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TOPICS IN PHASE-SHIFT ANALYSIS AND HIGHER SPIN FIELD THEORY

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CHAPTER I

INTRODUCTION

1. Preliminary remarks

In this section we sketch the way in which the topics of this thesis should be seen in the context of elementary particle physics. In this field one encounters for instance the following problems:
How can one establish the existence of a particle? What properties do these particles have?
Is it possible to decide whether such a particle is elementary or composite?
A part of these problems is how to extract information from experimental data, or in other words, how to translate this information into theoretical relevant quantities.

When one knows more about the existence and properties of certain particles and when one can decide whether those particles are elementary or composite, one can try to develop theories that describe these particles or predict their existence.

In this thesis two topics will be studied. The first topic is concerned with the extraction of information from the experimental data, the second one has to do with the construction of theories that describe massive or massless particles with spin.

We first sketch both problems more specifically. Later in sections 2 and 3 they will be discussed in more detail.

In elementary particle physics most of the information is obtained from scattering experiments. The quantities which one measures, such as the differential cross section and the polarization both are functions that depend on the energy and scattering angle. On the other hand these observables are also related to theoretically more relevant quantities like scattering amplitudes. These scattering amplitudes are complex valued functions which in quadratic combinations give rise to a differential cross section and polarization.

There exists a procedure, called phase-shift analysis or amplitude-analysis, which aims to determine the scattering amplitudes from the experimental data. From the properties of the amplitude one can for instance establish the existence of a particle or deduce what the interaction between the scattered particles has been. When an unstable particle, a so-called resonance, is produced, its mass, lifetime and spin also follow from the phase-shift analysis. Thus a large number of unstable particles were discovered by these analyses.

One may now ask, whether the amplitudes which are obtained from the experimental data are unique. Some ambiguity is expected, since these data necessarily have statistical errors. But even when these data are supposed to be exact, the mathematical question can be raised if the amplitudes are determined unambiguously.

A negative answer could mean that certain resonances do not exist and others have been overlooked. Because of this possibility it is interesting to study the mathematical question of the existence of so-called phase-shift ambiguities. In section 2 we shall discuss this problem in more detail.

→ The other topic has to do with the construction of theories with which certain particles can be described. Once the existence of particles with a specific mass and spin is known, one wants to set up a theory that describes them. One will most often try to develop a quantum field theory. In such a theory the various particles are represented by fields. The different properties of these particles are carried by the so-called Lagrangian, a quantity that depends on these fields and their derivatives. One part of this Lagrangian describes the free particles and another part the interactions between them.

The commonly used and accepted theories contain massive or massles particles with spin 0, $\frac{1}{2}$ or 1. However, when one wants to construct theories for higher spins one runs into problems. Even if only free particles with spin higher than 1 are considered, pecularities arise in the field theory. In this thesis the origin and characteristics of these phenomena will be traced. In section 3 this problem will be discussed in more detail.

2. The scattering amplitude, its physical meaning and possible ambiguities

Much of the knowledge about elementary particles and their interactions has been obtained by means of scattering experiments. In such experiments one measures quantities like the cross section $\frac{d\sigma}{d\Omega}$ and, when spin is involved, observables like the polarization.

In general, when the interaction between the particles is known, one can, within the framework of quantum theory, calculate a complex valued quantity, the so-called scattering amplitude $F(k,x)$ where $x = \cos\theta$, θ the scattering angle and k the centre of mass momentum. *This complex valued function is* related to the differential cross section in the well known way

$$k^2 \frac{d\sigma}{d\Omega} = |F(k,x)|^2 \ . \tag{2.1}$$

With this relation one can, of course, calculate the differential cross section. And thus one is able to predict and to check the results of a scattering experiment.

On the other hand, when knowledge about the interaction between the scattered particles is absent, one must reverse this procedure. First one tries to find the scattering amplitude from experimental quantities like the differential cross section. Because the differential cross section only involves the modulus of F(k,x), one has to determine its phase. This problem would be solved if experimental methods were available to obtain this phase.

Goldberger et al [1] proved that it is possible to determine the phase of F(k,x) by measuring intensity correlations of particles, scattered at different space time points. However, such experiments, especially in elementary particle physics are extremely difficult to perform.

Thus the phase of the scattering amplitude may be regarded as a quantity that cannot in general be measured. Therefore one must try to find the phase by means of additional principles such as the constraints of unitarity, which will be discussed more explicitly in the next chapter. When the amplitude is known one can deduce knowledge about the interaction from it and in particular one can establish the formation of a resonance.

The construction of the scattering amplitude from the differential cross section and the constraints of unitarity gives rise to some rather general questions. In particular, one may ask under which conditions the solution is unique. This question has been answered only partially [2]. It is well known that the solution is not always unique i.e. that there exist so-called phase-shift ambiguities. One may then ask how many unitary amplitudes correspond to the same differential cross section, and how these amplitudes can be explicitly constructed.

In this thesis we shall consider several aspects of the existence and construction of phase-shift ambiguities.

The outline of this part of the thesis is as follows. Before we start our discussion of phase-shift ambiguities, we want to introduce the subject by making a few remarks about scattering theory. We shall also summarize the main results that have been obtained in connection with the questions raised above. This will be presented in chapter II. In the same chapter we shall discuss the mathematical restrictions by which we are able to present our problems more clearly.

In chapter III we consider the construction of different unitary amplitudes, which correspond to the same differential cross section. Examples of such phase-shift ambiguities have been found only for rather special cases. We shall show that these results can be considerably generalized. Moreover this generalization reveals properties of phase-shift ambiguities that were absent in the previously known examples. These properties will be discussed in detail.

So far we mentioned only scattering of spinless particles. In case spin is involved, more experimental quantities, like for instance the polarization play a role.

In chapter IV we discuss the possibility of constructing different amplitudes, which give the same differential cross-section and polarization. Again we shall generalize the results that have been obtained for the case of spin-0 spin-$\frac{1}{2}$ scattering.

3. Field theories for particles with higher spin

As was noted in the previous section, much knowledge about the existence and properties of particles, elementary or composite, has been obtained from scattering experiments. On the other hand, when the existence of these particles is known, one wants to develop theories that describe the properties of the various particles and the interactions between them. Such theories are called quantum field theories.

In a quantum field theory each particle is associated with a so-called operator field. The fields satisfy differential equations or field equations which contain the dynamics of the system of particles.

In many cases one can in first instance forget about the operator character of the field. One then speaks about a classical field theory. The transition from a classical field theory to a quantum field theory is called quantization. For the various ways in which a quantization can be performed we refer to the textbooks [3].

Perhaps one of the main problems in field theory is the construction of field equations describing a system of interacting particles. So far, the only acceptable interacting field theories exist for particles with spin 0, $\frac{1}{2}$ and 1.

If one does not consider the interactions between the particles one speaks of free field theories and free field equations.

In this thesis we shall restrict ourselves to the construction of free field theories within the framework of classical field theory. The construction of a free field equation for a particle with mass m and spin s must be done in several steps. First we must make a choice for the field function. In case of a particle with integer spin s we choose a symmetric s rank tensor field $\phi_{\mu_1 \ldots \mu_s}(x)$, or shortly $\phi_\omega(x)$, where ω represents the set of indices $\mu_1 \ldots \mu_s$. Half integer spin particles can be represented by a symmetric tensor-spinor $\psi_{\mu_1 \ldots \mu_s}(x)$. Then we must construct the field equation which should be satisfied by the field $\phi_\omega(x)$.

Here we are restricted by several requirements:

i) According to the special theory of relativity physical laws should be the same in every inertial system. Since different inertial systems are connected by a Lorentz transformation, the field equation should be Lorentz covariant.

ii) It is well known that a massive particle with spin s gives rise to 2s+1 independent spin states s, s-1, ..., -s. Consequently the field ϕ_ω should have 2s+1 independent components. In case of massless particles this number of components is even reduced to two, according to the two different spin states ±s for a massless particle.
As we shall see in more detail in chapter V, the field ϕ_ω turns out to have a number of superfluous degrees of freedom. These superfluous components are eliminated by a set of subsidiary conditions.
This fact can also be understood from a different viewpoint. According to the theory of representations, the irreducible representation of the inhomogeneous Lorentz group can be labeled by two numbers m and s interpreted as the mass and the spin of the particle. Such a representation is carried by some field function ϕ_ω. Since a symmetric rank s tensor field does not carry in general an irreducible representation of the inhomogeneous Lorentz group, a set of subsidiary conditons is required to eliminate superfluous components of this field function. Then the remaining components carry the irreducible representation.

As is well known, massive particles with integer spin s are described by a symmetric tensor field $\phi_{\mu_1 \ldots \mu_s}(x)$, which satisfies the Klein-Gordon equation

$$(\Box-m^2)\phi_{\mu_1\ldots\mu_s}(x) = 0 \ , \qquad (3.1)$$

and the subsidiary conditions

$$\phi_{\lambda\lambda\mu_3\ldots\mu_s}(x) = 0 \ ,$$

and (3.2)

$$\partial_\lambda\phi_{\lambda\mu_2\ldots\mu_x}(x) = 0 \ .$$

For half-integer spin particles the symmetric tensor-spinor $\psi_{\mu_1\ldots\mu_s}(x)$ satisfies the Dirac equations

$$(i\gamma_\mu\partial_\mu-m)\psi_{\mu_1\ldots\mu_s}(x) = 0 \ , \qquad (3.3)$$

and the subsidiary condition

$$\gamma_\lambda\psi_{\lambda\mu_2\ldots\mu_s} = 0 \ ,$$

and (3.4)

$$\partial_\lambda\psi_{\lambda\mu_2\ldots\mu_s} = 0 \ .$$

The requirements mentioned before are satisfied in both cases.

On the other hand, Fierz and Pauli [4] and Dirac [5] investigated the possibility to construct a field equation, equivalent to (3.1) and (3.2). For instance, a spin-1 particle can be described by the Proca-equation

$$\Box\phi_\mu - \partial_\mu\partial_\nu\phi_\nu = m^2\phi_\mu \ . \qquad (3.5)$$

By contraction with ∂_μ we get the subsidiary condition

$$\partial_\mu\phi_\mu = 0 \qquad (3.2)$$

and the Proca-equation reduces to the Klein-Gordon equation

$$\Box\phi_\mu = m^2\phi_\mu \ . \qquad (3.1)$$

This shows (3.5) to be equivalent to (3.1) and (3.2).

For higher spins, however, the construction of the field equation becomes much more difficult. In this thesis we shall use the root method with which higher-spin field equations can be constructed.

Once we have a field equation it is useful to try to derive it from a more general principle which is called the principle of least action. The main facts concerning this principle can be found in the textbooks [3]. A central role is played by the Lagrangian $\mathcal{L}$, a Lorentz covariant scalar function, depending on the field and its derivatives

$$\mathcal{L} = \mathcal{L}\big(\phi_\omega(x),\ \partial_\mu\phi_\omega(x)\big) \ . \qquad (3.6)$$

From the action principle follow the field equations, or the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_\omega} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\omega)}\right) = 0 \ . \qquad (3.7)$$

Although the principle of least action leads to the same field equation, knowledge of the Lagrangian is very valuable for various theoretical considerations like invariances, quantization and inclusion of interactions.

It is difficult to introduce interactions in the system of equations (3.1) and (3.2), but in the Lagrangian an interaction can be introduced explicitly.

The outline of the remaining part of this thesis is organized as follows. In chapter V we explain the main characteristics of the root method. In particular we shall show that the root method leads to the free field equations (3.1) and the subsidiary conditions (3.2). Furthermore, we shall discuss the relation between the field equations of massive and massless particles.

In chapter VI the root method will be applied explicitly for the case of spin 1, 2 and 3. We shall construct a field equation and Lagrangian for massive particles. We shall also show how a massless field equation and Lagrangian can be obtained from massive field equation.

In chapter VII the relation between massive and massless field equations is investigated in more detail. In particular we shall compare the expression for the amplitude, describing exchange of a particle between two external sources, in both the massive and massless case. It will be shown that the $m\to 0$ limit leads to various problems.

References

[1] M.L. Goldberger, H.W. Lewis and K.M. Watson, Phys.Rev. 132, 2764 (1963).

[2] J.E. Bowcock and H. Burkhardt, Rep.Progr.Phys. 38, 1099 (1975), and references therein.

[3] C. Ytzikson and J.B. Zuber, Quantum Field Theory, MacGraw Hill Inc. (1980).

[4] M. Fierz, Helv.Phys. Acta 12, 3 (1936).
M. Fierz and W. Pauli, Proc.R.Soc. A173, 211 (1939).

[5] P.H.M. Dirac, Proc.R.Soc. A155, 447 (1936).

CHAPTER II

PHASE-SHIFT AMBIGUITIES, SUMMARY OF THE MAIN RESULTS

1. Some remarks about scattering theory

We first introduce some quantities with which we shall deal throughout this part of the thesis.

In a scattering experiment a target is bombarded with a beam of particles. One measures the number of particles, scattered in a given direction. This direction can be specified by two angles θ and ϕ. Let N_0 be the number of incident particles per unit area and per unit time, and N the number of scattering centres present in the target. Then one finds the following expression for the number of scattered particles per solid angle $d\Omega$ and per unit time:

$$N(\theta,\phi) = N_0 N \frac{d\sigma}{d\Omega} d\Omega, \tag{1.1}$$

where the factor $\frac{d\sigma}{d\Omega}$ is called the differential cross section. The total cross section σ_{tot} can be calculated by integrating over the whole solid angle

$$\sigma_{tot} = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega \ . \tag{1.2}$$

In the framework of scattering theory, the differential cross section is closely related to the interaction between the incident particles and the scattering centres. For instance in the case of non relativistic elastic scattering between spinless uncharged particles, where the interaction is described by a spherically symmetric or central potential $V(r)$, the asymptotic solution of the Schrödinger equation takes the form

$$\psi(r,\theta) \doteq e^{ikr} + F(k,\theta) \frac{e^{ikr}}{kr} \ . \tag{1.3}$$

This solution contains a part describing the incident particle and another part describing the scattered particle. The scattering amplitude $F(k,\theta)$ depends on the potential $V(r)$. In the centre of mass system (c.m.s.) the incident particle has a reduced mass μ, a kinetic energy E and momentum $k = \sqrt{2\mu E}$.

It was mentioned in the introduction that the dimensionless amplitude $F(k,\theta)$ is related to the differential cross section as follows:

$$k^2 \frac{d\sigma}{d\Omega}(k,\theta) = |F(k,\theta)|^2 . \tag{1.4}$$

The question of determining the potential that describes the interaction between the particles can be thought of as consisting of two parts. First one has to obtain the scattering amplitude from the differential cross section. Then one may try to determine the potential $V(r)$ from the amplitude $F(k,\theta)$. In this thesis we only discuss how $F(k,\theta)$ can be obtained from the differential cross section.

Of course, as (1.4) shows, $\frac{d\sigma}{d\Omega}$ only determines the modulus of $F(k,\theta)$. However, $F(k,\theta)$ must satisfy another relation. This relation results from an important principle in the theory of scattering, which is called conservation of flux or conservation of probability. In case one describes scattering within the framework of S-matrix theory rather than in terms of potential scattering this principle is equivalent to the unitarity of the S-matrix. In both cases, but only for purely elastic scattering, we get a relation for the scattering amplitude, which may be represented in integral form

$$\mathrm{Im}\, F(k,\theta_{12}) = \frac{1}{4\pi} \int d\Omega_3\, F(\theta_{13}) F^*(\theta_{23}) \tag{1.5}$$

where the integration has to be carried out over all values of the solid angle Ω_3 and where 1 and 2 denote the unit vectors in the initial and final direction, whereas θ_{12} denotes the angle between those vectors.

When we evaluate (1.5) for $\theta_{12} = 0$, we obtain a simple relation between the total differential cross section (1,2) and the imaginary part of the scattering amplitude in the forward direction

$$k^2 \sigma_{tot}(k) = 4\pi\, \mathrm{Im}\, F(k,0), \tag{1.6}$$

showing that in the forward direction the phase of the scattering amplitude can be determined by an experiment.

The problem we shall discuss in this thesis can be formulated as follows: is it possible to determine the phase of $F(k,\theta)$ by using the unitarity constraint (1.5), when one has perfect knowledge of the differential cross section, or, equivalently, of the modulus of $F(k,\theta)$?

It should be stressed here that this problem is formulated for an ideal situation, because a perfect knowledge of $|F(k,\theta)|$ is supposed here. In any actual experiment, this is not possible because every measurement of $|F(k,\theta)|$ is accompanied by experimental uncertainties.

For our purposes it will be convenient to reformulate slightly the problem of determining the phase of $F(k,\theta)$. Due to the spherical symmetry of the scattering potential $V(r)$ one also can express $F(k,\theta)$ in terms of Legendre polynomials or, equivalently, in angular momentum eigenfunctions.

Putting $x = \cos\theta$

$$F(k,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(x) \ , \qquad (1.7)$$

where the f_ℓ are the so-called partial waves.

It now is possible to rephrase the unitarity condition (1.5) in terms of the partial waves f_ℓ. For all $\ell = 0,1,2,\ldots$

$$|f_\ell(k)|^2 = \mathrm{Im}\, f_\ell(k) \ . \qquad (1.8)$$

Consequently, it is possible to show that f_ℓ is determined by only one real parameter δ_ℓ which is called the ℓ-th phase-shift

$$f_\ell(k) = \frac{1}{2i} (e^{2i\delta_\ell(k)} - 1). \qquad (1.9)$$

One easily verifies that (1.8) is satisfied by (1.9).
Thus we can restate our problem in terms of the phase-shifts δ_ℓ: find all different sets $(\delta_0, \delta_1, \ldots)$ of phase-shifts, each δ_ℓ being defined by (1.9), giving the same modulus $|F(k,\theta)|$ of the scattering amplitude $F(k,\theta)$.

A remark should be made about the case of inelastic scattering, which will not be discussed in this thesis. When the energy exceeds the first inelastic threshold, scattering experiments are specified by a different number of particles in the initial and final state. In this case the unitarity constraint for the scattering amplitude takes a different form.

In terms of the partial waves and phase-shifts, this condition reads

$$f_\ell(k) = \frac{1}{2i} (\eta_\ell\, e^{2i\delta_\ell(k)} - 1) \qquad (1.10)$$

with $0 \leq \eta_\ell \leq 1$, for all ℓ-values.
For purely elastic scattering $\eta_\ell = 1$ for all ℓ-values. In general, however, f_ℓ is determined by two real parameters: the inelasticity η_ℓ and the phase-shift δ_ℓ.

Finally we want to discuss the polynomial expansion (1.7) of $F(k,\theta)$. In the case of elastic scattering and in the presence of a central scattering potential one can prove quite generally that the higher partial waves behave exponentially as

$$f_\ell \sim e^{-\alpha\ell} \ . \qquad (1.11)$$

A condition for this property is a sufficiently fast decrease of the potential with the radial distance r, which is for instance the case for the Yukawa potential.

The most important contribution therefore comes from the lower partial waves.

In practical phase-shift analysis, i.e. the procedure to extract partial wave amplitudes from the experimental data, one usually assumes, that the partial waves are zero for values of ℓ greater than some L. Then the amplitude is a polynomial in $x = \cos\theta$ of degree L.

$$F(k,\theta) = \sum_{\ell=0}^{L} (2\ell+1) f_\ell(k) P_\ell(x) \ . \tag{1.12}$$

This expression is clearly an approximation of the exact behaviour of the partial waves mentioned above. Nevertheless one may expect it to be reasonably good in practice, particularly if α is sufficiently large (cf. equation (1.11)). Moreover it has the great practical advantage of reducing the number of parameters in a phase-shift analysis from an infinite to a finite number.

In this so-called "polynomial case" one may study the construction of phase-shift ambiguities for small L-values. Once one has obtained more insight in these lower L cases one may try to generalize these results to arbitrary L. However, it should be stressed again, that the case of a polynomial amplitude is only an approximation of the exact situation, where one has an analytical amplitude and therefore an infinite number of partial waves.

2. Existence and uniqueness of solutions

The problem of determining different amplitudes with the same modulus that satisfy the unitarity condition (1.5) has one trivial solution. Once an amplitude $F(k,\theta)$ is given, it is clear that both $F(k,\theta)$ and $F(k,\theta)^*$ have the same modulus and obey the condition (1.5) due to unitarity. This example is called the trivial ambiguity and it can be obtained by reversing the signs of all the phase-shifts $\delta_\ell \to -\delta_\ell$ (cf. eq. (1.9)). This ambiguity can be removed by using analiticity with respect to energy, which will not be discussed here.

We are only interested in non-trivial ambiguities. Therefore all our statements will be modulo the trivial ambiguity.

We consider again the unitarity condition in integral form (1.5)

$$\text{Im } F(x_{12}) = \frac{1}{4\pi} \int d\Omega_3 \; F(x_{13})F^*(x_{32})$$

with $x_{12} = \cos\theta_{12}$.

Let $\psi(x_{12})$ be the argument of $F(x_{12})$. Thus (1.5) takes a different form:

$$\sin\psi(x_{12}) = \frac{1}{4\pi|F(x_{12})|} \int d\Omega_3 |F(x_{13})\cdot F(x_{32})| \cos\big(\psi(x_{13}) - \psi(x_{32})\big) \; . \qquad (2.1)$$

This equation defines a non-linear mapping of a function space onto itself. In fact it can be written as follows

$$\psi' = A\psi, \qquad (2.2)$$

where the non-linear mapping A is defined by the inverse sine of the right-hand side of (2.1).

When the modulus of the scattering amplitude is known, one would like to answer the following questions:

1) Under which conditions does equation (2.1) have a solution?
2) When is this solution unique?
3) Furthermore, if there is more than one solution or a phase-shift ambiguity, how many different solutions do exist?

It has already been noticed in the introduction that the first two questions have only partially been answered.

By introducing the quantity

$$\sin\mu := \sup_{-1 \leq x_{12} \leq 1} \frac{1}{4\pi|F(x_{12})|} \int d\Omega_3 |F(x_{13})F(x_{32})| \qquad (2.3)$$

and by using convenient sets of functions and techniques from non-linear analysis, like Schauder's theorem and the contraction mapping principle, it is possible to prove the following results (Newton, Martin, Atkinson et all [1]).

(i) There exists at least one solution of (2.1) , if $\sin\mu < 1$ (2.4)

(ii) Equation (2.1) has a unique solution, if $\sin\mu < 0.79$. (2.5)

It must be stressed that the inequality (2.4) is very restrictive, because it implies for the phase-shifts

$$\delta_\ell < \frac{\pi}{6} , \quad \text{for all } \ell \; . \qquad (2.6)$$

Therefore this result may only be of interest in the case of purely elastic scattering at very low energies.

3. Phase-shift ambiguities in the polynomial case

In the last section existence and uniqueness of the solution of equation (2.1) were discussed. Here we want to list a number of results concerning the situation where the scattering amplitude is approximated by a polynomial of degree L.

It has been shown in this case that besides the trivial ambiguity also less trivial examples of ambiguities exist (see refs. [2] and [3]). Their existence was proved, however, only for the very special case L = 2,3.

It was also proved that for the polynomial case conditions exist under which the phase-shifts are uniquely determined by the differential cross section.

We summarize these results as follows:

(i) A non-trivial ambiguity was first discovered by Crichton for the polynomial case [2] .

He considered the case L=2 and he showed that the two sets of phase-shifts

$$\begin{array}{lll} \delta_0 = -23^\circ 20' & \delta_1 = -43^\circ 27' & \delta_2 = 20^\circ \\ \delta'_0 = 98^\circ 50' & \delta'_1 = -26^\circ 33' & \delta'_2 = 20^\circ \end{array} \qquad (3.1)$$

give exactly the same differential cross section.

Moreover two sets (δ_0,δ_1) and (δ'_0,δ'_1) correspond to any δ_2-value in the interval $[12^\circ 32', 24^\circ 9']$, varying continuously with δ_2, that still give different scattering amplitudes F and F' with the same modulus.

(ii) A method for constructing phase-shift ambiguities for higher L-values was given by Berends and Ruijsenaars [3] . With this method they were able to construct all the different phase-shift ambiguities for the L=3 case.

Again they found two different sets of phase-shifts $(\delta_0,\delta_1,\delta_2,\delta_3)$ and $(\delta'_0,\delta'_1,\delta'_2,\delta'_3)$, each phase-shift δ_ℓ or δ'_ℓ being a continuous function of only one parameter p, which can take values in some interval.

When p varies in this interval $(\delta_0,\delta_1,\delta_2,\delta_3)$ and $(\delta'_0,\delta'_1,\delta'_2,\delta'_3)$ both form curves in a 4-dimensional space, which are closed (mod π). A similar property holds for the Crichton ambiguity as well.

(iii) Moreover it was observed (see also ref. [3]), that there were never more than two solutions in the case L = 2,3.
In the L=4 case too it was shown by Cornille and Drouffe [4] that the maximum number of solutions is again 2.

