{3},\ldots,\underline{m-1}) \\
&\quad + s_{\underline{m-1}\,\underline{m}\,m+1} J(\underline{2},\underline{3},\ldots,\underline{m-2}) + \cdots + s_{\underline{34\cdots m}\,m+1} J(\underline{2}) \Big) \\
&\quad + K_{m+1}\cdot \Big(t_{\underline{m}\,m+1} J(\underline{2},\underline{3},\ldots,\underline{m-1}) + t_{\underline{m-1}\,\underline{m}\,m+1} J(\underline{2},\underline{3},\ldots,\underline{m-2}) \\
&\quad + \cdots + t_{\underline{34\cdots m}\,m+1} J(\underline{2}) \Big) \Big] \,.
\end{aligned}
\tag{5.3}
$$

From eqs. (IV.2.32) and (5.2) one finds

$$s_{1\underline{23\cdots m}} = \frac{-1}{K_1 \cdot \kappa(2,m)} K_1 \cdot \Big(J(\underline{2},\underline{3},\ldots,\underline{m}) + s_{1\underline{2}} J(\underline{3},\ldots,\underline{m}) + s_{1\underline{23}} J(\underline{4},\ldots,\underline{m}) + \cdots + s_{1\underline{23\cdots m-1}} J(\underline{m}) \Big) . \tag{5.4}$$

Just as one obtains eq. (4.6) one now has

$$J(1,\underline{2},\underline{3},\ldots,\underline{m},m+1) = s_{1\underline{23\cdots m}\,m+1}\, J(1,m+1) + \text{gauge terms} , \tag{5.5}$$

where the gauge terms are currents in which polarization vectors $J(i)$ are replaced by momenta K_i. These terms do no contribute to the subamplitudes. Instead of having exclusively gluons 1 and $(m+1)$ hard one can have more hard gluons like in eq. (4.8). This gives for the subamplitude

$$\mathcal{C}(1,\underline{2},\underline{3},\ldots,\underline{m},m+1,\ldots,n) = s_{1\underline{23\cdots m}\,m+1} \mathcal{C}(1,m+1,\ldots,n) . \tag{5.6}$$

The soft factor is a generalization of eq. (4.9)

$$s_{1\underline{23\cdots m}\,m+1} = s_{1\underline{23\cdots m}} + s_{1\underline{2}}\, s_{\underline{3\cdots m}\,m+1} + \cdots + s_{1\underline{2\cdots l}}\, s_{\underline{l+1\cdots m}\,m+1} + \cdots + s_{\underline{23\cdots m}\,m+1} . \tag{5.7}$$

This factor is gauge invariant i.e. the replacement of any polarization vector $J(i)$ by K_i ($1 < i < m+1$) gives zero. This is a consequence of eq. (5.6), where the l.h.s. possesses this property.

Certain sums lead to factorization as in eq. (4.12)

$$\sum\nolimits' s_{1\underline{23\cdots l\, l+1\cdots m}\,m+1} = s_{1\underline{23\cdots l}\,m+1}\, s_{1\underline{l+1\cdots m}\,m+1} , \tag{5.8}$$

where the sum runs over all those permutations of $(2\cdots m)$ which leave the order of the set $(2\cdots l)$ and the set $(l+1\cdots m)$ unaffected. A consequence of eq. (5.8) is

$$\sum_{P(2\cdots m)} s_{1\underline{2\cdots m}\,m+1} = \prod_{l=2}^{m} s_{1\,\underline{l}\,m+1} . \tag{5.9}$$

Note that the r.h.s. of eq. (5.9) corresponds to the factorization of eq. (1.2).

We will now prove eq. (5.8). According to the definition of $s_{1\underline{2\cdots m}\,m+1}$ we have

$$s_{1\underline{23\cdots m}\,m+1} = s_{1\underline{234\cdots m}} + s_{1\underline{2}}\, s_{\underline{34\cdots m}\,m+1} + s_{1\underline{23}}\, s_{\underline{4\cdots m}\,m+1} + \cdots + s_{\underline{234\cdots m}\,m+1} , \tag{5.10}$$

$$\begin{aligned} s_{1\underline{23\cdots l}\,m+1}\, s_{1\underline{l+1\cdots m}\,m+1} = {} & \big(s_{1\underline{23\cdots l}} + s_{1\underline{2}}\, s_{\underline{3\cdots l}\,m+1} + \cdots + s_{1\underline{23\cdots l-1}}\, s_{\underline{l}\,m+1} + s_{\underline{23\cdots l}\,m+1} \big) \\ & \times \big(s_{1\underline{l+1\, l+2\cdots m}} + s_{1\,\underline{l+1}}\, s_{\underline{l+2\cdots m}\,m+1} + \cdots + s_{1\underline{l+1\, l+2\cdots m-1}}\, s_{\underline{m}\,m+1} + s_{\underline{l+1\, l+2\cdots m}\,m+1} \big) . \end{aligned} \tag{5.11}$$

Suppose the following identities hold

$$\sum\nolimits' s_{1\underline{23\cdots l\, l+1\cdots m}} = s_{1\underline{23\cdots l}}\, s_{1\underline{l+1\cdots m}} , \tag{5.12}$$

$$\sum\nolimits' s_{\underline{23\cdots l\, l+1\cdots m}\,m+1} = s_{\underline{23\cdots l}\,m+1}\, s_{\underline{l+1\cdots m}\,m+1} , \tag{5.13}$$

then the terms in eq. (5.10) will be modified upon summation over the permutations as required in eq. (5.8). The changes are

$$s_{1\underline{23\cdots m}} \to s_{1\underline{23\cdots l}}\, s_{1\,\underline{l+1\cdots m}} \ , \tag{5.14}$$

$$s_{1\underline{2}}\, s_{\underline{3\cdots m}\,m+1} \to s_{1\underline{2}}\, s_{\underline{3\cdots l}\,m+1}\, s_{\underline{l+1\cdots m}\,m+1} + s_{1\,\underline{l+1}}\, s_{\underline{2\cdots l}\,m+1}\, s_{\underline{l+2\cdots m}\,m+1} \ , \tag{5.15}$$

$$\begin{aligned} s_{1\underline{2\cdots k}}\, s_{\underline{k+1\cdots m}\,m+1} \quad \to \quad & s_{1\underline{2\cdots k}}\, s_{\underline{k+1\cdots l}\,m+1}\, s_{\underline{l+1\cdots m}\,m+1} + s_{1\underline{2\cdots k-1}}\, s_{1\,\underline{l+1}} s_{\underline{k\cdots l}\,m+1}\, s_{\underline{l+2\cdots m}\,m+1} \\ & + s_{1\underline{2\cdots k-2}}\, s_{1\,\underline{l+1\,l+2}} s_{\underline{k-1\,k\cdots l}\,m+1}\, s_{\underline{l+3\cdots m}\,m+1} \\ & + \cdots + s_{1\,\underline{l+1\,l+2\cdots l+k-1}}\, s_{\underline{2\cdots l}\,m+1} s_{\underline{l+k\cdots m}\,m+1} \ . \\ & \qquad (0 < k+1 < l,\ l \leq \frac{1}{2} m) \end{aligned} \tag{5.16}$$

The second term in (5.16) originates from combining factors of the type $s_{1\underline{2\cdots k-1\,l+1}} + s_{1\underline{2\cdots l+1\,k-1}} + \cdots + s_{1\underline{l+1\,2\cdots k-1}}$ and then using eq. (5.12). The terms in (5.16) are characterized by having all the factors $s_{1i_2\cdots i_k}$ or $s_{1i_2\cdots i_n}\, s_{1i_{n+1}\cdots i_k}$. Thus we see that eqs. (5.14)-(5.16) give the terms of the r.h.s. of eq. (5.11) and therefore the validity of eq. (5.8) is proved.

What remains to be proved is the validity of eqs. (5.12) and (5.13). We only consider the case of eq. (5.12), the other proof goes in a similar fashion. From eq. (IV.2.34) one has

$$\sum J(1,\ldots,l,l+1,\ldots,m) = 0 \ , \tag{5.17}$$

where we sum over all those permutations of $(1,\ldots,m)$, which maintain the order in the sets $(1,\ldots,l)$ and $(l+1,\ldots,m)$. We can rewrite this sum in separate sums $\sum^{(k)}$ depending on the position of 1 :

$$\begin{aligned} & \sum\nolimits^{(1)} J(1,2,\ldots,l,l+1,\ldots,m) + \sum\nolimits^{(2)} J(l+1,1,2,\ldots,l,l+2,\ldots,m) \\ & \quad + \cdots + \sum\nolimits^{(k)} J(l+1,\ldots,l+k-1,1,2,\ldots,l,l+k,\ldots,m) \\ & \quad + \cdots + J(l+1,\ldots,m,1,2,\ldots,l) = 0 \ , \end{aligned} \tag{5.18}$$

where $\sum^{(k)}$ denotes the sum over all those permutations of $(2,\ldots,l,l+k,\ldots,m)$ which preserve the order in the sets $(2,\ldots,l)$ and $(l+k+1,\ldots,m)$. Upon taking the soft gluon limit for gluons $2,\ldots,m$ in eq. (5.18) we get, apart from gauge terms

$$\begin{aligned} & \sum\nolimits^{(1)} s_{1\underline{2\cdots l\,l+1\cdots m}} + \sum\nolimits^{(2)} s_{\underline{l+1}\,1}\, s_{1\underline{2\cdots l\,l+2\cdots m}} \\ & \quad + \cdots + \sum\nolimits^{(k)} s_{\underline{l+1\cdots l+k-1}\,1}\, s_{1\underline{2\cdots l\,l+k\cdots m}} + \cdots + s_{\underline{l+1\cdots m}\,1}\, s_{1\underline{2\cdots l}} = 0 \ . \end{aligned} \tag{5.19}$$

Using induction for proving (5.12) we can perform the sums $\sum^{(k)}$ for $k > 1$ giving

$$\begin{aligned} & \sum\nolimits^{(1)} s_{1\underline{2\cdots l\,l+1\cdots m}} + s_{\underline{l+1}\,1}\, s_{1\underline{2\cdots l}}\, s_{1\underline{l+2\cdots m}} + \cdots + s_{\underline{l+1\cdots l+k-1}\,1}\, s_{1\underline{2\cdots l}}\, s_{1\underline{l+k\ldots m}} \\ & \quad + \cdots + s_{\underline{l+1\cdots m}\,1}\, s_{1\underline{2\cdots l}} = 0 \ , \end{aligned} \tag{5.20}$$

or

$$\begin{aligned} & \sum\nolimits^{(1)} s_{1\underline{2\cdots l\,l+1\cdots m}} + s_{1\underline{2\cdots l}} \Big(s_{\underline{l+1}\,1}\, s_{1\underline{l+2\cdots m}} + \cdots + s_{\underline{l+1\cdots l+k-1}\,1}\, s_{1\,\underline{l+k\ldots m}} \\ & \quad + \cdots + s_{\underline{l+1\cdots m}\,1} \Big) = 0 \ . \end{aligned} \tag{5.21}$$

The factor in parentheses can be rewritten by use of

$$s_{\underline{i_1 \cdots i_k} 1} = (-1)^k s_{1 \underline{i_k \cdots i_1}} , \tag{5.22}$$

and the induction hypothesis for eq. (5.12) :

$$\begin{aligned} (\cdots) &= -\sum\nolimits^{(1)} s_{1 \underline{l+1}\, \underline{l+2 \cdots m}} + \sum\nolimits^{(2)} s_{1 \underline{l+2\, l+1}\, \underline{l+3 \cdots m}} \\ &\quad + \cdots + (-1)^k \sum\nolimits^{(k)} s_{1 \underline{l+k \cdots l+1}\, \underline{l+k+1 \cdots m}} + \cdots + (-1)^{m-l} s_{1 \underline{m \cdots l+1}} \\ &= -s_{1 \underline{l+1\, l+2 \cdots m}} \; . \end{aligned} \tag{5.23}$$

The sums $\sum^{(k)}$ now preserve the orderings in each of the sets $(l+k, \ldots, l+1)$ and $(l+k+1, \ldots, m)$. In the l.h.s. of eq. (5.23) all terms cancel except the first term in $\sum^{(1)}$. Combining eqs. (5.21) and (5.23) then proves eq. (5.12).

When $\omega_2 \ll \omega_3, \ldots, \omega_m$ we have from eq. (5.5)

$$J(1, \underline{2}, \underline{3}, \ldots, \underline{m}, m+1) = s_{1 \underline{2} \underline{3 \cdots m}\, m+1} J(1, m+1) \; . \tag{5.24}$$

Gluon 2 is soft compared to the other gluons, so we first apply eq. (3.7) on the current. Because gluons 3 through m are soft relative to gluons 1 and $(m+1)$, we use next eq. (5.5) to reduce the current further

$$\begin{aligned} J(1, \underline{2}, \underline{3}, \ldots, \underline{m}, m+1) &= s_{1\underline{2}3} J(1, \underline{3}, \ldots, \underline{m}, m+1) + \text{gauge terms} \\ &= s_{1\underline{2}3} s_{1 \underline{3 \cdots m}\, m+1} J(1, m+1) + \text{gauge terms} \; . \end{aligned} \tag{5.25}$$

From eqs. (5.24) and (5.25) we see that

$$s_{1 \underline{2} \underline{3 \cdots m}\, m+1} = s_{1\underline{2}3}\, s_{1 \underline{3 \cdots m}\, m+1} \; . \tag{5.26}$$

The gauge terms from eqs. (5.24) and (5.25) are irrelevant for physical amplitudes.

In the strong ordering case

$$\omega_2 \ll \omega_3 \ll \cdots \ll \omega_m \tag{5.27}$$

one finds by repeated application of eq. (5.26)

$$s_{1 \underline{23 \cdots m}\, m+1} = s_{1\underline{2}3}\, s_{1\underline{3}4} \cdots s_{1 \underline{m} m+1} \; . \tag{5.28}$$

The two soft gluon factors can be generalized in the equal helicity case to an arbitrary number of soft gluons. For the polarization choice of eq. (3.12) the explicit form of the current $J(2+, 3+, \ldots, m+)$ is known (V.3.10),

$$J_{\dot{A}B}(1+, \ldots, m+) = (\sqrt{2})^m \frac{\kappa_{\dot{A}C}(1, m)\, b^C b_B}{\langle b1 \rangle \langle 12 \rangle \cdots \langle mb \rangle} \; . \tag{5.29}$$

Using induction in eqs. (5.2) and (5.4) one then finds

$$s^{+++\cdots+}_{\underline{234 \cdots m}\, m+1} = (\sqrt{2})^{m-1} \frac{\langle m{+}1\; b \rangle}{\langle 2b \rangle} \frac{1}{\langle 23 \rangle \langle 34 \rangle \cdots \langle m\; m{+}1 \rangle} , \tag{5.30}$$

$$s^{+++\cdots+}_{1\underline{234\cdots m}} = (\sqrt{2})^{m-1}\,\frac{\langle 1b\rangle}{\langle mb\rangle}\,\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle m{-}1\; m\rangle}\,. \tag{5.31}$$

Inserting eqs. (5.30) and (5.31) in eq. (5.7) now leads to

$$s^{++\cdots+}_{1\underline{23\cdots m}\,m+1} = (\sqrt{2})^{m-1}\,\frac{\langle 1\; m{+}1\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle m\; m{+}1\rangle}\,, \tag{5.32}$$

which implies the factorization form

$$s^{++\cdots+}_{1\underline{23\cdots m}\,m+1} = s^{+}_{1\underline{2}3}s^{+}_{1\underline{3}4}\cdots s^{+}_{1\underline{m}\,m+1}\,. \tag{5.33}$$

Note that this factorization is the same as in the strong ordering case. As a consequence of the definition of the soft factor in eq. (5.5), we can derive this factor also directly from eq. (5.29) by taking gluons 2 through $(m-1)$ soft. This results in

$$\begin{aligned}
J_{\dot{A}B}(1+,\underline{2+,\ldots,(m-1)+},m+) &= (\sqrt{2})^{m}\frac{(K_1+K_m)_{\dot{A}C}\,b^C b_B}{\langle b1\rangle\langle 12\rangle\cdots\langle mb\rangle} \\
&= \left((\sqrt{2})^{m-2}\frac{\langle 1\,m\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle m-1\,m\rangle}\right) \\
&\quad\times\left((\sqrt{2})^{2}\frac{(K_1+K_m)_{\dot{A}C}\,b^C b_B}{\langle b1\rangle\langle 1m\rangle\langle mb\rangle}\right) \\
&= s^{++\cdots+}_{1\underline{2\cdots m-1}\,m}\,J_{\dot{A}B}(1+,m+)\,,
\end{aligned} \tag{5.34}$$

which gives a similar soft factor as in eq. (5.32). Note that the gauge dependent term is zero for this choice of the gauge spinors.

For an explicit evaluation of the soft gluon factors for other helicity combinations one has to calculate all currents up to $J(2,\ldots,m)$. That is, one needs the same information as one would need for an m-gluon hard scattering amplitude. So when one knows the currents for $m-1$ gluons one knows the expression for $2m-1$ gluon scattering where m gluons are hard and $m-1$ are soft.

6 Soft gluon emission in a process producing $q\bar{q}$ and n gluons

The matrix element for the production of a quark, antiquark and n gluons with outgoing momenta $Q, P, K_1, \ldots, K_n$ is given by (IV.4.11)

$$\mathcal{M}(Q;1,2,\ldots,n;P) = -ig^n \sum_{P(1,\ldots,n)} (a_1\cdots a_n)_{ij}\mathcal{D}(Q;1,2,\ldots,n;P)\,. \tag{6.1}$$

One may wonder whether we have for the subamplitudes $\mathcal{D}$ (generalized) subcyclic identities like eq. (IV.4.5). If one replaces T^{a_n} by a unit matrix we are actually calculating a process where gluon n has been replaced by a photon. We then have

$$\mathcal{M}(Q;1,2,\ldots,n{-}1,\tilde{n};P) = -ig^{n-1}eq \sum_{P(1,\ldots,n-1)} (a_1\cdots a_{n-1})_{ij}\mathcal{D}(Q;1,2,\ldots,n{-}1,\tilde{n};P)\,, \tag{6.2}$$

with

$$\begin{aligned}\mathcal{D}(Q;1,2,\ldots,n-1,\tilde{n};P) &= \mathcal{D}(Q;n,1,2,\ldots,n-1;P)\\ &+\mathcal{D}(Q;1,n,2,\ldots,n-1;P)+\cdots+\mathcal{D}(Q;1,2,\ldots,n,n-1;P)\\ &+\mathcal{D}(Q;1,2,\ldots,n-1,n;P)\ .\end{aligned} \tag{6.3}$$

The matrix element (6.2) describes the production of an outgoing quark pair with charge eq together with $(n-1)$ gluons and a photon with momentum K_n. In general, when m gluons and $(n-m)$ photons are produced one has

$$\begin{aligned}&\mathcal{M}(Q;1,2,\ldots,m,\widetilde{m+1},\ldots,\tilde{n};P) =\\ &-ig^m(eq)^{n-m}\sum_{P(1,\ldots,m)}(a_1\cdots a_m)_{ij}\mathcal{D}(Q;1,2,\ldots,m,\widetilde{m+1},\ldots,\tilde{n};P)\ ,\end{aligned} \tag{6.4}$$

where

$$\mathcal{D}(Q;1,2,\ldots,m,\widetilde{m+1},\ldots,\tilde{n};P) = \sum_{\text{perms}}\mathcal{D}(Q;1,2,\ldots,m,m+1,\ldots,n;P)\ , \tag{6.5}$$

with a sum running over all permutations of $(1,\ldots,n)$ which preserve the order of $(1,\ldots,m)$.

The soft gluon properties of the gluon current and the recursion relation (IV.3.9) are the basis for the discussion of the soft gluon behaviour of the subamplitudes $\mathcal{D}$. In the following gauge terms will occur i.e. currents $J(Q;1,2,\ldots,n)$ where one of the polarization vectors $J(i)$ has been replaced by K_i

$$J(Q;K_1,2,\ldots,n) = J(Q)\frac{\kappa(1,n)\cdot Y(K_1,2,\ldots,n)}{\kappa(1,n)^2}\ , \tag{6.6}$$

$$\begin{aligned}J(Q;1,\ldots,K_m,\ldots,n) &= J(Q)\frac{\kappa(1,n)\cdot Y(1,\ldots,K_m,\ldots,n)}{\kappa(1,n)^2}\\ &+\sum_{l=1}^{m-2}J(Q;1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,K_m,\ldots,n)}{\kappa(l+1,n)^2}\\ &+J(Q;1,\ldots,m-1)\frac{\kappa(m,n)\cdot Y(K_m,\ldots,n)}{\kappa(m,n)^2}\ ,\end{aligned} \tag{6.7}$$

$$\begin{aligned}J(Q;1,\ldots,n-1,K_n) &= J(Q)\frac{\kappa(1,n)\cdot Y(1,\ldots,n-1,K_n)}{\kappa(1,n)^2}\\ &+\sum_{l=1}^{n-2}J(Q;1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,n-1,K_n)}{\kappa(l+1,n)^2}\\ &-J(Q;1,\ldots,n-1)\ .\end{aligned} \tag{6.8}$$

These gauge terms give vanishing functions $\mathcal{D}(Q;1,\ldots,K_m,\ldots,n;P)$ since the inverse propagator is not cancelled in eq. (IV.4.12) by the propagator. Instead it becomes the Dirac equation for $v(P)$.

We will now prove eqs. (6.6)-(6.8). For the derivation of eq. (6.8) we need the identities

$$J(Q)\not{k}(1,n)\,[\not{Q}+\not{k}(1,n)-m]^{-1}=J(Q)\;, \tag{6.9}$$

$$\begin{aligned}[\not{Q}+\not{k}(1,k)-m]^{-1}\not{k}(k+1,n)\,[\not{Q}+\not{k}(1,n)-m]^{-1} &= \\ [\not{Q}+\not{k}(1,k)-m]^{-1}-[\not{Q}+\not{k}(1,n)-m]^{-1}\;, \end{aligned} \tag{6.10}$$

$$\begin{aligned} J(Q;1,\ldots,m)\not{k}(m+1,n)\,[\not{Q}+\not{k}(1,n)-m]^{-1} &= J(Q;1,\ldots,m) \\ +\left[J(Q)\not{J}(1,\ldots,m)+\sum_{k=1}^{m-1}J(Q;1,\ldots,k)\not{J}(k+1,\ldots,m)\right]&\left[\not{Q}+\not{k}(1,n)-m\right]^{-1}. \end{aligned} \tag{6.11}$$

The last equation follows from eqs. (IV.3.9), (6.9) and (6.10). In the recursion relation (IV.3.9) we replace $J(n)$ by K_n, insert eq. (6.9) and use eq. (6.11) to obtain

$$\begin{aligned} &J(Q;1,\ldots,n-1,K_n) \\ &\quad=-\left[J(Q)\not{J}(1,\ldots,n-1,K_n)+\sum_{l=1}^{n-1}J(Q;1,\ldots,l)\not{J}(l+1,\ldots,n-1,K_n)\right] \\ &\qquad\times[\not{Q}+\not{k}(1,n)-m]^{-1} \\ &\quad=\frac{\kappa(1,n)\cdot Y(1,\ldots,n-1,K_n)}{\kappa(1,n)^2}J(Q) \\ &\qquad+\sum_{l=1}^{n-2}J(Q;1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,K_n)}{\kappa(l+1,n)^2}-J(Q;1,\ldots,n-1) \\ &\qquad-\Bigg\{J(Q)\Bigg[\not{Y}(1,\ldots,n-1,K_n)+\not{J}(1,\ldots,n-1) \\ &\qquad\qquad-\sum_{l=1}^{n-2}\not{J}(1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,K_n)}{\kappa(l+1,n)^2}\Bigg] \\ &\qquad\quad+\sum_{k=1}^{n-3}J(Q;1,\ldots,k)\Bigg[\not{Y}(k+1,\ldots,n-1,K_n)+\not{J}(k+1,\ldots,n-1) \\ &\qquad\qquad-\sum_{l=k+1}^{n-2}\not{J}(k+1,\ldots,l)\frac{\kappa(l+1,n)\cdot Y(l+1,\ldots,K_n)}{\kappa(l+1,n)^2}\Bigg] \\ &\qquad\quad+J(Q;1,\ldots,n-2)\,[\not{Y}(n-1,K_n)+\not{J}(n-1)]\Bigg\}\;. \end{aligned} \tag{6.12}$$

One can show upon using eq. (2.4), that the term in curly brackets vanishes thus proving eq. (6.8).

We now turn to eq. (6.6). Using eq. (6.9) one has

$$J(Q;K_1)=-J(Q)\;. \tag{6.13}$$

Using this result, the explicit form $Y(K_1,2)$ and again eq. (6.9) one finds

$$J(Q;K_1,2) = J(Q)\frac{\kappa(1,2)\cdot Y(K_1,2)}{\kappa(1,2)^2}\ . \tag{6.14}$$

This then leads to the hypothesis

$$J(Q;K_1,\ldots,m) = J(Q)\frac{\kappa(1,m)\cdot Y(K_1,2,\ldots,m)}{\kappa(1,m)^2}\ , \tag{6.15}$$

which can be proven to be correct by induction. We prove eq. (6.15) for $n = m$, assuming its validity for $m < n$. The recursion relation leads to

$$\begin{aligned}
&J(Q;K_1,2,\ldots,n)\\
&\quad= -\Bigg[J(Q)\not{J}(K_1,2,\ldots,n) + J(Q;K_1)\not{J}(2,\ldots,n)\\
&\qquad\quad+\sum_{l=2}^{n-1} J(Q;K_1,2,\ldots,l)\not{J}(l+1,\ldots,n)\Bigg]\left[\not{Q}+\not{\kappa}(1,n)-m\right]^{-1}\\
&\quad= -J(Q)\Bigg[Y(K_1,2,\ldots,n) - \not{J}(2,\ldots,n)\\
&\qquad\qquad+\sum_{l=2}^{n-1}\frac{\kappa(1,l)\cdot Y(K_1,2,\ldots,l)}{\kappa(1,l)^2}\not{J}(l+1,\ldots,n)\Bigg]\\
&\qquad+J(Q)\frac{\kappa(1,n)\cdot Y(K_1,2,\ldots,n)}{\kappa(1,n)^2}\ .
\end{aligned}\tag{6.16}$$

The explicit form (2.3) for $Y(K_1,2,\ldots,n)$ then gives a vanishing term in the square brackets, giving eq. (6.15). For the proof of eq. (6.7) one again calculates some explicit cases, which can be generalized into eq. (6.7), which then is proven by induction. Also here eqs. (6.9) and (6.11) have to be used together with the explicit forms of $Y(1,\ldots,K_m,\ldots,k)$.

We discuss explicitly single and double soft gluon emission. From this discussion it will be clear how to generalize it to multiple soft gluon emission. For single soft gluon emission we need the $K_1\to 0$ limit of various expressions :

$$\begin{aligned}
J(Q;\underline{1}) &= -\frac{Q\cdot J(1)}{Q\cdot K_1}\,J(Q)\\
&= s_{Q\underline{1}}\,J(Q)\ ,
\end{aligned}\tag{6.17}$$

$$\begin{aligned}
J(Q;\underline{1},2) &= \left(s_{Q\underline{1}}+s_{\underline{1}2}\right)J(Q;2)+t_{\underline{1}2}\,J(Q;K_2)\\
&= s_{Q\underline{1}2}\,J(Q;2)+t_{\underline{1}2}\,J(Q;K_2)\ ,
\end{aligned}\tag{6.18}$$

where the previous definitions (3.4) and (3.9) now also incorporate the momentum Q. It is easy to extend (6.18) from two gluons to n gluons and to consider other positions of the soft gluons. The general results are

$$J(Q;\underline{1},2,\ldots,n) = s_{Q\underline{1}2}\,J(Q;2,\ldots,n)+t_{\underline{1}2}\,J(Q;K_2,3,\ldots,n) \tag{6.19}$$

$$\begin{aligned} J(Q;1,\ldots,\underline{m},\ldots,n) &= s_{m-1\,\underline{m}\,m+1}\, J(Q;1,\ldots,m-1,m+1,\ldots,n) \\ &+ t_{m-1\,\underline{m}}\, J(Q;1,\ldots,K_{m-1},m+1,\ldots,n) \\ &+ t_{\underline{m}\,m+1}\, J(Q;1,\ldots,m-1,K_{m+1},\ldots,n) \ , \end{aligned} \tag{6.20}$$

or in terms of the subamplitudes

$$\mathcal{D}(Q;\underline{1},\ldots,n;P) = s_{Q\underline{1}2}\,\mathcal{D}(Q;2,\ldots,n;P) \ , \tag{6.21}$$

$$\mathcal{D}(Q;1,\ldots,\underline{m},\ldots,n;P) = s_{m-1\,\underline{m}\,m+1}\,\mathcal{D}(Q;1,\ldots,m-1,m+1,\ldots,n;P) \ , \tag{6.22}$$

$$\mathcal{D}(Q;1,\ldots,\underline{n};P) = s_{n-1\,\underline{n}\,P}\,\mathcal{D}(Q;1,\ldots,n-1;P) \ . \tag{6.23}$$

The last expression can be obtained most conveniently by starting with a spinor current where the antiquark is on shell (IV.3.11).

We now turn to the colour summed matrix element squared. In leading order in N we find

$$|\mathcal{M}(Q;1,\ldots,n;P)|^2 = \alpha^n \frac{N^2-1}{N} \sum_{P(1\cdots n)} |\mathcal{D}(Q;1,\ldots,n;P)|^2 \ . \tag{6.24}$$

This approximation will turn out to be not as good as in the pure gluon case. When $(n-m)$ gluons are replaced by photons as in eq. (6.4) one has to replace $(n-m)$ factors α by

$$\alpha = (eq)^2 \ . \tag{6.25}$$

Moreover the sum then runs over all permutations of $(1,\ldots,m)$. When gluon n is soft eq. (6.24) becomes

$$\begin{aligned} |\mathcal{M}(Q;1,\ldots,\underline{n};P)|^2 &= \alpha^{n-1}\frac{N^2-1}{N} \\ &\times \sum_{P(1\cdots n-1)} S(Q,1,\ldots,n-1,P;n)\,|\mathcal{D}(Q;1,\ldots,n-1;P)|^2 \ , \end{aligned} \tag{6.26}$$

with

$$S(Q,1,\ldots,n-1,P;n) = \alpha\left(|s_{Q\underline{n}1}|^2 + \sum_{m=1}^{n-2} |s_{m\,\underline{n}\,m+1}|^2 + |s_{n-1\,\underline{n}\,P}|^2\right) \ . \tag{6.27}$$

This factor is similar to $S(1,2,\ldots,n-1;n)$ defined in eq. (3.20). It is not cyclic invariant in $(Q,1,\ldots,n-1,P)$ since it lacks a term $|s_{P\underline{n}Q}|^2$. Summed over the helicities of gluon n we have an expression similar to eq. (3.21)

$$S(Q,1,\ldots,n-1,P;n) = 2\alpha\left(\frac{(Q\cdot 1)}{(Q\cdot n)(n\cdot 1)} + \sum_{m=1}^{n-2}\frac{(m\cdot m+1)}{(m\cdot n)(n\cdot m+1)} + \frac{(n-1\cdot P)}{(n-1\cdot n)(n\cdot P)}\right) . \tag{6.28}$$

In eq. (6.28) we have neglected the masses of the quarks. Including them would give an additional term

$$-\alpha m^2\left[(Q\cdot n)^{-2}+(P\cdot n)^{-2}\right]\ . \tag{6.29}$$

For explicit calculations in the helicity formalism one needs the subamplitudes $\mathcal{D}(Q+;1,\ldots,n;P-)$. Consider the massless quark case. The subamplitudes vanish when all gluon helicities are the same. For $n=2,3$ it is sufficient to use (V.5.22)

$$\begin{aligned}\mathcal{D}(Q+;2+,&\ldots,m+,1-,(m+1)+,\ldots,n+;P-)\\ &=(\sqrt{2})^n\frac{\langle p1\rangle^3\langle q1\rangle}{\langle pq\rangle\langle q2\rangle\langle 23\rangle\cdots\langle m1\rangle\langle 1\, m+1\rangle\cdots\langle np\rangle}\ ,\end{aligned} \tag{6.30}$$

and the rules for changing all helicities and only the quark helicities. The former means complex conjugation of the whole expression (6.30), the latter the interchange of p and q in the numerator.

For $n=3$ eq. (6.26) leads to

$$\begin{aligned}|\mathcal{M}(Q;1,2,\underline{3};P)|^2 &= 8\alpha^3\frac{N^2-1}{N}\\ &\times\frac{\sum_{k=1}^{2}(P\cdot k)(Q\cdot k)\left((Q\cdot k)^2+(P\cdot k)^2\right)}{(P\cdot Q)}\\ &\times\sum_{P(123)}\frac{1}{(Q\cdot 1)(1\cdot 2)(2\cdot 3)(3\cdot P)}\ .\end{aligned} \tag{6.31}$$

In contrast to the 5-gluon case, eq. (3.26), the expression (6.31) is not exact in the colour N. In order to discuss the exact formulae we first give for $n=2,3,4$ the complete hard gluon result thus improving on eq. (6.24). The colour summed matrix elements squared are

$$|\mathcal{M}(Q;1,2;P)|^2=\alpha^2\frac{N^2-1}{N}\left[\sum_{P(12)}|\mathcal{D}(Q;1,2;P)|^2-\frac{1}{N^2}|\mathcal{D}(Q;\tilde{1},\tilde{2};P)|^2\right]\ , \tag{6.32}$$

$$\begin{aligned}|\mathcal{M}(Q;1,2,3;P)|^2 &= \alpha^3\frac{N^2-1}{N}\\ &\left[\sum_{P(123)}\left\{|\mathcal{D}(Q;1,2,3;P)|^2-\frac{1}{N^2}|\mathcal{D}(Q;1,2,\tilde{3};P)|^2\right\}\right.\\ &\left.+\left(\frac{1}{N^2}+\frac{1}{N^4}\right)|\mathcal{D}(Q;\tilde{1},\tilde{2},\tilde{3};P)|^2\right]\ ,\end{aligned} \tag{6.33}$$

$$\begin{aligned}|\mathcal{M}(Q;1,2,3,4;P)|^2 &= \alpha^4\frac{N^2-1}{N}\\ &\left[\sum_{P(1234)}\left\{|\mathcal{D}(Q;1,2,3,4;P)|^2-\frac{1}{N^2}|\mathcal{D}(Q;1,2,3,\tilde{4};P)|^2\right.\right.\end{aligned}$$

$$+\frac{1}{N^4}|\mathcal{D}(Q;1,2,\tilde{3},\tilde{4};P)|^2\Big\} + \left(\frac{1}{N^2}-\frac{3}{N^4}-\frac{1}{N^6}\right)|\mathcal{D}(Q;\tilde{1},\tilde{2},\tilde{3},\tilde{4};P)|^2$$
$$-\frac{1}{N^2}\sum_{P(1234)}\Big\{\mathcal{D}^*(Q;1,2,3,4;P)\Big(\mathcal{D}(Q;\underbrace{2,1}\underbrace{4,3};P)-\mathcal{D}(Q;4,2,3,1;P)$$
$$-\mathcal{D}(Q;4,1,3,2;P)-\mathcal{D}(Q;3,2,4,1;P)-\mathcal{D}(Q;3,1,4,2;P)\Big)\Big\}\Big] . \tag{6.34}$$

In eq. (6.33) the sum over permutations for the second term implies that there are terms where 3 is a photon, 2 is a photon etc. In eq. (6.34) the sum over permutations induces in the third term pairwise equal terms since the order of e.g. photons $\tilde{3}$ and $\tilde{4}$ does not matter. In the expression (6.34) interference terms between the subamplitudes arise like in the six-gluon process. The following notation has been introduced

$$\mathcal{D}(Q;\underbrace{2,1},\underbrace{4,3};P) = \sum_{\text{perms}} \mathcal{D}(Q;2,1,4,3;P) , \tag{6.35}$$

where all permutations of $(2,1,4,3)$ are taken which leave the order $(2,1)$ and $(4,3)$ unaffected. In contrast to the pure gluonic case we find for $n = 2,3$ non leading terms in N^{-2}. These terms spoil the quality of the approximation (6.24). The photonic terms and the terms described by eq. (6.35) are absent in the pure gluonic case because of the subcyclic identities (IV.4.5).

Taking gluon 3 soft one has

$$\sum_{\lambda_3}|\mathcal{M}(Q;1,2,\underline{3};P)|^2 = \alpha^2\frac{N^2-1}{N}\sum_{\lambda_3}\Big\{\sum_{P(12)} S(Q,1,2,P;3)|\mathcal{D}(Q;1,2;P)|^2$$
$$-\frac{\alpha}{N^2}\left[|s_{1\underline{3}P}|^2+|s_{2\underline{3}P}|^2+|s_{Q\underline{3}1}|^2+|s_{Q\underline{3}2}|^2\right]|\mathcal{D}(Q;\tilde{1},\tilde{2};P)|^2$$
$$-\frac{\alpha}{N^2}|s_{Q\underline{3}P}|^2\sum_{P(12)}|\mathcal{D}(Q;1,2;P)|^2+\frac{\alpha}{N^2}\left(1+\frac{1}{N^2}\right)|s_{Q\underline{3}P}|^2|\mathcal{D}(Q;\tilde{1},\tilde{2};P)|^2\Big\} , \tag{6.36}$$

with amongst others

$$\sum_{\lambda_3}|s_{Q\underline{3}P}|^2 = -\frac{m^2}{(Q\cdot 3)^2}+2\frac{(Q\cdot P)}{(Q\cdot 3)(3\cdot P)}-\frac{m^2}{(P\cdot 3)^2} . \tag{6.37}$$

When we neglect quark masses the required subamplitudes are simple, e.g.

$$\mathcal{D}(Q+;\tilde{1}-,\tilde{2}+;P-) = -2\frac{\langle p1\rangle^3\langle q1\rangle}{\langle q1\rangle\langle 1p\rangle\langle q2\rangle\langle 2p\rangle} . \tag{6.38}$$

Summing over the helicities of all partons we obtain in the massless quark case

$$\sum_{\text{hel.}}|\mathcal{M}(Q;1,2,\underline{3};P)|^2 = 8\alpha^3\frac{N^2-1}{N}$$

$$\times \sum_{k=1}^{2}(P\cdot k)(Q\cdot k)\left((Q\cdot k)^2+(P\cdot k)^2\right)$$
$$\times \left\{ \sum_{P(123)} \left(\frac{1}{\mathcal{N}(Q123P)} - \frac{1}{N^2}\frac{1}{\mathcal{N}(Q12P)}\frac{(Q\cdot P)}{(Q\cdot 3)(3\cdot P)} \right) \right.$$
$$\left. + \left(\frac{1}{N^2}+\frac{1}{N^4}\right)\frac{(Q\cdot P)}{(Q\cdot 1)(1\cdot P)(Q\cdot 2)(2\cdot P)(Q\cdot 3)(3\cdot P)} \right\}, \tag{6.39}$$

where

$$\mathcal{N}(12345) = (1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 5)(5\cdot 1)\,. \tag{6.40}$$

Taking gluon 4 soft in eq. (6.34) one finds

$$\sum_{\text{hel.}} |\mathcal{M}(Q;1,2,3,\underline{4};P)|^2 = 16\alpha^4\frac{N^2-1}{N}$$
$$\times \sum_{k=1}^{3}(P\cdot k)(Q\cdot k)\left((Q\cdot k)^2+(P\cdot k)^2\right)$$
$$\times \left\{ \sum_{P(1234)} \left(\frac{1}{\mathcal{N}(Q1234P)} - \frac{1}{N^2}\frac{1}{\mathcal{N}(Q123P)}\frac{(Q\cdot P)}{(Q\cdot 4)(4\cdot P)} \right.\right.$$
$$\left. + \frac{1}{N^4}\frac{1}{\mathcal{N}(Q12P)}\frac{(Q\cdot P)^2}{(Q\cdot 3)(3\cdot P)(Q\cdot 4)(4\cdot P)} \right)$$
$$\left. + \left(\frac{1}{N^2}-\frac{3}{N^4}-\frac{1}{N^6}\right)\frac{(Q\cdot P)^2}{(Q\cdot 1)(1\cdot P)(Q\cdot 2)(2\cdot P)(Q\cdot 3)(3\cdot P)(Q\cdot 4)(4\cdot P)} \right\}. \tag{6.41}$$

In this limit the interference term arising from the last curly bracket in eq. (6.34) has been neglected. Numerically this term turned out to be very small just as this happens in the six gluon case (3.27).

Double gluon bremsstrahlung can be discussed along similar lines as in sec. 4. One starts with

$$\begin{aligned} J(Q;\underline{1},\underline{2}) &= -[J(Q)V(\underline{1},\underline{2}) + J(Q;\underline{1})V(\underline{2})]\frac{1}{\not Q + \not\kappa(1,2) - m} \\ &= -J(Q)\frac{Q\cdot\left(J(\underline{1},\underline{2}) + s_{Q\underline{1}}J(\underline{2})\right)}{Q\cdot\kappa(1,2)} \\ &= s_{Q\underline{12}}J(Q)\,, \end{aligned} \tag{6.42}$$

from which follows

$$J(Q;\underline{1},\underline{2},3) = s_{Q\underline{123}}J(Q;3) + \left(t_{\underline{123}} + s_{Q\underline{1}}t_{\underline{23}}\right)J(Q;K_3)\,, \tag{6.43}$$

where

$$s_{Q\underline{123}} = s_{Q\underline{12}} + s_{Q\underline{1}}s_{\underline{23}} + s_{\underline{123}}\,, \tag{6.44}$$

which can be written as eq. (4.11). For the subamplitudes one finds in general

$$\mathcal{D}(Q;\underline{1},\underline{2},3,\ldots,n;P)=s_{Q\underline{12}3}\mathcal{D}(Q;3,\ldots,n;P)\ , \tag{6.45}$$

$$\mathcal{D}(Q;1,\ldots,\underline{m-1},\underline{m},\ldots,n;P)=s_{m-2\,\underline{m-1}\,\underline{m}\,m+1}\mathcal{D}(Q;1,\ldots,m-2,m+1,\ldots,n;P)\ , \tag{6.46}$$

$$\mathcal{D}(Q;1,\ldots,\underline{n-1},\underline{n};P)=s_{n-2\,\underline{n-1}\,\underline{n}P}\mathcal{D}(Q;1,\ldots,n-2;P)\ . \tag{6.47}$$

Using the leading order in N colour summed matrix element squared we see that taking gluon n and $(n-1)$ soft one needs

$$\mathcal{D}(Q;1,\ldots,i-1,\underline{n-1},i,\ldots,j-1,\underline{n},j,\ldots,n-2;P)= \\ s_{i-1\,\underline{n-1}\,i}s_{j-1\,\underline{n}\,j}\mathcal{D}(Q;1,\ldots,n-2;P)\ , \tag{6.48}$$

$$\mathcal{D}(Q;1,\ldots,i-1,\underline{n-1},\underline{n},i,\ldots,n-2;P)=s_{i-1\,\underline{n-1}\,\underline{n}\,i}\mathcal{D}(Q;1,\ldots,n-2;P)\ , \tag{6.49}$$

where $1<i,j<n-1$, for $i,j=2$ one takes for $i-1$ $(j-1)$ momentum Q and for $i,j=n-1$ one takes momentum P.

The factor multiplying $|\mathcal{D}(Q;1,\ldots,n-2;P)|^2$ in eq. (6.24) now becomes

$$\begin{aligned} &S(P,1,\ldots,n-2,Q;n-1,n) \\ &=\alpha^2\sum_{i=1}^{n-1}\left(|s_{i-1\,\underline{n-1}\,\underline{n}\,i}|^2+|s_{i-1\,\underline{n}\,\underline{n-1}\,i}|^2\right)+\alpha^2\sum_{i,j=1\,;i\neq j}^{n-1}|s_{i-1\,\underline{n-1}\,i}|^2|s_{j-1\,\underline{n}\,j}|^2 \\ &=S(Q,1,\ldots,n-2,P;n-1)S(Q,1,\ldots,n-2,P;n) \\ &-2\alpha^2 Re\Bigg[s_{Q\,\underline{n-1}\,\underline{n}\,1}s^*_{Q\,\underline{n}\,\underline{n-1}\,1}+\sum_{m=1}^{n-3}s_{m\,\underline{n-1}\,\underline{n}\,m+1}s^*_{m\,\underline{n}\,\underline{n-1}\,m+1} \\ &\quad+s_{n-2\,\underline{n-1}\,\underline{n}\,P}s^*_{n-2\,\underline{n}\,\underline{n-1}\,P}\Bigg]\ . \end{aligned} \tag{6.50}$$

The factor $S(P,1,\ldots,n-2,Q;n-1,n)$ is analogous to $S(1,2,\ldots,n-2;n-1,n)$ of eq. (4.29). The latter is cyclic invariant in the labels $(1,\ldots,n-2)$, the former would need in each sum one additional term like e.g. $|s_{Q\,\underline{n-1}\,\underline{n}\,P}|^2$ to become cyclic invariant in the labels $(P,1,\ldots,n-2,Q)$. The colour summed matrix element squared now becomes in the double soft limit

$$|\mathcal{M}(Q;1,\ldots,n-2\ \underline{n-1},\underline{n};P)|^2=\alpha^{n-2}\frac{N^2-1}{N} \\ \times\left[\sum_{P(1\cdots n-2)}S(Q,1,\ldots,n-2,P;n-1,n)|\mathcal{D}(Q;1,\ldots,n-2;P)|^2+\mathcal{O}(N^{-2})\right]\ . \tag{6.51}$$

Multiple soft gluon emission can be easily discussed in analogy with the above double radiation and with the help of sec. 5.