(iv) Finally it was shown by Martin [5], that the amplitude is uniquely determined by its modulus, if

$$\sin \mu < 1 \tag{3.2}$$

or if

$$\sigma_{tot} < \frac{4\pi}{k^2} \cdot 1{,}38 \tag{3.3}$$

4. Spin-0 - spin-½ elastic scattering. Summary of the main results

In high energy physics much attention has been given to scattering experiments involving mesons with spin 0 and nucleons with spin ½. In this case, one can measure besides the differential cross section different experimental observables such as the recoil nucleon polarization P.

Since we consider the strong interactions between those particles, parity is conserved. Therefore we have to specify an eigenstate by two quantum numbers: j for total angular momentum, and
ℓ for angular momentum.
We denote an angular momentum eigenstate with $\ell\pm$, when $j = \ell\pm\frac{1}{2}$.

In terms of these quantum numbers both differential cross section and polarization can be described by two complex functions of $x = \cos\theta$ and k where θ is the c.m.-scattering angle and k is the c.m. momentum. These two functions are in turn given by the following expansion

$$f(x) = \sum_{\ell=0}^{\infty} \left\{(\ell+1)f_{\ell+} + \ell f_{\ell-}\right\} P_\ell(x) \tag{4.1}$$

$$g(x) = i \sum_{\ell=0}^{\infty} \left\{f_{\ell+} - f_{\ell-}\right\} (1-x^2)^{\frac{1}{2}} \frac{dP_\ell(x)}{dx} \tag{4.2}$$

The so-called partial waves $f_{\ell\pm}$ satisfy

$$f_{\ell\pm} = \frac{1}{2i}(\zeta_{\ell\pm} - 1) = \frac{1}{2i}(\eta_{\ell\pm} e^{2i\delta_{\ell\pm}} - 1) \tag{4.3}$$

with $\delta_{\ell\pm}$ real and $0 \leq \eta_{\ell\pm} \leq 1$.

When one only considers elastic scattering, the unitarity condition simply implies that $\eta_{\ell\pm} = 1$.

Differential cross section and polarization are then given by

$$k^2 \frac{d\sigma}{d\Omega} = |f(x)|^2 + |g(x)|^2 \ , \tag{4.4}$$

$$P = \frac{2\text{Re}\ f(x)^* g(x)}{|f(x)|^2 + |g(x)|^2} \ . \tag{4.5}$$

The problem is to investigate how many different pairs of amplitudes f(x) and g(x), satisfying the unitarity condition, correspond to the same differential cross section and polarization.

Here we shall restrict ourselves again to the case where f(x) and g(x) are given by finite partial wave decompositions

$$f(x) = \sum_{\ell=0}^{L} \left\{(\ell+1)f_{\ell+} + \ell f_{\ell-}\right\} P_\ell(x) \ , \tag{4.6}$$

$$g(x) = i\sum_{\ell=0}^{L} (f_{\ell+} - f_{\ell-})(1-x^2)^{\frac{1}{2}} \frac{dP_\ell(x)}{dx} \ . \tag{4.7}$$

If we have a finite number of phase-shifts, one may ask the question whether different sets of phase-shifts $(\delta_0,\delta_1,\ldots,\delta_L)$ and $(\delta'_0,\delta'_1,\ldots\delta'_L)$ exist, which give the same differential cross section and polarization.

There exist three well-known examples of ambiguities for arbitrary L. They are characterized by the following transformations

(i) Reflection. $\delta'_\ell = -\delta_\ell$, or

$$f'(x) = -f^*(x);\ g(x)' = g^*(x). \tag{4.8}$$

(ii) Minami [6]. $\delta'_{\ell+} = \delta_{(\ell+1)-}$, $\delta'_{\ell-} = \delta_{(\ell-1)+}$, or

$$f'(x) = xf(x) + i(1-x^2)^{\frac{1}{2}}y(x) \tag{4.9}$$

$$g'(x) = -i(1-x^2)^{\frac{1}{2}}f(x) - xg(x).$$

(iii) Yang [7] . $f'(x) = f(x) \quad g'(x) = -g(x).$ (4.10)

Only the reflection and the Minami ambiguity satisfy the unitarity condition $|\zeta'_{\ell\pm}| = 1$. And therefore only these are of interest.

Furthermore, as one easily verifies, the experimental quantities transform in each of these cases in the following way:

$$\left(\frac{d\sigma}{d\Omega}\right)' = \frac{d\sigma}{d\Omega} \ , \quad \text{and } P' = -P.$$

From these observations it is clear that only simultaneous application of the Minami ambiguity and reflection leaves both differential cross section and polarization invariant [7]. This combination is called the modified Minami ambiguity.

The modified Minami ambiguity is given by the following transformation:

$$\delta'_{\ell\pm} = -\,\delta_{(\ell+1)\mp}\,, \text{ or}$$
$$f'(x) = -xf^*(x) + i(1-x^2)^{\frac{1}{2}}g^*(x) \qquad (4.11)$$
$$g'(x) = -xg^*(x) + i(1-x^2)^{\frac{1}{2}}f^*(x) \quad .$$

Besides this ambiguity less obvious examples have been constructed by Berends and Ruijsenaars [8]. In fact they found all possible ambiguities, giving the same cross section and polarization, for the case of S and P waves only.

References

[1] A. Martin, Nuov.Cim. A42, 930 (1966).
Nuov.Cim. A59, 131 (1969).
R.G. Newton, J.Math.Phys. 9, 2050 (1968).
D. Atkinson et al., Commun.Math.Phys. 28, 133 (1972).
Commun.Math.Phys. 33, 221 (1973).

[2] J.H. Crichton, Nuov.Cim. A45, 256 (1966).

[3] F.A. Berends and S.N.M. Ruijsenaars, Nucl.Phys. B56, 507 (1973).

[4] H. Cornille and J.M. Drouffe, Nuov.Cim. A20, 401 (1974).

[5] A. Martin, Proceedings of the Adriatic Summer Meeting on Particle Physics (1973).

[6] S. Minami, Progr.Theor.Phys. 11, 213 (1954).

[7] H.A. Bethe and F. Hoffmann, Mesons and Fields, Row Peterson (1955), p. 72.

[8] F.A. Berends and S.N.M. Ruijsenaars, Nucl.Phys. B56, 525 (1973).

CHAPTER III

PHASE-SHIFT AMBIGUITIES FOR SPINLESS ELASTIC SCATTERING

1. Introduction

As was mentioned in the previous chapter phase-shift ambiguities have been constructed for the cases $L=2,3$ [1]. In particular for the case $L=3$ two sets of phase-shifts $(\delta_0,\delta_1,\delta_2,\delta_3)$ and $(\delta_0',\delta_1',\delta_2',\delta_3')$ have been found, both giving two amplitudes F and F' with the same modulus. Moreover these phase-shifts δ_ℓ and δ_ℓ' are continuous functions on some real interval and at the endpoints of this interval δ_ℓ and δ_ℓ' become equal. Thus both sets $\{\delta_\ell\}$ and $\{\delta_\ell'\}$ form a curve in a 4-dimensional real space. Since at the endpoints δ_ℓ equals δ_ℓ' for all ℓ, the corresponding pair of curves forms one closed curve (mod π).

Of course an obvious question is, whether it is possible to construct phase-shift ambiguities for arbitrary values of L. Before discussing this problem, we first present a method by which, at least in principle, all different sets of phase-shifts can be obtained.

If the amplitude $F(\cos\theta) = F(x)$ is approximated by a polynomial of L-th degree

$$F(x) = \sum_{\ell=0}^{L} (2\ell+1) f_\ell P_\ell(x) , \tag{1.1}$$

one may also express F(x) in terms of its L complex roots $z_1,z_2,\ldots,z_L$:

$$F(x) = F(1) \prod_{k=1}^{L} \left(\frac{x-z_k}{1-z_k}\right) . \tag{1.2}$$

It was observed by Gersten [2], that all different amplitudes with the same modulus can be obtained by two types of transformations or by products of them

$$\begin{aligned} &(i)\ T_i : z_i \to z_i^* \\ &(ii)\ S : \mathrm{Re}\, F(1) \to -\mathrm{Re}\, F(1) . \end{aligned} \tag{1.3}$$

Two such transformations A and B are called equivalent, if their product equals either the identity transform, or the trivial transformation $F(x) \to -F(x)^*$. This trivial transformation amounts to reversing all signs of the phase-shifts:

$\delta_\ell \to -\delta_\ell$ for all ℓ. Therefore all non equivalent sets of phase-shifts can be obtained for instance by considering only the transformations T_i. For completeness we mention the unitarity condition again. According to this condition and in the case of elastic scattering the coefficients f_ℓ in equation (1.1) are of the form

$$f_\ell = \zeta_\ell - 1 = e^{2i\delta_\ell} - 1 \ . \tag{1.4}$$

This expression differs from the one in the previous chapters by a factor 2i. However, by a convenient redefinition this factor can be absorbed in the amplitude F(x).

Summarizing, the problem of constructing phase-shift ambiguities can be stated as follows:

(i) All different sets of phase-shifts $(\delta_0, \delta_1, \ldots, \delta_L)$ giving different amplitudes with the same modulus can be generated by complex conjugation of one or more roots of F(x).

(ii) It must be checked, whether in all these cases, the partial wave amplitudes f_ℓ have the form given by (1.4).

In the next section we consider the case of conjugation of only one root. A method will there be discussed with which it is, at least in principle, possible to construct all phase-shift ambiguities in case one root is conjugated. This method also suggests generalization to the case of conjugation of more than one root. However, the equations that define the ambiguities become partly by the increase of L and partly by conjugation of more roots, much more involved. In spite of this complexity it can be shown that rather simple examples exist for arbitrary L if only one root is conjugated.

Again the different sets of phase-shifts $(\delta_0, \delta_1, \ldots, \delta_L)$ and $(\delta'_0, \delta'_1, \ldots, \delta'_L)$ will form curves in an L+1 dimensional real space. These curves are parametrized by only one real variable, which can take values in some closed interval. However, these curves do not have the property of being closed (mod π) at the endpoints of this interval, as was noticed in ref. [1] .

It can be shown, however, that they will meet other curves which are again defined by different phase-shift ambiguities. At least one of these curves will appear to be closed. Therefore, in this extended sense, the examples constructed here will still form closed curves (mod π).

2. Formalism for the construction of ambiguities

In the case of complex conjugation of one root we first want to derive expressions for the coefficients f_ℓ or, equivalently, ζ_ℓ, which explicitly depend on the conjugated root z. Therefore we will write the amplitude F(x) as follows:

$$F(x) = \frac{x-z}{1-z} P(x), \tag{2.1}$$

P(x) being a polynomial of degree L-1.
Of course the transformed amplitude is given by

$$F'(x) = \frac{x-z^*}{1-z^*} P(x) \quad . \tag{2.2}$$

Obviously we have $|F(x)| = |F'(x)|$ and the forward scattering amplitude F(1) is unchanged.

Both F(x) and P(x) may be expanded in Legendre polynomials

$$F(x) = \sum_{\ell=0}^{L} (2\ell+1) f_\ell \, P_\ell(x) \tag{2.3}$$

$$P(x) = \sum_{\ell=0}^{L-1} (2\ell+1)(\alpha_\ell - 1) P_\ell(x) \; . \tag{2.4}$$

Using the property

$$xP_\ell(x) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(x) + \frac{\ell}{2\ell+1} P_{\ell-1}(x) \; , \tag{2.5}$$

we find

$$F(x) = \frac{x-z}{1-z} P(x) = \frac{x-z}{1-z} \sum_{\ell=0}^{L-1} (2\ell+1)(\alpha_\ell - 1) P_\ell(x) =$$

$$= \frac{1}{1-z} \sum_{\ell=0}^{L} \left\{ \ell\alpha_{\ell-1} + (\ell+1)\alpha_{\ell+1} - (2\ell+1)(1-z+z\alpha_\ell) \right\} P_\ell(x) \tag{2.6}$$

where $\alpha_\ell = 1$ for $\ell \geq L$ and $\ell < 0$. (2.7)

According to the equations (2.6) and (2.3) we express the coefficients f_ℓ in terms of the coefficients α_ℓ and the root z:

$$f_\ell = \frac{1}{1-z} \left(\frac{\ell+1}{2\ell+1} \alpha_{\ell+1} + \frac{\ell}{2\ell+1} \alpha_{\ell-1} - z\alpha_\ell \right) - 1 \; . \tag{2.8}$$

From (2.8) we get for the coefficients $\zeta_\ell = f_\ell + 1$:

$$\zeta_\ell = \frac{1}{1-z} \left(\frac{\ell+1}{2\ell+1} \alpha_{\ell+1} + \frac{\ell}{2\ell+1} \alpha_{\ell-1} - z\alpha_\ell \right) \; . \tag{2.9}$$

The coefficients $\zeta_\ell(z)$ and the transformed ones $\zeta_\ell(z^*)$ both have to obey the unitarity constraints:

$$|\zeta_\ell(z)| = |\zeta_\ell(z^*)| \tag{2.10}$$

$$|\zeta_\ell(z)| = 1 \ . \tag{2.11}$$

First we impose (2.10) and then (2.11). Equation (2.10) is satisfied, if, for all ℓ:

$$\mathrm{Im}\left[\alpha_\ell^*\left(\frac{\ell+1}{2\ell+1}\,\alpha_{\ell+1} + \frac{\ell}{2\ell+1}\,\alpha_{\ell-1}\right)\right] = 0 \ . \tag{2.12}$$

The coefficients α_ℓ can be parametrized in such a way that this condition is automatically satisfied for all ℓ.

As we shall see many different parametrizations are possible. These lead to different representations for the ζ_ℓ.

We start with 2L+2 parameters: $z,\alpha_0,\alpha_1,\ldots,\alpha_{L-1}$. There are L equations (2.12), and if they are satisfied for $\ell=1,2,\ldots,L$, then (2.12) is automatically satisfied for $\ell=0$.

Therefore, by imposing these L conditions, the number of 2L+2 parameters will reduce to L+2. These L+2 remaining parameters can be solved in terms of one real parameter by using the L+1 conditions (2.11). We shall now indicate how the representations in terms of the L+2 parameters are obtained.

We first consider (2.12) for $\ell=L$. In this case (2.12) reads

$$\mathrm{Im}\ \alpha_L^*\alpha_{L-1} = 0 \ . \tag{2.13}$$

Since $\alpha_L = 1$, we have either α_{L-1} is real or $\alpha_{L-1} = 0$. Obviously (2.12) for $\ell = L-1$ reduces to

$$\mathrm{Im}\ \alpha_{L-1}^*\alpha_{L-2} = 0 \ . \tag{2.14}$$

If in the former step two parameters were eliminated by choosing $\alpha_{\ell-1} = 0$, then this last equation (2.14) is satisfied for α_{L-2} complex. However, if we choose α_{L-1} to be real, this equation implies either α_{L-2} is real or $\alpha_{L-2} = 0$.

Continuing in this way and by using successively the equations (2.12) one can eliminate one parameter in each step. Suppose in this way (2.12) is satisfied for $\ell = L,L-1,\ldots,k+1$ by choosing $\alpha_{L-1}, \alpha_{L-2},\ldots,\alpha_{k+1}$ real, and $\alpha_k=0$. In case $\ell=k$ equation (2.12) reads

$$\mathrm{Im}\ \alpha_k^*\left(\frac{k+1}{2k+1}\,\alpha_{k+1} + \frac{k}{2k+1}\,\alpha_{k-1}\right) = 0 \ . \tag{2.15}$$

Obviously this equation is satisfied because $\alpha_k = 0$. Therefore there is no restriction on α_{k-1}.

In case $\ell = k-1$ we have

$$\text{Im } \alpha^*_{k-1} \left(\frac{k}{2k-1} \alpha_k + \frac{k-1}{2k-1} \alpha_{k-2}\right) = 2$$

or (2.16)

$$\text{Im } \alpha^*_{k-1}\alpha_{k-2} = 0 \ .$$

This equation is satisfied if either $\alpha_{k-2} = 0$ or α_{k-2} is complex with $\alpha_{k-2} = \lambda\alpha_{k-1}$ and λ real or if $\alpha_{k-1} = 0$. If we choose $\alpha_{k-1} = 0$, however, we eliminate two parameters by using only one equation. Therefore only the possibilities $\alpha_{k-2} = 0$ or $\alpha_{k-1} = \lambda\alpha_{k-1}$ with λ real and α_{k-1} complex are allowed.

Consequently, it is impossible to have two subsequent α's, that both equal zero. Moreover two or more subsequent complex α's will always have the same argument.

It should be noted that, when finally (2.12) is satisfied for $\ell = L, L-1, \ldots, 1$, then it is automatically satisfied for $\ell=0$. This fact forbids us to choose $\alpha_0 = 0$. It must be stressed too, that at least one of the coefficients α_ℓ has to be complex. If all the coefficients α_ℓ are real, we simply get the trivial ambiguity.

These arguments together with (2.12) give us a set of rules which enable us at each step to decide whether some coefficient α_ℓ is either complex or real or zero. Thus many different representations of the ζ_ℓ, depending on L+2 parameters, are obtained.

After imposing unitarity, $|\zeta_\ell| = 1$, all the partial waves will depend on only one parameter for which we will choose $|z|$. We first shall list the rules.

(1) $\alpha_L = \alpha_{L+1} = 1$, therefore either α_{L-1} is real or $\alpha_{L-1} = 0$.

(2) If for some ℓ, α_ℓ is real, then either $\alpha_{\ell-1}$ is real or $\alpha_{\ell-1} = 0$.

(3) If for some ℓ, $\alpha_\ell = 0$, then $\alpha_{\ell-1}$ is complex.

(4) If for some ℓ, α_ℓ is complex, then either $\alpha_{\ell-1} = 0$ or $\alpha_{\ell-1}$ is complex. In case of a complex $\alpha_{\ell-1}$, both α_ℓ and $\alpha_{\ell-1}$ have equal arguments.

(5) α_0 is complex.

It is convenient to use diagrams to distinguish between the different representations of the ζ_ℓ. These diagrams will be built up by means of three different arrows . Each arrow will be associated with one out of the three possibilities for some coefficient α_ℓ. Explicitly for the ℓ-th arrow

α_ℓ real $\longrightarrow$ α_ℓ complex $\downarrow$ $\alpha_\ell = 0 \rightsquigarrow$.

By using the rules (1) - (5) one can write down all the different representations for $\zeta_0, \zeta_1, \ldots, \zeta_L$. Below we list all the different possibilities up to L=5.

<u>L=2</u>

$$\zeta_2 = \frac{1}{1-z}\left(\frac{3}{5} + \frac{2}{5}\alpha_1 - z\right)$$
$$\zeta_1 = \frac{1}{1-z}\left(\frac{2}{3} + \frac{1}{3}\alpha_0 - z\alpha_1\right) \tag{2.17}$$
$$\zeta_0 = \frac{1}{1-z}\left(\alpha_1 - z\alpha_0\right) .$$

Representation: $\alpha_1 = 0, \alpha_0$ complex

$$\zeta_2 = \frac{1}{1-z}\left(\frac{3}{5} - z\right) \rightsquigarrow$$
$$\zeta_1 = \frac{1}{1-z}\left(\frac{2}{3} + \frac{1}{3}\alpha_0\right) \downarrow \tag{2.18}$$
$$\zeta_0 = \frac{1}{1-z}\left(-z\alpha_0\right) .$$

<u>L=3</u> General equations

$$\zeta_3 = \frac{1}{1-z}\left(\frac{4}{7} + \frac{3}{7}\alpha_2 - z\right)$$
$$\zeta_2 = \frac{1}{1-z}\left(\frac{3}{5} + \frac{2}{5}\alpha_1 - z\alpha_2\right)$$
$$\zeta_1 = \frac{1}{1-z}\left(\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_0 - z\alpha_1\right) \tag{2.19}$$
$$\zeta_0 = \frac{1}{1-z}\left(\alpha_1 - z\alpha_0\right).$$

a) $\alpha_2 = 0$; α_1, α_0 complex; $\frac{1}{3}\alpha_0 = \lambda\alpha_1$, λ real.

$$\zeta_3 = \frac{1}{1-z}\left(\frac{4}{7} - z\right) \rightsquigarrow$$
$$\zeta_2 = \frac{1}{1-z}\left(\frac{3}{5} + \frac{2}{5}\alpha_1\right) \downarrow$$
$$\zeta_1 = \frac{1}{1-z}\left(\lambda - z\right)\alpha_1 \downarrow \tag{2.20}$$
$$\zeta_0 = \frac{1}{1-z}\left(1 - 3\lambda z\right)\alpha_1 .$$

b) α_2 real, $\alpha_1 = 0, \alpha_0$ complex

$$\zeta_3 = \frac{1}{1-z} \left(\frac{4}{7} + \frac{3}{7}\alpha_2 - z\right) \quad \rightsquigarrow$$
$$\zeta_2 = \frac{1}{1-z} \left(\frac{3}{5} - z\alpha_2\right)$$
$$\zeta_1 = \frac{1}{1-z} \left(\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_0\right) \quad (2.21)$$
$$\zeta_0 = \frac{1}{1-z} \left(-z\alpha_0\right) .$$

L=4 General equations

$$\zeta_4 = \frac{1}{1-z} \left(\frac{5}{9} + \frac{4}{9}\alpha_3 - z\right)$$
$$\zeta_3 = \frac{1}{1-z} \left(\frac{4}{7} + \frac{3}{7}\alpha_2 - z\alpha_3\right)$$
$$\zeta_2 = \frac{1}{1-z} \left(\frac{3}{5}\alpha_3 + \frac{2}{5}\alpha_1 - z\alpha_2\right) \quad (2.22)$$
$$\zeta_1 = \frac{1}{1-z} \left(\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_0 - z\alpha_1\right)$$
$$\zeta_0 = \frac{1}{1-z} \left(\alpha_1 - z\alpha_0\right) .$$

a) $\alpha_3 = 0$, $\alpha_0, \alpha_1, \alpha_2$ complex; $\frac{2}{5}\alpha_1 = \lambda\alpha_2$

$$\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_0 = \mu\alpha_1$$

$$\zeta_4 = \frac{1}{1-z} \left(\frac{5}{9} - z\right)$$
$$\zeta_3 = \frac{1}{1-z} \left(\frac{4}{7} + \frac{3}{7}\alpha_2\right) \quad \rightsquigarrow$$
$$\zeta_2 = \frac{1}{1-z} \left(\lambda - z\right)\alpha_2 \quad (2.23)$$
$$\zeta_1 = \frac{1}{1-z} \left(\mu - z\right)\frac{5}{2}\alpha_2\lambda$$
$$\zeta_0 = \frac{1}{1-z} \left(\frac{5}{2}\lambda - \left(\frac{15}{2}\lambda\mu - 2\right) z\right)\alpha_2$$

b) α_3 real; $\alpha_2 = 0$; α_0, α_1 complex; $\frac{1}{3}\alpha_0 = \lambda\alpha_1$

$$
\begin{aligned}
\zeta_4 &= \frac{1}{1-z}\left(\frac{5}{9} + \frac{4}{9}\alpha_3 - z\right) \\
\zeta_3 &= \frac{1}{1-z}\left(\frac{4}{7} + z\alpha_3\right) \\
\zeta_2 &= \frac{1}{1-z}\left(\frac{3}{5}\alpha_3 + \frac{2}{5}\alpha_1\right) \\
\zeta_1 &= \frac{1}{1-z}\left(\lambda - z\right)\alpha_1 \\
\zeta_0 &= \frac{1}{1-z}\left(1 - 3\lambda z\right)\alpha_1 .
\end{aligned}
\tag{2.24}
$$

c) α_3, α_2 real, $\alpha_1 = 0, \alpha_0$ complex

$$
\begin{aligned}
\zeta_4 &= \frac{1}{1-z}\left(\frac{5}{9} + \frac{4}{9}\alpha_3 - z\right) \\
\zeta_3 &= \frac{1}{1-z}\left(\frac{4}{7} + \frac{3}{7}\alpha_2 - z\alpha_3\right) \\
\zeta_2 &= \frac{1}{1-z}\left(\frac{3}{5}\alpha_3 - z\alpha_2\right) \\
\zeta_1 &= \frac{1}{1-z}\left(\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_0\right) \\
\zeta_0 &= \frac{1}{1-z}\left(-z\alpha_0\right) .
\end{aligned}
\tag{2.25}
$$

d) $\alpha_1 = \alpha_3 = 0$; α_2, α_0 complex

$$
\begin{aligned}
\zeta_4 &= \frac{1}{1-z}\left(\frac{5}{9} - z\right) \\
\zeta_3 &= \frac{1}{1-z}\left(\frac{4}{7} + \frac{3}{7}\alpha_2\right) \\
\zeta_2 &= \frac{1}{1-z}\left(-z\alpha_2\right) \\
\zeta_1 &= \frac{1}{1-z}\left(\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_0\right) \\
\zeta_0 &= \frac{1}{1-z}\left(-z\alpha_0\right) .
\end{aligned}
\tag{2.26}
$$

From these examples it is clear that the representations of the coefficients ζ_ℓ are determined by the following general structure for the ζ_ℓ. The first set of α_ℓ : $\alpha_L, \alpha_{L-1}, \ldots, \alpha_{k+1}$ are real $(1 \le k \le L-1)$. Then $\alpha_k = 0$. Finally we get blocks of complex coefficients α_ℓ which all have the same arguments. These blocks alternate by coefficients α_ℓ which equal zero.

We like to conclude this section with some remarks.

(i) The L=2 case gives only one representation. By imposing $|\zeta_2| = |\zeta_1| = |\zeta_0| = 1$, one will find the ambiguity, that originally was found by Crichton [3].

(ii) In a similar way one can solve the case L=3. Then one gets some of the ambiguities constructed by Berends and Ruijsenaars [1].

(iii) Consdering the L=4 case one immediately observes an increasing number of representations. Although the equations $|\zeta_\ell| = 1$, will become more involved for increasing L, the problem of conjugating one root can in principle be solved.

(iv) In spite of the complexity of the equations, the representation (2.26) will appear to be relatively simple. In the next section we shall show, that it is possible to solve the unitarity relations $|\zeta_\ell| = 1$ for this representation and for all $L \geq 4$ and L even. If L is odd, this representation has to be slightly modified to be able to prove, at least locally, the existence of an ambiguity.

3. Existence of ambiguities for arbitrary $L \geq 4$

The case of even L

In this part of this section we consider for L even the coefficients ζ_ℓ, which in general are given by

$$\zeta_\ell = \frac{1}{1-z}\left(\frac{\ell+1}{2\ell+1}\,\alpha_{\ell+1} + \frac{\ell}{2\ell+1}\,\alpha_{\ell-1} - z\alpha_\ell\right) . \tag{3.1}$$

The two sets $\{\zeta_\ell(z)\}$ and $\{\zeta_\ell(z^*)\}$ both give different scattering amplitudes with the same modulus.

As was stressed earlier the $\zeta_\ell(z)$ and $\zeta_\ell(z^*)$ have to obey the unitarity constraints. In particular for the coefficients ζ_ℓ these constraints can be stated as follows: for all $\ell = 0,1,\ldots,L$

$$|\zeta_\ell(z)| = |\zeta_\ell(z^*)| , \tag{3.2}$$

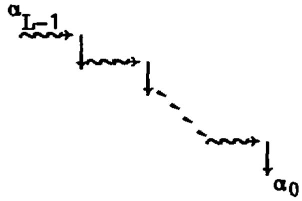
$$|\zeta_\ell(z)| = 1 . \tag{3.3}$$

We shall impose these conditions on the particular representation of the ζ_ℓ that was given for the case L=4 by equation (2.26) in the previous section. This representation can be generalized to arbitrary, even L by the following three conditions on the α_ℓ:

(i) $\alpha_L = \alpha_{L+1} = 1$

(ii) $\alpha_{2\ell+1} = 0$, for $\ell = 0,1,\ldots,\frac{1}{2}L-1$ (3.4)

(iii) $\alpha_{2\ell}$ complex, for $\ell = 0,1,\ldots, \frac{1}{2}L-1$.



This representation is characterized by the following diagram.
From now on we shall often refer to the corresponding ambiguity as the "staircase" ambiguity, as is suggested by its diagrammatic representation.

The equations for the ζ_ℓ read explicitly:

$$\zeta_L = \frac{1}{1-z}\left(\frac{L+1}{2L+1} - z\right)$$

$$\zeta_{L-1} = \frac{1}{1-z}\left(\frac{L}{2L-1} + \frac{L-1}{2L-1}\,\alpha_{L-2}\right) \tag{3.5}$$

$$\zeta_\ell = \begin{cases} \dfrac{1}{1-z}\left(\dfrac{\ell+1}{2\ell+1}\,\alpha_{\ell+1} + \dfrac{\ell}{2\ell+1}\,\alpha_{\ell-1}\right) & \ell = 1,3,5,\ldots,L-3 \\[2ex] -\dfrac{z\alpha_\ell}{1-z} & \ell = 0,2,4,\ldots,L-2\,. \end{cases}$$

In the previous section it was shown that the unitarity constraint (3.2) is satisfied by the staircase representation (3.5). We have still to impose, however, the second unitarity constraint (3.3): $|\zeta_\ell| = 1$.

According to the four different forms in (3.5) this unitarity condition gives four types of equations which we will treat successsively. As was remarked earlier, we choose $|z|$ as a parameter.

A. The case $\ell=L$.

Imposing $|\zeta_\ell| = 1$, we get

$$\left|\frac{L+1}{2L+1} - z\right| = |1-z|, \text{ from which we find}$$

$$x = \text{Re } z = \frac{3L+2}{4L+2}\;. \tag{3.6}$$

Obviously from (3.6) a lower bound for $|z|$ follows.

$$|z| \geq x = \frac{3L+2}{4L+2}\,. \tag{3.7}$$

Moreover one can now write $|1-z|$ as a function, that depends only on $|z|$

$$|1-z|^2 = |z|^2 - \frac{L+1}{2L+1}\,. \tag{3.8}$$

B. The case ℓ even.

Defining $\alpha_\ell = |\alpha_\ell| e^{i\phi_\ell}$, (3.9)

we get from $|\zeta_\ell| = 1$:

$$|\alpha_\ell| = |1-z|/|z| \ . \tag{3.10}$$

C. The case ℓ odd.

Imposing $|\zeta_\ell| = 1$ for ℓ odd, we get

$$(2\ell+1)^2|z|^2 = (\ell+1)^2 + 2\ell(\ell+1)\cos(\phi_{\ell+1} - \phi_{\ell-1}) + \ell^2 \ , \text{ or}$$

$$\cos(\phi_{\ell+1} - \phi_{\ell-1}) = \frac{1}{2\ell(\ell+1)} \left\{(2\ell+1)^2|z|^2 - (\ell+1)^2 - \ell^2\right\} . \tag{3.11}$$

One obviously has to require:

$$-1 \leq \cos(\phi_{\ell+1} - \phi_{\ell-1}) \leq +1 \ . \tag{3.12}$$

From the right-hand side of this inequality we get

$$|z| \leq 1 \ , \tag{3.13}$$

whereas the left-hand side of (3.12) defines another lower bound for the allowed $|z|$-region:

$$|z| \geq \frac{1}{2\ell+1} \ . \tag{3.14}$$

Of course the inequality (3.8) remains more restrictive than (3.14), for any ℓ-value.

Therefore the only allowed values $|z|$ can take are given by

$$\frac{3L+2}{4L+2} \leq |z| \leq 1 \ . \tag{3.15}$$

D. The case $\ell = L-1$.

In this case we find from (3.3) an expression for the argument ϕ_{L-2} of α_{L-2}

$$\cos \phi_{L-2} = \frac{(2L-1)^2|1-z|^2 - L^2 - (L-1)^2|\alpha_{L-2}|^2}{2L(L-1)|\alpha_{L-2}|}$$

or, with (3.10)

$$\cos \phi_{L-2} = \frac{(2L-1)^2|1-z|^2|z|^2 - L^2|z|^2 - (L-1)^2|1-z|^2}{2L(L-1)|1-z||z|} \ . \tag{3.16}$$

Again we have to require

$$-1 \leq \cos \phi_{L-2} \leq +1 \ . \tag{3.17}$$

This inequality is obviously satisfied for $|z| = 1$. But for $|z| = \frac{3L+2}{4L+2}$ $\cos \phi_{L-2}$ turns out to be less than -1. Therefore with respect to the lower bound of the allowed $|z|$-region (3.17) turns out to be more restrictive than (3.7) or (3.15).

The left-hand side of (3.17) defines another, L-dependent, lower bound $a_0(L)$ for $|z|$.

This lower bound can be determined for arbitrary L by the equation

$$\cos \phi_{L-2}\big(|z| = a_0(L)\big) = -1 \ . \tag{3.18}$$

This equation is equivalent to a polynomial equation of fourth degree in a_0:

$$a_0^4 + 2(1-A_{L-1})a_0^3 + \big((1-A_{L-1})^2 - A_{L-1}^2 - A_L\big)a_0^2 - \\ - 2(1-A_{L-1})A_L a_0 - (1-A_{L-1})^2 A_L = 0 \ , \tag{3.19}$$

where $A_L = \frac{L+1}{2L+1}$.

It can be shown that there always exists a solution of (3.19), which lies in the interval defined by (3.15). In particular for $L=4$ we found $a_0(4) = 0.836$. Furthermore $a_0(L)$ decreases for increasing L: if $L \to \infty$, then $a_0(L) \to 0.7693$.

Summarizing we conclude that for arbitrary but even L phase-shift ambiguities can be constructed. Moreover we can, at least in principle, find expressions for both $\zeta_\ell(z)$ and $\zeta_\ell(z^*)$, as functions of only one real parameter $|z|$. Modulus and argument of the α_ℓ are given by (3.10), (3.11) and (3.16).

The coefficients ζ_ℓ are in turn given by (3.5) and the amplitudes F(x) and F'(x) are defined by (1.1) and (1.4) in the previous section.

The case of odd L

To complete the proof that phase-shift ambiguities exist for arbitrary $L \geq 4$, we still have to consider the case of odd L.

In order to show the existence of ambiguities for odd L, we slightly modify the staircase representation, which was treated in the previous section. This so-called modified staircase representation is specified by the following conditions on the α_ℓ:

(i) $\alpha_{L+1} = \alpha_L = 1$,

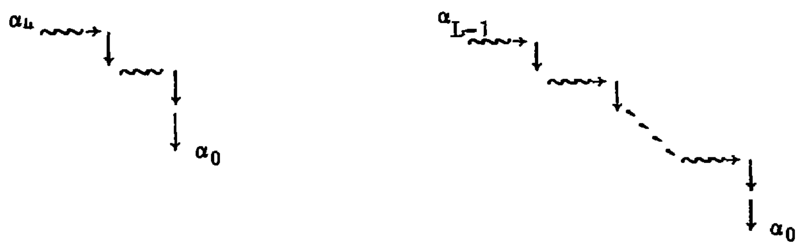
(ii) $\alpha_{2\ell} = 0$ for $\ell = 1,2,\ldots,\frac{1}{2}(L-1)$,

(iii) $\alpha_{2\ell-1}$ is complex for $\ell = 1,2,\ldots,\frac{1}{2}(L-1)$. (3.20)

(iv) α_0 is complex.

The diagram by which this representation is characterized, is:

for L=5 and in general



Most of the coefficients resemble exactly the ones given by (3.5). Only for $\ell = 0,1,2$ their expressions are different:

$$\zeta_2 = \frac{1}{1-z}\left(\frac{3}{5}\alpha_3 + \frac{2}{5}\alpha_1\right)$$

$$\zeta_1 = \frac{1}{1-z}(\lambda - z)\alpha_1 \tag{3.21}$$

$$\zeta_0 = \frac{1}{1-z}(1-3\lambda z)\alpha_1 \quad .$$

Here, we used

$$\alpha_0 = 3\lambda\alpha_1; \quad \lambda \text{ real} \; . \tag{3.22}$$

Equation (3.22) guarantees $|\zeta_\ell(z)| = |\zeta_\ell(z^*)|$ for $\ell = 0,1$.

After imposing $|\zeta_\ell| = 1$, we get the same equations for α_ℓ as in the previous case of even L. Only for $\ell = 0,1,2$ the condition $|\zeta_\ell| = 1$, will give rise to some additional complications, which we will discuss in a few steps.

A. The case $\ell = 0,1$.

Imposing $|\zeta_\ell| = 1$ for $\ell = 0,1$, we get

$$|\alpha_1| = \left|\frac{1-z}{\lambda-z}\right| \; ; \quad \alpha_0 = 3\lambda\alpha_1, \quad \lambda \text{ real} \tag{3.23}$$

$$|\lambda-z| = |1-3\lambda z| \; . \tag{3.24}$$

From (3.24) one observes that λ can be solved from an equation of second degree in λ:

$$\lambda^2(9|z|^2 - 1) - 4x\lambda + (1 - |z|^2) = 0 \tag{3.25}$$

where $x = \mathrm{Re}\, z$.

Defining $t = |z|^2$, we find

$$\lambda_\pm(t) = \frac{1}{9t-1}\left(2x \pm \left[9t^2 - 10t + 1 + 4x^2\right]^{\frac{1}{2}}\right). \tag{3.26}$$

In particular

$$\begin{aligned} \lambda_+(t)\big|_{t=1} &= \tfrac{1}{2}x \\ \lambda_-(t)\big|_{t=1} &= 0\ . \end{aligned} \tag{3.27}$$

First we study the case $\lambda = \lambda_-(t)$.

Then one can easily verify the following properties.

(i) The square root part of $\lambda_\pm(t)$ is well defined for all t, because its argument is positive definite for $x = 3L+2/4L+2$.

(ii) For the derivative with respect to t:

$$\frac{d}{dt}\lambda_-(t)\big|_{t=1} = -\frac{1}{4x} < 0\ . \tag{3.28}$$

B. The case $\ell=2$.

When we require the modulus of $\zeta_2 = \frac{1}{1-z}\left(\frac{3}{5}\alpha_3 - \frac{2}{5}\alpha_1\right)$ to equal 1, it follows

$$|1-z|^2 = \frac{9}{25}|\alpha_3|^2 + \frac{4}{25}|\alpha_1|^2 + \frac{12}{25}|\alpha_3||\alpha_1| \cos\phi_{13} \quad ,$$

or

$$\cos\phi_{13} = \frac{25|1-z|^2 - 9|\alpha_3|^2 - 4|\alpha_1|^2}{12|\alpha_3||\alpha_1|}\ , \tag{3.29}$$

where $\phi_{13} = \phi_3 - \phi_1$.

When we use: $|\alpha_3| = \left|\frac{1-z}{z}\right|$, $|\alpha_1| = \left|\frac{1-z}{\lambda-z}\right|$ we get for $\cos\phi_{13}$:

$$\cos\phi_{13} = \frac{25|\lambda-z|^2|z|^2 - 9|\lambda-z|^2 - 4|z|^2}{12|\lambda-z||z|}$$

or, with $t = |z|^2$

$$\cos\phi_{13}(t) = \frac{25t(t-2x\lambda+\lambda^2) - 9(t-2x\lambda+\lambda^2) - 4t}{12\sqrt{t(t-2x\lambda+\lambda^2)}}\ . \tag{3.30}$$

Now we put $\lambda(t) = \lambda_-(t)$.

Of course the inequalities

$$-1 \leq \cos \phi_{13} \leq +1 \tag{3.31}$$

prescribe which t-values are allowed. Therefore by means of the two inequalities defined by (3.30) and (3.31) we have to determine the boundary points of the allowed t-interval. Since these equations are rather complicated it is a priori not clear, whether they are more restrictive with respect to the allowed t-values from equations (3.19) and (3.13). Nevertheless it is possible to show the existence of this particular ambiguity in a neighbourhood of t=1.

We first like to remark that in t=1

$$\begin{aligned} \lambda_- &= 0 \\ |\alpha_1| &= |1-z| = 2(1-x) \\ |\alpha_0| &= 0 \ . \end{aligned} \tag{3.32}$$

Thus we get from (3.32) and (3.30)

$$\cos \phi_{13}(t)\Big|_{t=1} = 1 \ . \tag{3.33}$$

Obviously the right-hand side of the inequality (3.31) is satisfied and the modified staircase ambiguity exists in $|z|^2 = t = 1$.

However, we like to show its existence in a neighbourhood of t=1. Therefore we mention another property of $\cos \phi_{13}(t)$:

$$\frac{d}{dt} \cos \phi_{13}(t)\Big|_{t=1} = \frac{5}{12} \left(5 - 4x \cdot \frac{d}{dt} \lambda_-(t)\right)\Big|_{t=1} \ . \tag{3.34}$$

Since one of the properties of $\lambda_-(t)$ which we mentioned in (3.28) was

$$\frac{d}{dt} \lambda_-(t)\Big|_{t=1} = -\frac{1}{4x}$$

we get

$$\frac{d}{dt} \cos \phi_{13}(t)\Big|_{t=1} = \frac{5}{2} > 0 \ . \tag{3.35}$$

Because $\cos \phi_{13}(t)$ is an increasing function of t in t=1 and since it equals 1 in t=1, there exists a neighbourhood $(1-\varepsilon,1]$ in which $-1 \leq \cos \phi_{13}(t) \leq +1$ holds.

In this neighbourhood the unitarity constraints for $\ell=0,1,2$ are consistent with the constraints for other ℓ-values which were discussed in the previous section and which also hold for the case of odd L. Thus we have shown this ambiguity, defined by the representation given by (3.20), to exist in a neighbourhood of t=1.

Finally one remark should be made about the choice $\lambda=\lambda_+$. For this choice

too one can prove the existence of the corresponding ambiguity in a neighbourhood $(1-\varepsilon,1]$ of $t=1$. Here the condition (3.31) is automatically satisfied.
However, the choice $\lambda=\lambda_-$ is more interesting for us with respect to problems which we deal with in the next chapter.

4. Discussion and conclusions

In the previous sections it has been shown that for arbitrary $L \geq 4$ phase-shift ambiguities can be constructed. Moreover some pecularities of these ambiguities can be noticed.

(i) In the allowed parameter region from the equations (3.10) and (3.16) $|\alpha_\ell|$ and ϕ_{L-2} are known. By iteration (see (3.11)) the arguments ϕ_ℓ of α_ℓ can be calculated. Then the coefficients ζ_ℓ are given by (3.5) and the two amplitudes $F(x)$ and $F'(x)$ are determined by (1.1) and (1.4).
At the boundary $|z| = 1$ of the parameter interval all the expressions simplify:

$$|\alpha_\ell| = 2 \text{ Re } (1-z) \tag{4.1}$$

$$\phi_\ell = \phi_{\ell+2} \text{ for all } \ell = 0,2,4,\ldots,L-4 \ . \tag{4.2}$$

For the amplitude we get

$$F(x) = -\frac{\text{Im } z}{1-z} \left((\alpha_0-1)\cdot P'_{L-1}(x) - P'_L(x)\right). \tag{4.3}$$

(ii) From the last equation (4.3) we notice that the amplitude becomes infinite for $L\to\infty$. This occurs for all values of $x=\cos\theta$. In realistic situations, however, the amplitude is an analytic function in some ellipse. Therefore the different amplitudes constructed here do not elucidate the real situation, because they are divergent for $L\to\infty$.

(iii) For even L and for all values of $|z|$ the differences of the phase-shifts $\delta_\ell-\delta'_\ell$ are equal for $\ell = 0,2,4,\ldots,L-2$, and similarly for $\ell = 1,3,5,\ldots,L-1$, whereas $\delta_L = -\delta'_L$.
For $|z| = 1$ we even have:

$$\delta_0 = \delta_2 = \ldots = \delta_{L-2}$$

$$\delta_1 = \delta_3 = \ldots = \delta_{L-3} \ .$$

Similar results, but slightly different, hold for odd L.

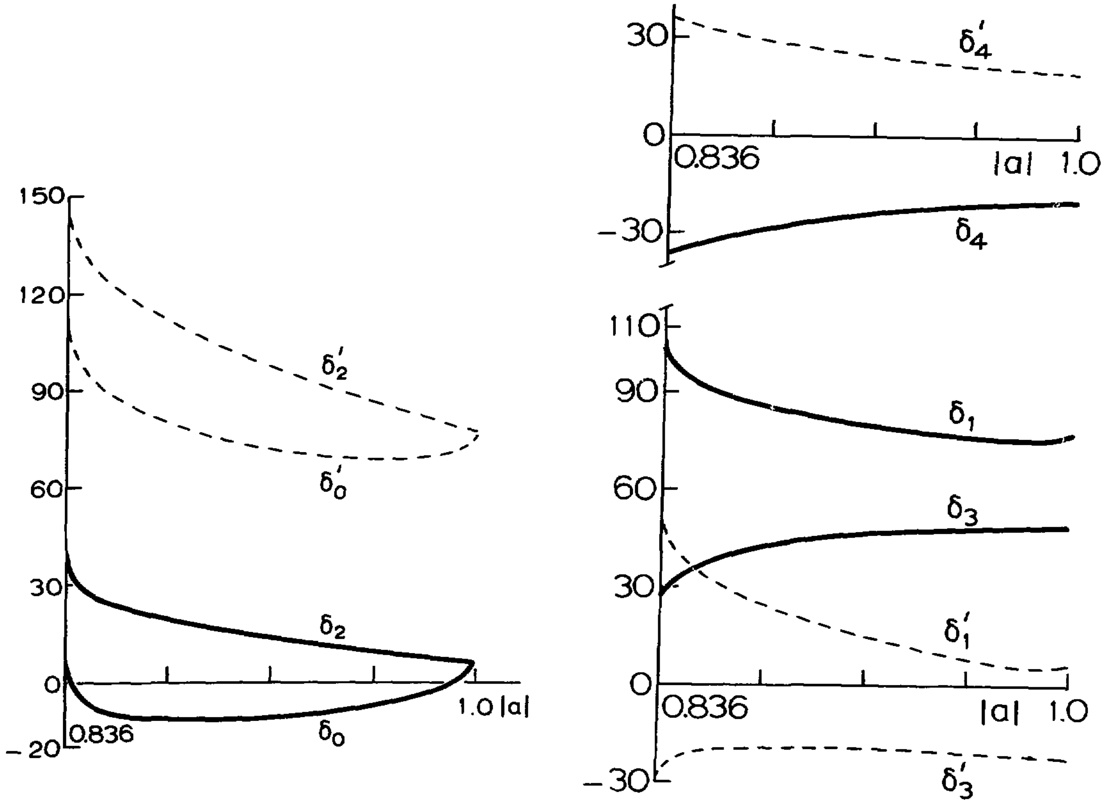


Fig. 1. The phase-shifts δ_ℓ (solid line) and δ_ℓ' (dashed line) for the allowed range of $|a|$-values in the case L=4.

(iv) In contrast to the known examples in the case L=2,3 [1] the two sets of phase-shift do not form a closed curve (mod π) in an L+1 dimensional real space, since not all δ_ℓ and δ'_ℓ become equal at the boundary points of the $|z|$-interval.

In the next section we shall show, that the curves constructed here, will intersect curves that correspond to different ambiguities. One of these curves will nonetheless be closed.

(v) Finally at the boundary point $|z| = 1$ all the coefficients α_ℓ, $\ell \leq L-2$, will have the same argument. Therefore in this point the staircase ambiguity is a special case of the representation in which all the α_ℓ, $\ell \leq L-2$ are complex numbers with the same argument.

Many of the properties mentioned here are illustrated for the case L=4 in figure 1.

5. On the intersection of curves

In the previous section we showed that it is possible to construct a phase-shift ambiguity for arbitrary L. This ambiguity was obtained by choosing a particular representation for the coefficients ζ_ℓ and it was called the staircase ambiguity. The phase-shifts δ_ℓ and δ'_ℓ both appeared to depend on only one parameter $|z|$. Thus both sets $(\delta_0,\delta_1,\ldots,\delta_L)$ and $(\delta'_0,\delta'_1,\ldots,\delta'_L)$ form two curves in a L+1-dimensional real space, when $|z|$ varies in some interval. All the ambiguities, which were found for the case L=2,3 [1], have the property that at the endpoints of this interval for all ℓ: $\delta_\ell = \delta'_\ell$. In such a case the two curves defined by the sets $\{\delta_\ell\}$ and $\{\delta'_\ell\}$ form one closed curve (mod π). In contrast to the ambiguities for L=2,3, the examples constructed here, do not have this property. For the staircase ambiguity we find a pair of curves that will not form one closed curve.

It is, however, possible that this pair of curves will intersect different pairs of curves of which at least one will form a closed curve. In fact it is suggested that this may happen, because for $|z| = 1$ the arguments of the coefficients α_ℓ become equal. A different ambiguity that can be constructed by choosing α_ℓ complex for $\ell \leq L-2$ and $\alpha_{L-1} = 0$, also has the property that the coefficients α_ℓ all have the same argument. Therefore in $|z| = 1$ the staircase ambiguity is a special case of the former kind of ambiguity and the corresponding two pairs of curves probably intersect each other in $|z| = 1$.

In this section we shall answer the question whether the curves defined by the staircase ambiguity will intersect curves corresponding to different ambiguities. We shall also consider the question, which of these curves are closed (mod π). We restrict ourselves to the case of even L and $L \geq 4$.

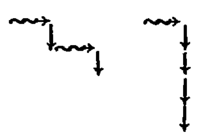
The representations in which we are interested can be obtained by replacing in the staircase diagram the subdiagram $\overset{\alpha_{2k+1}}{\rightsquigarrow}\downarrow\alpha_{2k}$ by the subdiagram $\begin{matrix}\downarrow\alpha_{2k+1}\\ \downarrow\alpha_{2k}\end{matrix}$. This means that in the staircase representation we choose α_{2k+1} complex instead of zero. Therefore α_{2k+1} and α_{2k} will have the same argument. This substitution can be performed for all α_{2k+1} except for α_{L+1}. In this way $2^{\frac{1}{2}L-1}$ different representations, including the staircase representation, are obtained.