7 Soft gluon emission in a process producing $q\bar{q}$, V and n gluons

For completeness we also briefly discuss soft gluon emission in a process where besides gluons and a quark pair also a vector boson is produced. The matrix element is written as (IV.4.19)

$$\mathcal{M}(Q,P;V;1,\ldots,n) = icg^n \sum_{P(1\cdots n)} (a_1\cdots a_n)_{ij}\, S_\mu(Q;1,\ldots,n;P)V^\mu \,. \tag{7.1}$$

The soft gluon behaviour of $S_\mu(Q;1,\ldots,n;P)$ is determined by that of the quark current and antiquark current e.g. eqs. (6.19), (6.20) for the former and similar ones for the latter. For double soft gluon emission relations like eq. (6.43) should be used. The gauge terms should cancel in S_μ, this we will now explicitly show.

In addition to eqs. (6.6)-(6.8) and (6.13) one needs the analogous relations

$$\begin{aligned}
J(K_1;P) &= J(P)\,, & (7.2)\\
J(1,\ldots,K_n;P) &= -\frac{\kappa(1,n)\cdot Y(1,\ldots,n-1,K_n)}{\kappa(1,n)^2}J(P)\,, & (7.3)\\
J(1,\ldots,K_m,\ldots,n;P) &= -\frac{\kappa(1,m)\cdot Y(1,\ldots,K_m)}{\kappa(1,m)^2}J(m+1,\ldots,n;P) \\
&\quad -\sum_{l=m+1}^{n-1}\frac{\kappa(1,l)\cdot Y(1,\ldots,K_m,\ldots,l)}{\kappa(1,l)^2}J(l+1,\ldots,n;P) \\
&\quad -\frac{\kappa(1,n)\cdot Y(1,\ldots,K_m,\ldots,n)}{\kappa(1,n)^2}J(P)\,, & (7.4)\\
J(K_1,2,\ldots,n;P) &= J(2,\ldots,n;P) \\
&\quad -\sum_{l=2}^{n-1}\frac{\kappa(1,l)\cdot Y(K_1,2,\ldots,l)}{\kappa(1,l)^2}J(l+1,\ldots,n;P) \\
&\quad -\frac{\kappa(1,n)\cdot Y(K_1,2,\ldots,n)}{\kappa(1,n)^2}J(P)\,. & (7.5)
\end{aligned}$$

For the simplest cases it is easy to see that the gauge terms cancel e.g.

$$\begin{aligned}
S_\mu(Q;K_1;P) &= J(Q)\Gamma_\mu J(K_1;P) + J(Q;K_1)\Gamma_\mu J(P) \\
&= J(Q)\Gamma_\mu J(P) - J(Q)\Gamma_\mu J(P) \\
&= 0\,. & (7.6)
\end{aligned}$$

For the most general case we split eq. (IV.4.20) into a number of terms

$$\begin{aligned}
S_\mu(Q;1,\ldots,K_m,\ldots,n;P) &= J(Q)\Gamma_\mu J(1,\ldots,K_m,\ldots,n;P) \\
&\quad +\sum_{l=1}^{m-2} J(Q;1,\ldots,l)\Gamma_\mu J(l+1,\ldots,K_m,\ldots,n;P) \\
&\quad +J(Q;1,\ldots,m-1)\Gamma_\mu J(K_m,\ldots,n;P) \\
&\quad +J(Q;1,\ldots,K_m)\Gamma_\mu J(m+1,\ldots,n;P) \\
&\quad +\sum_{l=m+1}^{n-1} J(Q;1,\ldots,K_m,\ldots,l)\Gamma_\mu J(l+1,\ldots,n;P) \\
&\quad +J(Q;1,\ldots,K_m,\ldots,n)\Gamma_\mu J(P)\,. & (7.7)
\end{aligned}$$

For all terms the previously derived gauge terms eqs. (6.7), (6.8), (7.4) and (7.5) should be inserted. Although the number of terms expand they cancel eventually.

Now we know the gauge terms cancel in the vector current we get for the single soft gluon emission

$$\mathcal{S}_\mu(Q;\underline{1},2,\ldots,n;P) = s_{Q\underline{1}2}\mathcal{S}_\mu(Q;2,\ldots,n;P)\ , \tag{7.8}$$

$$\mathcal{S}_\mu(Q;1,\ldots,\underline{m},\ldots,n;P) = s_{m-1\,\underline{m}\,m+1}\mathcal{S}_\mu(Q;1,\ldots,m-1,m+1,\ldots,n;P)\ , \tag{7.9}$$

$$\mathcal{S}_\mu(Q;1,\ldots,\underline{n};P) = s_{n-1\,\underline{n}\,P}\mathcal{S}_\mu(Q;1,\ldots,n-1;P)\ , \tag{7.10}$$

and for double soft gluon emission

$$\mathcal{S}_\mu(Q;\underline{1},\underline{2},3,\ldots,n;P) = s_{Q\underline{12}3}\mathcal{S}_\mu(Q;3,\ldots,n;P)\ , \tag{7.11}$$

$$\begin{aligned}\mathcal{S}_\mu(Q;1,\ldots,\underline{m-1},\underline{m},\ldots,n;P) &= s_{m-2\,\underline{m-1}\,\underline{m}\,m+1} \\ &\quad\times\mathcal{S}_\mu(Q;1,\ldots,m-2,m+1,\ldots,n;P)\ ,\end{aligned} \tag{7.12}$$

$$\mathcal{S}_\mu(Q;1,\ldots,\underline{n-1},\underline{n};P) = s_{n-2\,\underline{n-1}\,\underline{n}\,P}\mathcal{S}_\mu(Q;1,\ldots,n-2;P)\ . \tag{7.13}$$

For multiple soft gluon emission analogous expressions are obtained.

Since the matrix element (7.1) has exactly the same colour decomposition as the matrix element (6.1) the expressions for the matrix element squared such as eqs. (6.24), (6.26), (6.32)-(6.34), (6.36) and (6.51) also hold for the process with a vector boson. One just makes the replacement

$$\mathcal{D}(Q;1,2,\ldots,n;P) \longrightarrow \mathcal{S}_\mu(Q;1,2,\ldots,n;P)V^\mu\ , \tag{7.14}$$

to obtain the relevant expression. When the subamplitude $\mathcal{D}$ has some photons instead of gluons the subamplitude $\mathcal{S}_\mu V^\mu$ similarly contains photons.

The process with a single gluon, which was physically not relevant in the previous section can occur here and has the colour summed matrix element squared

$$|\mathcal{M}(Q,P;V;1)|^2 = \alpha\frac{N^2-1}{N}|\mathcal{S}_\mu(Q;1;P)V^\mu|^2\ . \tag{7.15}$$

In order to obtain explicit expressions for the soft gluon expression for a low number of hard gluons one needs for $\mathcal{S}_\mu$ expressions like eq. (6.30). These expressions are given by eqs. (V.5.30) and (V.5.31). In chap. 8 the current $\mathcal{S}_\mu$ will be calculuted explicitly for up to three gluons.

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Chapter VII

Exact expressions for multi gluon scattering

In this chapter we use the recursive method to obtain building blocks from which n-gluon scattering amplitudes can be constructed. The method is discussed in detail for up to 8 gluons. For 4, 5, 6, and 7 gluons exact analytic results are given.

1 Introduction

In $P\bar{P}$ collisions multi jet events have been seen, a maximum number of six jets has even been reported [1]. In future experiments multi jet events will be produced copiously. The calculation of the hard scattering processes is involved. The n-gluon scattering processes ($= 2g \to (n-2)g$) being one type of the many possible parton scattering processes illustrate the situation. A historical review of this calculation was given in chap. 2. For six gluon scattering [2] reasonable simple expressions can be obtained [3,4] when the following colour decomposition is used (IV.4.1)

$$\mathcal{M}(1,2,\ldots,n) = 2ig^{n-2} \sum_{P(1,2,\ldots,n-1)} Tr\,(T^{a_1} \cdot T^{a_2} \cdots T^{a_n})\; \mathcal{C}(1,2,\ldots,n)\,. \quad (1.1)$$

The number of Feynman diagrams contributing to a n-gluon process is rapidly growing with the number of external gluons [5], as can be seen from table 7.1. When one want to calculate processes with seven or more gluons one has to rely on a different technique. This is the recursive scheme [6] developed in chap. 4, which can be used for analytical as well as for numerical evaluations [5,7,8].

The actual outline of this chapter is as follows. In sec. 2 the tools of the recursive method are used to illustrate the analytical calculation of multi gluon scattering. Sec. 3 uses the recursive methods to find the analytic expressions for four and five gluon scattering, whereas in sec. 4 the six gluon subamplitudes are derived. In sec. 5 the analytical results for the subamplitudes of seven gluon scattering are given, being a result of 2485 Feynman diagrams.

n	number of diagrams
3	1
4	4
5	25
6	220
7	2485
8	34300
9	559405
10	10525900

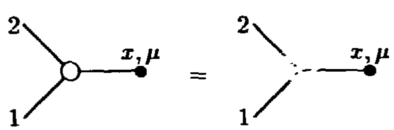
Table 7.1. The number of contributing Feynman diagrams to n-gluon scattering.

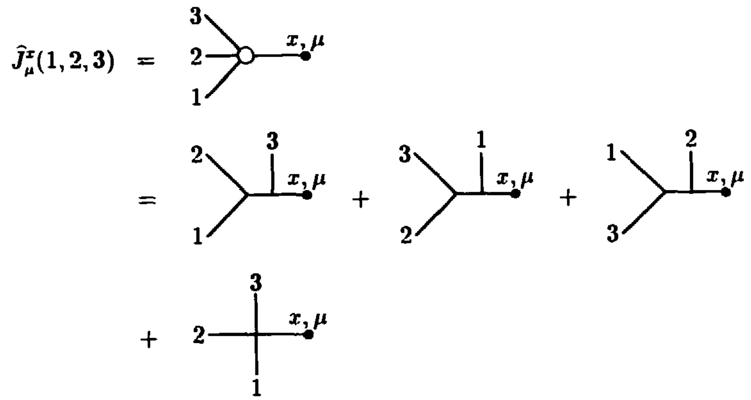
2 The building blocks for multi gluon scattering

In this section we use the recursive method of chap. 4 to show how certain building blocks can be used to obtain analytical results for up to 8 gluons. The recursive method makes it possible to calculate the n-gluon currents $J(1,2,\ldots,n)$ of eq. (IV.2.29) recursively in n. At first sight one needs the current $J(1,2,\ldots,n)$ in order to calculate $(n+1)$-gluon scattering. This is certainly sufficient to obtain the scattering amplitude and a straightforward numerical implementation would use the n-gluon current because the chance of making errors this way is minimal [5].

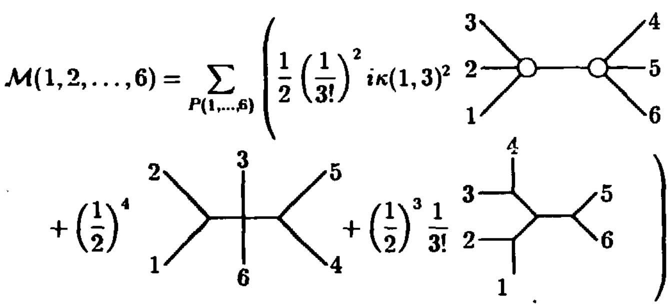
Here however we would like to use the diagrammatic structure of the amplitude $\mathcal{M}(1,\ldots,n+1)$ in order to find an expression for $\mathcal{M}$ in terms of currents $J(1,\ldots,m)$ with $m < n$. For the case of 6, 7 and 8 gluons these expressions are given in detail and they depend on at most the 4-current for $n = 8$ and the 3-current for $n = 6,7$. The purpose of this section is to derive these representations of $\mathcal{M}$ in terms of "short" currents. We shall also give explicit analytical expressions for the 3-currents, which will lead to the explicit 6- and 7-gluon amplitudes [7] in the following sections. For the 4-currents we do not use as yet analytical results but we can calculate them numerically and then use these currents in the above mentioned representation of $\mathcal{M}(1,\ldots,8)$. This gives an evaluation of $\mathcal{M}(1,\ldots,8)$ which is numerically faster than the full recursive method [8].

The building blocks which we require are the 2-current $\hat{J}^x_\mu(1,2)$, the 3-current $\hat{J}^x_\mu(1,2,3)$ and the 4-current $\hat{J}^x_\mu(1,2,3,4)$. In terms of Feynman diagrams $\hat{J}^x_\mu(1,2)$ and $\hat{J}^x_\mu(1,2,3)$ are given by

$\hat{J}^x_\mu(1,2) =$ 



The black dot denotes the off shell gluon, the numbered lines represent on shell gluons. All momenta are outgoing. In the definition of the current $\hat{J}^x_\mu(1,\ldots,n)$ a propagator $-i/\kappa(1,n)^2$ has been included for the off shell gluon. The 4-current consists of 25 diagrams which would be needed for 5-gluon scattering. The amplitude $\mathcal{M}(1,2,\ldots,6)$ can be expressed in 2- and 3-currents in the following diagrammatic expression



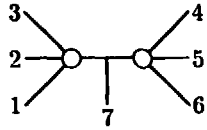
(2.1)

In this formula the sum over all permutations of the six gluon labels should be taken. The factors in front of the diagrams correct for multiple counting, or in other words give the number of different diagrams when multiplied by 6!. For instance the first term gives $6!/(2 \times 3! \times 3!) = 10$ diagrams involving 3-currents i.e. 160 Feynman diagrams, the second gives $6!/(2^4) = 45$ and the third $6!/(3! \times 2^3) = 15$ Feynman diagrams. Thus in total we have the usual 220 Feynman diagrams for 6-gluon scattering, which we can summarize as follows

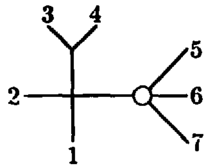
$$6!\left(\frac{4^2}{2(3!)^2} + \frac{1}{2^4} + \frac{1}{2^3 3!}\right) = 220 \ .$$

In the same way the 7-gluon amplitude can be written as

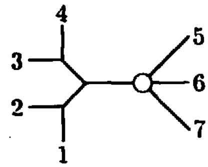
$$\mathcal{M}(1,2,\ldots,7) = \sum_{P(1,\ldots,7)} \Bigg(\frac{1}{2}\left(\frac{1}{3!}\right)^2$$



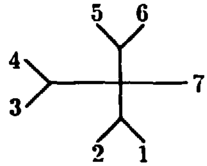
$$+\left(\frac{1}{2}\right)^2 \frac{1}{3!}$$



$$+\left(\frac{1}{2}\right)^3 \frac{1}{3!}$$



$$+\left(\frac{1}{2}\right)^3 \frac{1}{3!}$$



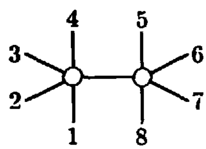
$$\Bigg) . \tag{2.2}$$

A similar counting of Feynman diagrams gives

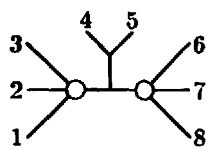
$$7!\left(\frac{4^2}{2(3!)^2}+\frac{4}{2^2 3!}+\frac{4}{2^3 3!}+\frac{1}{2^3 3!}\right) = 2485 .$$

Use of the 4-current gives the following representation of the 8-gluon amplitude

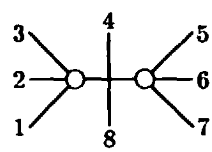
$$\mathcal{M}(1,2,\ldots,8) = \sum_{P(1,\ldots,8)} \Bigg(\frac{1}{2}\left(\frac{1}{4!}\right)^2 i\kappa(1,4)^2$$



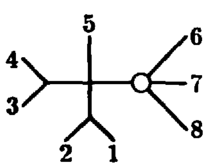
$$+\left(\frac{1}{2}\right)^2\left(\frac{1}{3!}\right)^2$$



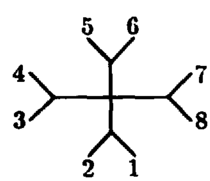
$$+\left(\frac{1}{2}\right)^2\left(\frac{1}{3!}\right)^2$$



$$+\left(\frac{1}{2}\right)^3 \frac{1}{3!}$$



$$+\left(\frac{1}{2}\right)^4 \frac{1}{4!}$$



$$\Bigg) . \tag{2.3}$$

The corresponding number of Feynman diagrams is

$$8!\left(\frac{25^2}{2(4!)^2}+\frac{4^2}{2^2(3!)^2}+\frac{4^2}{2^2(3!)^2}+\frac{4}{2^3 3!}+\frac{1}{2^4 4!}\right) = 34300 .$$

In the first term of both eq. (2.1) and (2.3) one propagator of the current had to be removed.

These examples show that the currents $\hat{J}(1,\ldots,m)$ are useful as building blocks for n-gluon amplitudes. We next show that the colour free subamplitudes $\mathcal{C}(1,\ldots,n)$, defined in (IV.4.2), can be expressed through the representations (2.1)-(2.3) in terms of the colourless current $J(1,\ldots,n)$. In the following we will use a shorthand notation for the Lorentz structure of the 3- and 4-vertex in which two or three colourless currents come together. This notation is given in eq. (IV.2.30) and (IV.2.31) respectively.

We express the diagrammatic representations in terms of the currents $\hat{J}$. As an example we give the translation of the third term $\mathcal{M}_3$ in eq. (2.2). There are three currents $\hat{J}(1,2)$, $\hat{J}(3,4)$ and $\hat{J}(5,6,7)$ joined by a 3-vertex. Using eq. (IV.2.28) we find

$$
\begin{aligned}
\mathcal{M}_3 &= \frac{-g}{3!2^3}\sum_{P(1,\ldots,7)} f^{x_1x_2x_3}V_{\alpha_1\alpha_2\alpha_3}\hat{J}^{x_1\alpha_1}(1,2)\hat{J}^{x_2\alpha_2}(3,4)\hat{J}^{x_3\alpha_3}(5,6,7) \\
&= \frac{-g}{3!2^3}\sum_{P(1,\ldots,7)}\left(f^{x_1x_2x_3}\sum_{P(1,2)}\sum_{P(3,4)}\sum_{P(5,6,7)}(a_1a_2x_1)(a_3a_4x_2)(a_5a_6a_7x_3)\right. \\
&\qquad\qquad \left.\times\, 2^3g^4\big[J(1,2),J(3,4)\big]\cdot J(5,6,7)\right) . \qquad (2.4)
\end{aligned}
$$

Carrying out the colour sums over x_i results in

$$
\mathcal{M}_3 = 2ig^5\sum_{P(1,\ldots,7)}(a_1a_2\cdots a_7)\big[J(1,2),J(3,4)\big]\cdot J(5,6,7)\,, \qquad (2.5)
$$

where we have used eq. (IV.2.21). So the contribution to $\mathcal{C}(1,2,\ldots,7)$ of this particular term is

$$
\mathcal{C}_3 = \sum_{C(1,\ldots,7)}\big[J(1,2),J(3,4)\big]\cdot J(5,6,7)\,, \qquad (2.6)
$$

where the sum runs over the cyclic permutations of labels $1,\ldots,7$.

Treating every term in this way eqs. (2.1)-(2.3) lead to

$$
\begin{aligned}
\mathcal{C}(1,2,\ldots,6) = \sum_{C(1,\ldots,6)}&\left(\frac{1}{2}\kappa(1,3)^2J(1,2,3)\cdot J(4,5,6)\right. \\
&+\big\{J(1),J(2),J(3,4)\big\}\cdot J(5,6)+\frac{1}{2}\big\{J(1),J(2,3),J(4)\big\}\cdot J(5,6) \\
&\left.+\frac{1}{3}\big[J(1,2),J(3,4)\big]\cdot J(5,6)\right)\,, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}(1,2,\ldots,7) = \sum_{C(1,\ldots,7)}&\left(\big[J(1,2,3),J(4,5,6)\big]\cdot J(7)\right. \\
&+\big\{J(1,2),J(3,4),J(5,6)\big\}\cdot J(7)+\big\{J(1,2),J(3),J(4)\big\}\cdot J(5,6,7)
\end{aligned}
$$

$$+\Big\{J(1),J(2),J(3,4)\Big\}\cdot J(5,6,7)+\Big\{J(1),J(2,3),J(4)\Big\}\cdot J(5,6,7)$$
$$+\Big[J(1,2),J(3,4)\Big]\cdot J(5,6,7)\Big)\ , \tag{2.8}$$

$$C(1,2,\ldots,8)=\sum_{C(1,\ldots,8)}\Big(\frac{1}{2}\kappa(1,4)^2 J(1,2,3,4)\cdot J(5,6,7,8)$$
$$+J(1,2)\cdot\Big[J(3,4,5),J(6,7,8)\Big]+\Big\{J(1,2,3),J(4,5,6),J(7)\Big\}\cdot J(8)$$
$$+\frac{1}{2}\Big\{J(1,2,3),J(4),J(5,6,7)\Big\}\cdot J(8)+\Big\{J(1),J(2,3),J(4,5)\Big\}\cdot J(6,7,8)$$
$$+\Big\{J(1,2),J(3),J(4,5)\Big\}\cdot J(6,7,8)+\Big\{J(1,2),J(3,4),J(5)\Big\}\cdot J(6,7,8)$$
$$+\frac{1}{4}\Big\{J(1,2),J(3,4),J(5,6)\Big\}\cdot J(7,8)\Big)\ . \tag{2.9}$$

In order to get explicit analytic results for these subamplitudes one requires analytic expressions for the 3- and 4-current. For the 3-current we have a compact form. The explicit form of the 3-current contains the abelian part of the gluon field strengths

$$F_i^{\mu\nu}=K_i^\mu e_i^\nu-K_i^\nu e_i^\mu\ , \tag{2.10}$$

and is separated into a gauge invariant part and a non-invariant part (VI.2.8)

$$J_\mu(1,2,3)=\frac{G_\mu(1,2,3)}{\kappa(1,3)^2}+\frac{\kappa(1,3)^\nu}{\kappa(1,3)^2}\left[\kappa(1,3)_\nu Y_\mu(1,2,3)-\kappa(1,3)_\mu Y_\nu(1,2,3)\right]\ , \tag{2.11}$$

$$G_\mu(1,2,3)=-\frac{\kappa(1,3)^\nu\sum_{P(123)}\Big[(F_1\cdot F_2\cdot F_3)_{\nu\mu}-\frac{1}{4}Tr(F_1\cdot F_2)F_{3\nu\mu}\Big]}{(K_1\cdot K_2)(K_2\cdot K_3)}\ , \tag{2.12}$$

$$\begin{aligned}Y_\mu(1,2,3)=\frac{1}{2(K_1\cdot K_2)(K_2\cdot K_3)}\Big[\ &(J_3\cdot F_2\cdot F_1)_\mu+(J_1\cdot F_2\cdot F_3)_\mu\\ -\ &K_2\cdot J_1\,(J_3\cdot F_2)_\mu-K_2\cdot J_3(J_1\cdot F_2)_\mu\\ +\ &K_2\cdot J_3\,K_1\cdot J_2\,J_{1\mu}+K_2\cdot J_1\,K_3\cdot J_2\,J_{3\mu}\Big]\ ,\end{aligned} \tag{2.13}$$

with obvious notations, for example $(F_1\cdot F_2)_{\mu\nu}=F_{1\mu\alpha}F_2{}^\alpha{}_\nu$, $(J_1\cdot F_2\cdot F_3)_\mu=J_{1\alpha}F_2^{\alpha\beta}F_{3\beta;}$ and $Tr(F_1\cdot F_2)=F_{1\mu\nu}F_2^{\nu\mu}$.