We shall show that in a half-open interval $1-\epsilon < |z| \leq 1$, each of these representations defines two sets of phase-shifts $(\delta_0,\delta_1,\ldots,\delta_L)$ and $(\delta_0',\delta_1',\ldots,\delta_L')$, both giving the same differential cross section and depending on only one parameter $|z|$. Then we will demonstrate that all the curves defined by these sets $\{\delta_\ell\}$ and $\{\delta_\ell'\}$ will intersect the curves belonging to the staircase ambiguity in $|z| = 1$.

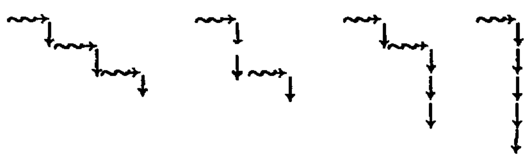
For any L-value, we also find only one representation that gives two sets of phase-shifts which may become equal (mod π) at the endpoints of their parameter interval. All the other ambiguities will give curves that are not closed.

Below we list for the cases L=4,6,8 the ambiguities that intersect each other in $|z| = 1$.

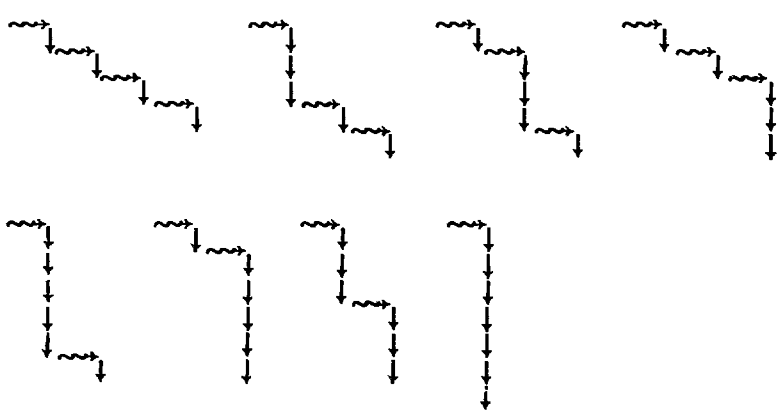
L=4



L=6



L=8



In particular it can be proved, that the representation, given by $\alpha_{L-1} = 0$ and α_ℓ complex, for all $\ell \leq L-2$, defines an ambiguity for $1-\varepsilon < |z| < 1+\varepsilon$. Moreover this ambiguity forms a closed curve, e.g. the two curves defined by the sets $\{\delta_\ell\}$ and $\{\delta'_\ell\}$ become equal (mod π) at the endpoints of the corresponding $|z|$-interval.

In order to prove the existence of the ambiguities discussed above, in a neighbourhood of $|z| = 1$, we shall apply the implicit function theorem.

Implicit function theorem:

Let U be an open subset of $\mathbb{R}^2$ and let $F(x,y)$ be continuously differentiable with respect to x and y on U. If for $(x_0,y_0) \in U$ $F(x_0,y_0) = 0$, and $\frac{\partial}{\partial y} F(x_0,y_0) \neq 0$, then there exist an $\varepsilon>0$ such that for $x_0-\varepsilon < x < x_0+\varepsilon$ we have a continuously differentiable function $y = f(x)$ with the following properties:

(i) $y_0 = y(x_0)$,

(ii) $F(x,f(x)) = 0$ for all x: $x_0-\varepsilon < x < x_0+\varepsilon$,

(iii) the derivative f' of f is given by $f'(x) = -\frac{F_x}{F_y}$.

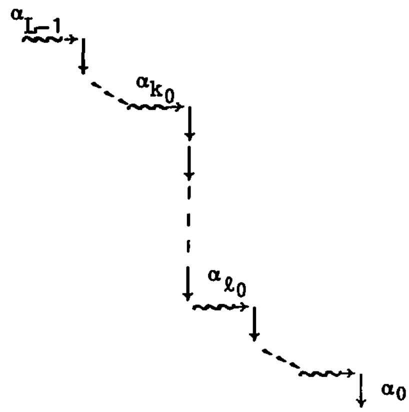
Instead of studying all possible kinds of $2^{\frac{1}{2}L-1}$ representations, it is sufficient to apply this theorem to only one, rather general, kind of representation. Let L be even and $L \geq 4$. Let k_0 and ℓ_0 be odd numbers satisfying $L-1 > k_0 \geq \ell_0 \geq 1$. Then we shall investigate the following modification of the staircase representation: $\alpha_{k_0-1}, \alpha_{k_0-2}, \ldots, \alpha_{\ell_0+1}$ are complex. (The case, where $k_0 = L-1$ and $\ell_0+1 = 0$, will be discussed in more detail at the end of this section.)

Consequently, as was noticed before, all these coefficients will have the same arguments, e.g.

$$\phi_{k_0-1} = \phi_{k_0-2} = \ldots = \phi_{\ell_0+1} \; . \tag{5.1}$$

We noticed that the staircase representation also has this property in $|z| = 1$. Therefore, for $|z| = 1$, the staircase ambiguity is a special case of some of these modified staircase ambiguities. We shall show this in more detail below.

The representation under consideration can be characterized with the following diagram:



It is convenient to introduce the numbers

$$b_\ell = \frac{\ell+1}{2\ell+1} \; ; \quad c_\ell = 1-b_\ell = \frac{\ell}{2\ell+1} \quad . \tag{5.2}$$

Imposing unitarity, e.g. $|\zeta_\ell| = |\zeta'_\ell| = 1$, we distinguish between the cases

<u>A</u> $\ell \geq k_0+1$.

$\ell \leq \ell_0-1$,

<u>B</u> $\ell_0+1 \leq \ell \leq k_0-1$,

<u>C</u> $\ell = k_0, \ell_0$.

Case A: The unitarity constraints in this case will be called the staircase equations, since this part of the representations equals exactly the staircase representation. Because these equations have been studied before we only list the results

$$\alpha_\ell = \left|\frac{1-z}{z}\right| e^{i\phi_\ell}, \quad \ell \text{ even} , \tag{5.3}$$

$$\cos(\phi_{\ell+1} - \phi_{\ell-1}) = 1 - \frac{|z|^2 - 1}{2b_\ell c_\ell} , \quad \ell \text{ odd} , \tag{5.4}$$

$$\mathrm{Re}(1-z) = 1-x = \tfrac{1}{2}c_L , \tag{5.5}$$

$$\cos \phi_{L-2} = \frac{1}{2b_{L-1}c_{L-1}} \left|\frac{z}{1-z}\right| \left(|1-z|^2 - b_{L-1}^2 - c_{L-1}^2 \left|\frac{1-z}{z}\right|^2\right) . \tag{5.6}$$

Case B: In this case we have for the coefficients ζ_ℓ:

$$\zeta_\ell = \frac{1}{1-z}\left(b_\ell|\alpha_{\ell+1}| + c_\ell|\alpha_{\ell-1}| - z|\alpha_\ell|\right) e^{i\phi} , \tag{5.7}$$

where ϕ is the argument of $\alpha_{k_0-1}, \alpha_{k_0-2}, \ldots, \alpha_{\ell_0+1}$.

Obviously we have: $|\zeta_\ell(x)| = |\zeta_\ell(z^*)|$.

We define:

$$\mu_k = \left|\frac{\alpha_k}{1-z}\right| \quad . \tag{5.8}$$

Then we get from (5.7) and (5.8)

$$\zeta_\ell = \frac{|1-z|}{1-z}\,(b_\ell\mu_{\ell+1} + c_\ell\mu_{\ell-1} - z\mu_\ell)e^{i\phi} \; . \tag{5.9}$$

The coefficients must also obey: $|\zeta_\ell| = 1$.

From this constraint it follows that

$$b_\ell\mu_{\ell+1} + c_\ell\mu_{\ell-1} = x\mu_\ell - \varepsilon_\ell[(x^2 - |z|^2)\mu_\ell^2 + 1]^{\frac{1}{2}} \; , \tag{5.10}$$

where $\varepsilon_\ell = \pm 1$.

The choice $\varepsilon_\ell = (-1)^\ell$, (i.e. $\varepsilon_{\ell_0} = -1$) (5.11)

guarantees, as we shall see, that this representation equals the staircase representation in $|z| = 1$.

Introducing the parameter $t = |z|^2$, the coefficients μ_ℓ are given by

$$b_\ell\mu_{\ell+1} + c_\ell\mu_{\ell-1} = x\mu_\ell - (-1)^\ell[(x^2-t)\mu_\ell^2 + 1]^{\frac{1}{2}} \; , \tag{5.12}$$

for $\ell = \ell_0+1, \ell_0+2,\ldots,k_0-1$,

and with

$$\mu_{\ell_0} = \mu_{k_0} = 0 \; . \tag{5.13}$$

<u>Case C:</u>

In this case we get from the unitarity constraints relations for $\cos(\phi_{k_0+1} - \phi_{k_0-1})$ and $\cos(\phi_{\ell_0+1} - \phi_{\ell_0-1})$.

When we use from (5.5) $|\alpha_{k_0+1}| = |\alpha_{\ell_0-1}| = \left|\frac{1-z}{z}\right|$, we get:

$$\cos(\phi_{k_0+1} - \phi_{k_0-1}) = \frac{t^2 - b_{k_0}^2 - c_{k_0}^2\mu_{k_0-1}^2t^2}{2b_{k_0}c_{k_0}\mu_{k_0-1}t} =: c_1(t) \; , \tag{5.14}$$

$$\cos(\phi_{\ell_0+1} - \phi_{\ell_0-1}) = \frac{t^2 - b_{\ell_0}^2\mu_{\ell_0+1}t^2 - c_{\ell_0}^2}{2b_{\ell_0}c_{\ell_0}\mu_{\ell_0+1}t} =: c_2(t) \; . \tag{5.15}$$

Of course in accceptable cases the functions $c_1(t)$ and $c_2(t)$ must obey

$$|c_1(t)| \le 1 \; , \qquad |c_2(t)| \le 1 \; . \tag{5.16}$$

Next we study the equations (5.12) and (5.13) in more detail. If all the square root parts in (5.12) are well defined, it is possible to express successively, starting with $\ell = \ell_0+1$, every μ_ℓ, $\ell = \ell_0+2,\dots,k_0$ in terms of only two variables $\mu = \mu_{\ell_0+1}$ and $t = |z|^2$.

Moreover the $\mu_\ell(\mu,t)$ are then differentiable functions with respect to both μ and t. Furthermore we must have $\mu_{k_0}(\mu,t) = 0$ according to the choice of the representation. From this equation one can, at least in principle, obtain μ as a function of t. In general this cannot be done explicitly. As will be shown below, one can solve μ as a function of t from (5.13) only in a neighbourhood of t=1, by applying the implicit function theorem.

However, due to the choice (5.11), and by using an induction argument, it can easily be proved that, if $\mu = t = 1$,

$$\mu_\ell = \begin{cases} 0, & \ell = \ell_0+2,\ \ell_0+4\ \dots,\ k_0-2,\ k_0 \\ 1, & \ell = \ell_0+1,\ \ell_0+3,\dots,\ k_0-3 \end{cases} \tag{5.17}$$

More explicitly, in order to prove that all the coefficients μ_k, including μ, are continuously differentiable functions of t on some neighbourhood of t=1, we study equation (5.12) successsively for $\ell = \ell_0+1,\ \ell_0+2,\dots,\ k_0-1$. From the first equation, $\ell = \ell_0+1$, we solve μ_{ℓ_0+2} as a continuously differentiable function of μ and t. From the second one, $\ell = \ell_0+2$, we find μ_{ℓ_0+3} as a continuously differentiable function of μ_{ℓ_0+2}, μ and t. Since μ_{ℓ_0+2} depends only on μ and t, μ_{ℓ_0+3} also does, and, moreover it is continuously differentiable with respect to these variables.

By studying all the equations (5.12) up to $\ell = k_0-1$, we finally can solve μ_{k_0} as a function of μ and t alone.

Since in the representation under consideration we choose $\alpha_{k_0} = 0$ we still have to put $\mu_{k_0} = 0$ (5.13). From this equation it is, at least in principle, possible to obtain μ as a function of t. In fact, however, this problem turns out to be too complicated. Because, $\mu_{k_0} = 0$, for $\mu = t = 1$, we shall instead apply the implicit function theorem. By using this theorem we shall demonstrate that an $\varepsilon>0$ exist, such that for $1-\varepsilon < t < 1+\varepsilon$ $\mu = \mu(t)$ is a continuously differentiable function of t and that also $\mu_{k_0} = \mu_{k_0}\{\mu(t),t\}$ for these t-values.

According to this theorem we have to show that the partial derivative $\frac{\partial}{\partial\mu}\mu_{k_0}(\mu,t)$ does not vanish for $\mu = t = 1$. In order to prove this we study again the equation (5.12). Obviously this equation gives in a natural way a relation between the partial derivatives $\frac{\partial}{\partial\mu}\mu_\ell(\mu,t)$.

If $\mu = t = 1$, the partial derivatives $\frac{\partial\mu_\ell}{\partial\mu}$ satisfy for $\ell = \ell_0+2,\dots,k_0$:

$$b_\ell \frac{\partial\mu_{\ell+1}}{\partial\mu} + c_\ell \frac{\partial\mu_{\ell-1}}{\partial\mu} = \begin{cases} x^{-1} \dfrac{\partial\mu_\ell}{\partial\mu} & \ell \text{ even} \\ x \dfrac{\partial\mu_\ell}{\partial\mu} & \ell \text{ odd} \end{cases} . \tag{5.18}$$

For $\ell = \ell_0+2, \ell_0+3, \ldots k_0$, we define the real numbers σ_ℓ:

$$\sigma_\ell = \begin{cases} x \dfrac{\partial\mu_\ell}{\partial\mu} (\mu,t)\Big|_{\mu=t=1} & \ell \text{ odd} \\ \dfrac{\partial\mu_\ell}{\partial\mu} (\mu,t)\Big|_{\mu=t=1} & \ell \text{ even}; \quad \sigma_{\ell_0+1} = 1 \end{cases} . \tag{5.19}$$

From this definition, and from equation (3.18) the following inequalities for the σ_ℓ can easily be proved:

$$\sigma_{k_0} > \sigma_{k_0-1} > \ldots > \sigma_{\ell_0+2} > \sigma_{\ell_0-1} = 1 . \tag{5.20}$$

<u>Proof</u>

This statement can be proved by induction.

(i) $\ell = \ell_0+1$.

From equation (5.18)

$$b_{\ell_0+2} \frac{\partial\mu_{\ell_0+2}}{\partial\mu} = x^{-1} \frac{\partial\mu_{\ell_0+1}}{\partial\mu} = x^{-1} \frac{\partial\mu_{\ell_0+1}}{\partial\mu} .$$

According to the definition (5.19):

$$b_{\ell_0+1}\sigma_{\ell_0+2} = 1 \rightarrow \sigma_{\ell_0+2} = b_{\ell_0+1}^{-1} > 1 .$$

(ii) $\ell = \ell_0+2$.

From (5.18) we get:

$$b_{\ell_0+2} \frac{\partial\mu_{\ell_0+3}}{\partial\mu} + c_{\ell_0+2} \frac{\partial\mu_{\ell_0+1}}{\partial\mu} = x \frac{\partial\mu_{\ell_0+2}}{\partial\mu} .$$

According to the definition (5.19):

$$b_{\ell_0+2} \frac{\sigma_{\ell_0+3}}{\sigma_{\ell_0+2}} + c_{\ell_0+2} \frac{\sigma_{\ell_0+1}}{\sigma_{\ell_0+2}} = 1 .$$

Since $b_{\ell_0+2} + c_{\ell_0+2} = 1$,

and

$$\frac{\sigma_{\ell_0+1}}{\sigma_{\ell_0+2}} < 1 \ ,$$

we find $\dfrac{\sigma_{\ell_0+3}}{\sigma_{\ell_0+2}} > 1$.

(iii) By repeating this argument for $\ell_0+3 \le \ell \le k_0$ we get

$$b_\ell \frac{\sigma_{\ell+1}}{\sigma_\ell} + c_\ell \frac{\sigma_{\ell-1}}{\sigma_\ell} = 1 \ . \tag{5.21}$$

Suppose we proved $\dfrac{\sigma_\ell}{\sigma_{\ell-1}} > 1$, then consequently $\dfrac{\sigma_{\ell+1}}{\sigma_\ell} > 1$ and the inequality (5.20) is proved.

In particular for $\ell = k_0$ we have:

$$\frac{\partial \mu_{k_0}}{\partial \mu}(\mu,t)\Big|_{\mu=t=1} = x^{-1}\sigma_{k_0} > 0 \ . \tag{5.22}$$

Then from the implicit function theorem it follows immediately: there exist a real $\varepsilon>0$, such that $\mu = \mu(t)$ is a continuously differentiable function of t, if

$$1-\varepsilon < t < 1+\varepsilon \ , \tag{5.23}$$

and, moreover: $\mu_{k_0}(\mu(t),t) = 0$.

This last statement (5.23) shows, that a solution of (5.12) exists not only in t=1, but also in some open neighbourhood of t=1.

In order to show the existence of an ambiguity, defined by the representation under consideration we still have to verify, whether the constraints (5.16) from case <u>C</u> are satisfied. Due to the constraints from case <u>A</u> it only makes sense to consider t-values in the half open interval $(1-\varepsilon,1]$. It was mentioned that for t=1 both $c_1(t)$ and $c_2(t)$ equal 1. Therefore we have to determine whether these functions are less than 1 and greater than -1 for t-values in $(1-\varepsilon,1]$.

We shall prove this by showing that the $c_1(t)$ and $c_2(t)$ are increasing functions of t for t=1. Because the derivatives of these functions with respect to t are determined by the derivatives $\frac{d\mu_\ell}{dt}(t)$, we shall first study the derivatives $\frac{d\mu_\ell}{dt}(t)$ in t=1.

Again equation (5.12) gives in a natural way relations between the partial derivatives $\frac{\partial\mu_\ell}{\partial t}(\mu,t)$ for $\ell = \ell_0+2,\ldots,k_0$ and $\mu=t=1$:

$$b_\ell \frac{\partial\mu_{\ell+1}}{\partial t} + c_\ell \frac{\partial\mu_{\ell-1}}{\partial t} = \begin{cases} x\,\frac{\partial\mu_\ell}{\partial t} & \ell \text{ even} \\ x^{-1}\left(\frac{\partial\mu_\ell}{\partial t} + 1\right) & \ell \text{ odd} \end{cases} , \tag{5.24}$$

whereas

$$\frac{\partial\mu_{\ell_0+1}}{\partial t} = \frac{\partial\mu}{\partial t} = 0 .$$

For $\ell = \ell_0+1,\ \ \ell_0+2,\ldots,k_0$ we define the real numbers ρ_ℓ:

$$\rho_\ell = \begin{cases} x\,\frac{\partial\mu_\ell}{\partial t}(\mu,t)\Big|_{\mu=t=1} & \ell \text{ odd} , \\ \frac{\partial\mu_\ell}{\partial t}(\mu,t)\Big|_{\mu=t=1} & \ell \text{ even} , \end{cases} \tag{5.25}$$

$$\rho_{\ell_0+1} = 0 .$$

Then from (5.24) and (5.25) we obtain the following set of inequalities:

$$\rho_{k_0} > \rho_{k_0-1} > \ldots > \rho_{\ell_1+2} > 1; \quad \rho_{\ell_0+1} = 0 . \tag{5.26}$$

<u>Proof</u>

(i) $\ell = \ell_0+1$.

From (5.24)

$$b_{\ell_0+1} \frac{\partial\mu_{\ell_0+2}}{\partial t} = x^{-1}\left(\frac{\partial\mu_{\ell_0+1}}{\partial t} + 1\right) .$$

According to the definition (5.25):

$$b_{\ell_0+1}\rho_{\ell_0+2} = 1 \Rightarrow \rho_{\ell_0+2} = b_{\ell_0+1}^{-1} > 1 .$$

(ii) $\ell = \ell_0+2$.

From (5.24)

$$b_{\ell_0+2} \frac{\partial\mu_{\ell_0+3}}{\partial t} + c_{\ell_0+2} \frac{\partial\mu_{\ell_0+1}}{\partial t} = x\,\frac{\partial\mu_{\ell_0+2}}{\partial t}$$

and by using definition (5.25) we find

$$b_{\ell_0+2}\,\frac{\rho_{\ell_0+3}}{\rho_{\ell_0+2}} = 1\ .$$

Since $b_{\ell_0+2} < 1$, we have $\rho_{\ell_0+3}/\rho_{\ell_0+2} > 1$.

(iii) In general we can prove: for $\ell_0+3 \le \ell \le k_0$

$$b_\ell\,\frac{\rho_{\ell+1}}{\rho_\ell} + c_\ell\,\frac{\rho_{\ell-1}}{\rho_\ell} = 1 + \frac{1}{2\rho_\ell}\left(1 + (-1)^\ell\right)\ . \tag{5.27}$$

Consequently

$$b_\ell\,\frac{\rho_{\ell+1}}{\rho_\ell} + c_\ell\,\frac{\rho_{\ell-1}}{\rho_\ell} \ge 1\ .$$

Suppose it has been proved that $\dfrac{\rho_\ell}{\rho_{\ell-1}} > 1$. Since $b_\ell + c_\ell = 1$ and $\rho_{\ell-1}/\rho_\ell < 1$, we find $\dfrac{\rho_{\ell+1}}{\rho_\ell} > 1$, and the inequality (5.26) is proved.

Now we shall study the total derivatives $\dfrac{d\mu_\ell}{dt}$ for $\ell = \ell_0+1, \ell_0+2,\dots,k_0$. Of course we have

$$\frac{d\mu_\ell}{dt} = \left(\frac{\partial\mu_\ell}{\partial\mu}\right)\left(\frac{d\mu}{dt}\right) + \frac{\partial\mu_\ell}{\partial t}\ . \tag{5.28}$$

According to the definitions (5.19) and (5.25) for μ=t=1 these total derivatives can be expressed in terms of the numbers σ_ℓ and ρ_ℓ:

$$\frac{d\mu_\ell}{dt}(t)\Big|_{t=1} = \begin{cases} x^{-1}\left(\sigma_\ell\cdot\mu'(t)\big|_{t=1} + \rho_\ell\right), & \ell \text{ odd}\ ,\\[2ex] \left(\sigma_\ell\cdot\mu'(t)\big|_{t=1} + \rho_\ell\right), & \ell \text{ even}\ . \end{cases} \tag{5.29}$$

In particular for $\ell=k_0$ we must impose $\mu_{k_0} = 0$ for all t, because in our representation we have $\alpha_{k_0} = 0$. This in turn implies

$$\frac{d\mu_{k_0}}{dt}(t) = 0, \quad \text{for } 1-\varepsilon < t < 1+\varepsilon\ . \tag{5.30}$$

Therefore equations (5.29) for $\ell=k_0$ and (5.30) imply:

Proposition 1.

For $\ell = \ell_0+1,\ell_0+2,\dots,k_0$, the coefficients μ_ℓ are continuously differentiable functions of t for $1-\varepsilon < t < 1+\varepsilon$. Their derivatives with respect to t at t=1 are given by

$$\frac{d\mu_\ell}{dt}(t)\Big|_{t=1} = \begin{cases} -\dfrac{\rho_{k_0}}{\sigma_{k_0}} < 0 \ , & \text{for } \ell = \ell_0+1, \\ x^{-1}\left(\rho_\ell - \dfrac{\rho_{k_0}}{\sigma_{k_0}}\sigma_\ell\right), & \ell > \ell_0+1 \text{ and odd}, \\ \left(\rho_\ell - \dfrac{\rho_{k_0}}{\sigma_{k_0}}\sigma_\ell\right) \ , & \ell > \ell_0+1 \text{ and even.} \end{cases} \tag{5.31}$$

All these derivatives can be proved to be less than zero. In order to show this, we use the following lemma:

Lemma: The coefficients ρ_ℓ and σ_ℓ satisfy the following inequality:

$$\frac{\rho_\ell}{\rho_{\ell-1}} > \frac{\sigma_\ell}{\sigma_{\ell-1}} \ , \quad \ell = \ell_0+3, \ \ \ell_0+4,\ldots,k_0. \tag{5.32}$$

Proof: by induction.