The gauge independent part G of the current J determines $C(1,2,3,4)$ as can be seen from eq. (IV.4.2). For our calculations we need the full expression including the Y-terms. It is convenient to have the explicit forms for G for various helicity combinations. Using the Weyl-van der Waerden spinor calculus [3] of chap. 3 we define

$$G_{\dot{A}B}=\sigma^\mu_{\dot{A}B}G_\mu \tag{2.14}$$

so that

$$G_{\dot{A}B}(1+,2+,3+)=0\ , \tag{2.15}$$

$$G_{\dot{A}B}(1+,2+,3-) = \frac{1}{\sqrt{2}} \frac{\langle 12\rangle^{*2}\left(K_1+K_2+K_3\right)_{\dot{A}C} k_3^C\, k_{3B}}{(K_1\cdot K_2)(K_2\cdot K_3)}\,, \tag{2.16}$$

$$G_{\dot{A}B}(1+,2-,3+) = \frac{1}{\sqrt{2}} \frac{\langle 13\rangle^{*2}\left(K_1+K_2+K_3\right)_{\dot{A}C} k_2^C\, k_{2B}}{(K_1\cdot K_2)(K_2\cdot K_3)}\,, \tag{2.17}$$

$$G_{\dot{A}B}(1-,2+,3+) = \frac{1}{\sqrt{2}} \frac{\langle 23\rangle^{*2}\left(K_1+K_2+K_3\right)_{\dot{A}C} k_1^C\, k_{1B}}{(K_1\cdot K_2)(K_2\cdot K_3)}\,. \tag{2.18}$$

The other helicity combinations follow from complex conjugation

$$G_{\dot{A}B}(1\lambda_1,2\lambda_2,3\lambda_3) = \left(G_{\dot{B}A}(1-\lambda_1,2-\lambda_2,3-\lambda_3)\right)^*\,. \tag{2.19}$$

With these 3-currents we will determine in the next section 4- and 5-gluon scattering. In sec. 4 we will calculate with the aid of eq. (2.1) and the above 3-currents the 6-gluon amplitude. While in sec. 5 we will calculate in a similar fashion the 7-gluon amplitude, using again the 3-currents and eq. (2.2).

3 The 4- and 5-gluon amplitudes

We already determined the 4- and 5-gluon helicity amplitudes in chap. 5 (eq. (V.5.11)). For completeness we give here the spin and colour summed matrix elements squared of 4- and 5-gluon scattering. The 4-gluon scattering matrix element squared is given by

$$\begin{aligned}|\mathcal{M}(1,2,3,4)|^2 &= 4\left(\frac{g^2N}{2}\right)^2\left(N^2-1\right)\left(\sum_{i=1}^{3}\sum_{j=i+1}^{4}(i\cdot j)^4\right)\\ &\quad\times\left(\sum_{P(123)}\frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 1)}\right)\end{aligned} \tag{3.1}$$

and the 5-gluon squared matrix element is

$$\begin{aligned}|\mathcal{M}(1,2,3,4,5)|^2 &= 8\left(\frac{g^2N}{2}\right)^3\left(N^2-1\right)\left(\sum_{i=1}^{4}\sum_{j=i+1}^{5}(i\cdot j)^4\right)\\ &\quad\times\left(\sum_{P(1234)}\frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 5)(5\cdot 1)}\right)\,,\end{aligned} \tag{3.2}$$

with $(i\cdot j) = K_i\cdot K_j$. One could also derive the above amplitudes with the 3-currents of eqs. (2.15)-(2.19) and using eqs. (IV.4.1) and (2.11). The terms containing $Y_\mu(1,2,3)$ cancel in the 4-gluon case, because of momentum conservation.

An alternative formula for the 4-gluon scattering can be obtained by using the 3-current of (2.12). This leads to the subamplitude

$$C(1,2,3,4) = -\frac{1}{2}\frac{\sum_{P(123)}\left(Tr(F_1\cdot F_2\cdot F_3\cdot F_4) - \frac{1}{4}Tr(F_1\cdot F_2)Tr(F_3\cdot F_4)\right)}{K_1\cdot K_2\; K_2\cdot K_3}\,. \tag{3.3}$$

This equation is explicitly gauge invariant and is valid in any space-time dimension.

4 The 6-gluon amplitude

The 6-gluon matrix element can also be determined without the recursive scheme as was done in refs. [2,3,4]. Here, however, we will use the recursive scheme and use the decomposition of the amplitude in 3-currents according to eq. (2.1). With this technique the pole structure of the 6-gluon amplitude will become transparent and without much calculational effort the answer of refs. [3,4] is obtained.

Inserting the expression (2.11) into eq. (2.7) we find

$$C(1,2,\ldots,6) = \sum_{C(1,\ldots,6)} \left(\frac{1}{2}\frac{G(1,2,3)\cdot G(4,5,6)}{\kappa(1,3)^2} + X(1,2,3,4,5,6)\right) \tag{4.1}$$

with

$$\begin{aligned} X(1,\ldots,6) &= G(1,2,3)\cdot Y(4,5,6) + \kappa(1,3)^2 Y(1,2,3)\cdot Y(4,5,6) \\ &+ \kappa(1,3)\cdot Y(1,2,3)\kappa(4,6)\cdot Y(4,5,6) + \Big\{J(1),J(2),J(3,4)\Big\}\cdot J(5,6) \\ &+ \frac{1}{2}\Big\{J(1),J(2,3),J(4)\Big\}\cdot J(5,6) + \frac{1}{3}\Big[J(1,2),J(3,4)\Big]\cdot J(5,6)\ . \end{aligned} \tag{4.2}$$

The first term in eq. (4.1) contains the poles with three momenta. It is gauge invariant in itself. The term X is also gauge invariant but the invariance is not manifest since the quantities Y and the 2-currents are not gauge invariant objects. The factorization of the pole terms with three momenta is obvious. There are three 3-pole terms of this type

$$P = \frac{G(1,2,3)\cdot G(3,4,5)}{\kappa(1,3)^2} + \frac{G(2,3,4)\cdot G(5,6,1)}{\kappa(2,4)^2} + \frac{G(3,4,5)\cdot G(6,1,2)}{\kappa(3,5)^2}\ . \tag{4.3}$$

In a specific helicity amplitude not all three terms are necessarily present. For instance, whenever a contraction occurs of the type

$$\kappa(1,3)_{\dot{A}C}\, k_i^C\, \kappa(1,3)^{\dot{A}D}\, k_{jD} = \kappa(1,3)^2\langle k_j k_i\rangle \tag{4.4}$$

the pole $\kappa(1,3)^2$ drops out. This will be the case for every amplitude with 4 equal helicities and two opposite ones. This leads to the general form (V.5.9) for this type of amplitude, e.g.

$$C(1-,2-,3+,4+,5+,6+) = \frac{(\sqrt{2})^6}{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\ . \tag{4.5}$$

Thus the 3-pole terms can only be present when three helicities are equal and the rest opposite e.g. $C(1+,2+,3+,4-,5-,6-)$ contains $\kappa(2,4)^2$ and $\kappa(3,5)^2$ poles. The terms of eq. (4.3) give directly the terms of refs. [3,4] corresponding to the 3-poles. Take as an example the term containing the 3-pole $\kappa(1,3)^2$ in the subamplitude $C(1+,2+,3-,4+,5-,6-)$. The pole term is given by the first term in eq.

(4.3). For this specific helicity amplitude it results in

$$\begin{aligned}
&\frac{G(1+,2+,3-)\cdot G(4+,5-,6-)}{2\kappa(1,3)^2}\\
&=\frac{1}{2}\frac{1}{\kappa(1,3)^2}\left(\frac{1}{\sqrt{2}}\frac{\langle 12\rangle^{*2}(\check{K}_1+\check{K}_2)_{\dot{A}C}k_3^C k_{3B}}{(\check{K}_1\cdot\check{K}_2)(\check{K}_2\cdot\check{K}_3)}\right)\times\left(\frac{1}{\sqrt{2}}\frac{\langle 56\rangle^2(\check{K}_5+\check{K}_6)_{\dot{C}B}k_4^{\dot{C}}k_{4\dot{A}}}{(\check{K}_4\cdot\check{K}_5)(\check{K}_5\cdot\check{K}_6)}\right)\\
&=-\frac{1}{4}\frac{1}{\kappa(1,3)^2}\frac{\langle 12\rangle^{*2}\langle 56\rangle^2\langle 4|1+2|3\rangle^2}{(\check{K}_1\cdot\check{K}_2)(\check{K}_2\cdot\check{K}_3)(\check{K}_4\cdot\check{K}_5)(\check{K}_5\cdot\check{K}_6)}\,.
\end{aligned}$$

(See for the notation eq. (5.6).)

The quantity X which gives the term with only 2-poles leads to the result of [3,4], after some effort. The term X is most easily calculated using the helicity formalism of chap. 3.

The resulting subamplitudes with 3 positive and 3 negative helicities are given by a set of three C-functions. With this basic set of C-functions one can derive all the other C-functions with the use of the reflective and cyclic properties. The subamplitudes have a basic pole stucture as can be readily seen from eq. (4.1), only the numerators are helicity dependent. This general structure is given by [4]

$$\begin{aligned}
&C(1,\lambda_1;2,\lambda_2;3,\lambda_3;4,\lambda_4;5,\lambda_5;6,\lambda_6)=\\
&\quad\frac{(\sqrt{2})^6}{2}\left(\frac{A^2}{(\check{K}_1+\check{K}_2)^2(\check{K}_2+\check{K}_3)^2(\check{K}_1+\check{K}_2+\check{K}_3)^2(\check{K}_4+\check{K}_5)^2(\check{K}_5+\check{K}_6)^2}\right.\\
&\qquad+\frac{B^2}{(\check{K}_2+\check{K}_3)^2(\check{K}_3+\check{K}_4)^2(\check{K}_2+\check{K}_3+\check{K}_4)^2(\check{K}_5+\check{K}_6)^2(\check{K}_6+\check{K}_1)^2}\\
&\qquad+\frac{C^2}{(\check{K}_3+\check{K}_4)^2(\check{K}_4+\check{K}_5)^2(\check{K}_3+\check{K}_4+\check{K}_5)^2(\check{K}_6+\check{K}_1)^2(\check{K}_1+\check{K}_2)^2}\\
&\qquad\left.+\frac{BC(\check{K}_1+\check{K}_2+\check{K}_3)^2+CA(\check{K}_2+\check{K}_3+\check{K}_4)^2+AB(\check{K}_3+\check{K}_4+\check{K}_5)^5}{(\check{K}_1+\check{K}_2)^2(\check{K}_2+\check{K}_3)^2(\check{K}_3+\check{K}_4)^2(\check{K}_4+\check{K}_5)^2(\check{K}_5+\check{K}_6)^2(\check{K}_6+\check{K}_1)^2}\right)\,.
\end{aligned}\tag{4.6}$$

For $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)=(+,+,+,-,-,-)$ we have

$$A=0$$

$$\begin{aligned}
B&=\langle 56\rangle\langle 23\rangle^*\langle 1|2+3|4\rangle\\
C&=\langle 45\rangle\langle 12\rangle^*\langle 3|1+2|6\rangle\,,
\end{aligned}$$

see for notation eq. (5.6). Note that in this helicity subamplitude the $(\check{K}_1+\check{K}_2+\check{K}_3)^2$ pole drops out. This can directly be understood with the help of eqs. (2.15) and (4.3). For $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)=(+,+,-,-,+,-)$ we find

$$A=\langle 46\rangle\langle 12\rangle^*\langle 5|1+2|3\rangle$$

$$\begin{aligned}
B&=\langle 34\rangle\langle 15\rangle^*\langle 2|3+4|6\rangle\\
C&=\langle 34\rangle\langle 12\rangle^*\langle 5|3+4|6\rangle\,.
\end{aligned}$$

Finally for $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (+, -, +, -, +, -)$ we find

$$A = \langle 46\rangle\langle 13\rangle^* \langle 5|1 + 3|2\rangle$$

$$B = \langle 24\rangle\langle 51\rangle^* \langle 3|2 + 4|6\rangle$$

$$C = \langle 26\rangle\langle 35\rangle^* \langle 1|3 + 5|4\rangle \; .$$

From the cyclic symmetry and the property that complex conjugation leads to opposite helicities we have the relations

$$\begin{aligned} \mathcal{C}^{+++---}(123456) &= \left(\mathcal{C}^{+++---}(456123)\right)^*, \\ \mathcal{C}^{++--+-}(123456) &= \left(\mathcal{C}^{++--+-}(432165)\right)^*, \\ \mathcal{C}^{+-+-+-}(123456) &= \left(\mathcal{C}^{+-+-+-}(234561)\right)^* . \end{aligned} \tag{4.7}$$

With these relations all helicity combinations can be obtained from the above set of three subamplitudes.

The squaring of the matrix element can be done in leading order in the colour approximation, given by eq. (IV.4.10)

$$\sum_{\text{col.}} |\mathcal{M}(123456)|^2 = \left(\frac{g^2 N}{2}\right)^4 \left(N^2 - 1\right) \left(\sum_{P(12345)} |\mathcal{C}(123456)|^2\right) . \tag{4.8}$$

The exact squared matrix element is given by

$$\begin{aligned} \sum_{\text{col.}} |\mathcal{M}(123456)|^2 &= \left(\frac{g^2 N}{2}\right)^4 \left(N^2 - 1\right) \\ &\times \sum_{P(12345)} \Bigg(|\mathcal{C}(123456)|^2 \\ &+ \frac{2}{N^2} \mathcal{C}(123456) \times \left[\mathcal{C}(135264) + \mathcal{C}(153624) + \mathcal{C}(513642)\right]^* \Bigg) \end{aligned} \tag{4.9}$$

as was shown in refs. [3,4]. A detailed analysis of squaring procedures, the numerical implementation and the effects of the leading order in colour approximations is given in ref. [7].

5 The 7-gluon amplitude

In this section the analytic results giving rise to helicity amplitudes of seven gluon scattering are presented. The helicity combinations with five or more equal helicities were already calculated for arbitrary number of gluons in chap. 5. Specifying $n = 7$ in eqs. (V.5.2) and (V.5.10) give the equations for these helicity combinations. The helicity combination with three positive and four negative polarized gluons will be determined in this section. For this helicity configuration we need four independent subamplitudes. From these the other subamplitudes can be obtained by cyclic permutations, reflections and relabeling. The subamplitudes are obtained with the method of sec. 2. Taking the large number of Feynman diagrams in consideration the end result is still reasonably compact.

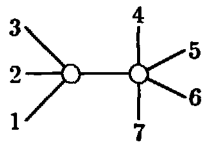
The formulae below show a specific pole structure which we like to discuss. The pole terms one would expect in the subamplitude $C(1234567)$ are the 2-poles

$$(K_1+K_2)^2,(K_2+K_3)^2,(K_3+K_4)^2,(K_4+K_5)^2,(K_5+K_6)^2,(K_6+K_7)^2,(K_7+K_1)^2 \tag{5.1}$$

and the 3-poles

$$\begin{gathered}(K_1+K_2+K_3)^2,(K_2+K_3+K_4)^2,(K_3+K_4+K_5)^2,(K_4+K_5+K_6)^2,\\(K_5+K_6+K_7)^2,(K_6+K_7+K_1)^2,(K_7+K_1+K_2)^2.\end{gathered} \tag{5.2}$$

However, sometimes 3-poles do not occur as we have already seen in the previous section for the 6-gluon case. This depends on the helicity configuration as can be readily understood by looking at all the diagrams contributing to a certain 3-pole. Let us take as an example the 3-pole $(K_1+K_2+K_3)^2$, the contributing diagrams to this pole are given by



$$= \kappa(1,3)^2 J(1,2,3)\cdot J(4,5,6,7)\ . \tag{5.3}$$

If gluon 1, 2 and 3 have the same helicity the pole $(K_1+K_2+K_3)^2$ will be cancelled because of eqs. (2.11) and (2.15). Such an argument can easily be extended to an arbitrary number of gluons. Consider as an example the subamplitude with the following helicity configuration $C(1+,2+,\ldots,m+,(m+1)\lambda_{m+1},\ldots,n\lambda_n)$. This subamplitude will *not* contain the poles $(K_i+K_{i+1}+\cdots+K_j)^2$ with $1\le i\le m-2$, $i+2\le j\le m$ because of the vanishing of the gauge invariant part of the current $J_\mu(i+,\ldots,j+)$, i.e. $G_\mu(i+,\ldots,j+)=0$.

The subamplitudes with five (or more) equal helicities are already calculated in chap. 5, eqs. (V.5.1) and (V.5.9). For instance

$$C(1\pm,2+,3+,4+,5+,6+,7+)=0 \tag{5.4}$$

and

$$C(1-,2-,3+,4+,5+,6+,7+)=\frac{(\sqrt{2})^7}{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 71\rangle}\ . \tag{5.5}$$

We will now give and discuss one of the four different helicity configurations of the subamplitudes which determine the matrix element. The others are given in ref. [7]. The following abbreviations are used

$$\begin{aligned}\{i,j\} &= \langle ij\rangle\langle ij\rangle^* = (K_i+K_j)^2\\(i+j+k)^2 &= (K_i+K_j+K_k)^2\\\langle a|b+c|d\rangle &= \langle ab\rangle^*\langle db\rangle+\langle ac\rangle^*\langle dc\rangle = k_{a\dot A}k_{dB}(K_b+K_c)^{\dot AB}\ .\end{aligned} \tag{5.6}$$

The subamplitude is given by

$$
\begin{aligned}
&\mathcal{C}(1+,2+,3+,4-,5-,6-,7-) = \\
\frac{(\sqrt{2})^7}{2} \Bigg[&- \frac{\langle 56\rangle\langle 67\rangle\langle 71\rangle\langle 1|6+7|5\rangle A^2}{\{2,3\}\{3,4\}(2+3+4)^2\{5,6\}\{6,7\}\{7,1\}(6+7+1)^2} \\
&+ \frac{\langle 12\rangle^*\langle 45\rangle^2\langle 56\rangle\langle 67\rangle B(\langle 56\rangle^*\langle 3|1+2|6\rangle + \langle 57\rangle^*\langle 3|1+2|7\rangle)}{\{1,2\}\{3,4\}\{4,5\}(3+4+5)^2\{5,6\}\{6,7\}\{7,1\}} \\
&- \frac{\langle 12\rangle^*\langle 23\rangle^*\langle 45\rangle^2\langle 67\rangle^2(\langle 34\rangle^*\langle 1|6+7|4\rangle + \langle 35\rangle^*\langle 1|6+7|5\rangle)}{\{3,4\}\{4,5\}(3+4+5)^2\{6,7\}\{7,1\}(6+7+1)^2} \\
&- \frac{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 3|4+5|6\rangle B^2}{\{3,4\}\{4,5\}(3+4+5)^2\{5,6\}\{7,1\}\{1,2\}(7+1+2)^2} \\
&- \frac{\langle 23\rangle^*\langle 45\rangle\langle 56\rangle\langle 67\rangle^2 A(\langle 64\rangle^*\langle 1|2+3|4\rangle + \langle 65\rangle^*\langle 1|2+3|5\rangle)}{\{2,3\}\{3,4\}\{4,5\}\{5,6\}\{6,7\}\{7,1\}(6+7+1)^2} \\
&+ \frac{\langle 45\rangle\langle 56\rangle\langle 67\rangle AB(1+2+3)^2}{\{1,2\}\{2,3\}\{3,4\}\{4,5\}\{5,6\}\{6,7\}\{7,1\}} \Bigg]
\end{aligned} \tag{5.7}
$$

with $A = \langle 23\rangle^*\langle 1|2+3|4\rangle$ and $B = \langle 12\rangle^*\langle 3|1+2|7\rangle$.

Considering the number of 2485 Feynman diagrams this is a surprisingly short expression. Because of the helicity configuration a large number of poles are absent as was explained above. This is demonstrated in the expression, namely the only 3-poles that appear are the expected ones, $(K_2 + K_3 + K_4)^2$, $(K_3 + K_4 + K_5)^2$, $(K_6 + K_7 + K_1)^2$ and $(K_7 + K_1 + K_2)^2$, i.e. a 3-pole must contain the momenta of gluons which have different helicities. For the other helicity combinations one will find more different propagators and subsequently the expressions will be longer [7].

In the case of the helicity combination of all but two equal helicities it was possible to generalize the helicity amplitude to a arbitrary number of gluons [9] and could subsequently be proven [6] in chap. 5. One may wonder whether the case with all but three equal helicities can be generalized in a similar way. Comparing the six and seven gluon helicity amplitudes of eqs. (4.6) and (5.7) no systematics are revealed. So how to generalize this particular helicity combination to an arbitrary number of gluons is not yet clear.

The squaring of the matrix element can again be done in leading order in the colour approximation, given by eq. (IV.4.10)

$$
\sum_{\text{col.}} |\mathcal{M}(1234567)|^2 = \left(\frac{g^2 N}{2}\right)^5 \left(N^2 - 1\right) \left(\sum_{P(123456)} |\mathcal{C}(1234567)|^2 \right) . \tag{5.8}
$$

Details on the exact squared matrix element are given in ref. [7].

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Chapter VIII

Processes involving a vector boson and up to five partons

In this chapter exact expressions are derived for the matrix elements of parton processes relevant for jet production in $e^+e^-, \gamma\gamma, e^-P$ and $P\bar{P}$ collisions. For the latter the formulae describe the production of a vector boson in conjunction with up to three jets. The possibility to evaluate cross sections in an arbitrary dimension and for massive quarks is kept open for the most relevant processes. Explicit results for helicity amplitudes of massless partons are given for all processes. The general structure of the process $V \to q\bar{q} + n$ gluons emerges from the explicit calculations up to $n = 3$ [1]. A systematic use is made of recursive techniques [2], the abelian part of gluon field strengths and spinor calculus [3].

1 Introduction

The development of exact calculations for processes, where besides partons a virtual or real vector boson participates is less advanced. This can partly be understood when we compare the number of diagrams for some typical processes like

$$\begin{aligned} gg &\to n\,g && (1.1)\\ q\bar{q} &\to n\,g && (1.2)\\ V &\to q\bar{q} + n\,g && (1.3)\\ V &\to q\bar{q}q'\bar{q}' + (n-2)g\,. && (1.4) \end{aligned}$$

Up to $n = 5$ the number of diagrams is listed in table 8.1. We see that process (1.3) contains more diagrams than process (1.1). Moreover, processes (1.3) and (1.4) contain less symmetry than (1.1) and (1.2). The virtuality of the vector boson also complicates the evaluation. From the experimental point of view the process

$$V \to (n+2)\ \text{partons} \qquad (1.5)$$

is as interesting as the pure parton processes. The matrix elements are needed for jet production in the following collisions

$$e^+(P_+) + e^-(P_-) \;\to\; (n+2)\ \text{jets}\,, \qquad (1.6)$$

Process	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$gg \to n\,g$	-	1	4	25	220	2485
$q\bar{q} \to n\,g$	-	1	3	16	123	1240
$V \to q\bar{q}\,n\,g$	1	2	8	50	428	4670
$q\bar{q} \to q'\bar{q}'\,(n-2)\,g$	-	-	1	5	36	341
$V \to q\bar{q}\,q'\bar{q}'\,(n-2)\,g$	-	-	4	24	196	2040

Table 8.1. Number of contributing Feynman diagrams.

$$\gamma + P \;\to\; X + (n+1)\,\text{jets}\,, \tag{1.7}$$

$$e^-(P) + P \;\to\; \begin{pmatrix} e^-(P') \\ \nu(P') \end{pmatrix} + X + (n+1)\,\text{jets}\,, \tag{1.8}$$

$$P + \bar{P} \;\to\; X + V + n\,\text{jets}\,, \tag{1.9}$$

where the momenta of the leptons are P_+, P_-, P and P'. In reaction (1.6) up to five jet events have been seen [4]. Muon scattering experiments have seen jets [5], whereas in $P\bar{P}$ scattering process (1.9) has been established, with $n = 1$ [6]. For all these processes exact numerical (and sometimes analytical) calculations exist up to $n = 2$. Besides tree level calculations [7] for (1.6) also QCD corrections [8] have been performed up to $n = 1$. The reactions (1.7) and (1.8) will be measured at HERA, the first one occurring through the Weiszäcker-Williams approximation in small angle electron scattering, the second one in deep inelastic scattering. Numerical studies have been carried out in ref. [9]. For reaction (1.9) analytical [10] and numerical [11] evaluations have been performed as well.