It was shown that the ρ_ℓ and σ_ℓ for $\ell_0+3 \le \ell \le k$ are given by

$$b_\ell \frac{\rho_{\ell+1}}{\rho_\ell} + c_\ell \frac{\rho_{\ell-1}}{\rho_\ell} = 1 + \frac{1}{2\rho_\ell}\left(1 + (-1)^\ell\right) \ , \tag{5.27}$$

$$b_\ell \frac{\sigma_{\ell+1}}{\sigma_\ell} + c_\ell \frac{\sigma_{\ell-1}}{\sigma_\ell} = 1 \ . \tag{5.21}$$

Therefore we find:

$$b_\ell\left(\frac{\rho_{\ell+1}}{\rho_\ell} - \frac{\sigma_{\ell+1}}{\sigma_\ell}\right) + c_\ell\left(\frac{\rho_{\ell-1}}{\rho_\ell} - \frac{\sigma_{\ell-1}}{\sigma_\ell}\right) \ \ge \ 0 \ .$$

(i) $\ell = \ell_0+2.$

$$b_{\ell_0+2}\left(\frac{\rho_{\ell_0+3}}{\rho_{\ell_0+2}} - \frac{\sigma_{\ell_0+3}}{\sigma_{\ell_0+2}}\right) + c_{\ell_0+2}\left(\frac{\rho_{\ell_0+1}}{\rho_{\ell_0+2}} - \frac{\sigma_{\ell_0+1}}{\sigma_{\ell_0+2}}\right) \ \ge \ 0 \ ,$$

Since $\rho_{\ell_0+1} = 0$, $\sigma_{\ell_0+1} = 1$; $\sigma_{\ell_0+2} > 1$, we find

$$\frac{\rho_{\ell_0+1}}{\rho_{\ell_0+2}} - \frac{\sigma_{\ell_0+1}}{\sigma_{\ell_0+2}} < 0 \ .$$

And consequently $\dfrac{\rho_{\ell_0+3}}{\rho_{\ell_0+2}} - \dfrac{\sigma_{\ell_0+3}}{\sigma_{\ell_0+2}} > 0$.

(ii) For some fixed n, $\ell_0+3 < n \le k_0$ we have

$$b_n\left(\frac{\rho_{n+1}}{\rho_n} - \frac{\sigma_{n+1}}{\sigma_n}\right) + c_n\left(\frac{\rho_{n-1}}{\rho_n} - \frac{\sigma_{n-1}}{\sigma_n}\right) \geq 1 .$$

Suppose the lemma is proved for $\ell_0+3 \leq \ell \leq n$:

$$\frac{\rho_n}{\rho_{n-1}} - \frac{\sigma_n}{\sigma_{n-1}} > 0 \quad \text{then} \quad \frac{\rho_{n-1}}{\rho_n} - \frac{\sigma_{n-1}}{\sigma_n} < 0 ,$$

and consequently $\frac{\rho_{n+1}}{\rho_n} - \frac{\sigma_{n+1}}{\sigma_n} > 0$.

Thus the lemma is proved.

The following proposition is an obvious consequence of the last lemma and proposition 1.

Proposition 2.

For all $\ell = \ell_0+1, \ \ell_0+2,\dots,k_0$, the total derivatives $\frac{d\mu_\ell}{dt}(t)$ satisfy in t=1

$$\frac{d\mu_\ell}{dt}(1) < 0 . \tag{5.33}$$

Proof: From (5.31) one can see that it is sufficient to show that

$$\rho_\ell - \sigma_\ell \frac{\rho_{k_0}}{\sigma_{k_0}} < 0, \quad \text{for } \ell = \ell_0+1, \ \ell_0+2,\dots,k_0-1 .$$

For $\ell < k_0$, we have, according to the lemma

$$\frac{\sigma_{\ell-1}}{\sigma_\ell} > \frac{\rho_{\ell-1}}{\rho_\ell}, \quad \text{therefore:}$$

$$\frac{\sigma_\ell}{\sigma_{k_0}} = \frac{\sigma_\ell}{\sigma_{\ell+1}} \frac{\sigma_{\ell+1}}{\sigma_{\ell+2}} \cdots \frac{\sigma_{k_0-1}}{\sigma_{k_0}} > \frac{\rho_\ell}{\rho_{\ell+1}} \frac{\rho_{\ell+1}}{\rho_{\ell+2}} \cdots \frac{\rho_{k_0-1}}{\rho_{k_0}} = \frac{\rho_\ell}{\rho_{k_0}} .$$

From this we get $\rho_\ell - \rho_{k_0} \frac{\sigma_\ell}{\sigma_{k_0}} < 0$. □

Finally, we must consider the constraints (5.16) from case C.

$|c_1(t)| \leq 1$ and $|c_2(t)| \leq 1$.

It was noticed, that for t=1

$c_1(t) = c_2(t) = 1$.

From the expressions (5.14) and (5.15) for $c_1(t)$ and $c_2(t)$, it easily can be proved that:

$$\frac{d}{dt}\, c_1(t)\Big|_{t=1} = \frac{1}{c_{k_0}} - \frac{1}{b_{k_0}} \cdot \frac{d\mu_{k_0-1}}{dt}\,(1)\Big|_{t=1} \,,$$
$$\frac{d}{dt}\, c_2(t)\Big|_{t=1} = \frac{1}{b_{\ell_0}} - \frac{1}{c_{\ell_0}} \cdot \frac{d\mu}{dt}\,(t)\Big|_{t=1} \,. \qquad (5.34)$$

Because, according to the last proposition, both μ'_{k_0-1} and μ' are less than zero if $t=1$, we get from (5.34):

$$\frac{d}{dt}\, c_1(t)\Big|_{t=1} > 0 \text{ and } \frac{d}{dt}\, c_2(t)\Big|_{t=1} > 0 \,. \qquad (5.35)$$

Since $c_1(t)\Big|_{t=1} = c_2(t)\Big|_{t=1} = 1$,

we obviously have:

there exist a $\delta > 0$ such that, if $1-\delta \leq t \leq 1$: (5.36)

$|c_1(t)| \leq 1$ and $|c_2(t)| \leq 1$.

Summarizing we conclude:

(i) An $\varepsilon > 0$ exists, for which the representation under consideration gives an ambiguity, if $1-\varepsilon < t \leq 1$.
Moreover the two curves associated with this ambiguity, intersect the pair of curves defined by the staircase ambiguity at t=1.

(ii) In fact this can also be proved for different modifications of the staircase representation by using similar arguments. In this way $2^{\frac{1}{2}L-1}$ ambiguities, including the staircase ambiguity, are obtained.

(iii) The representation, defined by $\alpha_{L-1} = 0$, and α_ℓ complex for $\ell \leq L-2$, needs special attention.
Modulus invariance:

$|\zeta_\ell| = |\zeta'_\ell|$

and unitarity

$|\zeta_\ell| \doteq 1$

imply in this case

$\alpha_\ell = |1-z|\mu_\ell\, e^{i\phi}$, (see (5.8))

$b_\ell\mu_{\ell+1} + c_\ell\mu_{\ell-1} = x\mu_\ell - (-1)^\ell\,[(x^2-t)\mu_\ell^2 + 1]^{\frac{1}{2}}$, (see (5.12))

and

$$\cos\phi = \frac{|1-z|^2 - b_{L-1}^2 - c_{L-1}^2|\alpha_{L-2}|^2}{2b_{L-1}c_{L-1}|\alpha_{L-2}|} \quad . \tag{5.37}$$

One easily shows that, if $|z|^2 = t = 1$

$$|\cos\phi| \le 1 \; . \tag{5.38}$$

Since $\cos\phi$ is a continuous function of t, this inequality will also be satisfied in a neighbourhood of t=1.
There exist an $\varepsilon > 0$, such that

$$|\cos\phi| < 1, \quad \text{for } 1-\varepsilon < t < 1+\varepsilon \; . \tag{5.39}$$

Because the square root parts in (5.12) are well defined in a neighbourhood of t=1, this representation gives an ambiguity for a $1-\varepsilon < t < 1+\varepsilon$. In this case t=1 is not an endpoint of the ambiguity. Such an endpoint is determined in this case by the following types of conditions.

(a) $\cos\phi = \pm 1$, or (5.40)

$$t = x^2 = (\mathrm{Re}\; z)^2. \tag{5.41}$$

(b) $(x^2-t)\mu_\ell^2 + 1 = 0$, and

$$\frac{d}{dt}(x^2-t)\mu_\ell^2 \neq 0. \tag{5.42}$$

In the first case either the α_ℓ all become real (if $\cos\phi = \pm 1$), or z becomes real, (if $|z| = \mathrm{Re}\; z$). Then all the phase-shifts δ_ℓ and δ'_ℓ are equal (mod π) and both sets $\{\delta_\ell\}$ and $\{\delta'_\ell\}$ will form one closed curve. In all other representations, with $\alpha_\ell = 0$ for at least two ℓ-values, this situation does not occur. Therefore the sets $\{\delta_\ell\}$ and $\{\delta'_\ell\}$ do not form one closed curve.
However, if only $\alpha_{L-1} = 0$, it is not a priori clear, whether we get a closed curve. Since we have also the second type of condition (5.42), it is also possible that this constraint determines both end points of the ambiguity. A closed curve will only occur, if the inequality (5.40) or (5.41) is more restrictive than (5.42).

(iv) In a straightforward way one can generalize these results for L odd. Instead of the staircase representation we must modify then the representation, discussed in section 3 (B).

References

[1] F.A. Berends and S.N.M. Ruijsenaars, Nucl.Phys. B56, 507 (1973).

[2] A. Gersten, Nucl. Phys. B12, 537 (1969).

[3] J.H. Crichton, Nuov. Cim. A45, 256 (1966).

CHAPTER IV

PHASE-SHIFT AMBIGUITIES FOR SPIN-0 - SPIN-$\frac{1}{2}$ ELASTIC SCATTERING

1. Introduction

In this chapter we discuss the scattering of spinless particles and spin-$\frac{1}{2}$ particles at energies below the first inelastic threshold.

As was mentioned before, more experimental observables besides the differential cross section $\frac{d\sigma}{d\Omega}$ can be measured, when spin is involved. Such a quantity is for instance the recoil polarization P of the spin-$\frac{1}{2}$ particle.

As is well known, these experimental quanties can be specified by two complex valued functions $f(\theta)$ and $g(\theta)$, $0 \leq \theta \leq \pi$

$$k^2 \frac{d\sigma}{d\Omega}(\theta) = |f(\theta)|^2 + |g(\theta)|^2 = \tfrac{1}{2}\left(|f(\theta) + g(\theta)|^2 + |f(\theta) - g(\theta)|^2\right) \tag{1.1}$$

$$P(\theta) = \frac{2\,\mathrm{Re}\, f(\theta)^* g(\theta)}{|f(\theta)|^2 + |g(\theta)|^2} = \frac{|f(\theta)+g(\theta)|^2 - |f(\theta)-g(\theta)|^2}{|f(\theta)+g(\theta)|^2 + |f(\theta)-g(\theta)|^2} \tag{1.2}$$

In principle we have the following partial wave decomposition for $f(\theta)$ and $g(\theta)$:

$$\begin{aligned} f(\theta) &= \sum_{\ell=0}^{\infty} \left((\ell+1)f_{\ell+} + \ell f_{\ell-}\right)P_\ell(\cos\theta) \\ g(\theta) &= i \sum_{\ell=1}^{\infty} \left(f_{\ell+} - f_{\ell-}\right)\sin\theta \,\frac{dP_\ell(\cos\theta)}{d\cos\theta} \end{aligned} \tag{1.3}$$

where the $f_{\ell\pm}$ are the partial wave amplitudes corresponding with orbital angular momentum ℓ and total angular momentum $\ell \neq \frac{1}{2}$.

In practical phase-shift analysis, however, one approximates the functions $f(\theta)$ and $g(\theta)$ by

$$f(\theta) = \sum_{\ell=0}^{L} \left((\ell+1)f_{\ell+} - \ell f_{\ell-}\right)P_\ell(\cos\theta) \tag{1.4}$$

$$g(\theta) = i \sum_{\ell=1}^{L} (f_{\ell+} - f_{\ell-})\sin\theta \,\frac{dP_\ell(\cos\theta)}{d\cos\theta} \quad . \tag{1.5}$$

In this case of purely elastic scattering the partial wave amplitudes $f_{\ell\pm}$ are given by

$$f_{\ell\pm} = \frac{1}{2i}(\zeta_{\ell\pm}-1) \ , \quad \zeta_{\ell\pm} = e^{2i\delta_{\ell\pm}} \ . \tag{1.6}$$

where, according to the unitarity constraints, the parameters $\delta_{\ell\pm}$ are real.

Using the implications of unitarity our problem can be stated as follows: Is it possible to construct for arbitrary L different sets of phase-shifts $(\delta_0, \delta_{\pm}, \ldots, \delta_{\pm})$ and $(\delta'_0, \delta'_{\ell\pm}, \ldots, \delta'_{\ell\pm})$ giving the same, differential cross section and polarization?

It should be mentioned that, due to the optical theorem, Im f(0) is connected with the total cross section, according to:

$$\mathrm{Im}\ f(0) = \frac{k^2}{4\pi}\sigma_{tot} \ . \tag{1.7}$$

From (1.1) and (1.2) it follows that cross section and polarization do not change when $\delta_{\ell\pm}$ are changed into $\delta'_{\ell\pm}$, if and only if $|f(\theta) + g(\theta)|$ and $|f(\theta) - g(\theta)|$ do not transform.

Moreover $f(\theta) + g(\theta)$ and $f(\theta) - g(\theta)$ are simply connected by

$$f(\theta) - g(\theta) = f(-\theta) + g(-\theta) \tag{1.8}$$

Therefore, as was first observed by Gersten [1], knowledge of $\frac{d\sigma}{d\Omega}$ and P is equivalent to knowledge of $|f(\theta) + g(\theta)|$.

We follow Gersten by introducing the new variable

$$t = \tan \tfrac{1}{2}\theta \ . \tag{1.9}$$

Therefore:

$$\sin\theta = \frac{2t}{1+t^2} \quad \text{and} \quad \cos\theta = \frac{1-t^2}{1+t^2} \ . \tag{1.10}$$

In terms of this new variable t, the expression

$$G(t) = f(t) + g(t) \tag{1.11}$$

turns out to be a rational function of t:

$$G_L(t) = \frac{f(0)}{(1+t^2)^L} \prod_{k=1}^{2L} \left(\frac{t-z_k}{z_k}\right) \ . \tag{1.12}$$

From this expression we can easily find all the transformations that leave G(t) invariant. By means of these transformations we can in principle construct all different ambiguities for any L. We can indicate the three different classes of transformations:

(i) $S : \mathrm{Re}\ f(0) \to -\mathrm{Re}\ f(0)$. (1.13)

Recall that Im f(0) is determined by an experiment, since it is related to σ_{tot} (see (1.7)).

(ii) $T_k : z_k \to z_k^*$; (1.14)

(iii) A special case of (ii) occurs, when G(t) has pairs of roots +i, and -i. In this case the value of L changes. If one of these roots (either +i or -i) is conjugated

$$R_\pm : \pm i \to \mp i \ , \tag{1.15}$$

$G_L(t)$ changes into $G_{L'}(t)$, $L' = L+1$, according to

$$G_{L'}(t) = e^{\pm i\theta}\, G_L(t) \ . \tag{1.16}$$

In terms of the partial waves this transformation is given by the following relations.

In case $R = R_+$,

$$\begin{aligned} f'_{\ell+} &= \frac{1}{2\ell+1}\left(2\ell f_{(\ell-1)+} + f_{(\ell+1)-}\right) , \\ f'_{\ell-} &= \frac{1}{2\ell+1}\left(-f_{(\ell-1)+} + 2(\ell+1) f_{(\ell+1)-}\right) . \end{aligned} \tag{1.17}$$

As one can easily verify, this change of the partial waves correspond to successsive application of the Yang and the Minami transformations, both mentioned in section 4 of chapter II.

In case $R = R_-$,

$$\begin{aligned} f'_{\ell+} &= \frac{1}{2\ell+3}\left(2(\ell+2) f_{(\ell+1)+} - f_{(\ell+1)-}\right) , \\ f'_{\ell-} &= \frac{1}{2\ell-1}\left(f_{(\ell-1)+} + 2(\ell-1) f_{(\ell-1)-}\right) . \end{aligned} \tag{1.18}$$

It can be shown that this case is equivalent to successsive application of the Minami and the Yang transformation.

It should also be noted that repeated application of either R_+ or R_-, in case more pairs of roots ±i of G(t) occur, will increase the number of partial waves of the transformed amplitudes f'(t) and g'(t).

Of course also combinations of the three different types of transformation are allowed.

Since differential cross section and polarization in terms of G(t) are given by

$$k^2 \frac{d\sigma}{d\Omega} = \tfrac{1}{2}\left(|G(t)|^2 + |G(-t)|^2\right) \tag{1.19}$$

and

$$k^2 \cdot \frac{d\sigma}{d\Omega} \cdot P = \frac{1}{2}\left(|G(t)|^2 - |G(-t)|^2\right) , \tag{1.20}$$

it is obvious that both observables remain unchanged, when one of the transformations (i), (ii), (iii) or a combination of them is performed.

Such a combination is for instance $S \prod_{k=1}^{2L} T_k$. Under this transformation the amplitudes $f(\theta)$ and $g(\theta)$ change according to

$$f'(\theta) = -f(\theta)^* \text{ and } g'(\theta) = -g(\theta)^* , \tag{1.21}$$

If we combine this transformation with a class (iii) type transformation

$$G_{L'}(t) = e^{-i\theta} G_L(t) \quad \text{(see (1.16))},$$

we get the modified Minami-ambiguity (section 4, chapter II). This ambiguity is defined by the following transformation property of the amplitudes:

$$f(\theta) = -\cos\theta \cdot f(\theta)^* + i \sin\theta \cdot g(\theta)^* ,$$

$$g'(\theta) = -\cos\theta \cdot g(\theta)^+ + i \sin\theta \cdot f(\theta)^* . \tag{1.22}$$

In terms of the partial waves this gives

$$f'_{\ell\pm} = -f^*_{(\ell\pm1)\mp} . \tag{1.23}$$

In the remaining part of this chapter we exclude the special case of pairs of roots $\pm i$ of $G(t)$. Therefore we shall only consider the transformations of class (i) and (ii).

In principle all different sets of partial waves $\zeta_{\ell\pm}$ and $\zeta'_{\ell\pm}$ can be constructed. First one has to establish the connection between the quantities $\zeta_{\ell\pm}$, the roots z_k and the forward amplitude $f(0)$.

A different set of partial waves $\zeta'_{\ell\pm}$ can then be found by performing one of the transformations of class (i) and (ii) or by a combination of them.

Both sets will give the same differential cross section and polarization. Due to elastic unitarity, the quantities $\zeta_{\ell\pm}$ and $\zeta'_{\ell\pm}$ must also obey:

$$|\zeta_{\ell\pm}| = |\zeta'_{\ell\pm}| = 1 . \tag{1.24}$$

In the next section we shall show that ambiguities can be constructed according to this program. In particular we shall construct ambiguities that arise from the transformation $T = \prod_{k=1}^{2L} T_k$.

2. Existence and construction of ambiguities

In the previous section we have seen how in principle different amplitudes $f(\theta)$ and $g(\theta)$ can be constructed, both giving the same differential cross section and polarization. It was also stressed there that we shall restrict ourselves to transformations that do not change the number of partial waves.

In this section we discuss one particular transformation

$$T = \prod_{k=1}^{2L} T_k \quad . \tag{2.1}$$

This transformation changes $f(\theta)$ and $g(\theta)$ into $f'(\theta)$ and $g'(\theta)$ such that

$$|f(\theta) + g(\theta)| = |f'(\theta) + g'(\theta)| \quad . \tag{2.2}$$

We want also to express the partial wave amplitudes $f_{\ell\pm}$ in terms of quantities, related to the roots $z_1, z_2, \ldots, z_{2L}$ of $f(\theta) + g(\theta)$, in order to be able to write down the transformed amplitudes explicitly.

It is easy to see that the set of amplitudes $(f(\theta), g(\theta))$ transforms under T into a different set $(f'(\theta), g'(\theta))$ according to

$$f'(\theta) = \frac{f(0)}{f(0)^*} f(\theta)^* \; , \quad g'(\theta) = \frac{f(0)}{f(0)^*} g(\theta)^* \; . \tag{2.3}$$

Defining new coefficients

$$\begin{aligned} A &= f(0) \; , \\ a_\ell &= \frac{1}{A}\left((\ell+1) f_{\ell+} + \ell f_{\ell-}\right) \quad \text{for } \ell \geq 0 \; , \\ b_\ell &= \frac{i}{A}\left(f_{\ell+} - f_{\ell-}\right) \qquad \text{for } \ell \geq 1 \; , \end{aligned} \tag{2.4}$$

we get much simpler expressions for the amplitudes:

$$\begin{aligned} f(\theta) &= A \sum_{\ell=0}^{L} a_\ell P_\ell(\cos\theta) \; , \\ g(\theta) &= A \sum_{\ell=1}^{L} b_\ell \sin\theta \frac{dP_\ell(\cos\theta)}{d\cos\theta} \; . \end{aligned} \tag{2.5}$$

From (2.3) we see that the coefficients a_ℓ and b_ℓ transform under T according to

$$a'_\ell = a^*_\ell \; , \quad b'_\ell = b^*_\ell \; . \tag{2.6}$$

From the definitions (1.6) and (2.4) we can express the coefficients $\zeta_{\ell\pm}$ in terms of a_ℓ and b_ℓ:

$$\zeta_{\ell+} = 1 + \frac{2A}{2\ell+1}\,(ia_\ell + \ell b_\ell)\ , \quad 0 \le \ell \le L\ ,$$

and (2.7)

$$\zeta_{\ell-} = 1 + \frac{2A}{2\ell+1}\,(ia_\ell - (\ell+1)b_\ell)\ ,\ 1 \le \ell < L\ .$$

as can be seen from (2.5) by putting θ=0 the coefficients a satisfy

$$\sum_{\ell=0}^{L} a_\ell = 1\ . \tag{2.8}$$

Our problem can now be stated as follows:
The coefficients $\zeta_{\ell\pm}$ are given by (2.7). The substitution (2.6) transform $\zeta_{\ell\pm}$ into $\zeta'_{\ell\pm} = \zeta_{\ell\pm}(a^*_\ell, b^*_\ell)$. Both sets give still the same differential cross section and polarization. Then we have to find coefficients a_ℓ and b_ℓ such that for all ℓ:

$$|\zeta_{\ell\pm}(a_\ell,b_\ell)| = |\zeta_{\ell\pm}(a^*_\ell,b^*_\ell)|,$$

and (2.9)

$$|\zeta_{\ell\pm}(a_\ell,b_\ell)| = 1\ .$$

Using the expressions (2.7) for the coefficients $\zeta_{\ell\pm}$, one easily derives from the unitarity constraints (2.9) a set of conditions in terms of the quantities A, a_ℓ and b_ℓ. Defining

$$A = X + iY\ , \tag{2.10}$$

we get for each ℓ:

$$X\,\mathrm{Im}\ a_\ell = 0\ , \tag{2.11}$$

$$\mathrm{Im}\,b_\ell\left(|A|^2\ \mathrm{Re}\ a_\ell - \tfrac{1}{2}(2\ell+1)Y\right) - |A|^2\ \mathrm{Im}\,a_\ell\,\mathrm{Re}\ b_\ell = 0\ , \tag{2.12}$$

$$|A|^2|b_\ell|^2 - (2\ell+1)X\,\mathrm{Re}\,b_\ell = 0\ , \tag{2.13}$$

$$|A|^2\left(|a_\ell|^2 + \ell(\ell+1)|b_\ell|^2\right) - (2\ell+1)Y\,\mathrm{Re}\,a_\ell = 0\ . \tag{2.14}$$

The possible ambiguities can be classified as follows:

<u>A</u>. X=0

From (2.12), (2.13) and (2.14) we get for each ℓ :

$$b_\ell = 0\ ,$$

$$a_\ell = |a_\ell|e^{i\psi_\ell}\ , \quad \text{with} \tag{2.15}$$

$$\cos\ \psi_\ell = \frac{Y}{2\ell+1}\ |a_\ell|\ .$$

Of course $|\cos\psi_\ell| \le 1$ implies

$$\frac{Y}{2\ell+1} |a_\ell| \leq 1 \ . \tag{2.16}$$

Since $b_\ell = 0$ for each ℓ, we get, according to (2.7):

$$\zeta_{\ell+} = \zeta_{\ell-} \ , \tag{2.17}$$

which in turn implies that the polarization P vanishes. Moreover changing a_ℓ into a_ℓ^* corresponds in this case to the trivial ambiguity, where $\delta_\ell' = -\delta_\ell'$. Therefore this possibility is not of any interest for us and it will be omitted in the rest of this section.

The remaining possible classes of ambiguities are then

B. $X \neq 0$, a_ℓ real

This condition gives four different classes of ambiguities, which result from the remaining set of equations (2.12), (2.13) and (2.14). In all these cases a_ℓ and b_ℓ will be parametrized in terms of X and $s = Y/X$.