In future experiments multi jet events in reactions (1.6)-(1.9) are expected, so an extension above $n = 2$ is called for. This chapter gives all tree level matrix elements up to $n = 3$ in analytical form. Beyond $n = 3$ the process (1.3) can still be calculated numerically in a straightforward way by means of recursion relations of chap. 4. Since this parton subprocess is expected to be dominant in (1.5) one should get a reasonable estimate for multi jet events in reactions (1.6)-(1.9). Our results indicate that the difference in computational speed between the numerical evaluation of analytic expressions and the recursive numerical calculation decreases in such a way that for $n = 4$ it becomes less urgent to perform analytical calculations for process (1.3). For process (1.4) we only give the result for $n = 2, 3$. In principle, one could introduce also here a recursive method, but we have not done so.

As we will see, our result allows a straightforward replacement of a gluon by a photon. This means that our formulae also apply to 4 jet production in photon-photon collisions, where one photon can also be off shell.

Ref. [12] also addresses the problem of the processes (1.6)-(1.9) ($n \leq 3$). Their approach is the straightforward approach of writing out Feynman diagrams, which can subsequently be programmed. Their colour decomposition is the same as in eq. (IV.4.19). No attempt is made to simplify the matrix elements before numerical evaluation. With the results of this chapter and of refs. [1,12] numerical studies have been performed in ref. [13] of process (1.9). In ref. [14] process (1.6) is studied for a virtual photon exchange. The matrix element was obtained using the standard spin summation techniques with a algebraic manipulation program.

In this paper the systematics of process (1.3) is emphasized. The general structure for arbitrary n becomes clear. Up to $n = 3$ explicit compact formulae are given. For reaction (1.4) the gain in simplicity is far less than for (1.3). For completeness analytical formulae are given.

In order to describe all processes (1.6)-(1.9) at the same time we calculate $\widehat{S}_\mu$ (see eq. (IV.4.19)) in

$$\mathcal{M}(Q;1,\ldots,n;P) = V^\mu \widehat{S}_\mu(Q;1,\ldots,n;P) \tag{1.10}$$

for (1.3) and $\widehat{T}_\mu$ in

$$\mathcal{M}(Q_1,Q_2,Q_3,Q_4;1,\ldots,n) = V^\mu \widehat{T}_\mu(Q_1,Q_2,Q_3,Q_4;1,\ldots,n) \tag{1.11}$$

for process (1.4). In these expressions V^μ is the polarization vector of the vector boson, whereas $\widehat{S}_\mu$ and $\widehat{T}_\mu$ are currents, containing quarks and gluons. These currents depend on the momenta $K_1,\ldots,K_n$ of the outgoing gluons, Q, Q_1, Q_3 (P, Q_2, Q_4) of the outgoing quarks (antiquarks), the helicities and colours of the partons. The currents also contain the QCD coupling constant g and the electroweak vertex which we denote by $ie\delta_{ij}\Gamma_\mu^{V,f_1f_2}$, given by eq. (IV.4.15). It will be usefull in the following sections to translate Γ_μ^{V,f_1f_2} in the Weyl-van der Waerden formalism of chap. 3. Using eqs. (III 2.23) and (III.2.24) we find

$$\begin{aligned}\Gamma_\mu^{V,f_1f_2} &= L^V_{f_1f_2}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right) + R^V_{f_1f_2}\gamma_\mu\left(\frac{1+\gamma_5}{2}\right) \\ &= \begin{pmatrix} 0 & -iL^V_{f_1f_2}\sigma_{\mu\dot{B}A} \\ iR^V_{f_1f_2}\sigma_\mu^{\dot{A}B} & 0 \end{pmatrix},\end{aligned} \tag{1.12}$$

where the left-handed coupling $L^V_{f_1f_2}$ and the right-handed couplings $R^V_{f_1f_2}$ are given in eqs. (IV.4.16)-(IV.4.18) for various combinations of fermions and vector bosons.

The lepton couplings are needed for reactions (1.6) and (1.8). For (1.6) we have

$$V_\mu^\gamma = e\bar{v}(P_+)\Gamma_\mu^{\gamma,ee}\frac{1}{s}u(P_-)\,, \tag{1.13}$$

$$V_\mu^Z = e\bar{v}(P_+)\Gamma_\mu^{Z,ee}\frac{1}{s-M_Z^2}u(P_-)\,, \tag{1.14}$$

and for (1.8)

$$V_\mu^\gamma = e\bar{u}(P')\Gamma_\mu^{\gamma,ee}\frac{1}{Q^2}u(P)\,, \tag{1.15}$$

$$V_\mu^Z = e\bar{u}(P')\Gamma_\mu^{Z,ee}\frac{1}{Q^2-M_Z^2}u(P)\,, \tag{1.16}$$

$$V_\mu^W = e\bar{u}(P')\Gamma_\mu^{W,\nu e}\frac{1}{Q^2-M_W^2}u(P)\,, \tag{1.17}$$

where $s = (P_+ + P_-)^2$ and $Q^2 = (P - P')^2$. The charged current case is described by eq. (1.17). The matrix elements for reactions (1.6) and (1.8) are given by

$$\mathcal{M} = V_\gamma^\mu \hat{S}_\mu^\gamma + V_Z^\mu \hat{S}_\mu^Z \tag{1.18}$$

and

$$\mathcal{M} = V_W^\mu \hat{S}_\mu^W \tag{1.19}$$

and similar expressions with $\hat{T}_\mu$. The labels γ, Z or W on $\hat{S}_\mu$ or $\hat{T}_\mu$ refer to the type of vector boson coupling to the current. Of course, $\hat{S}_\mu$ and $\hat{T}_\mu$ which will be subsequently given for reaction (1.5) should be crossed appropriately for reactions (1.7)-(1.9). This will be indicated in sec. 5.

In the case of massless leptons one can introduce helicity states in V^μ. Using the Weyl-van der Waerden formalism one obtains for e^+e^- annihilation

$$V_\gamma^\mu(e^+ +; e^- -) = e\,\sigma^\mu_{\dot{A}B}\, p_-^{\dot{A}}\, p_+^B\, \frac{L_{ee}^\gamma}{s}\ ,\ V_Z^\mu(e^+ +; e^- -) = e\,\sigma^\mu_{\dot{A}B}\, p_-^{\dot{A}}\, p_+^B\, \frac{L_{ee}^Z}{s - M_Z^2}\ , \tag{1.20}$$

$$V_\gamma^\mu(e^+ -; e^- +) = e\,\sigma^\mu_{\dot{A}B}\, p_+^{\dot{A}}\, p_-^B\, \frac{R_{ee}^\gamma}{s}\ ,\ V_Z^\mu(e^+ -; e^- +) = e\,\sigma^\mu_{\dot{A}B}\, p_+^{\dot{A}}\, p_-^B\, \frac{R_{ee}^Z}{s - M_Z^2}\ . \tag{1.21}$$

For electron scattering we find

$$V_\gamma^\mu(+;+) = e\,\sigma^\mu_{\dot{A}B}\, p'^{\dot{A}}\, p^B\, \frac{R_{ee}^\gamma}{Q^2}\ ,\ V_Z^\mu(+;+) = e\,\sigma^\mu_{\dot{A}B}\, p'^{\dot{A}}\, p^B\, \frac{R_{ee}^Z}{Q^2 - M_Z^2}\ , \tag{1.22}$$

$$V_\gamma^\mu(-;-) = e\,\sigma^\mu_{\dot{A}B}\, p^{\dot{A}}\, p'^B\, \frac{L_{ee}^\gamma}{Q^2}\ ,\ V_Z^\mu(-;-) = e\,\sigma^\mu_{\dot{A}B}\, p^{\dot{A}}\, p'^B\, \frac{L_{ee}^Z}{Q^2 - M_Z^2}\ , \tag{1.23}$$

whereas the charged current reaction gives

$$V_W^\mu(+;+) = 0\ ,\ V_W^\mu(-;-) = e\,\sigma^\mu_{\dot{A}B}\, p^{\dot{A}}\, p'^B\, \frac{L_{e\nu}^W}{Q^2 - M_W^2}\ . \tag{1.24}$$

For translation of eqs. (1.18) and (1.19) in the spinor formalism relation (III.2.12) is useful.

The actual outline of the chapter is as follows. In sec. 2 two of the steps to obtain S_μ are carried out analytically, such that the building blocks for $\hat{S}_\mu$ are obtained. This leads in sec. 3 to the explicit $\hat{S}_\mu$ expressions up to 3 gluons. A general structure for n gluons is conjectured. The quantities $\hat{T}_\mu$ for 0 and 1 gluon are discussed in sec. 4. Sec. 5 explains how to square the amplitudes and discusses crossing, which is necessary to obtain (1.7)-(1.9) from process (1.5). Some helicity currents are given in sec. 6.

2 Analytic results for the gluon and spinorial currents

The central quantity to be calculated in this chapter is $V^\rho \hat{S}_\rho(Q; 1, \ldots, n; P)$, the amplitude for the decay of a real or virtual vector boson into $q\bar{q}$ and n gluons

$$V \to q(Q) + \bar{q}(P) + g(K_1) + \cdots + g(K_n)\ ,$$

where V^ρ is the polarization vector of V. For the explicit analytical results the maximum number of gluons will be three, for numerical results there is no limit other than computer time. The quantity $\widehat{S}_\rho$ is evaluated in three steps. In the first step we evaluate recursively pure gluonic currents $\hat{J}_\mu(1,2,\ldots,m)$ for all $m \leq n$. Contracting those currents with the polarization vector of gluon $(m+1)$ would give the $(m+1)$-gluon scattering amplitude, see eq. (IV.4.2). However here they are used as building blocks in recursion relations for the spinorial currents $\hat{J}(Q;1,2,\ldots,\ell)$ and $\hat{J}(1,\ldots,\ell;P)$ with $\ell \leq n$. Again, contracting these currents with $v(P)$ or $\bar{u}(Q)$ would give amplitudes for $q\bar{q}+n$ gluon production from the vacuum as indicated in eq. (IV.4.12). They can also be used as building blocks for $\widehat{S}_\rho$. This current is given by

$$\widehat{S}_\mu(Q;1,2,\ldots,n;P) = ieg^n \sum_{P(1,2,\ldots,n)} (T^{a_1}\cdot T^{a_2}\cdots T^{a_n})_{ij}\, S_\mu(Q;1,2,\ldots,n;P) \quad (2.1)$$

where $S_\mu(Q;1,\ldots,n;P)$ is the colourless current of eq. (IV.4.20).

In chap. 5 we have shown that in the case of massless quarks the expression (IV.4.20) for S_μ can be calculated explicitly for a specific helicity configuration, given in eqs. (V.5.30) and (V.5.31). These equations are in the notation of this chapter given by

$$S_\mu(Q+;1+,2+,\ldots,n+;P-) = R^V_{f_1 f_2}(\sqrt{2})^n \sigma_\mu^{\dot{A}B} \frac{[Q+\kappa(1,n)]_{\dot{A}C}\, p^C p_B}{\langle q1\rangle\langle 12\rangle\cdots\langle np\rangle}\,,$$

$$S_\mu(Q-;1+,2+,\ldots,n+;P+) = -L^V_{f_1 f_2}(\sqrt{2})^n \sigma_\mu^{\dot{A}B} \frac{[P+\kappa(1,n)]_{\dot{A}C}\, q^C q_B}{\langle q1\rangle\langle 12\rangle\cdots\langle np\rangle}\,. \quad (2.2)$$

In these expressions the notation of Weyl-van der Waerden spinor calculus has been used (see chap. 3). The above derivation is feasible since the gauge invariance of S_μ makes it possible to choose specific gauge spinors in eq. (III.2.55) for the gluon polarizations which simplify the recursion relations. For other helicity combinations the simplifications are not so great that one can evaluate S_μ for all n. One has to build up the expressions by increasing n step by step. This has to be done for every helicity combination. Although this procedure will give the required answers, we follow here a different path. We postpone the specific choice of the helicities. Instead, we express gauge invariant objects in terms of the abelian parts of the gluon field strengths for which we already showed their usefulness in chaps. 6 and 7. The abelian part of the field strength is given by

$$F_{i\mu\nu} = K_{i\mu}J_{i\nu} - K_{i\nu}J_{i\mu} \quad (2.3)$$

which will often be contracted with γ matrices

$$\not{F}_i = F_{i\mu\nu}\gamma^\mu\gamma^\nu = 2\not{K}_i\not{J}_i\,. \quad (2.4)$$

In a later stage the helicities can be chosen, such that from one expression for S_μ in terms of F_i's the various helicity amplitudes are obtained. In the helicity formalism, and using the Weyl-van der Waerden notation, the abelian field strength get a simple form

$$\not{F}^+ = \begin{pmatrix} 0 & 0 \\ 0 & 2\sqrt{2}k^{\dot{A}}k_{\dot{B}} \end{pmatrix} , \tag{2.5}$$

$$\not{F}^- = \begin{pmatrix} -2\sqrt{2}k_A k^B & 0 \\ 0 & 0 \end{pmatrix} , \tag{2.6}$$

as can be seen by using eqs. (III.2.10), (III.2.18), (III.2.23) and (III.2.55). Note that due to the gauge invariance the gauge spinor drops out. These forms lead to simplifying relations like

$$\bar{u}_+(Q)\not{F}_1^- = 0 , \tag{2.7}$$

$$\not{F}_1^+\not{F}_2^- = 0 , \tag{2.8}$$

$$\not{K}_1\not{F}_1^\pm = \not{F}_1^\pm\not{K}_1 = 0 ,$$

$$\not{F}_1^+\not{Q}\not{F}_2^+ = 0 , \tag{2.9}$$

where we used eqs. (III.2.10), (III.2.23) and (III.2.38).

One could refrain altogether from inserting the helicities and proceed in the standard way to obtain the squared matrix element. Then one can keep the fermion masses and one can give a result in an arbitrary dimension. The S_μ expression in terms of F_i's is reasonably compact since the gauge cancellations have already been performed.

For the pure gluonic part of the calculation one needs the currents $J(1)$, $J(12)$ and $J(123)$. The last one is non trivial. It gives through eq. (IV.4.2) the gauge invariant subamplitude $C(1234)$. Therefore, one can write

$$C(1234) = \alpha_{\mu\nu}(F_1, F_2, F_3)\, F_4^{\nu\mu}|_{\kappa(1,4)=0} , \tag{2.10}$$

with

$$\alpha_{\mu\nu}(F_1, F_2, F_3) = -\alpha_{\nu\mu}(F_1, F_2, F_3) . \tag{2.11}$$

This implies for $J_\mu(123)$, see eqs. (VI.2.1)-(VI.2.8).

$$J_\mu(123) = \frac{G_\mu(123)}{\kappa(1,3)^2} + \frac{\kappa^\nu(1,3)}{\kappa(1,3)^2}\left[\kappa_\nu(1,3)Y_\mu(123) - \kappa_\mu(1,3)Y_\nu(123)\right] \tag{2.12}$$

with

$$G_\mu(123) = 2\kappa^\nu(1,3)\alpha_{\nu\mu}(F_1, F_2, F_3) . \tag{2.13}$$

In other words, $J_\mu(123)$ contains a gauge invariant part $G_\mu(123)/\kappa(1,3)^2$ which contributes to the subamplitude C, the gauge dependent part $Y_\mu(123)$ does not contribute to the subamplitude C because of momentum conservation and transversality of the fourth gluon. The explicit expressions for $G_\mu(123)$ and $Y_\mu(123)$ are given in eqs. (VII.2.12) and (VII.2.13).

The spinorial currents are similarly divided into a gauge invariant part G_0 and a gauge dependent part Y. The explicit forms are derived from the recursion relations (IV.3.9) and (IV.3.13). In general we have

$$J(Q; 1, 2, \ldots, n) = G_0(Q; 1, 2, \ldots, n) + Y(Q; 1, 2, \ldots, n) ,$$

$$J(1,2,\dots,n;P) = G_0(1,2,\dots,n;P) + Y(1,2,\dots,n;P)\,, \tag{2.14}$$

with a priori different masses m and $\overline{m}$ for quark and antiquark. Up to $n = 3$ we obtain

$$G_0(Q) = J(Q) = \bar{u}(Q)\,, \tag{2.15}$$

$$G_0(Q;1) = \frac{1}{4}\frac{G_0(Q)\not{J}_1}{Q\cdot K_1}\,, \tag{2.16}$$

$$Y(Q;1) = -\frac{Q\cdot J_1}{Q\cdot K_1}G_0(Q)\,, \tag{2.17}$$

$$G_0(Q;12) = G(Q;12)\left[\not{Q} + \not{k}(1,2) - m\right]^{-1} + \frac{1}{16}\frac{G_0(Q)\not{J}_1\not{K}_1\not{J}_2}{Q\cdot K_1\, K_1\cdot K_2}\,, \tag{2.18}$$

$$G(Q;12) = \frac{1}{16}G_0(Q)\frac{\not{J}_1\not{Q}\not{J}_2 + \not{J}_2\not{Q}\not{J}_1 - (\not{J}_1\not{J}_2 + \not{J}_2\not{J}_1)\not{Q}}{Q\cdot K_1\, K_1\cdot K_2}\,, \tag{2.19}$$

$$Y(Q;12) = -\frac{K_1\cdot J_2}{K_1\cdot K_2}G_0(Q;1) - \frac{\frac{1}{2}Q\cdot K_1\, J_1\cdot J_2 - Q\cdot J_1\, K_1\cdot J_2}{Q\cdot K_1\, K_1\cdot K_2}G_0(Q)\,, \tag{2.20}$$

$$G_0(Q;123) = G(Q;123)\left[\not{Q} + \not{k}(1,3) - m\right]^{-1} + \frac{1}{64}\frac{G_0(Q)\not{J}_1\not{K}_1\not{J}_2\not{K}_2\not{J}_3}{Q\cdot K_1\, K_1\cdot K_2\, K_2\cdot K_3}\,, \tag{2.21}$$

$$\begin{array}{rcl} G(Q;123) & = & -G_0(Q)\dfrac{\not{C}(123)}{\kappa(1,3)^2} - \dfrac{1}{4}G(Q;12)\left[\not{Q} + \not{k}(1,2) - m\right]^{-1}\dfrac{(\not{K}_2\not{J}_3 - \not{J}_3\not{K}_2)}{K_2\cdot K_3} \\ & + & \dfrac{1}{64}\dfrac{G_0(Q)}{Q\cdot K_1\, K_1\cdot K_2\, K_2\cdot K_3}\Big[\not{J}_1(\not{J}_2\not{K}_1\not{J}_3 + \not{J}_3\not{K}_1\not{J}_2) \\ & & \quad + (\not{J}_1\not{Q}\not{J}_2 + \not{J}_2\not{Q}\not{J}_1)\not{J}_3 - \not{J}_3(\not{J}_1\not{Q}\not{J}_2 + \not{J}_2\not{Q}\not{J}_1) \\ & & \quad + \not{J}_3\not{Q}\not{J}_1\not{J}_2 - \not{J}_2\not{J}_1\not{Q}\not{J}_3 + (\not{J}_3\not{J}_2\not{J}_1 - \not{J}_1\not{J}_2\not{J}_3)\not{Q}\Big]\,. \end{array} \tag{2.22}$$

$$\begin{array}{rcl} Y(Q;123) & = & -\dfrac{K_2\cdot J_3}{K_2\cdot K_3}G_0(Q;12) \\ & - & \dfrac{\frac{1}{2}K_1\cdot K_2\, J_2\cdot J_3 - K_1\cdot J_2\, K_2\cdot J_3}{K_1\cdot K_2\, K_2\cdot K_3}G_0(Q;1) \\ & - & \Bigg[\dfrac{\frac{1}{2}Q\cdot F_1\cdot K_2\, J_2\cdot J_3 - \left(\frac{1}{2}Q\cdot K_1\, J_1\cdot J_2 - Q\cdot J_1\, K_1\cdot J_2\right)K_2\cdot J_3}{Q\cdot K_1\, K_1\cdot K_2\, K_2\cdot K_3} \\ & & -\dfrac{\kappa(1,3)\cdot Y(123)}{\kappa(1,3)^2}\Bigg]G_0(Q)\,. \end{array} \tag{2.23}$$

In the case of massless quarks the last term of eqs. (2.19) and (2.22) vanishes. In order to see this one uses a relation due to Kahane [17] (corollary A_4)

$$(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{2n}} + \gamma_{\mu_{2n}}\cdots\gamma_{\mu_2}\gamma_{\mu_1})\,\gamma_\alpha = \gamma_\alpha\,(\gamma_{\mu_2}\cdots\gamma_{\mu_{2n}}\gamma_{\mu_1} + \gamma_{\mu_1}\gamma_{\mu_{2n}}\cdots\gamma_{\mu_2})\,. \tag{2.24}$$

The above expressions give the amplitudes $\mathcal{D}$ for the production of $q\bar{q}$ and n gluons through eq. (IV.4.12). Thus one finds for $n = 2$

$$\begin{aligned}
\mathcal{D}(Q;12;P) &= G_0(Q;12)\left[\not{Q} + \not{J}(1,2) - m\right] v(P) \\
&= G(Q;12) v(P) \\
&= \frac{1}{16}\bar{u}(Q)\frac{\not{J}_1\not{Q}\not{J}_2 + \not{J}_2\not{Q}\not{J}_1 - (\not{J}_1\not{J}_2 + \not{J}_2\not{J}_1)\not{Q}}{Q\cdot K_1\, K_1\cdot K_2} v(P)\ , \qquad (2.25)
\end{aligned}$$

and for massless quarks

$$\mathcal{D}(Q;12;P) = \frac{1}{16}\bar{u}(Q)\frac{\not{J}_1\not{Q}\not{J}_2 + \not{J}_2\not{Q}\not{J}_1}{Q\cdot K_1\, K_1\cdot K_2} v(P)\ . \qquad (2.26)$$

The case of $J(1,2,\ldots,n;P)$ is related to the above expressions (2.15)-(2.23) by charge conjugation. The corresponding quantities for the second spinorial currents in eq. (2.14) can be obtained from the charge conjugation operation given in eqs. (IV.3.14)-(IV.3.17). For example from eq. (2.17) one derives

$$Y(Q;1)C^{-1} = -Y^T(1;Q)\ , \qquad (2.27)$$

or

$$Y^T(1;P) = \frac{P\cdot J_1}{P\cdot K_1} v^T(P)\ , \qquad (2.28)$$

or

$$Y(1;P) = \frac{P\cdot J_1}{P\cdot K_1} J(P)\ . \qquad (2.29)$$

Similarly from eqs. (2.18) and (2.19)

$$G_0(P;21)C^{-1} = G_0^T(12;P)\ , \qquad (2.30)$$

$$G_0^T(12;P) = G(P;21)\left[\not{P} + \not{J}(1,2) - \overline{m}\right]^{-1} C^{-1} + \frac{1}{16} J(P)\frac{\not{J}_2\not{J}_1}{P\cdot K_2\, K_2\cdot K_1} C^{-1}\ , \qquad (2.31)$$

$$G_0(12;P) = \left[\not{P} + \not{J}(1,2) + \overline{m}\right]^{-1} G(12;P) + \frac{1}{16}\frac{\not{J}_1\not{J}_2}{K_1\cdot K_2\, K_2\cdot P} J(P)\ , \qquad (2.32)$$

with

$$G(12;P) = \frac{1}{4}\frac{\not{J}_1\not{P}\not{J}_2 + \not{J}_2\not{P}\not{J}_1 - \not{P}(\not{J}_1\not{J}_2 + \not{J}_2\not{J}_1)}{K_1\cdot K_2\, K_2\cdot P} J(P)\ . \qquad (2.33)$$

These examples show that the quantities $Y(1,2,\ldots,n;P)$, $G_0(1,2,\ldots,n;P)$ and $G(1,2,\ldots,n;P)$ are obtained from the known quantities $Y(P;n,\ldots,2,1)$, $G_0(P;n,\ldots,2,1)$ and $G(P;n,\ldots,2,1)$ by reading the latter backward. Every gluon and every $\not{J}_i$ introduces a minus sign and in the propagator $-\overline{m}$ changes into $+\overline{m}$. We are now ready to construct the current $S_\mu(Q;1,\ldots,n;P)$.