Case 1: b_ℓ complex

Defining

$$b_\ell = |b_\ell| e^{i\phi_\ell} \ ,$$

we get

$$|b_\ell| = \frac{(2\ell+1)s}{2\sqrt{\ell(\ell+1)}X(1+s^2)} \ ,$$

$$\cos\phi_\ell = \frac{s}{2\sqrt{\ell(\ell+1)}} \ , \tag{2.18}$$

$$a_\ell = \tfrac{1}{2}(2\ell+1)\frac{s}{X(1+s^2)} \ .$$

Obviously the condition $|\cos\phi_\ell| \leq 1$ implies

$$|s| \leq 2\sqrt{\ell(\ell+1)} \ . \tag{2.19}$$

Case 2: b_ℓ real

In this case we find:

$$b_\ell = \frac{2\ell+1}{X(1+s^2)} \ ,$$

$$a_\ell = \tfrac{1}{2}(2\ell+1)\frac{s}{X(1+s^2)}\left(1 \pm \left[1 - 4\frac{\ell(\ell+1)}{s^2}\right]^{\frac{1}{2}}\right) \ . \tag{2.20}$$

Here a_ℓ will be well defined, if:

$$|s| > 2\sqrt{\ell(\ell+1)} \ . \tag{2.21}$$

This case consists of two subclasses according to the ± sign in a_ℓ. Finally we have two subclasses according to the choice $b_\ell=0$.

Case 3: $b_\ell=0$

$$a_\ell = (2\ell+1)\,\frac{s}{X(1+s^2)} \tag{2.22}$$

Case 4: $b_\ell = a_\ell = 0$. (2.23)

We see that only in the first two cases we have restrictions on the possible values s can take. Moreover we observe that for any ℓ-values the two cases 1 and 2 become equivalent, if $s = 2\sqrt{\ell(\ell+1)}$. At this value of s all the parameters a_ℓ, b_ℓ and ϕ_ℓ are the same, in particular $\phi_\ell=0$.

An ambiguity can be obtained by choosing one of the four possibilities, successively for each $\ell=0,1,\ldots,L$. We shall specify any ambiguity by a row of L+1 numbers $(n_0,n_1,\ldots,n_L)$. For each ℓ n_ℓ = either 1, ±2, 3 or 4, according to the choice of the possibility one has made in the ℓ-th step. Here we denote the two subclasses of case 2 by the ± sign.

Not every row, however, will give an ambiguity. Several restrictions determine all the possible allowed rows. We list these restrictions below.

(i) In order to ensure that the transformation T is not the identity, at least one of the coefficients a_ℓ or b_ℓ has to be complex. Therfore:

$n_\ell = 1$ for at least one ℓ-value . (2.24)

(ii) Because for $\ell=0$ we have $b_0 = 0$, we have

n_0 equals either 3 or 4. (2.25)

(iii) For different ℓ and k we have the possibility $n_\ell = 1$; $n_k = \pm 2$. In both cases s has to satisfy different constraints (see (2.19) and (2.21)), which have to be compatible with each other. Therefore:

If for some ℓ,k ($\ell \neq k$) $n_\ell = 1$ and $n_k = \pm 2$, then $k<\ell$. (2.26)

Keeping these restrictions in mind all the different ambiguities, due to conjugation T of all the roots, can be obtained. Thus we can list the possible forms which $\zeta_{\ell\pm}$ can take.

Case 1:
$$\zeta_{\ell+} = \frac{1}{1-is}\left(1 + \frac{s^2}{2(\ell+1)} \pm i\,\frac{s}{2(\ell+1)}\left(4\ell(\ell+1) - s^2\right)^{\frac{1}{2}}\right) ,$$
$$\zeta_{\ell-} = \frac{1}{1-is}\left(1 - \frac{s^2}{2\ell} \mp i\,\frac{s}{2\ell}\left(4\ell(\ell+1) - s^2\right)^{\frac{1}{2}}\right) . \qquad (2.27)$$

Here the change $b_\ell \to b_\ell^*$ is expressed by the ± sign; the upper sign corresponds to $\zeta_{\ell\pm}$, whereas the lower sign corresponds to $\zeta'_{\ell\pm}$.

Case 2:
$$\zeta_{\ell+} = \zeta'_{\ell+} = \frac{1}{1-is}\left(2\ell+1 \pm is\left(1 - \frac{4\ell(\ell+1)}{s^2}\right)^{\frac{1}{2}}\right) ,$$
$$\zeta_{\ell-} = \zeta'_{\ell-} = \frac{1}{1-is}\left(-(2\ell+1) \pm is\left(1 - \frac{4\ell(\ell+1)}{s^2}\right)^{\frac{1}{2}}\right) . \qquad (2.28)$$

Here the ± sign corresponds to the two subclasses $n_\ell = \pm 2$.

Again it should be noticed here, that for any ℓ case 1 and 2 are equivalent if $s = 2\sqrt{\ell(\ell+1)}$. In particular we observe that in case 1 we have also $\zeta_{\ell\pm} = \zeta'_{\ell\pm}$ for this s-value.

Case 3:
$$\zeta_{\ell\pm} = \zeta'_{\ell\pm} = \frac{1+is}{1-is} \quad . \qquad (2.29)$$
$$\zeta_{\ell\pm} = \zeta'_{\ell\pm} = 1 \ . \qquad (2.30)$$

Notice that all these expressions depend on one single parameter s only. The remaining equation (2.8) is automatically satisfied, as one can easily show. By using this equation one is able to determine X as a function of s. In the following s is assumed to be positive. Exactly the same results can be obtained for negative s-values. Obviously Y is positive, since it determines the total cross section. Therefore positivity of s implies positivity of X.

Thus all possible ambiguities due to transformation T have been constructed for arbitrary L. It is clear that the number of different ambiguities becomes larger with increasing L. In case L=1, we find two different representations, according to the rules (2.24), (2.25) and (2.26). For $\ell=0$, n_0 equals either 3 or 4 and for $\ell=1$, the only possibility is $n_1=1$. Therefore the two possible representations for the L=1 case can be represented by the rows (3,1) and (4,1). These examples were already constructed by Berends and Ruijsenaars [2].

As was observed before this coefficients $\zeta_{\ell\pm}$ and the transformed ones $\zeta'_{\ell\pm}$ are analytical expressions in terms of only one parameter s. Consequently the two sets of phase-shifts $(\delta_0, \delta_{1\pm}, \ldots, \delta_{L\pm})$ and $(\delta'_0, \delta'_{1\pm}, \ldots, \delta_{L\pm})$ form both a curve in a 2L+1-dimensional real space. However, in contrast to the examples discussed in [2], these pairs of curves do not necessarily form one closed curve (mod π).

In the next section the endpoints of the different representations will be determined. There we shall show that some of these curves are not closed (mod π) in these endpoints. Moreover it will be shown that all the ambiguities, constructed here, are connected with each other. In fact each ambiguity appears to belong to some chain of connected ambiguities. Such a chain of ambiguities again forms a curve which is closed (mod π).

3. Chains of ambiguities

In the previous section it was shown that all the ambiguities, due to the transformation T, conjugation of all the roots of G(t), can be constructed. For all ℓ the different phase-shifts $\delta_{\ell\pm}$ and $\delta'_{\ell\pm}$ are functions of only one real-parameter s. We noticed also in the last section that the two sets of phase-shifts did not necessarily become equal (mod π) at the endpoints of the corresponding parameter interval.

In this section we shall study this pecularity in more detail. Some rules are given, in order to determine the endpoints of the various parameter intervals, for which the different ambiguities are defined. These points will be called "endpoints of an ambiguity". It will be discussed too, whether the different sets of phase-shifts $\{\delta_{\ell\pm}\}$ and $\{\delta'_{\ell\pm}\}$ will become equal in these endpoints. As we shall see, the following pattern will be observed. There exists an ambiguity of which the different sets of phase-shifts are equal at its first endpoint s=0. If at its second endpoint s_1 this is not the case, a second ambiguity can be found that is continuously connected to the first. The different sets of phase-shifts of this second ambiguity may differ in its second endpoint s_2. In this case a third ambiguity appears to be connected to the second one, and so on.

Continuing in this way one finally will obtain an ambiguity where the sets $\{\delta_{\ell\pm}\}$ and $\{\delta'_{\ell\pm}\}$ become equal at its final endpoint s_N, as is illustrated by fig. 2.

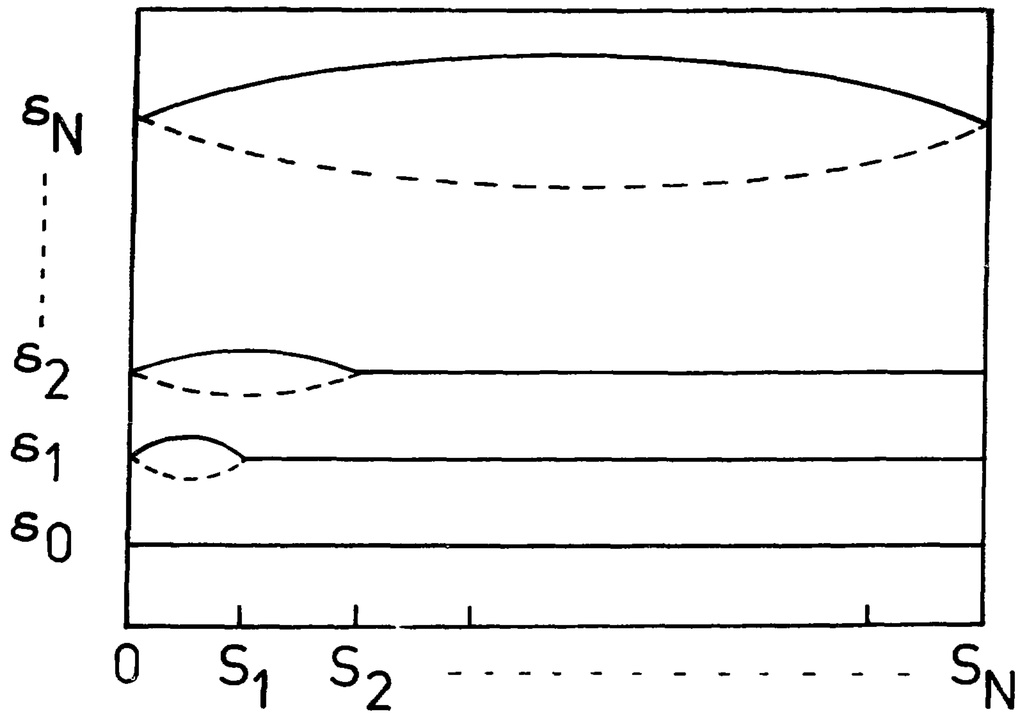


Fig. 2. The general pattern of a chain of ambiguities in terms of δ_ℓ (solid line) and δ'_ℓ (dashed line).

Thus a so-called chain of ambiguities is obtained. All these ambiguities are successively connected in their endpoints in a continuous way. To this chain correspond two different sets $\{\delta_{\ell\pm}\}$ and $\{\delta'_{\ell\pm}\}$ of phase-shifts which become equal at the first endpoint s=0 of the first ambiguity and at the second endpoint $s=s_N$ of the last ambiguity.

We first list the different rules, by which the endpoints of the various ambiguities can be determined. As was noticed before an ambiguity can be specified by some n-tuple $(n_0,n_1,\ldots,n_L)$.. It was also shown that only a choice $n_\ell=1$ or ± 2 gives rise to constraints on the allowed parameter values.

(a1) If for some ℓ $n_\ell=1$, we have according to (2.19) an upper bound $s \leq s_{max} = 2\sqrt{\ell(\ell+1)}$.
If an ambiguity is characterized by $n_\ell=1$ for more than one ℓ-value, this ambiguity has an upper endpoint

$$s = 2\sqrt{\ell_1(\ell_1+1)} , \qquad (3.1)$$

where ℓ_1 is the lowest ℓ-value, for which $n_\ell=1$ holds.

(a2) If for some ℓ-value $n_\ell=\pm 2$, we get according to (3.1) a lower bound for the allowed t-values e.g. $s \geq s_{min} = 2\sqrt{\ell(\ell+1)}$. If an ambiguity is specified by a choice $n_\ell=\pm 2$ for different ℓ-values, then this ambiguity

ambiguity has a lower endpoint

$$s = 2\sqrt{\ell_2(\ell_2+1)}\ , \tag{3.2}$$

where ℓ_2 is the highest ℓ-value, for which we have $n_\ell=\pm2$.

(a3) If no $n_\ell=\pm2$ case occurs in an ambiguity it has a lower endpoint

$$s=0. \tag{3.3}$$

For instance the possible ambiguities, starting with $n_0=3$, are in the case L=2:

(3,1,1), which is defined for $0 \leq s \leq 2\sqrt{2}$,
(3, 2,1), which is defined for $2\sqrt{2} \leq s \leq 2\sqrt{6}$,
(3,3,1), which is defined for $0 \leq s \leq 2\sqrt{6}$.

As a next step we investigate whether different sets of phase-shifts become equal in the endpoints discussed above.

(b1) Suppose for some ℓ $n_\ell=1$. As we noticed before we have $\zeta_{\ell\neq} = \zeta'_{\ell\pm}$, only if the parameter value $s = 2\sqrt{\ell(\ell+1)}$.
Therefore, if an ambiguity has $n_\ell=1$ for only one ℓ-value, both sets of partial waves $(\zeta_0,\zeta_{1\pm},\ldots,\zeta_{L\pm})$ and $(\zeta'_0,\zeta'_{1\pm},\ldots,\ '_{L\pm})$ become equal at the upper endpoint $s=2\sqrt{\ell(\ell+1)}$ of the parameter interval.
An ambiguity which has $n_\ell=2$ for different ℓ-values will not have this property anymore.

(b2) If an ambiguity contains no $n_\ell=2$ type partial waves then both sets $(\zeta_0,\zeta_{1\pm},\ldots,\zeta_{\ell\pm})$ and $(\zeta'_0,\zeta'_{1\pm},\ldots,\zeta'_{L\pm})$ are equal only at the lower endpoint s=0.

We illustrate these two rules for the L=2 case.

- When we consider, for instance the representation (3,1,1), we have at the lower endpoint of this ambiguity s=0 $\zeta_{\ell\pm}=\zeta'_{\ell\pm}$ for all ℓ. At the upper endpoint $s=2\sqrt{2}$, however, we have $\zeta_{2\pm}\neq\zeta'_{2\pm}$.
- The representation (3,±2,1) has two endpoints $s = 2\sqrt{2}$ and $s = 2\sqrt{6}$. Here $\zeta_{\ell\pm} = \zeta'_{\ell\pm}$, only if $s=2\sqrt{6}$. In its lower endpoint $s=2\sqrt{2}$ $\zeta_{1\pm}$ and $\zeta'_{1\pm}$ are different.
- The representation (3,3,1) shows the property $\zeta_{\ell\pm} = \zeta'_{\ell\pm}$ in both its endpoints s=0 and $s=2\sqrt{6}$.

As we showed, the endpoints of all different ambiguities can be determined in this way. Moreover, we can decide whether or not the two sets $\{\zeta_{\ell\pm}\}$ and $\{\zeta'_{\ell\pm}\}$ become equal in these endpoints.

We shall discuss now the way in which the various ambiguities can be connected with each other. Consider for some L a representation in which ℓ_1 is the smallest ℓ-value for which $n_\ell=1$ occurs. Then the upper endpoint of the corresponding ambiguity is $s=2\sqrt{\ell_1(\ell_1+1)}$, according to rule (a1). At this s-value we have also $\delta_{\ell_1\pm}=\delta'_{\ell_1\pm}$. It was noted before, that for $s=2\sqrt{\ell_1(\ell_1+1)}$ the partial waves corresponding with $n_{\ell_1}=1$ and $n_{\ell_1}=\pm2$ are equal. Consequently two ambiguities which only differ in the choice of n_{ℓ_1} – one ambiguity with $n_{\ell_1}=1$ and the other with $n_{\ell_1}=\pm2$ – will be equal in $s=2\sqrt{\ell_1(\ell_1+1)}$. Thus two ambiguities are continuously connected with each other in such an endpoint.

In this way a chain of ambiguities can be obtained. We shall illustrate this idea with an example.

Consider for instance the representation given by the L±1 row (3,1,1,...,1). To this representation corresponds an ambiguity which is defined for $0 \leq s \leq 2\sqrt{2}$. According to the rules (a3) and (b2) we have $\delta_{\ell\pm}=\delta'_{\ell\pm} \pmod{\pi}$ at s=0. In $s=2\sqrt{2}$ we have $\delta_0=\delta'_0$, $\delta_{1\pm}=\delta'_{1\pm}$, whereas for all other ℓ-values $\delta_{\ell\pm}\neq\delta'_{\ell\pm}$.

In the same point, $s=2\sqrt{2}$, this ambiguity is connected with two other ambiguities, given by the L+1-tuple (3,±2,1,...,1). Both ambiguities are defined for $2\sqrt{2} \leq s \leq 2\sqrt{6}$. For all s-values these ambiguities have the property $\delta_0=\delta'_0$, $\delta_{1\pm}=\delta'_{1\pm}$. In its upper endpoint $s=2\sqrt{6}$ we have also $\delta_{2\pm}=\delta'_{2\pm}$, but all the other phase-shifts $\delta_{\ell\pm}$ and $\delta'_{\ell\pm}$ are different, for $\ell\geq3$. Let us consider one of these two ambiguities, say the one with $n_1=+2$. Then the first ambiguity given by (3,1,1,...,1) is at $s=2\sqrt{2}$ continuously connected with a second ambiguity (3,+2,1,...,1). Of course all partial waves but the first for ℓ=0,1, maintain the same representation, whereas $\delta_{1\pm}\rightarrow\delta^+_{1\pm}$ and $\delta'_{1\pm}\rightarrow\delta^+_{1\pm}$. By using the same arguments, it can be shown that at $s=2\sqrt{6}$ the second representation (3,+2,1,1,...,1) is connected with a third one: (3,+2,-2,1,...,1) or (3,+2,+2,1,...,1). It should be stressed here that these transitions from one ambiguity to another are continuous, but not necessarily differentiable. Only transitions $\delta_{\ell\pm}\rightarrow\delta^+_{\ell\pm}$ and $\delta'_{\ell\pm}\rightarrow\delta^-_{\ell\pm}$ are differentiable.

Continuing in this way the starting representation (3,1,1,...,1) gives rise to 2^{L-1} chains, each chain consisting of L different ambiguities which are continuously connected. Such a chain is for instance:

(3,1,1,...,1); (3,+2,1,...,1); (3,+2,-2,1,...,1); ...;
(3,+2,-2,-2,...,+2,1).

Fig. 2 shows the general structure of the way in which L different ambiguities form a chain.

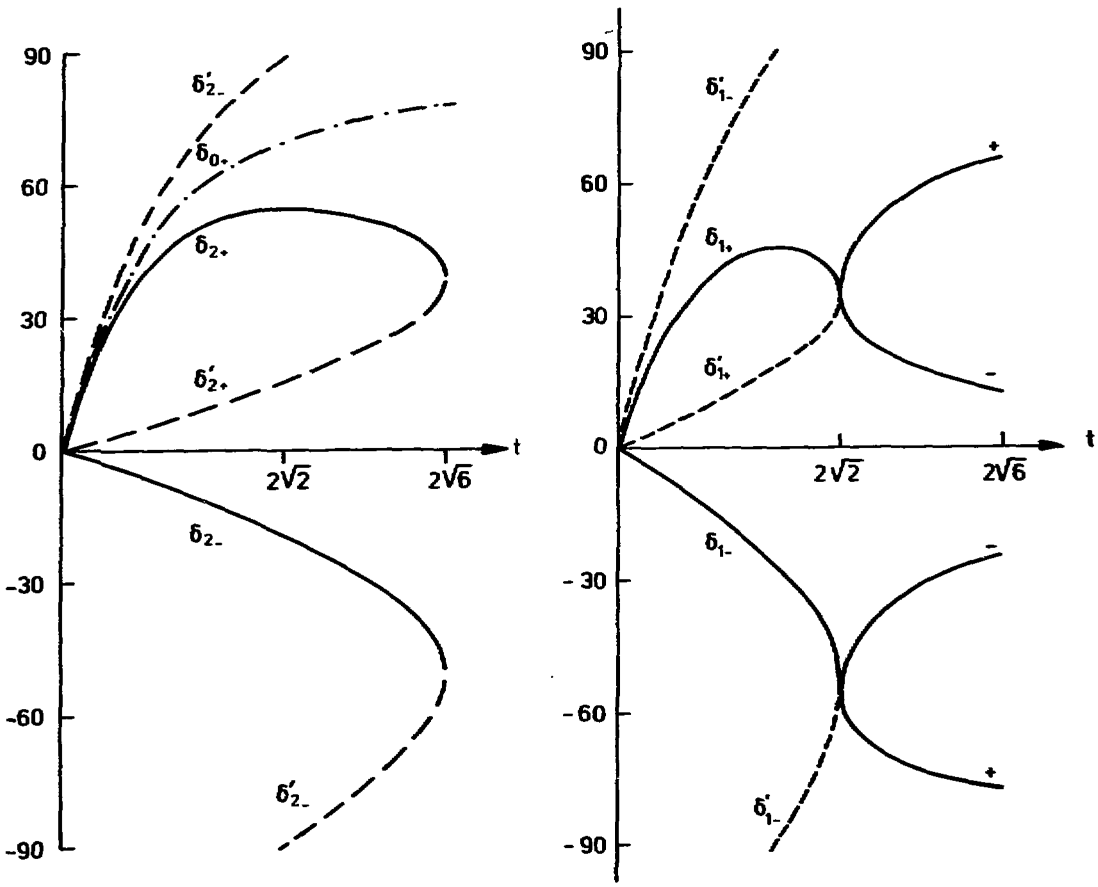


Fig. 3A and B. The phase-shift $\delta_{0+} = \delta'_{0+}$ (dashed-dotted line), $\delta_{1\pm}$ (solid line), $\delta'_{1\pm}$ (when $\neq \delta_{1\pm}$ dashed line), $\delta_{2\pm}$ (solid line), $\delta'_{2\pm}$ (dashed line) of a chain consisting of two ambiguities. The $\delta_{1\pm}$ are marked by a $\pm$ sign according to the choice $n_1 = \pm 2$.

s	0 – $2\sqrt{2}$	$2\sqrt{2}$ – $2\sqrt{6}$	$2\sqrt{6}$ – $2\sqrt{12}$	$2\sqrt{12}$ – $2\sqrt{L(L-1)}$	$2\sqrt{L(L-1)}$ – $2\sqrt{L(L+1)}$
n_0	3 or 4	3 or 4	3 or 4	- - - - -	3 or 4
n_1	1	±2	±2		±2
n_2	1	1	±2		±2
n_3	1	1	1		±2
⋮	⋮	⋮	⋮		⋮
n_{L-1}	1	1	1		±2
n_L	1	1	1		1

Table 1. The different possible choices for n_ℓ are listed for the allowed range of s-values.

Starting at s=0 the coefficients $b_1, b_2, \ldots, b_{L-1}$ and b_L successively will become zero for increasing s. Each time when a b_ℓ equals zero (at $s = 2\sqrt{\ell(\ell+1)}$) we have to choose between either $n_\ell=+2$ or $n_\ell=-2$. Therefore the starting representation (3,1,1,...,1) is the first ambiguity of 2^{L-1} chains.

If $n_\ell=1$, b_ℓ is complex and we have two phase-shifts δ_ℓ and δ'_ℓ which are represented by a solid line (δ_ℓ) and a dashed line (δ'_ℓ). At the boundary points $s = 2\sqrt{\ell(\ell+1)}$ b_ℓ is zero, and consequently $\delta_\ell=\delta'_\ell$. In this point we notice a continuous transition from the case $n_\ell=2$ into either $n_\ell=+2$ or $n_\ell=-2$. The other representations do not change at this point. It should be stressed that we always have $\delta_0=\delta'_0$ (according to the choices $n_0=3,4$), whereas for the highest phase-shift we get $\delta_\ell \neq \delta'_\ell$.

In table 1 the different possible choices for n_ℓ are listed for all possible chains.

Figs. 3A and 3B show two chains of two ambiguities (3,1,1) and (3, 2,1). These ambiguities are represented by the phase-shifts $\delta_1=\delta'_0$ (dashed dotted), $\delta_{1\pm}$ (solid) and $\delta'_{1\pm}$ (dashed if $\delta'_{1\pm} \neq \delta_{1\pm}$). The solid lines $\delta_{1\pm}$ are marked with a ± sign, according to the choice $n_1=\pm2$.

4. Discussion and conclusions

In this chapter we discussed techniques by which it is possible to construct, at least in principle, all different sets of phase-shift ambiguities in the case of elastic spin-0 - spin-$\frac{1}{2}$ scattering.

As was pointed out by Gersten, knowledge of the differential cross seciton and polarization is equivalent with knowledge of some rational function G(t). It was also shown that a special class of ambiguities could be obtained by conjugating one or more roots of G(t).

Here we restricted ourselves to the particular class of ambiguities due to the conjugation of all the roots of G(t). This transformation changed the partial wave amplitudes $f_{\ell\pm}$ into new amplitudes $f'_{\ell\pm}$. These coefficients have also to obey the constraints due to unitarity. By imposing these constraints the amplitudes were solved in terms of one single parameter s.

Thus all the ambiguities due to complex conjugation of all the roots of G(t) have been constructed. In a similar way one can, in principle, find other ambiguities, corresponding to conjugation of an arbitrary number of roots of G(t).

The class of ambiguities constructed here has some remarkable properties. First and in contrast to the examples found by Berends and Ruijsenaars, we noticed that the phase-shifts $\delta_{\ell\pm}$ and $\delta'_{\ell\pm}$ are not equal (mod π) at the end-points of the corresponding ambiguity. However, we showed that it is possible to indicate a number of L ambiguities, which are continuously connected with each other at their endpoints. In this way this set of ambiguities together do form a chain. The most interesting point is probably that at the remaining endpoints of the first and the last ambiguity the phase-shifts $\delta_{\ell\pm}$ and $\delta'_{\ell\pm}$ are equal (mod π).

Of course, when one only considers S- and P-waves (L=1) one will find chains consisting of one ambiguity : in this case the concepts of ambiguity and chain are clearly equivalent.

Secondly, we noticed that the number of different ambiguities and the number of chains increases for higher L-values, even if one considers only conjugation of all the L+1 roots of G(t). Obviously the number of transformations, leaving the modulus of G(t) invariant, will also increase for higher L-values. Therefore the examples that occur in an actual phase-shift analysis become probably more numerous.

Finally we mention an important difference between ambiguities in the spinless case and in spin-0 - spin-$\frac{1}{2}$ scattering. In case of spinless elastic scattering it was found that the phase-shifts δ_ℓ are uniquely determined, if for all ℓ, $\delta_\ell \leq \frac{\pi}{6}$. In spin-0 - spin-$\frac{1}{2}$ elastic scattering we don't have such a region of uniqueness. There we solved the partial wave amplitudes $f_{\ell\pm}$ in terms of one real parameter s and for same L-value ambiguities were constructed for $s^2 \leq 4L(L+1)$.

However, for $s \to 0$ all phase-shifts $\delta_{\ell\pm}$ become zero and consequently differential cross-section and polarization vanish. Therefore we still can have phase-shift ambiguities for arbitrary small values of all $\delta_{\ell\pm}$. This fact could be of interest for realistic scattering, since phase-shifts always grow from zero if the energy increases.

References

[1] A. Gersten, Nucl.Phys. B12, 537 (1969).

[2] F.A. Berends and S.N.M. Ruijsenaars, Nucl.Phys. B56, 525 (1973).

CHAPTER V

FIELD THEORIES FOR PARTICLES WITH ARBITRARY SPIN

1. Introduction

As was noticed in the last chapters, extraction of information from experimental data and translation of this information into theoretically relevant quantities lead to various kinds of problems. Some of these problems have been discussed.

On the other hand one can be concerned with the construction of theories, which describe particles with a certain mass and spin and from which experimental quantities can be predicted.

In the following chapters we shall discuss some problems which are connected with the construction of those theories.

The various elementary particles and their properties are described within the framework of quantum field theory. In this theory the physical particles are associated with so-called operator fields. To each kind of particle there corresponds such a field.

These fields satisfy differential equations, or field equations, which describe the dynamics of such a system of particles. These field equations may contain non-linear terms which give rise to self interactions of the same particles or to interactions between different particles. In absence of these non-linear terms, there are no interactions. In this case we have a free field theory, describing free particles.

In many cases one considers field functions instead of operator fields. One then speaks of classical fields, satisfying classical field equations. The transition from a classical theory to a quantum field theory is called quantization. Various quantization procedures are known, and can be found in the textbooks [1] .

The simplest example of a free classical field is given by $\phi(x)=\phi(\vec{r},t)$, and it satisfies the Klein Gordon equation

$$(\Box-m^2)\phi(x) = 0, \qquad (1.1)$$

where $\Box = \vec{\nabla}^2 - \frac{\partial^2}{\partial t^2}$.

The general solution of this equation is given by

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k_0}} \left(a(\vec{k})\, e^{ik\cdot x} + a^*(\vec{k})\, e^{-ik\cdot x} \right) \tag{1.2}$$

where

$$k\cdot x = -k_0 t + \vec{k}\cdot\vec{r} \quad \text{and} \quad k_0^2 = \vec{k}^2 + m^2 .$$

This solution is clearly a superposition of infinitely many plane waves.

After quantization the classical field $\phi(x)$ is replaced by an operator field $\hat{\phi}(x)$ which describes spinless particles with mass m.

The coefficients $a(\vec{k})$ and $a^*(\vec{k})$ in eq. (1.2) are then also operators, acting on a Hilbert space. They satisfy the following commutation relations

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^*(\vec{k}), \hat{a}^*(\vec{k}')] = 0 ,$$
$$[\hat{a}(\vec{k}), \hat{a}^*(\vec{k}')] = \delta(\vec{k}-\vec{k}') . \tag{1.3}$$

The element $|0\rangle$ of the Hilbert space represents the vacuum state. By repeated application of the operator $\hat{a}^*(\vec{k})$, the following set of basis vectors can be obtained

$$|k_1\rangle = \hat{a}^*(\vec{k}_1)|0\rangle \;, \quad |\vec{k}_1,\vec{k}_2\rangle = \hat{a}^*(\vec{k}_1)\hat{a}^*(\vec{k}_2)|0\rangle \;,\ldots,$$
$$|\vec{k}_1,\ldots,\vec{k}_n\rangle = \hat{a}^*(\vec{k}_1)\ldots\hat{a}^*(\vec{k}_n)|0\rangle \quad , \quad \text{etc.} \tag{1.4}$$

where $|\vec{k}_1,\ldots,\vec{k}_n\rangle$ represents a state with n particles with momenta $\vec{k}_1,\ldots,\vec{k}_n$.

Since these states cannot be normalized,they are strictly speaking not Hilbert space vectors. It is, however, not necessary here to use the proper "smeared-out" version.

From (1.4) it can be seen, that the number of particles increases by repeated application of $\hat{a}^*(\vec{k})$. From the commutation rules (1.3) it follows that application of $\hat{a}(\vec{k})$ decreases the number of particles represented by a state. For this reason the operators $\hat{a}^*(k)$ and $\hat{a}(\vec{k})$ are called creation and annihilation operators.

In this way one may understand how the operator field $\phi(x)$ is associated with a system of particles.

Although we shall only discuss classical fields, it is important to keep in mind that the coefficients $a^*(\vec{k})$ and $a(\vec{k})$ of the plane wave expansion (1.2) create or annihilate the many particle states in the quantum theory.

In this example no spin was present. The creation and annihilation operators only depend on the momentum variable $\vec{k}$.

If we consider particles with spin, the coefficients of the plane waves must also contain information about the spin. In the corresponding quantum field theory we then have creation and annihilation operators for particles with momentum $\vec{k}$ and a specific spin component along some prescribed direction. The spin component along the direction of the momentum $\vec{k}$ is called helicity. For instance, in the case of massive particles with spin 1, the coefficients of the plane wave solution consist for each $\vec{k}$ value of three independent coefficients. After quantization they can be combined to creation and annihilation operators for particles with momentum $\vec{k}$ and spin components +1, 0 and -1. If the particles are massless, however, the plane wave solution should have only two coefficients corresponding to the two helicity states ±1. In the general case of massive particles with spin s for each $\vec{k}$ we should have 2s+1 independent coefficients of the plane waves associated with the spin components s, s-1, s-2, ..., -s. In case of massless particles there are always two helicity states ±s, and therefore we should only have two independent coefficients.

The construction of a theory for interacting particles with arbitrary spin s can be done by the following four steps:

1. Choose a field which describes the particular spin s under consideration.
2. Construct the free field equation.
3. Construct additional terms in the field equation which are responsible for interactions between different particles and/or particles of the same kind.
4. Use a quantization procedure to obtain a quantum field theory.

In the following we shall discuss only the steps 1 and 2. Moreover we shall investigate the relation between theories in the massive and massless case.

In taking the first two steps one is restricted by the following facts:

1. The field equation has to be Lorentz covariant. According to the special theory of relativity physical laws are the same in every inertial system. Since inertial frames are connected by a Lorentz transformation, the field equation should take the same form after application of such a transformation.
2. Since we consider free fields only the field equation is required to be linear in the field and its derivatives.
   Moreover we restrict ourselves to field equations which are at most second order differential equations. This is a reasonable restriction since second order wave equations for spin 0 and spin 1 are known to give satisfactory physical theories. We want to treat higher spin field theories on the same footing.

3. In case of arbitrary spin the field has to describe more degrees of freedom than the spinless field $\phi(x)$ (1.2). Therefore we have to deal with fields which have more components. If the number of components turns out to be greater than the number of degrees of freedom, the field has to satisfy, besides the field equation, a set of subsidiary conditions. These subsidiary conditions eliminate the superfluous degrees of freedom of the field.

In order to satisfy the first requirement we choose for the description of massive particles with spin s a symmetric tensor field of rank s, $\phi_{\mu_1\ldots\mu_s}(x)$. By using the operator ∂_μ and $\Box$ we can get a Lorentz covariant field equation. If it is assumed that $\phi_{\mu_1\ldots\mu_s}$ satisfies the Klein Gordon equation

$$(\Box-m^2)\phi_{\mu_1\ldots\mu_s}(x) = 0 , \qquad (1.5)$$

the requirements 1 and 2 are satisfied and, moreover, the general solution

$$\phi_{\mu_1\ldots\mu_s} = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k_0}} \left(a_{\mu_1\ldots\mu_s}(\vec{k})\, e^{ik\cdot x} + a^*_{\mu_1\ldots\mu_s}(\vec{k})\, e^{-ik\cdot x}\right) \qquad (1.6)$$

is again a superposition of plane waves and the relativistic energy momentum relation

$$k_0^2 = \vec{k}^2+m^2 \qquad (1.7)$$

holds.

Note, that the plane wave coefficients $a_{\mu_1\ldots\mu_s}(k)$ are complex functions of k.

However, for the simplest case of spin 1, the corresponding vector field $\phi_\mu(x)$ has more components, 4, than required for the three helicity states. In case of higher spin this problem becomes even more serious. It can be shown that a symmetric tensor field of rank s in 4-dimensional space has $\binom{s+3}{3}$ components, whereas the spin s case only gives rise to 2s+1 helicities. Therefore we need a set of subsidiary conditions in order to eliminate these superfluous degrees of freedom. These subsidiary conditions turn out to be

$$\delta_{\mu_1\mu_2}\phi_{\mu_1\ldots\mu_s}(x) = 0$$

and (1.8)

$$\partial_{\mu_1}\phi_{\mu_1\mu_2\ldots\mu_s}(x) = 0 .$$

The meaning of these conditions can be understood in terms of the representations of the Poincaré group. We shall show that a field, satisfying

(1.5) and (1.8) transforms according to an irreducible representation of the Poincaré group [2]. All different irreducible representations can be obtained with the general theory of induced representations [2,3] and they are characterized by the two quantities m and s.

Evidently $\phi_{\mu_1 \ldots \mu_s}(x)$ and therefore $a_{\mu_1 \ldots \mu_s}$ carries a representation of the Poincaré group. We shall show that this representation is irreducible, if the plane wave coefficients $a_{\mu_1 \ldots \mu_s}$ satisfy the following set of Lorentz covariant subsidiary conditions

$$a_{\lambda\lambda\mu_3 \ldots \mu_s}(k) = 0 \; ,$$

and (1.9)

$$k_\lambda a_{\lambda\mu_2 \ldots \mu_s}(k) = 0 \; ,$$

which, of course, follows directly from eqs. (1.6) and (1.8). If we restrict ourselves to the proper Lorentz group $\mathcal{L}_0$, the transformation of the coefficients $a_{\mu_1 \ldots \mu_s}(k)$ is given by

$$\tilde{a}_{\mu_1 \ldots \mu_s}(k) = \Lambda_{\mu_1\lambda_1} \ldots \Lambda_{\mu_s\lambda_s} a_{\lambda_1 \ldots \lambda_s}(\Lambda^{-1}k) \; , \tag{1.10}$$

where Λ is an element of $\mathcal{L}_0$.

For each fixed four-momentum $\underline{k}_\mu$ the elements of $\mathcal{L}_0$, leaving $\underline{k}_\mu$ invariant form a subgroup called the little group, associated with $\underline{k}_\mu$.

According to the theory of induced representations, the irreducible unitary representations of the Poincaré group are uniquely determined by the irreducible unitary representation of its little group [3]. If we chôose $\underline{k}_\mu = (0,0,0,im)$ the corresponding little group is SO(3), the group of rotations in 3-dimensional space. Thus the question is whether or not $a_{\mu_1 \ldots \mu_s}(\underline{k})$, satisfying conditions (1.9), carries an irreducible representation of SO(3). From

$$\underline{k}_\lambda a_{\lambda\mu_2 \ldots \mu_s}(\underline{k}) = 0$$

it follows for $\underline{k}_\mu = (0,0,0,im)$

$$a_{4\mu_2 \ldots \mu_s}(\underline{k}) = 0 \tag{1.11}$$

which corresponds to $\binom{s+2}{3}$ equations. As a consequence of (1.11) the original 4-dimensional tensor $a_{\mu_1 \ldots \mu_s}$ is reduced to the 3-dimensional $a_{j_1 \ldots j_s}$ with $\binom{s+2}{2}$ independent components.

Such a tensor still carries a reducible representation D of SO(3). Since it contains spin s, s-2, s-4, etc. it transforms according to

$$D = \begin{cases} D(s) \oplus D(s-2) \oplus \ldots \oplus D(0) & s \text{ even}, \\ D(s) \oplus D(s-2) \oplus \ldots \oplus D(1) & s \text{ odd}, \end{cases} \tag{1.12}$$

where D(k) denotes the irreducible representation of dimension 2k+1 of SO(3). However, the second condition (1.9)

$$a_{\lambda\lambda\mu_3\ldots\mu_s}(k) = 0,$$

takes, as a consequence of (1.11), the following form:

$$a_{jj\ell_3\ldots\ell_s}(k) = 0 \;. \tag{1.13}$$

The last equation (1.13) corresponds with $\binom{s}{2}$ equations. Obviously the number of $\binom{s+2}{2}$ components in $a_{\ell_1\ldots\ell_s}$ is reduced to 2s+1 by the last $\binom{s}{2}$ equations (1.13).

Moreover it is a well-known fact that a traceless 3-dimensional symmetric rank s tensor is associated with an (2s+1)-dimensional irreducible representation of SO(3).

All the facts mentioned above can be summarized as follows: The original representation (1.10) of the Poincaré groups is irreducible, since, as a consequence of the subsidiary conditions (1.9) it corresponds to an irreducible representation of its little group SO(3). In configuration space the field $\phi_{\mu_1\ldots\mu_s}(x)$ satisfying (1.5) and (1.8) transforms also according to an irreducible representation of the Poincaré group.

For completeness we give the field equations plus subsidiary conditions for massive particles with half integer spin $s+\frac{1}{2}$ (s=0,1,2,...). Such particles can be represented by a symmetric tensor-spinor $\psi_{\mu_1\ldots\mu_s}(x)$ satisfying the Dirac equation

$$(\gamma_\lambda \partial_\lambda + m)\psi_{\mu_1\ldots\mu_s}(x) = 0 \;, \tag{1.14}$$

and the subsidiary conditions

$$\begin{aligned} \gamma_{\mu_1}\psi_{\mu_1\ldots\mu_s}(x) &= 0 \;, \\ \partial_{\mu_1}\psi_{\mu_1\ldots\mu_s}(x) &= 0 \;. \end{aligned} \tag{1.15}$$

The main issue now is, whether it is possible to construct a field equation for a field $\phi_{\mu_1\ldots\mu_s}$ which is equivalent to (1.5) and (1.9). In order to do this we shall use the so-called root method which we shall discuss in the next section.

Once we have such a field equation we require it to follow from a principle of least action. In classical field theory this is a special kind of variational principle, which is postulated for a Lagrangian (density) $\mathcal{L}$. The Lagrangian is a Lorentz invariant expression depending on both the fields $\phi_\omega(x)$ and their partial derivatives $\partial_\mu\phi_\omega(x)$.

From the principle of least action the following relativistically invariant field equations - usually called the Euler-Lagrange equations - are obtained

$$\frac{\partial\mathcal{L}}{\partial\phi_\omega} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_\omega}\right) = 0 \; . \tag{1.16}$$

A more detailed treatment of the least action principle can be found in [1].

For instance the Lagrangian for the spin-0 field reads

$$\mathcal{L} = \tfrac{1}{2}\left((\partial_\mu\phi)^2 + m^2\phi^2\right) \tag{1.17}$$

In this case it can easily be verified that the Euler-Lagrange equations (1.16) give the usual Klein-Gordon equation

$$(\Box - m^2)\phi(x) = 0 \; .$$

Although we get the same field equation as before (1.1), knowledge of the Lagrangian is valuable for various theoretical considerations like invariances, quantization and construction of theories with interaction. Therefore it is useful to construct a field equation which is equivalent to (1.5) and (1.9) and which is derivable from a Lagrangian.

In the following we shall also check a condition on the propagator. When a free field equation for arbitrary spin is obtained

$$\mathcal{O}\phi = 0 \; , \tag{1.18}$$

we can introduce an interaction with an external source t. Then the field equation becomes an inhomogeneous second order differential equation

$$\mathcal{O}\phi = t \; . \tag{1.19}$$

Here $\mathcal{O}$ denotes a second order differential operator.

The propagator is a Green function of eq. (1.19) and can - for our purposes - be defined in a heuristic way as an inverse of $\mathcal{O}$. The propagator plays a very important role in quantum field theory; it enables us to calculate the probability amplitudes of the various processes.

In particular when the field is coupled to an external source we shall consider the amplitude for the exchange of a particle between two sources $t\pi t$, where π is the propagator. This quantity is required to have only a single pole at the mass and the corresponding residue should be positive definite. This last condition follows from the unitarity of the S-matrix or - equivalently - from conservation of probability.

In the last chapter we shall study again the amplitude $t\pi t$ both in the massive and in the massless case. There we shall discuss whether or not the massless amplitude can be obtained from the massive amplitude by taking the $m\to 0$ limit.

The problem of constructing field equations for higher spins was first considered by Dirac [4] in trying to generalize his well-known spin-$\frac{1}{2}$ equation. A field theoretical appoach of this problem was undertaken by Fierz and Pauli [5]. They tried to get a field equation from a Lagrangian in order to introduce interactions in a more consistent way. They also noted that in order to get equations (1.5), (1.8), (1.11) and (1.12) from a Lagrangian one needs, besides the original tensor field, a set of auxiliary fields. A procedure for introducing auxiliary fields was developed by Chang [6]. However, he only constructed Lagrangians for the cases s=2,3 and 4.

Finally Hagen and Singh [7] constructed massive field equations for arbitrary s, which were derivable from a Lagrangian. Their field equations were, however, not homogeneous in second order derivatives, if integer spin was considered. Their work formed the starting point for Fronsdal [8], who constructed Lagrangians for massless particles with arbitrary spin. Here the interesting point was that auxiliary fields no longer were necessary in order to describe massless particles with spin s.

In the following section we shall describe a different method to construct higher spin field theories. We shall use there the root method, developed by Ogievetski and Sokatchev [9] . This approach has already been used for the construction of a spin-$\frac{5}{2}$ free field theory [10].

2. The root method

In this section we shall first discuss the main characteristics of the root method. It will be shown that with this method a field equation can be obtained, which describes free massive particles with arbitrary spin. Moreover we shall show that, if a field satisfies this equation, it also satisfies the Klein-Gordon equation plus the set of subsidiary conditions which were mentioned in the previous section.

Then we shall consider the problems which arise when one wants to construct a Lagrangian from which this field equation can be derived.

It should be stressed again that we restrict ourselves to the case of integer spin and uncharged particles.

For the presentation of the root method it is useful to introduce a set of spin projection and spin transition operators P^J_{ij} which act on the field and satisfy

$$P^J_{ij} P^L_{k\ell} = \delta^{J,L} \delta_{jk} P^J_{i\ell} \quad . \tag{2.1}$$

The superscripts J and L denote the spin subspace in which these operators act and the subscripts i,j and k, refer to the number of independent spins in one subspace. The operator P^J_{ij} is a projection operator if i=j and will be called a transition operator if i≠j.

In cases where only one projection operator exists, we sometimes omit the subscripts like in the spin-1 case.

This case can be described by a vector field $\phi_\mu(x)$. As is well known the spin content of this field is both spin-1 and spin-0 once. Consequently we have only two projection operators P^1 and P^0, which are defined as follows

$$\begin{aligned} P^1_{\mu\nu} &= \theta_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{\Box} \partial_\mu \partial_\nu \ , \\ P^0_{\mu\nu} &= \omega_{\mu\nu} = \frac{1}{\Box} \partial_\mu \partial_\nu \quad . \end{aligned} \tag{2.2}$$

One can easily verify that these operators satisfy relation (2.1).

By going to the rest frame in the k-representation as in equation (1.11) one sees that (2.2) projects out ϕ_j and ϕ_4 from ϕ_μ. For higher spins the projection and transition operator are combinations of products of the quantities $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$.

An advantage of the use of projection and transition operators is, that many algebraic manipulations simplify, since (2.1) is satisfied. If we, for instance, multiply the operators

$$A = \sum_J \sum_{ij} a^J_{ij} P^J_{ij} \quad \text{and} \quad B = \sum_L \sum_{k\ell} b^L_{k\ell} P^L_{k\ell} \ ,$$

we get

$$A \cdot B = \sum_J \sum_{ij} (a^J b^J)_{ij} P^J_{ij} \quad . \tag{2.3}$$

Consequently the new coefficients are simply the matrix products of the original matrices a^J_{ij} and $b^J_{k\ell}$.

The projection and transition operators form a complete set in which any Lorentz invariant differential operator, acting on the field, can be expanded. More specifically, a differential operator of even order can be written as

$$A_{\omega,\omega'} = \sum_J \sum_{ij} a^J_{ij} (P^J_{ij})_{\omega,\omega'} \quad ,$$

where ω stands for a set of indices depending on the kind of tensor field on which $A_{\omega,\omega'}$ acts.

The coefficients can generally be expressed as a power series in the d' Alembertion $\Box$. But since we are only interested in an operator of second degree we take these coefficients as constant multiples of $\Box$.

In order to construct a theory for massive particles with integer spin s we first choose a symmetric tensor field of rank s $\phi_{\mu_1\mu_2\ldots\mu_s}$, or simply ϕ_ω. When the spin content of this field is determined, one can construct the complete set of projection and transition operators P^J_{ij}. According to the spin content of ϕ_ω we always have one spin-s and one spin-(s-1) projection operator.

Since the lower spin sectors (s-2, s-3, etc.) are higher dimensional there exist more projection and transition operators in those cases. Then the most general homogeneous, second order field equation can be written as *)

$$(\mathcal{O}_{\omega,\omega'} - m^2\delta_{\omega,\omega'})\phi_{\omega'} = 0 \; ,$$

where

$$\mathcal{O} = (P^s + a^{s-1}P^{s-1} + \sum_{J\leq s-2} \sum_{i,j} a^J_{ij} P^J_{ij})\Box \tag{2.4}$$

and where the a^J_{ij} are real numbers.

However, because the projection and transition operators contain terms proportional to $\Box^{-s}$, $\Box^{-(s-1)}$ etc., eq. (2.4) contains singular expressions $\Box^{-(s-1)}$, $\Box^{-(s-2)}$, etc.

Since the field equation (2.4) is required to be regular, we have to eliminate these terms which of course imposes a set of conditions on the parameters a^J_{ij}.

The wave equation (2.4) has to describe massive particles with spin s. According to the root method developed by Ogievetsky and Sokatchev [9], a certain power of the wave operator should be proportional to the highest projection operator. If

$$\mathcal{O}^N = P^s\Box^N \; , \tag{2.5}$$

then the corresponding field equation describes spin s alone. Note, that if (2.5) holds for some integer N, it is automatically satisfied for larger N values.

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*) Repeated indices denote summation over these indices.

Equation (1.5) implies that the Klein Gordon equation plus subsidiary conditions must be a consequence of condition (2.5). To show this, we note that from (2.4) and (2.5) follows

$$(\mathcal{O}^N)_{\omega,\omega'}\phi_{\omega'} = m^{2N}\phi_\omega = P^s_{\omega,\omega'}\Box^N\phi_{\omega'} \quad . \tag{2.6}$$

From the last identity in (2.6) we get:

$$m^{2N}P^s_{\omega,\omega'}\phi_{\omega'} = P^s_{\omega,\omega'}\Box^N\phi_{\omega'} = m^{2N}\phi_\omega \ ,$$

which implies

$$P^s_{\omega,\omega'}\phi_{\omega'} = \phi_\omega \ . \tag{2.7}$$

From (2.7) it follows

$$P^J_{\omega,\omega'}\phi_{\omega'} = 0 \ , \quad \text{for } J<s \ . \tag{2.8}$$

When one substitutes (2.7) and (2.8) in the general field equation (2.4), one gets the Klein Gordon equation

$$(\Box-m^2)\phi_\omega = 0 \ , \tag{2.9}$$

whereas (2.8) is equivalent to the subsidiary conditions

$$\phi_{\lambda\lambda\mu_1\ldots\mu_{s-2}} = \phi'_{\mu_1\ldots\mu_{s-2}} = 0$$

and (2.10)

$$\partial_\lambda\phi_{\lambda\mu_1\ldots\mu_{s-1}} = (\partial\cdot\phi)_{\mu_1\ldots\mu_{s-1}} = 0 \ .$$

Thus it is shown, that condition (2.5) from the root method guarantees description of spin s alone.