3 The vector current for $V \to q\bar{q} + n$ gluons

The spinorial currents of the previous section will now be used to construct S_μ as indicated in eq. (IV.4.20). The division of the spinorial currents into a gauge independent part G_0 and gauge dependent part Y now pays of. The vector current S_μ is gauge independent and therefore the terms Y will combine into a gauge independent part.

The trivial $n = 0$ result is

$$S_\mu(Q; P) = J(Q)\Gamma_\mu J(P) = G_0(Q)\Gamma_\mu G_0(P)\,. \tag{3.1}$$

For $n = 1$ we show the steps explicitly, for $n = 2$ and 3 we omit them.

$$\begin{aligned} S_\mu(Q;1;P) &= J(Q;1)\Gamma_\mu J(P) + J(Q)\Gamma_\mu J(1,P) \\ &= G_0(Q;1)\Gamma_\mu G_0(P) + G_0(Q)\Gamma_\mu G_0(1;P) \\ &\quad + \left(\frac{P\cdot J_1}{P\cdot K_1} - \frac{Q\cdot J_1}{Q\cdot K_1}\right) G_0(Q)\Gamma_\mu G_0(P) \qquad (3.2) \\ &= G_0(Q)\Gamma_\mu G_0(1;P) + f_{QP}(1)G_0(Q)\Gamma_\mu G_0(P) + G_0(Q;1)\Gamma_\mu G_0(P)\,. \qquad (3.3) \end{aligned}$$

Here and in the following we use the notation

$$f_{AB}(m, m+1, \ldots, n) = \frac{A\cdot F_m \cdot F_{m+1}\cdots F_n \cdot B}{(A\cdot K_m)(K_m\cdot K_{m+1})\cdots(K_n\cdot B)}\,. \tag{3.4}$$

The quantities A and B are momenta like Q, P or K_i. In the latter case we write e.g. f_{iP}. The quantities f_{AB} can easily be evaluated for specific helicities using the Weyl-van der Waerden formalism, e.g.

$$\begin{aligned} f_{AB}(1+,2+,\ldots,n+) &= \left(\sqrt{2}\right)^n \frac{\langle ab\rangle}{\langle a1\rangle\langle 12\rangle\cdots\langle nb\rangle}\,, \qquad (3.5) \\ f_{AB}(1+,2-) &= \left(\sqrt{2}\right)^2 \frac{\langle a1\rangle^*\langle 1b\rangle^*\langle a2\rangle\langle 2b\rangle}{\{A,1\}\{1,2\}\{2,B\}}\,, \qquad (3.6) \\ f_{AB}(1+,2+,3-,4+,5-,6+) &= \left(\sqrt{2}\right)^6 \frac{\langle a1\rangle^*\langle 12\rangle^*\langle 24\rangle^*\langle 6b\rangle^*\langle a3\rangle\langle 35\rangle\langle 5b\rangle}{\{A,1\}\{1,2\}\cdots\{5,6\}\{6,B\}}\,, \qquad (3.7) \end{aligned}$$

where we have used eqs. (2.5) and (2.6). The vertex Γ_μ is a shorthand for the vertex Γ_μ^{V,f_1f_2} given in eq. (1.12).

For $n = 2, 3$ we find

$$\begin{aligned} S_\mu(Q;12;P) = {}& G_0(Q)\Gamma_\mu G_0(12;P) + f_{Q2}(1)G_0(Q)\Gamma_\mu G_0(2;P) \\ &+ f_{QP}(12)G_0(Q)\Gamma_\mu G_0(P) + G_0(Q;1)\Gamma_\mu G_0(2;P) \\ &+ f_{1P}(2)G_0(Q;1)\Gamma_\mu G_0(P) + G_0(Q;12)\Gamma_\mu G_0(P)\,, \qquad (3.8) \end{aligned}$$

$$
\begin{aligned}
S_\mu(Q;123;P) \;=\;& G_0(Q)\Gamma_\mu G_0(123;P) + f_{Q2}(1)G_0(Q)\Gamma_\mu G_0(23;P) \\
&+f_{Q3}(12)G_0(Q)\Gamma_\mu G_0(3;P) + f_{QP}(123)G_0(Q)\Gamma_\mu G_0(P) \\
&+G_0(Q;1)\Gamma_\mu G_0(23;P) + f_{13}(2)G_0(Q;1)\Gamma_\mu G_0(3;P) \\
&+f_{1P}(23)G_0(Q;1)\Gamma_\mu G_0(P) + G_0(Q;12)\Gamma_\mu G_0(3;P) \\
&+f_{2P}(3)G_0(Q;12)\Gamma_\mu G_0(P) + G_0(Q;123)\Gamma_\mu G_0(P)\ . \qquad (3.9)
\end{aligned}
$$

From these explicit evaluations we are led to a conjecture for the n-gluon vector boson current. Introduce

$$
\Gamma_\mu(m;m+1,\ldots,m+k;m+k+1) = \begin{cases} \Gamma_\mu & \text{for } k=0\ , \\ f_{m,m+k+1}(m+1,\ldots,m+k)\Gamma_\mu & \text{for } k\geq 1\ , \end{cases} \qquad (3.10)
$$

with the understanding that $m=0$ and $m+k+1=n+1$ correspond to Q and P respectively. Then eqs. (3.3), (3.8) and (3.9) generalize to

$$
\begin{aligned}
S_\mu(Q;1,\ldots,n;P) \;=\;& \sum_{m=0}^{n}\sum_{k=0}^{n-m} G_0(Q;1,\ldots,m)\Gamma_\mu(m;m+1,\ldots,m+k;m+k+1) \\
&\times G_0(m+k+1,\ldots,n;P)\ . \qquad (3.11)
\end{aligned}
$$

Note that also eqs. (2.16), (2.18) and (2.21) suggest a generalization for the gauge independent part $G_0(Q;1,\ldots,n)$:

$$
\begin{aligned}
G_0(Q;1,\ldots,n) \;=\;& G(Q;1,\ldots,n)[\not{Q}+\not{\kappa}(1,n)-m]^{-1} \\
&+\left(\frac{1}{4}\right)^n G_0(Q)\frac{\not{K}_1\not{K}_2\cdots\not{K}_n}{(Q\cdot K_1)(K_1\cdot K_2)\cdots(K_{n-1}\cdot K_n)}\ . \qquad (3.12)
\end{aligned}
$$

Although the general formula (3.11) requires the knowledge of (3.12) for a specific evaluation, it offers an insight in the general structure. For instance, the pole structure of S_μ becomes transparent. The object $\Gamma_\mu(m;m+1,\ldots,m+k;m+k+1)$ contains short poles as can be seen from eq. (3.4), whereas $G_0(Q;1,\ldots,m)$ and $G_0(m+k+1,\ldots,n;P)$ contains besides the short poles also the longer pole terms like $[(Q+\kappa(1,m))^2-m^2]^{-1}$ and $[(\kappa(m+k+1,n)+P)^2-\overline{m}^2]^{-1}$. The further structure depends on functions like $G(Q;1,\ldots,m)$ which determines $q\bar{q}\to m$ gluons (cf. eqs. (IV.4.11) and (2.25)). These functions in turn are determined by the gauge invariant part of the gluon currents $G_\mu(1,\ldots,m)$, which give m gluon scattering (cf. eqs. (IV.4.1) and (2.10)). Although we cannot prove the conjectures (3.11) and (3.12) we can perform a consistency check. In fact, the equations of (2.2) should also follow from the conjectured formulae. This is clear from the following argument. From the relations

$$
D(Q+;1+,2+,\ldots,n+;P-)=0\ , \qquad (3.13)
$$

$$
D(Q-;1+,2+,\ldots,n+;P+)=0\ , \qquad (3.14)
$$

follow

$$
G(Q+;1+,\ldots,n+)=0\ , \qquad (3.15)
$$

$$
G(1+,\ldots,n+;P-)=0\ . \qquad (3.16)
$$

This then determines the functions G_0

$$
\begin{aligned}
G_0(Q+;1+,2+,\ldots,n+) &= \left(\frac{1}{4}\right)^n G_0(Q+)\frac{\not{F}_1^+\not{F}_2^+\cdots\not{F}_n^+}{(Q\cdot K_1)\cdots(K_{n-1}\cdot K_n)} \\
&= -i\left(\sqrt{2}\right)^n \frac{k_{n\dot{A}}}{\langle q1\rangle\langle 12\rangle\cdots\langle n-1\,n\rangle}\,, \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
G_0(1+,2+,\ldots,n+;P-) &= \left(\frac{1}{4}\right)^n \frac{\not{F}_1^+\not{F}_2^+\cdots\not{F}_n^+}{(K_1\cdot K_2)\cdots(K_n\cdot P)}G_0(P-) \\
&= 0\,. \qquad (3.18)
\end{aligned}
$$

Because of eq. (3.18), the general formula (3.11) now simplifies to

$$
\begin{aligned}
&S_\mu(Q+;1+,\ldots,n+;P-) = G_0(Q+;1+,\ldots,n+)\Gamma_\mu G_0(P-) \\
&\quad+\sum_{m=1}^{n-1} f_{mP}((m+1)+,\ldots,n+)G_0(Q+;1+,\ldots,m+)\Gamma_\mu G_0(P_-) \\
&\quad+f_{QP}(1+,\ldots,n+)G_0(Q+)\Gamma_\mu G_0(P-)\,. \qquad (3.19)
\end{aligned}
$$

Expressing Γ_μ, the currents and f_{mP} in spinor notation and using eq. (3.17) one finds

$$
\begin{aligned}
&S_\mu(Q+;1+,\ldots,n+;P-) = R^V_{f_1f_2} i\sigma_\mu^{\dot{A}B}\left[G_{0\dot{A}}(Q+;1+,\ldots,n+)G_{0B}(P^-)\right. \\
&\quad+\sum_{m=1}^{n-1} f_{mP}((m+1)+,\ldots,n+)G_{0\dot{A}}(Q+;1+,\ldots,m+)G_{0B}(P_-) \\
&\quad\left.+f_{QP}(1+,\ldots,n+)G_{0\dot{A}}(Q+)\Gamma_\mu G_{0B}(P-)\right] \\
&= R^V_{f_1f_2}\sigma_\mu^{\dot{A}B}\left(\sqrt{2}\right)^n\left[\frac{k_{n\dot{A}}p_B}{\langle q1\rangle\langle 12\rangle\cdots\langle n-1\,n\rangle}\right. \\
&\quad\left.+\sum_{m=1}^{n-1}\frac{\langle m\,p\rangle}{\langle m\,m+1\rangle\cdots\langle n\,p\rangle}\frac{k_{m\dot{A}}p_B}{\langle q1\rangle\cdots\langle m-1\,m\rangle}+\frac{\langle q\,p\rangle}{\langle q1\rangle\cdots\langle n\,p\rangle}q_{\dot{A}}p_B\right] \\
&= R^V_{f_1f_2}\left(\sqrt{2}\right)^n\sigma_\mu^{\dot{A}B}\frac{[Q+\kappa(1,n)]_{\dot{A}C}\,p^C p_B}{\langle q1\rangle\langle 12\rangle\cdots\langle np\rangle}\,, \qquad (3.20)
\end{aligned}
$$

which is indeed the first equation of (2.2). In a similar fashion one obtains the other helicity amplitude.

The matrix elements of the vector current are thus given by eqs. (3.1)-(3.9) for up to three gluons. At this point everything is expressed in terms of F's and the spinors $\bar{u}(Q)$ and $v(P)$.

Eventually one likes to obtain a colour summed matrix element squared. For this it is useful to introduce symmetrizations in certain gluon variables like

$$
\begin{aligned}
&S_\mu(Q;1,2,\ldots,n-1,\tilde{n};P) = S_\mu(Q;n,1,2,\ldots,n-1;P) \\
&\quad+S_\mu(Q;1,n,2,\ldots,n-1;P)+\cdots+S_\mu(Q;1,2,\ldots,n-1,n;P)\,, \qquad (3.21)
\end{aligned}
$$

$$
S_\mu(Q;1,2,\ldots,m,\widetilde{m+1},\ldots,\tilde{n};P) = \sum_{\text{Perms}} S_\mu(Q;1,2,\ldots,m,m+1,\ldots,n;P)\,. \qquad (3.22)
$$

where the sum runs over all permutations of $(1,\ldots,n)$ which preserve the order of $(1,\ldots,m)$. The amplitudes containing $\widetilde{m+1},\ldots,\tilde{n}$ are in essence amplitudes in which gluons $m+1,\ldots,n$ are replaced by photons, as has been discussed in eqs. (VI.6.2)-(VI.6.5).

It should be noted that eq. (3.21) offers the possibility to calculate jet production in $\gamma\gamma$ physics. The electron which scatters over a sizeable angle gives a V_γ^μ (cf. eq. (1.13)) to be contracted with $\widehat{S}_\mu$. The electron which has only small angle scattering is treated in the Weiszäcker-Williams approximation, which gives a photon $\tilde{n}$. In eq. (2.1) one should then replace one coupling constant g by eQ_f, i.e. the quark charge.

The colour summed matrix elements squared now read (no averaging)

$$|\mathcal{M}(Q;1;P)|^2 = e^2\alpha\frac{N^2-1}{N}\,|V^\mu S_\mu(Q;1;P)|^2\ , \qquad (3.23)$$

$$\begin{aligned}|\mathcal{M}(Q;1,2;P)|^2 &= e^2\alpha^2\frac{N^2-1}{N}\left[\sum_{P(12)}|V^\mu S_\mu(Q;1,2;P)|^2\right.\\ &\left.\quad-\frac{1}{N^2}\left|V^\mu S_\mu(Q;\tilde{1},\tilde{2};P)\right|^2\right]\ ,\end{aligned} \qquad (3.24)$$

$$\begin{aligned}|\mathcal{M}(Q;1,2,3;P)|^2 &= e^2\alpha^3\frac{N^2-1}{N}\left[\sum_{P(123)}\left\{|V^\mu S_\mu(Q;1,2;P)|^2\right.\right.\\ &\left.\left.-\frac{1}{N^2}\left|V^\mu S_\mu(Q;1,2,\tilde{3};P)\right|^2\right\}+\left(\frac{1}{N^2}+\frac{1}{N^4}\right)\left|V^\mu S_\mu(Q;\tilde{1},\tilde{2},\tilde{3};P)\right|^2\right]\ .\end{aligned} \qquad (3.25)$$

The charge e which has been extracted from Γ_μ and thus from S_μ (cf. eqs. (1.12) and (2.1)) is explicitly shown. The quantity α contains the QCD coupling constant g

$$\alpha = \frac{g^2N}{2}\ , \qquad (3.26)$$

where N denotes the number of colours in the $SU(N)$ gauge group. At this point one can still choose for a calculation in an arbitrary dimension. Moreover the mass of the quarks has not been neglected. With standard polarization sums for quark spins and gluon spins (in fact $-g_{\mu\nu}$ is sufficient due to the use of the gauge invariant F's) one obtains the required expressions. However one can also choose for 4 dimensions and massless quarks. In that case the translation into Weyl-van der Waerden spinor calculus is worthwile. The helicity amplitudes can be easily obtained, they are given in sec. 6.

Using these explicit helicity amplitudes in a numerical calculation or using the recursion relations of chap. 4 makes a difference in computing time. From table 8.2 one concludes that for an increasing number of gluons the advantage of analytic expressions decreases.

n	Recursive	Analytic	Ratio
2	0.0708	0.0193	3.68
3	0.745	0.251	2.96
4	11.8	-	-
5	247.6	-	-

Table 8.2. CPU time in seconds needed on a VAX 750 for the calculation of the matrix element squared of $e^+e^- \to \gamma^* \to q\bar{q} + n$ gluons.

4 The process $V \to q\bar{q}q\bar{q} + n$ gluons

In this section the process (1.4) will be considered for $n = 0, 1$. The quarks are massless, the extension to massive quarks is straightforward. The outline of the calculation is given, specific helicity amplitudes are listed in sec. 6.

First we analyze the colour decomposition along the lines of chap. 4. (see also ref. [18]). We start with the process (equal or unequal flavours)

$$V \to q_1\bar{q}_2q_3\bar{q}_4 \tag{4.1}$$

where the quark (antiquark) i has momentum Q_i, helicity λ_i, flavour f_i and colour c_i. The coloured current is given by

$$\widehat{T}_\mu(Q_1, Q_2, Q_3, Q_4) = A_\mu(1234) - A_\mu(1432) + A_\mu(3412) - A_\mu(3214)\ , \tag{4.2}$$

where $A_\mu(1234)$ is given by the two diagrams of fig. 8.1. Note that $A_\mu(1234)$ does not contain any quark permutations. Once it is known $\widehat{T}_\mu$ can be obtained. In A_μ we factorize out the colour and flavour factor

$$A_\mu(1234) = ieg^2\left(\delta_{c_1c_4}\delta_{c_2c_3} - \frac{1}{N}\delta_{c_1c_2}\delta_{c_3c_4}\right)\delta_{f_3f_4}B_\mu^{f_1f_2}(1234) \tag{4.3}$$

with

$$\begin{aligned} B_\mu^{f_1f_2}(1234) \;=\; & \frac{1}{2}\frac{\left[\bar{u}(Q_1)\Gamma_\mu^{f_1f_2}(\not{Q}_2+\not{Q}_3+\not{Q}_4)\gamma_\nu v(Q_2)\right]\left[\bar{u}(Q_3)\gamma^\nu v(Q_4)\right]}{(Q_3+Q_4)^2(Q_2+Q_3+Q_4)^2} \\ & - \frac{1}{2}\frac{\left[\bar{u}(Q_1)\gamma_\nu(\not{Q}_1+\not{Q}_3+\not{Q}_4)\Gamma_\mu^{f_1f_2}v(Q_2)\right]\left[\bar{u}(Q_3)\gamma^\nu v(Q_4)\right]}{(Q_3+Q_4)^2(Q_1+Q_3+Q_4)^2}\ . \end{aligned} \tag{4.4}$$

These expressions easily give helicity amplitudes once spinor calculus is used. The results are collected in sec. 6.

As a check on the above calculation one can take the collinear limit of quark q_3 and antiquark $\bar{q}_4$. Also current conservation serves as a check. The collineair limit relates $B_\mu^{f_1f_2}(1234)$ to $S_\mu(Q_1; K; Q_2)$ according to

$$B_\mu^{f_1f_2}(1\lambda_1, 2\lambda_2, 3\lambda_3, 4\lambda_4) = \sum_{\lambda=\pm1} h_\lambda(\lambda_3, \lambda_4)\, S_\mu(Q_1\lambda_1; K\lambda; Q_2\lambda_2) \tag{4.5}$$

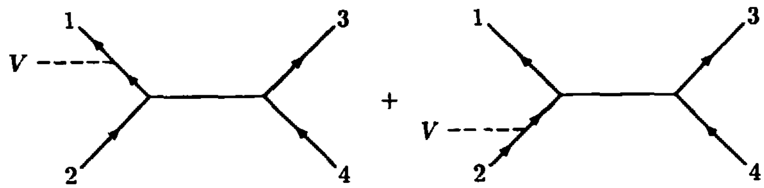

Fig. 8.1. The process $V \to q\bar{q}q\bar{q}$.

where K is a gluon null vector, $Q_3 = zK$, $Q_4 = (1-z)K$. The functions $h_\lambda(\lambda_3, \lambda_4)$ read

$$h_+(+,-) = \frac{\sqrt{2}z}{\langle q_3 q_4\rangle^*}\,, \qquad h_+(-,+) = -\frac{\sqrt{2}(1-z)}{\langle q_3 q_4\rangle^*}\,,$$
$$h_-(+,-) = -\frac{\sqrt{2}(1-z)}{\langle q_3 q_4\rangle}\,, \qquad h_-(-,+) = \frac{\sqrt{2}z}{\langle q_3 q_4\rangle}\,. \tag{4.6}$$

This collinear limiting behaviour has been used as a numerical check on all hecility amplitudes of this process. The squared matrix elements numerically agree with those of ref. [10].

For the process

$$V \to q_1\bar{q}_2q_3\bar{q}_4g \tag{4.7}$$

we again write

$$\begin{aligned}\hat{T}_\mu(Q_1Q_2Q_3Q_4;K_1) &= A_\mu(Q_1Q_2Q_3Q_4;1) - A_\mu(Q_1Q_4Q_3Q_2;1)\\ &+ A_\mu(Q_3Q_4Q_1Q_2;1) - A_\mu(Q_3Q_2Q_1Q_4;1)\,.\end{aligned} \tag{4.8}$$

The amplitude $A_\mu(Q_1Q_2Q_3Q_4;1)$ originates from $A_\mu(1234)$ by adding one gluon. This leads to the 12 diagrams of fig. 8.2. Thus we have

$$\begin{aligned}A_\mu(Q_1Q_2Q_3Q_4;1) &= \sum_{i=1}^{12} \hat{E}_{i\mu}(Q_1Q_2Q_3Q_4;1)\\ &= \sum_{i=1}^{12} C_i(c_1c_2c_3c_4;a_1)\,E_{i\mu}(Q_1Q_2Q_3Q_4;1)\,,\end{aligned} \tag{4.9}$$

where the factors C_i have the form

$$\begin{aligned}C_1 = C_2 = C_3 &= (T^{a_1}T^x)_{c_1c_2} T^x_{c_3c_4}\delta_{f_3f_4}\\ &= \frac{1}{2}\left(T^{a_1}_{c_1c_4}\delta_{c_2c_3} - \frac{1}{N}T^{a_1}_{c_1c_2}\delta_{c_3c_4}\right)\delta_{f_3f_4}\,,\\ C_4 = C_5 = C_6 &= \frac{1}{2}\left(T^{a_1}_{c_3c_2}\delta_{c_1c_4} - \frac{1}{N}T^{a_1}_{c_1c_2}\delta_{c_3c_4}\right)\delta_{f_3f_4}\,,\\ C_7 = C_8 &= i\left(T^x_{c_1c_2}f^{xa_1y}T^y_{c_3c_4}\right)\delta_{f_3f_4}\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(T^{a_1}_{c_1c_4}\delta_{c_2c_3} - T^{a_1}_{c_3c_2}\delta_{c_1c_4}\right)\delta_{f_3f_4} \ , \\
C_9 = C_{10} &= T^{x}_{c_1c_2}\left(T^{a_1}T^{x}\right)_{c_3c_4}\delta_{f_3f_4} \\
&= \frac{1}{2}\left(T^{a_1}_{c_3c_2}\delta_{c_1c_4} - \frac{1}{N}T^{a_1}_{c_3c_4}\delta_{c_1c_2}\right)\delta_{f_3f_4} \ , \\
C_{11} = C_{12} &= \frac{1}{2}\left(T^{a_1}_{c_1c_4}\delta_{c_2c_3} - \frac{1}{N}T^{a_1}_{c_3c_4}\delta_{c_1c_2}\right)\delta_{f_3f_4} \ .
\end{aligned}
$$

Thus A_μ decomposes into four gauge invariant parts

$$
\begin{aligned}
&A_\mu(Q_1Q_2Q_3Q_4;1) = ieg^3\delta_{f_3f_4}\Big[\delta_{c_1c_4}T^{a_1}_{c_3c_2}B^{f_1f_2}_{1\mu}(Q_1Q_2Q_3Q_4;1) \\
&+\delta_{c_3c_2}T^{a_1}_{c_1c_4}B^{f_1f_2}_{2\mu}(Q_1Q_2Q_3Q_4;1) - \frac{1}{N}\delta_{c_1c_2}T^{a_1}_{c_3c_4}B^{f_1f_2}_{3\mu}(Q_1Q_2Q_3Q_4;1) \\
&-\frac{1}{N}\delta_{c_3c_4}T^{a_1}_{c_1c_2}B^{f_1f_2}_{4\mu}(Q_1Q_2Q_3Q_4;1)\Big] \ ,
\end{aligned} \tag{4.11}
$$

with

$$
\begin{aligned}
B^{f_1f_2}_{1\mu} &= \frac{1}{2}(E_4 + E_5 + E_6 - E_7 - E_8 + E_9 + E_{10})_\mu \ , \\
B^{f_1f_2}_{2\mu} &= \frac{1}{2}(E_1 + E_2 + E_3 + E_7 + E_8 + E_{11} + E_{12})_\mu \ , \\
B^{f_1f_2}_{3\mu} &= \frac{1}{2}(E_9 + E_{10} + E_{11} + E_{12})_\mu \ , \\
B^{f_1f_2}_{4\mu} &= \frac{1}{2}(E_1 + E_2 + E_3 + E_4 + E_5 + E_6)_\mu \ .
\end{aligned} \tag{4.12}
$$

From these functions the helicity amplitudes are derived, see sec. 6. Again numerical checks have been carried out based on the collinear limit of quark q_3 and antiquark $\bar{q}_4$, on the soft gluon limit, on current conservation and on the gauge invariance of the external gluon.

The collinear limiting relations are

$$
\begin{aligned}
B^{f_1f_2}_{1\mu}(Q_1\lambda_1,Q_2\lambda_2,Q_3\lambda_3,Q_4\lambda_4;1\sigma_1) &= \sum_{\lambda=\pm1} h_\lambda(\lambda_3,\lambda_4)S_\mu(Q_1\lambda_1;K\lambda,1\sigma_1;Q_2\lambda_2) \ , \\
B^{f_1f_2}_{2\mu}(Q_1\lambda_1,Q_2\lambda_2,Q_3\lambda_3,Q_4\lambda_4;1\sigma_1) &= \sum_{\lambda=\pm1} h_\lambda(\lambda_3,\lambda_4)S_\mu(Q_1\lambda_1;1\sigma_1,K\lambda;Q_2\lambda_2) \ , \\
B^{f_1f_2}_{3\mu}(Q_1\lambda_1,Q_2\lambda_2,Q_3\lambda_3,Q_4\lambda_4;1\sigma_1) &= 0 \ , \\
B^{f_1f_2}_{4\mu}(Q_1\lambda_1,Q_2\lambda_2,Q_3\lambda_3,Q_4\lambda_4;1\sigma_1) &= \sum_{\lambda=\pm1} h_\lambda(\lambda_3,\lambda_4)\Big[S_\mu(Q_1\lambda_1;K\lambda,1\sigma_1;Q_2\lambda_2) \\
&\quad +S_\mu(Q_1\lambda_1;1\sigma_1,K\lambda;Q_2\lambda_2)\Big] \ .
\end{aligned} \tag{4.13}
$$

On the other hand, when the gluon becomes soft we get the limits

$$
\begin{aligned}
B^{f_1f_2}_{1\mu}(Q_1,Q_2,Q_3,Q_4;1) &= s_{Q_31Q_2}B^{f_1f_2}_{\mu}(Q_1,Q_2,Q_3,Q_4) \ , \\
B^{f_1f_2}_{2\mu}(Q_1,Q_2,Q_3,Q_4;1) &= s_{Q_11Q_4}B^{f_1f_2}_{\mu}(Q_1,Q_2,Q_3,Q_4) \ , \\
B^{f_1f_2}_{3\mu}(Q_1,Q_2,Q_3,Q_4;1) &= s_{Q_31Q_4}B^{f_1f_2}_{\mu}(Q_1,Q_2,Q_3,Q_4) \ , \\
B^{f_1f_2}_{4\mu}(Q_1,Q_2,Q_3,Q_4;1) &= s_{Q_11Q_2}B^{f_1f_2}_{\mu}(Q_1,Q_2,Q_3,Q_4) \ ,
\end{aligned} \tag{4.14}
$$

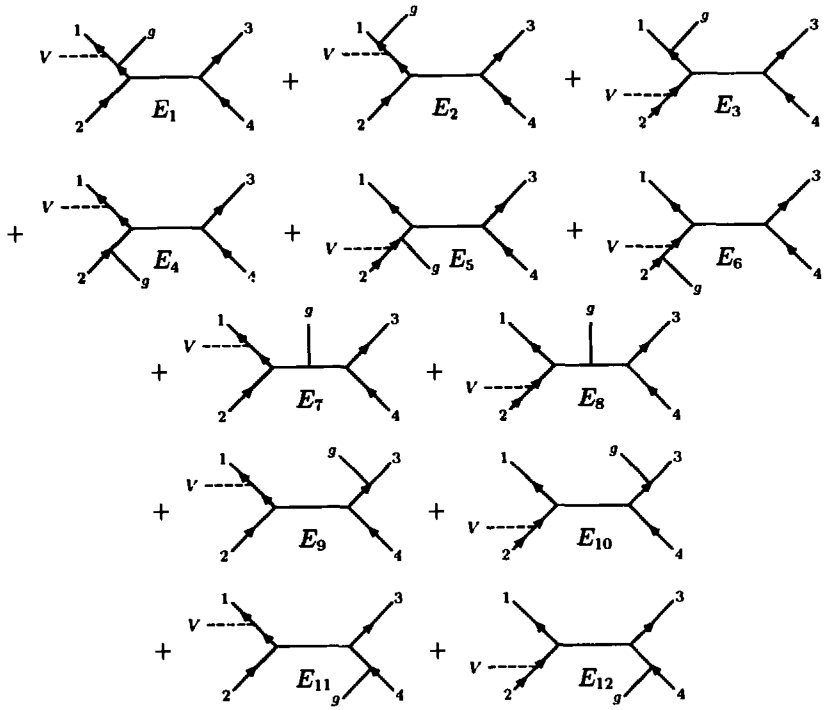

Fig. 8.2. The process $V \to q\bar{q}q\bar{q}g$.

with

$$s^{+}_{\underline{Q}1P} = \sqrt{2}\frac{\langle qp\rangle}{\langle q1\rangle\langle 1p\rangle} \qquad \text{for gluon helicity } + \,,$$

$$s^{-}_{Q1P} = \sqrt{2}\frac{\langle qp\rangle^*}{\langle q1\rangle^*\langle 1p\rangle^*} \qquad \text{for gluon helicity } - \,. \tag{4.15}$$

When more gluons are emitted the number of different amplitudes B increases since the colour structures generalize to terms like $(a_1\cdots a_m)_{c_1c_4}(a_{m+1}\cdots a_n)_{c_3c_2}$. One can again develop a recursive scheme to evaluate these amplitudes.

5 Squaring and crossing

In this section we briefly indicate the results of the colour summation in the squared matrixelements (1.18) or (1.19) with $\widehat{T}_\mu$ as given by eqs. (4.2) and (4.8). Moreover, we indicate the crossing rules in order to obtain from reaction (1.5) the reaction (1.7)-(1.9).

For the colour summation we introduce for eq. (4.2) the notation

$$\hat{T}_\mu = \sum_{i=1}^{4} \eta_i A_{i\mu} , \tag{5.1}$$

where

$$\eta_1 = \delta_{c_1c_4}\delta_{c_2c_3} - \frac{1}{N}\delta_{c_1c_2}\delta_{c_3c_4} , \ A_{1\mu} = ieg^2\delta_{f_3f_4}\ B_\mu^{f_1f_2}(1234) , \tag{5.2}$$

$$\eta_2 = \eta_1(2 \leftrightarrow 4) , \ A_{2\mu} = -A_{1\mu}(2 \leftrightarrow 4) , \tag{5.3}$$

$$\eta_3 = \eta_1(1 \leftrightarrow 3, 2 \leftrightarrow 4) = \eta_1 , \ A_{3\mu} = A_{1\mu}(1 \leftrightarrow 3, 2 \leftrightarrow 4) , \tag{5.4}$$

$$\eta_4 = \eta_1(1 \leftrightarrow 3) = \eta_2 , \ A_{4\mu} = -A_{1\mu}(1 \leftrightarrow 3) . \tag{5.5}$$

The colour summed matrix element squared reads

$$\sum_{\text{colour}} |\mathcal{M}|^2 = \sum_{i,j=1}^{4} C_{ij}\,(V^\mu A_{i\mu})(V^\nu A_{j\nu})^* \tag{5.6}$$

with the matrix C

$$C = \begin{pmatrix} \beta_1 & \beta_2 & \beta_1 & \beta_2 \\ \beta_2 & \beta_1 & \beta_2 & \beta_1 \\ \beta_1 & \beta_2 & \beta_1 & \beta_2 \\ \beta_2 & \beta_1 & \beta_2 & \beta_1 \end{pmatrix} \tag{5.7}$$

where

$$\begin{aligned} \beta_1 &= \sum_{\text{colour}} |\eta_1|^2 = N^2 - 1 , \\ \beta_2 &= \sum_{\text{colour}} \eta_1\eta_2^* = -N + \frac{1}{N} . \end{aligned} \tag{5.8}$$

In a similar fashion eq. (4.8) is written as a sum over 16 amplitudes

$$\hat{T}_\mu = \sum_{i=1}^{16} \varepsilon_i A_{i\mu} , \tag{5.9}$$

$$A_{i\mu} = ieg^3\delta_{f_3f_4}B_{i\mu}(Q_1, Q_2, Q_3, Q_4; 1) , \ i = 1,4$$

$$\begin{aligned} &\varepsilon_1 = \delta_{c_1c_4}T^{a_1}_{c_3c_2} \quad , \ \varepsilon_2 = \delta_{c_3c_2}T^{a_1}_{c_1c_4} \\ &\varepsilon_3 = -\frac{1}{N}\delta_{c_1c_2}T^{a_1}_{c_3c_4} \quad , \ \varepsilon_4 = -\frac{1}{N}\delta_{c_3c_4}T^{a_1}_{c_1c_2} \end{aligned} \tag{5.10}$$

$$\varepsilon_{i+4} = \varepsilon_i(2 \leftrightarrow 4) , \ A_{i+4} = -A_i(2 \leftrightarrow 4) ,$$

$$\varepsilon_{i+8} = \varepsilon_i(1 \leftrightarrow 3, 2 \leftrightarrow 4) , \ A_{i+8} = A_i(1 \leftrightarrow, 2 \leftrightarrow 4) ,$$

$$\varepsilon_{i+12} = \varepsilon_i(1 \leftrightarrow 3) , \ A_{i+12} = -A_i(1 \leftrightarrow 3) .$$

Note that

$$\varepsilon_9 = \varepsilon_2 , \ \varepsilon_{10} = \varepsilon_1 , \ \varepsilon_{11} = \varepsilon_4 , \ \varepsilon_{12} = \varepsilon_3 ,$$

$$\varepsilon_{13} = \varepsilon_6 \ , \ \varepsilon_{14} = \varepsilon_5 \ , \ \varepsilon_{15} = \varepsilon_8 \ , \ \varepsilon_{16} = \varepsilon_7 \ . \tag{5.11}$$

The colour summed matrix element squared is written as eq. (5.6) with $i, j = 1, 16$. The 16×16 matrix C can be written in terms of 4×4 matrices

$$C = \begin{pmatrix} c_A & c_B & c_C & c_D \\ c_B & c_A & c_D & c_C \\ c_C & c_D & c_A & c_B \\ c_D & c_C & c_B & c_A \end{pmatrix} , \tag{5.12}$$

with

$$c_A = \begin{pmatrix} \delta_1 & \delta_4 & \delta_2 & \delta_2 \\ \delta_4 & \delta_1 & \delta_2 & \delta_2 \\ \delta_2 & \delta_2 & \delta_6 & \delta_4 \\ \delta_2 & \delta_2 & \delta_4 & \delta_6 \end{pmatrix} , \ c_B = \begin{pmatrix} \delta_3 & \delta_3 & \delta_7 & \delta_4 \\ \delta_3 & \delta_3 & \delta_4 & \delta_7 \\ \delta_7 & \delta_4 & \delta_5 & \delta_5 \\ \delta_4 & \delta_7 & \delta_5 & \delta_5 \end{pmatrix} ,$$

$$c_C = \begin{pmatrix} \delta_4 & \delta_1 & \delta_2 & \delta_2 \\ \delta_1 & \delta_4 & \delta_2 & \delta_2 \\ \delta_2 & \delta_2 & \delta_4 & \delta_6 \\ \delta_2 & \delta_2 & \delta_6 & \delta_4 \end{pmatrix} , \ c_D = \begin{pmatrix} \delta_3 & \delta_3 & \delta_4 & \delta_7 \\ \delta_3 & \delta_3 & \delta_7 & \delta_4 \\ \delta_4 & \delta_7 & \delta_5 & \delta_5 \\ \delta_7 & \delta_4 & \delta_5 & \delta_5 \end{pmatrix} . \tag{5.13}$$

The constants δ_i are given by

$$\begin{aligned} \delta_{i=1,7} &= \sum_{\text{colours}} \left(|\varepsilon_1|^2 , \varepsilon_1\varepsilon_3^*, \varepsilon_1\varepsilon_5^*, \varepsilon_1\varepsilon_2^*, \varepsilon_3\varepsilon_7^*, |\varepsilon_3|^2 , \varepsilon_1\varepsilon_7^*, \right) \\ &= \frac{1}{2}\left(N^2 - 1\right)\left(N, -\frac{1}{N}, 1, 0, \frac{1}{N^2}, \frac{1}{N}, -1\right) . \end{aligned} \tag{5.14}$$

As to the momentum dependent part of eq. (5.6) one should note that for the contraction of V^μ and $\widehat{T}_\mu$ one could use eq. (III.2.12).

We now turn to the problem of crossing. All the explicit currents for helicity states are given for outgoing partons as in eq. (1.5). In order to obtain $\widehat{S}$ or $\widehat{T}$ for reactions (1.7)-(1.9) one has to change one or more partons from outgoing into incoming.

When the parton is an outgoing gluon it is characterized by a momentum K and polarization vector e^λ. The matrix element is a function $\mathcal{M}(K, e^\lambda(K))$ and in spinor language a function $\widetilde{\mathcal{M}}(k_A, k_{\dot{A}})$. When the gluon is incoming with momentum K' and helicity $-\lambda$ the matrix element is obtained from the previous one by taking $\mathcal{M}(-K', e^{\lambda^*}(K')) = \mathcal{M}(-K', e^{-\lambda}(K'))$. The spinors k_A and $k_{\dot{A}}$ are related to the corresponding momentum by eq. (III.2.18). The amplitude for the process with an incoming gluon takes the form $\widetilde{\mathcal{M}}(ik'_A, ik'_{\dot{A}})$. For the spinors arising from momenta this prescription is obviously correct. For the spinors arising from the polarization vectors this is also the case, since under the replacement

$$k_A \to ik'_A \ , \ k_{\dot{A}} \to ik'_{\dot{A}} \tag{5.15}$$

and consequently

$$\langle pk \rangle \to i\langle pk' \rangle \ , \ \langle pk \rangle^* \to i\langle pk' \rangle^* \ , \tag{5.16}$$

we have according to eq. (III.2.55)

$$e^{+}_{\dot{A}B}(K) \rightarrow \sqrt{2}\frac{k'_{\dot{A}}g_B}{(gk')} = e^{-}_{\dot{A}B}(K') \ , \tag{5.17}$$

$$e^{-}_{\dot{A}B}(K) \rightarrow \sqrt{2}\frac{g_{\dot{A}}k'_B}{(gk')^*} = e^{+}_{\dot{A}B}(K') \ . \tag{5.18}$$

For the quarks one can use a similar replacement as in eq. (5.15) i.e.

$$q_A \rightarrow iq'_A \ , \ q_{\dot{A}} \rightarrow iq'_{\dot{A}} \ , \tag{5.19}$$

when the incoming antiquark has a physical momentum Q', whereas the outgoing quark had momentum Q. Denoting the known amplitude by $\mathcal{M}(Q, \bar{u}_\lambda(Q))$ we have as amplitude for the crossed process $\mathcal{M}(-Q', \bar{v}_\lambda(Q'))$. For the substitution $Q \rightarrow -Q'$ eq. (5.19) is adequate, however when we make the replacement

$$\bar{u}_\lambda(Q) \rightarrow \bar{v}_{-\lambda}(Q') \tag{5.20}$$

the spinors transform as (eq.(III.2.38))

$$q_A \rightarrow q'_A \ , \ q_{\dot{A}} \rightarrow q'_{\dot{A}} \tag{5.21}$$

which differ from eq. (5.19) by a factor i. Thus one can apply the replacements (5.19) for quarks when at the same time one multiplies the amplitude with a factor $(-i)^{n_q}$, where n_q is the number of crossed quarks. The reason that we don't want the occurence of a complex phase factor is that in the evaluation of the helicity amplitudes we use complex conjugation which would lead in eqs. (4.2) and (4.8) relative phases in some of the amplitudes. For instance we have the replacement

$$\widetilde{\mathcal{M}}(q_A, q_{\dot{A}}) \rightarrow -i\widetilde{\mathcal{M}}(iq_A, iq_{\dot{A}}) \tag{5.22}$$

if we make a outgoing quark (antiquark) an incoming antiquark (quark).

6 The helicity currents

In this section helicity currents for massless quarks are given for process (1.3) for $n = 0, 1, 2, 3$ and for process (1.4) for $n = 0, 1$. These currents are suited for direct numerical computations and gives the analytic expressions for eqs. (1.6)-(1.9) up to 5 partons.

The current S_μ as defined in eq. (IV.4.20) is given by

$$S_\mu(Q+; 1\lambda_1, \ldots, n\lambda_n; P-) = R^V_{f_1 f_2}(\sqrt{2})^n \sigma_\mu^{\dot{A}B} S_{\dot{A}B}(Q+; 1\lambda_1, \ldots, n\lambda_n; P-) \ , \tag{6.1}$$

$$S_\mu(Q-; 1\lambda_1, \ldots, n\lambda_n; P+) = L^V_{f_1 f_2}(\sqrt{2})^n \sigma_\mu^{\dot{A}B} S_{\dot{A}B}(Q-; 1\lambda_1, \ldots, n\lambda_n; P+) \ . \tag{6.2}$$

The factor $\left(\sqrt{2}\right)^n$ is made explicit like in eq. (2.2). We list here the quantities $S_{\dot{A}B}(Q+; 1\lambda_1, \ldots, n\lambda_n; P-)$, the other current with the quark helicities flipped follows from the relation

$$S_{\dot{A}B}(Q-; 1\lambda_1, \ldots, n\lambda_n; P+) = (S_{\dot{B}A}(Q+; 1(-\lambda_1), \ldots, n(-\lambda_n); P-))^* \ . \tag{6.3}$$

The quantities in the expressions are first of all the spinorial inner products of chap. 3. We also introduce the notation

$$\langle a|B+C|d\rangle = a_{\dot{E}}d_F(B+C)^{\dot{E}F} = \langle ab\rangle^*\langle db\rangle + \langle ac\rangle^*\langle dc\rangle \tag{6.4}$$

where the last step only holds for null-vectors. Because we have specified the helicities of the quarks and use a fixed order of the gluons, we adopt the following shorthand notation

$$S_{\dot{A}B}(Q+;1\lambda_1,\ldots,n\lambda_n;P-) = S_{\dot{A}B}(+;\lambda_1\cdots\lambda_n;-)\,. \tag{6.5}$$

First we have the trivial $n=0$ result

$$S_{\dot{A}B}(+;-) \;=\; q_{\dot{A}}p_B\,.$$

Secondly, the $n=1$ results are

$$S_{\dot{A}B}(+;+;-) \;=\; \frac{(Q+K_1)_{\dot{A}D}p^D p_B}{\langle qk_1\rangle\langle k_1p\rangle}\,,$$

$$S_{\dot{A}B}(+;-;-) \;=\; -\frac{q_{\dot{A}}(K_1+P)_{\dot{C}B}q^{\dot{C}}}{\langle qk_1\rangle^*\langle k_1p\rangle^*}\,.$$

Thirdly, for $n=2$ we have 4 helicity combinations

$$\begin{aligned}
S_{\dot{A}B}(+;++;-) &= \frac{(Q+K_1+K_2)_{\dot{A}D}p^D p_B}{\langle qk_1\rangle\langle k_1k_2\rangle\langle k_2p\rangle}\\
S_{\dot{A}B}(+;+-;-) &= -\frac{\langle qk_1\rangle^*\langle qk_2\rangle(Q+K_1)_{\dot{A}D}k_2^D p_B}{\langle qk_1\rangle(K_1+K_2)^2(Q+K_1+K_2)^2}\\
&\quad -\frac{\langle pk_1\rangle^*\langle pk_2\rangle q_{\dot{A}}(K_2+P)_{\dot{C}B}k_1^{\dot{C}}}{\langle pk_2\rangle^*(K_1+K_2)^2(K_1+K_2+P)^2}\\
&\quad -\frac{(Q+K_1)_{\dot{A}D}k_2^D(K_2+P)_{\dot{C}B}k_1^{\dot{C}}}{\langle qk_1\rangle\langle k_2p\rangle^*(K_1+K_2)^2}\\
S_{\dot{A}B}(+;-+;-) &= -\frac{\langle pk_1\rangle^2 q_{\dot{A}}(K_1+P)_{\dot{C}B}k_2^{\dot{C}}}{\langle pk_2\rangle(K_1+K_2)^2(K_1+K_2+P)^2}\\
&\quad -\frac{\langle qk_2\rangle^{*2}(Q+K_2)_{\dot{A}D}k_2^D p_B}{\langle qk_1\rangle^*(K_1+K_2)^2(Q+K_1+K_2)^2}\\
&\quad +\frac{\langle pk_1\rangle\langle qk_2\rangle^* q_{\dot{A}}p_B}{\langle qk_1\rangle^*\langle pk_2\rangle(K_1+K_2)^2}\\
S_{\dot{A}B}(+;--;-) &= -\frac{q_{\dot{A}}(K_1+K_2+P)_{\dot{C}B}q^{\dot{C}}}{\langle qk_1\rangle^*\langle k_1k_2\rangle^*\langle k_2p\rangle^*}\,.
\end{aligned}$$

Last we list the two of the eight helicity combinations for the $n=3$ case. The others are given in ref. [1]. The two helicity combinations are

$$S_{\dot{A}B}(+;+++;-) \;=\; \frac{(Q+K_1+K_2+K_3)_{\dot{A}D}p^D p_B}{\langle qk_1\rangle\langle k_1k_2\rangle\langle k_2k_3\rangle\langle k_3p\rangle}$$

$$
\begin{aligned}
S_{\dot{A}B}(+;++-;-) = & \frac{\langle k_1k_2\rangle^*\langle q|K_1+K_2|k_3\rangle(Q+K_1+K_2)_{\dot{A}D}k_3^Dp_B}{\langle k_1k_2\rangle(K_2+K_3)^2(K_1+K_2+K_3)^2(Q+K_1+K_2+K_3)^2} \\
& -\frac{\langle k_2|Q+K_1|k_3\rangle(Q+K_1+K_2)_{\dot{A}D}k_3^Dp_B}{\langle qk_1\rangle\langle k_1k_2\rangle(K_2+K_3)^2(Q+K_1+K_2+K_3)^2} \\
& -\frac{\langle pk_3\rangle\langle k_1k_2\rangle^*q_{\dot{A}}(K_1+K_2+K_3+P)_{\dot{C}B}(K_1+K_2+K_3)^{\dot{C}D}k_{3D}}{\langle k_1k_2\rangle(K_2+K_3)^2(K_1+K_2+K_3)^2(K_1+K_2+K_3+P)^2} \\
& -\frac{\langle pk_3\rangle^2\langle pk_2\rangle^*q_{\dot{A}}(K_2+K_3+P)_{\dot{C}B}k_1^{\dot{C}}}{\langle k_1k_2\rangle(K_2+K_3)^2(K_2+K_3+P)^2(K_1+K_2+K_3+P)^2} \\
& +\frac{\langle pk_3\rangle\langle pk_2\rangle^*(Q+K_1)_{\dot{A}D}k_2^D(K_3+P)_{\dot{C}B}k_2^{\dot{C}}}{\langle qk_1\rangle\langle k_1k_2\rangle\langle k_3p\rangle^*(K_2+K_3)^2(K_2+K_3+P)^2} \\
& -\frac{(Q+K_1+K_2)_{\dot{A}D}k_3^D(K_3+P)_{\dot{C}B}k_2^{\dot{C}}}{\langle qk_1\rangle\langle k_1k_2\rangle\langle k_3p\rangle^*(K_2+K_3)^2}\,.
\end{aligned}
$$

Next we give the results for the current $\widehat{T}$ of eqs. (4.2) and (4.8) for respectively $V \to q\bar{q}q\bar{q}$ and $V \to q\bar{q}q\bar{q}+g$. First we look at $V \to q\bar{q}q\bar{q}$. Here $\widehat{T}$ is decomposed in the four A_μ-functions of eq. (4.2). These functions have no quark permutations. The colour and flavour factor is split off resulting in a colourless B_μ-function, eq. (4.4). There are again two sets of helicity amplitudes, namely the right-handed coupling

$$B_\mu(Q_1+,Q_2-,Q_3\lambda_3,Q_4\lambda_4) = R^V_{f_1f_2}\sigma_\mu^{\dot{A}B}H_{\dot{A}B}(Q_1+,Q_2-,Q_3\lambda_3,Q_4\lambda_4) \qquad (6.6)$$

and the left-handed coupling

$$B_\mu(Q_1-,Q_2+,Q_3\lambda_3,Q_4\lambda_4) = L^V_{f_1f_2}\sigma_\mu^{\dot{A}B}H_{\dot{A}B}(Q_1-,Q_2+,Q_3\lambda_3,Q_4\lambda_4)\,. \qquad (6.7)$$

We list the two possible combinations of $H_{\dot{A}B}(Q_1+,Q_2-,Q_3\lambda_3,Q_4\lambda_4)$. The other H-functions follow from complex conjugation

$$H_{\dot{A}B}(Q_1-,Q_2+,Q_3\lambda_3,Q_4\lambda_4) = \left(H_{\dot{B}A}(Q_1+,Q_2-,Q_3(-\lambda_3),Q_4(-\lambda_4))\right)^*. \qquad (6.8)$$

The two possible H-functions are

$$
\begin{aligned}
H_{\dot{A}B}(+-+-) = & -\frac{\langle q_1q_3\rangle^*(Q_1+Q_3)_{\dot{A}D}q_4^Dq_{2B}}{(Q_3+Q_4)^2(Q_1+Q_3+Q_4)^2} \\
& +\frac{\langle q_2q_4\rangle q_{1\dot{A}}(Q_2+Q_4)_{\dot{C}B}q_3^{\dot{C}}}{(Q_3+Q_4)^2(Q_2+Q_3+Q_4)^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
H_{\dot{A}B}(+--+) = & -\frac{\langle q_1q_4\rangle^*(Q_1+Q_4)_{\dot{A}D}q_3^Dq_{2B}}{(Q_3+Q_4)^2(Q_1+Q_3+Q_4)^2} \\
& +\frac{\langle q_2q_3\rangle q_{1\dot{A}}(Q_2+Q_3)_{\dot{C}B}q_4^{\dot{C}}}{(Q_3+Q_4)^2(Q_2+Q_3+Q_4)^2}\,.
\end{aligned}
$$

The process $V \to q\bar{q}q\bar{q} + g$ is more involved. Again we have four A_μ-functions (eq. (4.8)) which build up the current $\hat{T}_\mu(Q_1, Q_2, Q_3, Q_4; K_1)$. After the colour decomposition we arrived at eq. (4.11) which contained four gauge invariant functions B_{i_μ}. If we specify the helicities of quark 1 and antiquark 2, which couple to the vector boson, we can again devide the current in a left-handed and right-handed part as was done in eqs. (6.6) and (6.7). We only list the right-handed part, the left-handed part follows from complex conjugation of the associated H-function similar to relation (6.8). The right-handed B_{i_μ}-functions are given by

$$\begin{aligned}
B_{1\mu} &= \sqrt{2}R^V_{f_1 f_2}\sigma_\mu^{\dot{A}B}\left[H_{1\dot{A}B} + H_{2\dot{A}B} + H_{3\dot{A}B} + H_{4\dot{A}B}\right], & (6.9)\\
B_{2\mu} &= \sqrt{2}R^V_{f_1 f_2}\sigma_\mu^{\dot{A}B}\left[H_{5\dot{A}B} + H_{6\dot{A}B} - H_{3\dot{A}B} - H_{4\dot{A}B}\right], & (6.10)\\
B_{3\mu} &= \sqrt{2}R^V_{f_1 f_2}\sigma_\mu^{\dot{A}B}H_{7\dot{A}B}, & (6.11)\\
B_{4\mu} &= \sqrt{2}R^V_{f_1 f_2}\sigma_\mu^{\dot{A}B}\left[H_{8\dot{A}B} + H_{9\dot{A}B}\right], & (6.12)
\end{aligned}$$

where we have suppressed the obvious function arguments. The functions H_i arise from the further decomposition of the B_i functions in gauge invariant substructures. Of the four right-handed helicity combinations of the nine different $H_{i\dot{A}B}$-functions we give one helicity combination. See for the other helicity combinations ref. [1].

The expressions for the $H_i(+;-+-;+)$ functions are given by

$$\begin{aligned}
H_{1\dot{A}B}(+-+-;+) &= -\frac{\langle q_2q_4\rangle q_{1\dot{A}}(Q_2+Q_3+Q_4+K_1)_{\dot{C}B}(Q_3+K_1)^{\dot{C}D}q_{2D}}{\langle q_3k_1\rangle\langle k_1q_2\rangle(Q_3+Q_4+K_1)^2(Q_2+Q_3+Q_4+K_1)^2}\\
&\quad+\frac{\langle q_2q_4\rangle\langle q_1|Q_3+Q_4|q_2\rangle q_{1\dot{A}}(Q_2+Q_4+K_1)_{\dot{C}B}q_3^{\dot{C}}}{\langle k_1q_2\rangle(Q_3+Q_4)^2(Q_3+Q_4+K_1)^2(Q_2+Q_3+Q_4+K_1)^2}\\
H_{2\dot{A}B}(+-+-;+) &= \frac{\langle q_1q_3\rangle^*\langle k_1|Q_1+Q_3|q_4\rangle(Q_1+Q_3+Q_4+K_1)_{\dot{A}D}q_2^Dq_{2B}}{\langle q_2k_1\rangle(Q_3+Q_4)^2(Q_1+Q_3+Q_4)^2(Q_1+Q_3+Q_4+K_1)^2}\\
&\quad+\frac{\langle q_1|Q_3+K_1|q_2\rangle(Q_1+Q_3+K_1)_{\dot{A}D}q_4^Dq_{2B}}{\langle q_3k_1\rangle\langle k_1q_2\rangle(Q_3+Q_4+K_1)^2(Q_1+Q_3+Q_4+K_1)^2}\\
&\quad+\frac{\langle q_1q_3\rangle^*\langle k_1|Q_3+Q_4|q_2\rangle(Q_1+Q_3+K_1)_{\dot{A}D}q_4^Dq_{2B}}{\langle q_2k_1\rangle(Q_3+Q_4)^2(Q_3+Q_4+K_1)^2(Q_1+Q_3+Q_4+K_1)^2}\\
H_{3\dot{A}B}(+-+-;+) &= \frac{\langle q_2q_4\rangle\langle q_3k_1\rangle^*q_{1\dot{A}}(Q_2+Q_3+Q_4)_{\dot{C}B}k_1^{\dot{C}}}{(Q_3+Q_4)^2(Q_3+Q_4+K_1)^2(Q_2+Q_3+Q_4+K_1)^2}\\
H_{4\dot{A}B}(+-+-;+) &= -\frac{\langle q_1k_1\rangle^*\langle q_3k_1\rangle^*(Q_1+Q_3+K_1)_{\dot{A}D}q_4^Dq_{2B}}{(Q_3+Q_4)^2(Q_3+Q_4+K_1)^2(Q_1+Q_3+Q_4+K_1)^2}\\
H_{5\dot{A}B}(+-+-;+) &= \frac{\langle q_2q_4\rangle^2\langle q_2q_3\rangle^*q_{1\dot{A}}(Q_2+Q_3+Q_4)_{\dot{C}B}k_1^{\dot{C}}}{\langle q_4k_1\rangle(Q_3+Q_4)^2(Q_2+Q_3+Q_4)^2(Q_2+Q_3+Q_4+K_1)^2}\\
&\quad+\frac{\langle q_2q_4\rangle(Q_1+K_1)_{\dot{A}D}q_4^D(Q_2+Q_4)_{\dot{C}B}q_3^{\dot{C}}}{\langle q_1k_1\rangle\langle k_1q_4\rangle(Q_3+Q_4)^2(Q_2+Q_3+Q_4)^2}\\
&\quad+\frac{\langle q_2q_4\rangle\langle q_3k_1\rangle^*q_{1\dot{A}}(Q_2+Q_4+K_1)_{\dot{C}B}q_3^{\dot{C}}}{\langle q_3q_4\rangle^*\langle q_4k_1\rangle(Q_3+Q_4+K_1)^2(Q_2+Q_3+Q_4+K_1)^2}\\
H_{6\dot{A}B}(+-+-;+) &= -\frac{\langle q_3|Q_1+K_1|q_4\rangle(Q_1+Q_3+K_1)_{\dot{A}D}q_4^Dq_{2B}}{\langle q_1k_1\rangle\langle k_1q_4\rangle(Q_3+Q_4)^2(Q_1+Q_3+Q_4+K_1)^2}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\langle q_1 q_3\rangle^* \langle q_3 k_1\rangle^* (Q_1+Q_3+K_1)_{\dot{A}D} q_4^D q_{2B}}{\langle q_4 k_1\rangle \langle q_3 q_4\rangle^* (Q_3+Q_4+K_1)^2 (Q_1+Q_3+Q_4+K_1)^2} \\
H_{7\dot{A}B}(+-+-;+) \;=\; & \frac{\langle q_1|Q_3+K_1|q_4\rangle (Q_1+Q_3+K_1)_{\dot{A}D} q_4^D q_{2B}}{\langle q_3 k_1\rangle \langle k_1 q_4\rangle (Q_3+Q_4+K_1)^2 (Q_1+Q_3+Q_4+K_1)^2} \\
& -\frac{\langle q_2 q_4\rangle q_{1\dot{A}} (Q_2+Q_3+Q_4+K_1)_{\dot{C}B} (Q_3+K_1)^{\dot{C}D} q_{4D}}{\langle q_3 k_1\rangle \langle k_1 q_4\rangle (Q_3+Q_4+K_1)^2 (Q_2+Q_3+Q_4+K_1)^2} \\
H_{8\dot{A}B}(+-+-;+) \;=\; & \frac{\langle q_2 q_4\rangle (Q_1+K_1)_{\dot{A}D} q_2^D (Q_2+Q_4)_{\dot{C}B} q_3^{\dot{C}}}{\langle q_1 k_1\rangle \langle k_1 q_2\rangle (Q_3+Q_4)^2 (Q_2+Q_3+Q_4)^2} \\
& +\frac{\langle q_2 q_4\rangle^2 q_{1\dot{A}} (Q_2+Q_3+Q_4)_{\dot{C}B} k_1^{\dot{C}}}{\langle q_2 k_1\rangle \langle q_3 q_4\rangle (Q_2+Q_3+Q_4)^2 (Q_2+Q_3+Q_4+K_1)^2} \\
H_{9\dot{A}B}(+-+-;+) \;=\; & -\frac{\langle q_3|Q_1+K_1|q_2\rangle (Q_1+Q_3+K_1)_{\dot{A}D} q_4^D q_{2B}}{\langle q_1 k_1\rangle \langle k_1 q_2\rangle (Q_3+Q_4)^2 (Q_1+Q_3+Q_4+K_1)^2} \\
& +\frac{\langle q_1 q_3\rangle^* \langle k_1|Q_1+Q_3|q_4\rangle (Q_1+Q_3+Q_4+K_1)_{\dot{A}D} q_2^D q_{2B}}{\langle q_2 k_1\rangle (Q_3+Q_4)^2 (Q_1+Q_3+Q_4)^2 (Q_1+Q_3+Q_4+K_1)^2} \,.
\end{aligned}
$$

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Samenvatting

De eigenschappen en het berekenen van multiparton processen

Op CERN te Genève en FNAL (Fermilab nabij Chicago) worden protonen en antiprotonen met zeer hoge energie aan elkaar vestrooid . Uit zo'n botsing resulteren soms een of meer zogenaamde "jets", eventueel tezamen met leptonen. Een "jet" is een stroom hadronen die zich min of meer in dezelfde richting bewegen. De hadronen hebben als kenmerk dat ze zijn opgebouwd uit partonen, dit zijn quarks en gluonen. Het proton en antiproton zijn voorbeelden van hadronen. Tijdens een botsing tussen een proton en antiproton komt het soms voor dat de twee botsende partonen onder een grote hoek worden verstrooid. Zo'n parton, dat zeer veel energie heeft, zal gaan fragmenteren naar minder energetische partonen. Al deze partonen zullen, als hun energie laag genoeg geworden is door het fragmenteren, gebonden worden tot hadronen. Deze hadronen zullen min of meer dezelfde richting hebben als het oorspronkelijke parton. Ook de som van de energieën van de hadronen zal ongeveer gelijk zijn aan de energie van het oorspronkelijke parton. Dus uit het bestuderen van de "jets" leren we iets van de interactie tussen de individuele partonen. Deze interactie wordt beschreven door de sterke wisselwerking, ook wel quantum chromo dynamica (QCD) genoemd. Met behulp van QCD kunnen we de verstrooiingsprocessen tussen de partonen berekenen en voorspellingen doen over de te verwachten meetresultaten bij de experimenten. Ten eerste kunnen we hiermee toetsen of QCD de interacties tussen de partonen correct beschrijft. Ten tweede is het belangrijk om deze processen goed te kunnen voorspellen zodat nieuwe verschijnselen kunnen worden waargenomen. Deze nieuwe verschijnselen zijn in het experiment niet zonder meer te onderscheiden van de reeds bekende processen. Door deze bekende processen, de zogenaamde achtergrondprocessen, nauwkeurig te berekenen, kunnen we nieuwe verschijnselen waarnemen als afwijking op de achtergrondprocessen. Het is nu duidelijk dat het voor de experimenten belangrijk is dat botsingen waarbij "jets" en leptonen ontstaan theoretisch goed beschreven zijn. Deze berekeningen zijn gecompliceerd, vooral als bij een parton-parton botsing vele partonen ontstaan. Bijvoorbeeld een parton-parton botsing kan resulteren in vijf uitgaande hoog-energetische partonen, ieder evoluerend in een aparte "jet".

In dit proefschrift worden nieuwe methoden geintroduceerd om bovenstaande multipartôn processen uit te rekenen. Deze methoden zijn geschikt om zowel numerieke als analytische resultaten te verkrijgen. In hoofdstuk 2 wordt het theoretische model, bestaande uit het parton model tezamen met QCD, verder toegelicht. Ook wordt een historisch overzicht gegeven van het meest gecompliceerde multiparton proces, gluon-gluon verstrooiing naar n gluonen. Uit dit overzicht wordt het duidelijk welke problemen men ondervond bij het berekenen van dit proces en welke methoden ontwikkeld werden om verder te komen. Met name het zogenaamde heliciteitsformalisme was een belangrijke stap. Hoofdstuk 3 vertaalt dit heliciteitsformalisme in de Weyl-van der Waerden spinor notatie. Deze spinorrekening zal onze basis rekentechniek worden bij het berekenen van de multi parton processen.

Het heeft vele voordelen boven de conventionele Dirac spinorrekening, met name voor massaloze deeltjes. In hoofdstuk 4 behandelen we de recursierelaties. Deze relaties maken het mogelijk een proces recursief uit te rekenen, dus bij het toevoegen van een nieuw parton maken we gebruik van al eerder gedane berekeningen van processen met minder partonen. Hierdoor zijn we in staat processen uit te rekenen met zeer veel gluonen, zoals zal blijken uit de overige hoofdstukken. Het eerste resultaat wordt gegeven in hoofdstuk 5. Hier worden processen berekend waarbij het aantal gluonen willekeurig is, zij het voor specifiek gekozen heliciteiten van de gluonen. Dit is een deel van het totale antwoord voor het proces. Voor het totaal moeten we sommeren over alle mogelijke heliciteitscombinaties van de gluonen. De berekende speciale heliciteitscombinaties kunnen dienen als basis voor het formuleren van benaderende formules voor het totale proces. Deze heliciteitsbotsingsdoorsnedes waren voordien alleen bekend als postulaten. Met behulp van de recursierelaties kunnen in dit hoofdstuk deze postulaten met volledige inductie bewezen worden. Dit wordt gedaan voor processen waarbij naast een willekeurig aantal gluonen ook aanwezig kunnen zijn een quark-antiquark paar en eventueel een vectorboson dat kan vervallen in leptonen. Hoofdstuk 6 bestudeert het gedrag van "zachte" gluonen, hetgeen van belang is voor hogere orde correcties op het proces. Explicite analytische antwoorden voor multigluon processen worden met behulp van de recursierelaties berekend in hoofdstuk 7. De recursierelaties stellen ons in staat om de analytische antwoorden uit te breiden tot processen met zeven gluonen. Dit is relevant voor botsingsprocessen met vijf "jets". Met de technieken uit dit hoofdstuk zijn we ook in staat het proces met acht gluonen numeriek te evalueren. In hoofdstuk 8 worden parton processen berekend waarin ook een vectorboson voorkomt. Deze processen zijn belangrijk voor experimenten om verscheidene redenen. Zo is het vinden van een top quark een belangrijk doel van de huidige experimenten. Het "top quark" is een theoretisch voorspeld deeltje dat noodzakelijk is om het theoretische model intern consistent te houden. De processen berekend in hoofdstuk 8 vormen hierop een achtergrondproces.

De in dit proefschrift verkregen resultaten kunnen binnenkort worden vergeleken met de nieuwe experimentele gegevens van met name Fermilab. Hierdoor is het mogelijk QCD te toetsen bij hogere energieën en voor processen met veel "jets". Ook naar mogelijke afwijkingen kan worden gezocht. Dit kan resulteren in het vinden van het "top quark" en/of het vinden van nieuwe onverwachte verschijnselen.

Curriculum Vitae

van

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geboren op 14 september 1957 te Den Haag

Aan het Sint Willibrord-college te Goes behaalde ik in 1974 het MAVO diploma, in 1976 het HAVO diploma en in 1978 het VWO diploma. Daarna vervolgde ik mijn studie aan de Rijksuniversiteit te Leiden. Het kandidaatsexamen natuurkunde met bijvakken wiskunde en sterrenkunde werd afgelegd in mei 1983. Gedurende mijn experimentele stage heb ik gewerkt in de groep Quantumvloeistoffen o.l.v. Prof. Dr. R. de Bruyn Ouboter. In mei 1986 volgde het doctoraalexamen theoretische natuurkunde met bijvak wiskunde. Het doctoraal onderzoek is verricht onder begeleiding van Prof. Dr. F.A. Berends aan het onderwerp "Toepassingen van het Weyl van der Waerden formalisme op het berekenen van boom diagrammen in de hoge energie fysica". Direct na het doctoraalexamen begon ik aan mijn promotie onderzoek. Dit stond onder leiding van Prof. Dr. F.A. Berends en vond plaats binnen de werkgroep H-th-L van de Stichting voor Fundamenteel Onderzoek der Materie.

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