However, if one applies this method for spin $s>2$, some difficulties arise. When we first consider condition (2.5), realizing that P^s contains terms proportional to $\Box^{-s}$, then the regularity of $\mathcal{O}$ implies that at least

$$N \geq s \ , \tag{2.11}$$

in order to make the right-hand side of (2.5) regular as well. If (2.5) is satisfied for some N, $N \geq s$, then

$$(a^J)^N = 0 \quad \text{for all } J \quad (J<s) \ , \tag{2.12A}$$

but

$$(a^J)^{N-1} \neq 0 \text{ for at least one } J \text{ value} \ , \tag{2.12B}$$

since otherwise (2.5) would hold for a smaller N-value. Therefore a matrix, a^J, which satisfies the condition (2.12) should be at least a N×N-matrix.

On the other hand if we determine directly the dimension of a^J in $\mathcal{O}$, a^0 turns out to be the largest matrix since the spins-0 sector has the highest dimension. More specifically, a^0 is a

$$\begin{aligned}&(\tfrac{1}{2}s+1)\times(\tfrac{1}{2}s+1)\text{-matrix, if } s \text{ is even, and a}\\&\tfrac{1}{2}(s+1)\times\tfrac{1}{2}(s+1)\text{-matrix, if } s \text{ is odd.}\end{aligned} \tag{2.13}$$

Clearly the dimension of a^0 and therefore of all a^J is too small (i.e. smaller than a N×N-matrix, with $N\geq s$) and condition (2.12) cannot be satisfied.

One possibility to solve this problem is to allow higher order derivatives in the field equation. However, as was mentioned before, we only consider differential operators of second degree. Instead we can solve this problem by enlarging the dimension of the lower spin matrices a^J. This can be done by introducing additional field quantities, which we call auxiliary fields.

In order to satisfy the root method conditon (2.25) for a certain smallest N-value ($N\geq s$) we have to enlarge the dimension of the matrices a^J until there exists at least one matrix a^J for which (2.12) holds:

$$(a^J)^N = 0, \quad \text{but} \quad (a^J)^{N-1} \neq 0.$$

Then the difficulty which arose from (2.13) is solved.

It should be noted here that in general there is a certain arbitrariness in choosing the auxiliary fields, since there are different ways to enlarge the matrices a^J in order to satisfy (2.12).

In case we started with a symmetric tensor field of rank s the only possible auxiliary fields are tensor fields of rank s-2, s-4, s-6, etc., since the projection and transition operators are always combinations of products of $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ (see (2.2)).

The new wave operator $\mathcal{O}$ acts on the field configurations

$$\phi_\omega = \left(\phi^{(s)}_{\mu_1\ldots\mu_s},\ \phi^{(s-2)}_{\mu_1\ldots\mu_{s-2}},\ \phi^{s-4}_{\mu_1\ldots\mu_{s-4}},\ \ldots\right) . \tag{2.14}$$

Of course the introduction of auxiliary fields leads to an additional set of projection and transition operators. The field equation can still be written as

$$(\mathcal{O}_{\omega,\omega'} + m^2\delta_{\omega,\omega'})\phi_{\omega'} = 0$$

with (2.15)

$$\mathcal{O} = \left(P^s + \sum_{J\leq s-2}\ \sum_{i,j} a^J_{ij}P^J_{ij}\right)\Box .$$

Here we used the fact that, according to (2.5), we get $a^{s-1} = 0$.

Concerning the terminology we define here auxiliary fields as the additional fields besides the originally choosen symmetric tensor field. This implies that for a different original choice (for instance fields with spinorial instead of tensorial indices) different auxiliary fields will be required. The auxiliary fields introduce more spin degrees of freedom, just like the original tensor field also contains more spin degrees of freedom than the components of the highest spin. However, the wave equation will eliminate all those superfluous degrees of freedom as is demonstrated by eq. (2.8).

Thus a field equation (2.15) can be obtained, not containing negative powers of d'Alembertian $\Box$, and which describes massive particles with spin s only. This field equation, however, is in general not symmetric in the tensor indices ω and ω', in contradistinction to field equations derived from Lagrangian.

So one tries to symmetrize the field equation with a non-singular symmetric transformation V, which does not contain derivatives and terms proportional to $\Box^{-k}$. With this transformation the field and auxiliary fields are redefined as follows

$$\phi_\omega = V_{\omega,\omega'}\phi'_{\omega}. \tag{2.16}$$

Since V acts in the spin-s and spin-(s-1) sector as unity the relations (2.7) and (2.8) still hold with respect to ϕ'_ω. Therefore the physical content of ϕ_ω and ϕ'_ω is still the same. After redefining the fields, the field equation reads

$$(\mathcal{O} V - m^2 V)_{\omega,\omega'}\phi'_{\omega'} = 0 . \tag{2.17}$$

Expanding V in terms of projection and transition operators

$$V = P^s + P^{s-1} + \sum_{J \le s-2} \sum_{ij} v^J_{ij} P^J_{ij} , \tag{2.18}$$

equation (2.17) becomes:

$$\Big(P^s + \sum_{J \le s-1} \sum_{ij} (a^J v^J)_{ij} P^J_{ij} - m^2 \sum_J \sum_{ij} v^J_{ij} P^J_{ij}\Big)_{\omega,\omega'}\phi'_{\omega'} = 0 . \tag{2.19}$$

The matrix v^J being symmetric we only have to require $\hat{a}^J = a^J v^J$ to be symmetric for all J. Obviously the mass term has become more complicated now.

The symmetrized field equation (2.19) can be derived from a Lagrange function

$$\mathcal{L} = \phi^T(\hat{O}V - m^2 V)\phi \quad . \tag{2.20}$$

Summarizing, a field equation has been constructed which takes the general form given by equation (2.19).

This equation satisfies three requirements

1. the wave operator is regular, i.e. it does not contain terms proportional to $\Box^{-k}$;
2. the root method condition (2.5) for O is satisfied;
3. the wave operator is symmetric in ω and ω'.

These requirements lead to a set of equations for the coefficients a^J_{ij} and $v^L_{k\ell}$. For each case one has to look for a solution of this set of equations. If such a solution exists, then the field equation according to the root method is equivalent to the Klein Gordon equation (2.9) plus the subsidiary conditions (2.10). Thus a theory has been constructed describing massive particles with spin s only.

As was mentioned in the previous section we still have to discuss a condition on the propagator. When the propagator is evaluated in the k-representation it should only have a first order pole in k^2+m^2. Moreover, due to the unitarity of the S-matrix the residue of this pole should be positive definite.

For our purposes the propagator is defined as an inverse of the operator $(O - m^2 \mathbf{1})$. When we first consider the case of a non-symmetric field equation (2.4)

$$(O_{\omega,\omega'} - m^2 \delta_{\omega,\omega'})\phi_{\omega'} = 0$$

the propagator is given by

$$\pi(m) = \left\{ \left(\frac{\Box^s P^s}{m^{2s}(\Box^s - m^{2s})} - \frac{\mathbf{1}}{m^{2s}} \right) \left(O^{s-1} + m^2 O^{s-2} + \ldots + m^{2(s-1)} \mathbf{1} \right) \right\} . \tag{2.21}$$

One easily verifies that

$$\pi(m)(O - m^2 \mathbf{1}) = \mathbf{1} \tag{2.22}$$

holds.

In case of the symmetrized field equation (2.17)

$$((OV)_{\omega,\omega'} - m^2 V_{\omega,\omega'})\phi'_{\omega'} = 0$$

the symmetric propagator $\hat{\pi}(m)$ is related to the former in the following way

$$\hat{\pi}(m) = V^{-1}\pi(m) \quad . \tag{2.23}$$

In this case we obviously have

$$\hat{\pi}(m)(\mathcal{O}V - m^2V) = \mathbf{1} \quad . \tag{2.24}$$

If we sandwich the propagator (2.23) between two external sources, $T_\omega \hat{\pi}_{\omega\omega'}(m)T_{\omega'}$, we can determine the poles and the corresponding residues of this expression. We first decompose the non-symmetric wave operator $\mathcal{O}$:

$$\mathcal{O} = P^{(s)}\Box + \tilde{\mathcal{O}} \tag{2.25}$$

where $\tilde{\mathcal{O}}$ is the part of the field operator, affecting only the lower spin sectors.

Then the propagator $\hat{\pi}(m)$ can be written as follows:

$$\hat{\pi}(m) = \left\{ V^{-1}\left(\frac{P^s}{\Box - m^2} - \frac{1}{m^{2s}}\left(\tilde{\mathcal{O}}^{s-1} + m^2\tilde{\mathcal{O}}^{s-2} + \ldots + m^{2(s-1)}(\mathbf{1} - P^{(s)})\right)\right)\right\} . \tag{2.26}$$

From (2.26) one immediately observes that the expression $T\hat{\pi}(m)T$ has a first order pole in $(\Box - m^2)$ or in k^2+m^2. Furthermore presence of P^s in this pole part guarantees that propagation of only the highest spin part has this pole. Finally we show that the residue $TV^{-1}P^sT = TP^sT = (P^sT)^2$ is a positive definite expression.

It is shown by Chang [6] that in general the highest spin projection P^s consists of a product of s times $\theta_{\mu\nu}$, where $\theta_{\mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}$. This expression has the property that $\partial_\mu\theta_{\mu\nu} = 0$. Therefore, if we introduce the quantity

$$J_{\mu_1\mu_2\ldots\mu_s} = (P^s T)_{\mu_1\mu_2\ldots\mu_s} \tag{2.27}$$

the following property is obvious:

$$\partial_{\mu_1} J_{\mu_1\mu_2\ldots\mu_s} = (\partial\cdot J)_{\mu_2\mu_3\ldots\mu_s} = 0 \; . \tag{2.28}$$

When we evaluate (2.28) in the momentum representation, we get

$$k_\ell J_{\ell\mu_2\ldots\mu_s} + k_4 J_{4\mu_2\ldots\mu_s} = 0 \; ,$$

or

$$k_\ell J_{\ell\mu_2\ldots\mu_s} = k_o J_{o\mu_2\ldots\mu_s} \; . \tag{2.29}$$

Here repeated latin indices denote a summation over the space components alone, whereas repeated greek indices always denote a summation of all space and time components. The residue TP^sT can be expressed as follows

$$TP^sT = J^2 = J_{\ell_1\ell_2\ldots\ell_s} \prod_{i=1}^{s} \left(\delta_{\ell_i m_i} - \frac{k_{\ell_i} k_{m_i}}{k_o^2} \right) J_{m_1 m_2 \ldots m_s} \, . \tag{2.30}$$

Since each factor in $\prod_{i=1}^{s}\left(\delta_{\ell_i m_i} - \frac{k_{\ell_i} k_{m_i}}{k_o^2}\right)$ represents a positive definite matrix, TP^sT is also a positive definite expression. This can easily been understood, by going to a frame in which $\vec{k}$ is given by $k_1 = k_2 = k_3 = \frac{1}{3}\sqrt{3\vec{k}^2}$. In this case (2.30) turns into

$$TP^sT = J'_{\ell_1\ldots\ell_s} \prod_{i=1}^{s} \delta_{\ell_i m_i} \left(1 - \frac{\vec{k}^2}{3k_o^2}\right) J'_{m_1\ldots m_s} \, , \tag{2.31}$$

which is clearly positive definite, since $k_o^2 > \vec{k}^2$.

3. Field equations for massless particles

In the previous section we discussed the construction of field equations and Lagrangians for massive particles with integer spin s. These field equations were obtained by using the root method and a set of projection operators. Since in the massless case the operators are not projection operators, it is not possible to construct a theory for massless particles by the method discussed before. Instead, as a starting point, we shall use the massive field equation (2.19):

$$\left(P^s + \sum_{J<s-2} \sum_{ij} a^J_{ik} v^J_{kj} P^J_{ij} - m^2 \sum_J \sum_{ij} v^J_{ij} P^J_{ij}\right)_{\omega,\omega'} \phi_{\omega'} = 0 \, .$$

Here the coefficients a^J_{ij} and v^J_{ij} are subjected to a number of conditions, the origin of which has been discussed in the previous section.

In the $m\to 0$ limit equation (2.19) should lead to the two highest helicity states $\pm s$ only. The degrees of freedom which are associated with other helicities must vanish. The transition to the m=0 theory becomes more convenient, if a solution of the coefficients a^J_{ij} and v^J_{ij} exists such that the conditions mentioned before are satisfied and such that the coupling between the original field and the auxiliary fields takes place only through the mass term. If in this case the $m\to 0$ limit is taken the equations for the original field and the auxiliary fields will decouple directly. By taking the auxiliary fields zero, one gets an equation in terms of the original field $\phi_{\mu_1\ldots\mu_s}(x)$. The wave operator, thus obtained, is determined by the remaining matrices

$$\hat{a}^J = a^J v^J , \tag{3.1}$$

which are only related to the projection and transition operators of the original field $\phi_{\mu_1\ldots\mu_s}(x)$. The field equation reads, in presence of an external source T

$$\big(P^s + \sum_{J\leq s-2} \hat{a}^J_{ij} P^J_{ij}\big)_{\omega\omega'} \phi(x)_{\omega'} = T_\omega \ . \tag{3.2}$$

In the next chapter we shall show for the case s=1, 2 and 3, that the field equation (3.2) in the presence of an external source takes the following form. (see also [11]) :

$$W_{\mu_1\ldots\mu_s} - \tfrac{1}{2} \textstyle\sum_2 \delta_{\mu_1\mu_2} W'_{\mu_3\ldots\mu_s} = T_{\mu_1\ldots\mu_s} \ , \tag{3.3}$$

where

$$W_{\mu_1\ldots\mu_s} = \Box\phi_{\mu_1\ldots\mu_s} - \textstyle\sum_1 \partial_{\mu_1}(\partial\cdot\phi)_{\mu_2\ldots\mu_s} + \textstyle\sum_2 \partial_{\mu_1}\partial_{\mu_2}\phi'_{\mu_3\ldots\mu_s} \ , \tag{3.4}$$

and where $\phi_{\mu_1\ldots\mu_s}$ is a symmetric tensor field of rank s.

The summations are made over all independent permutations of the indices $\mu_1\ldots\mu_s$.*) For some contractions a short-hand notation is used, i.e.

$$\begin{aligned} (\partial\cdot\phi)_{\mu_2\ldots\mu_s} &= \partial_\lambda \phi_{\lambda\mu_2\ldots\mu_s} \ , \\ \phi'_{\mu_3\ldots\mu_s} &= \phi_{\lambda\lambda\mu_3\ldots\mu_s} \ . \end{aligned} \tag{3.5}$$

The fields in (3.3) and (3.4) are chosen such that their double trace vanishes, i.e.

$$\phi''_{\mu_5\ldots\mu_s} = 0 \ . \tag{3.6}$$

Obviously this condition is only important in case $s\geq 4$. The equations (3.3), (3.4) and (3.6) were originally found by Fronsdal [8]. He showed that they give a correct description of massless particles with integer spin. These equations were also discussed by de Wit and Freedman [11].

It should be noted that the matrices $\hat{a}^J$ in equation (3.2) have zero eigen-

*) In particular $\sum_1$ denotes a sum of s, and $\sum_2$ denotes a sum of $\frac{1}{2}s(s-1)$ independent permutations of the indices $\mu_1\ldots\mu_s$.

values. The singularity of the matrices $\hat{a}^J$ follows from the nilpotency of the matrices a^J, which in turn is a consequence of the root method.

Such a singular matrix $\hat{a}^J$ gives use to a certain number of left and right null vectors, which are automatically null vectors of the wave operator $\mathcal{O}$.

In the presence of a source T, the field equation (3.2) can be written as

$$\mathcal{O}_{\omega,\omega'}\phi_{\omega'} = T_{\omega} \ . \tag{3.7}$$

Suppose we transform ϕ according to

$$\phi_{\omega} \to \phi_{\omega} + \Big(\sum_{\ell} x^J_{\ell} P^J_{\ell m}\Big)_{\omega,\omega'}\phi_{\omega'} \ , \tag{3.8}$$

where $x^J = (x^J_1, \ldots, x^J_{k_J})$ is a right null vector of $\hat{a}^J$. It can easily be verified that the field equation (3.7) remains unchanged under the transformation of (3.8). Thus every right null vector gives rise to a gauge invariance of the massless field equation (3.7).

In the next chapter we shall show that for the case s=1,2 and 3 the various gauge transformations (3.8) can be combined in the following one

$$\delta\phi_{\mu_1\ldots\mu_s} = \sum\nolimits_1 \partial_{\mu_1}\varepsilon_{\mu_2\ldots\mu_s} \ , \tag{3.9}$$

where $\varepsilon_{\mu_2\ldots\mu_s}$ is a symmetric traceless tensor of rank s-1.

In the same way it can be understood that each left null vector y^J of $\hat{a}^J$ causes a source constraint

$$\sum_{\ell} (y^J_{\ell} P^J_{m\ell})_{\omega,\omega'} T_{\omega'} = 0 \ . \tag{3.10}$$

Again for the cases s = 1,2,3 it will be shown that the various source constraints (3.10) can be put in the form

$$\partial\mu_1\Big(T_{\mu_1\ldots\mu_s} - \frac{1}{2(s-1)}\,\delta_{\mu_2\mu_3}T'_{\mu_1\mu_4\ldots\mu_s}\Big) = 0 \ . \tag{3.11}$$

Sources obeying these constraints are called physical sources. Also the gauge invariance (3.9) and the source constraint (3.11) were found for the general case by Fronsdal [8].

If physical sources are absent, the free field equation turns out to be

$$\Box\phi_{\mu_1\ldots\mu_s} - \sum\nolimits_1 \partial_{\mu_1}(\partial\cdot\phi)_{\mu_2\ldots\mu_s} + \sum\nolimits_2 \partial_{\mu_1}\partial_{\mu_2}\phi'_{\mu_3\ldots\mu_s} = 0 \ , \tag{3.12}$$

with ϕ symmetric,

$$\phi''_{\mu_5\ldots\mu_s} = 0 \ . \tag{3.6}$$

and with gauge invariance under

$$\delta\phi_{\mu_1\ldots\mu_s} = \sum_1 \partial_{\mu_1} \varepsilon_{\mu_2\ldots\mu_s} ; \quad \varepsilon'_{\mu_4\ldots\mu_s} = 0 . \tag{3.9}$$

We shall show for this general case that the system of equations (3.12), (3.6) and (3.9) describes massless particles with *only* two helicity states $\pm s$. By using gauge transformations the field ϕ can be transformed such that it satisfies the following Lorentz covariant condition:

$$(\partial\cdot\phi)_{\mu_2\ldots\mu_s} = \frac{1}{2} \sum_1 \partial_{\mu_2} \phi'_{\mu_3 \ldots\mu_s} . \tag{3.13}$$

Note that for fields, satisfying (3.13) the gauge invariance (3.9) still exists, but with an ε such that $\Box\varepsilon = 0$. For fields satisfying this condition the field equation turns into the massless Klein Gordon equation

$$\Box\phi_{\mu_1\ldots\mu_s}(x) = 0 , \tag{3.14}$$

the general solution of which can then be written as

$$\phi_{\mu_1\ldots\mu_s}(x) = \int \frac{d\vec{k}}{\sqrt{2k_0}} \left(a_{\mu_1\ldots\mu_s}(k)\ e^{ik\cdot x} + a^*_{\mu_1\ldots\mu_s}(k)\ e^{-ik\cdot x}\right) . \tag{3.15}$$

Here the $a_{\mu_1\ldots\mu_s}(k)$ are complex functions of k_μ which accoding to (3.14) satisfy

$$k^2 = -k_0^2 + \vec{k}^2 = 0 . \tag{3.16}$$

Of course the double trace of $a_{\mu_1\ldots\mu_s}$ vanishes, i.e.

$$a''_{\mu_5\ldots\mu_s} = 0 . \tag{3.17}$$

Clearly the functions a have too many components; we need only two components for the description of two helicity states. As we shall see gauge invariance can again be used to eliminate these superfluous components as well.

From the gauge invariance (3.9) it follows that the functions $a_{\mu_1\ldots\mu_s}$ can be redefined

$$a_{\mu_1\ldots\mu_s} \rightarrow a_{\mu_1\ldots\mu_s} + \sum_1 k_{\mu_1} \chi_{\mu_2\ldots\mu_s}(k) ,$$

with (3.18)

$$\chi'_{\mu_4\ldots\mu_s}(k) = 0 .$$

Obviously $a_{\mu_1 \ldots \mu_s}$ also satisfies the gauge condition (3.13)

$$k_\lambda a_{\lambda\mu_2 \ldots \mu_s} = \frac{1}{2} \sum_1 k_{\mu_2} a'_{\mu_3 \ldots \mu_s} . \tag{3.19}$$

The set of functions $a_{\mu_1 \ldots \mu_s}(k)$, satisfying (3.17), forms an infinite dimensional space A. We consider the following Lorentz invariant subspaces of A:

$\mathcal{H}_0$ consisting of all elements of A, which have the form $\sum_1 k_{\mu_1} \chi_{\mu_2 \ldots \mu_s}(k)$, with $\chi'_{\mu_4 \ldots \mu_s} = 0$;

$\mathcal{H}_1$ consisting of all elements of A, which satisfy (3.19).

Clearly $\mathcal{H}_0$ is a linear subspace of $\mathcal{H}_1$, since each $a \in \mathcal{H}_0$ satisfies (3.19).
We consider the following hermitian form on $\mathcal{H}_1$:

$$(a,b) = \int \frac{d\vec{k}}{2k_0} \left(a^*_{\mu_1 \ldots \mu_s} b_{\mu_1 \ldots \mu_s} - \frac{1}{4} s(s-1) a^*_{\lambda\lambda\mu_3 \ldots \mu_s} b_{\nu\nu\mu_3 \ldots \mu_s} \right) . \tag{3.20}$$

The following properties can easily be proved.

(i) for all $a \in \mathcal{H}_1$ we have

$$(a,a) \geq 0 , \tag{3.21}$$

(ii) but $(a,a) = 0$
if and only if $a \in \mathcal{H}_0$ (3.22)

(iii) for all $a,b \in \mathcal{H}_1$ and $c \in \mathcal{H}_0$ we have

$$(a,b+c) = (a,b) . \tag{3.23}$$

So it follows from (3.21) that the hermitian form (3.20) is positive but, according to (3.22), it is not positive definite. However, as a consequence of property (3.23), it is possible to define the hermitian form (3.20) on the quotient space $\mathcal{H}_1/\mathcal{H}_0$. On this quotient space the expression (3.20) is positive definite and consequently it gives a well defined inner product on $\mathcal{H}_1/\mathcal{H}_0$. With this inner product $\mathcal{H}_1/\mathcal{H}_0$ is an infinitely dimensional Hilbert space. This space can be considered to consist of infinitely many spaces V_k of finite dimension. Here the label k represents a point on the light-cone.

A simple counting argument shows V_k to be two-dimensional for any k on the light-cone. Or stated in other words: any function $a_{\mu_1 \ldots \mu_s}(k)$, representing an element of $\mathcal{H}_1/\mathcal{H}_0$, has only two independent components for any k with $k^2=0$. Indeed any symmetric s rank tensor, the double trace of which vanishes, has $2s^2+2$ components. If we restrict ourselves to $\mathcal{H}_1$, $a_{\mu_1 \ldots \mu_s}$ has to satisfy (3.19), which corresponds to s^2 conditions. So s^2+2 components are left. The $a_{\mu_1 \ldots \mu_s}$ can be redefined - still representing the same element of $\mathcal{H}_1/\mathcal{H}_0$ - by using a gauge transformation (3.18). By redefining $a_{\mu_1 \ldots \mu_s}$ again s^2 components can be eliminated and therefore two independent components are left. This shows that the many different components of $a_{\mu_1 \ldots \mu_s}(k)$ are reduced to two by using the gauge invariance of the theory for any k on the light cone.

In order to understand that these two components describe the two helicity states $\pm s$, it must be shown that the quotient space $\mathcal{H}_1/\mathcal{H}_0$ carries an irreducible lightlike **representation** of the Poincaré group associated with the two spin components $\pm s$.

From the way the field function ϕ transforms under an element a, of the Poincaré group, it follows that the functions $a_{\mu_1 \ldots \mu_s}(k)$ transform according to:

$$\tilde{a}_{\mu_1 \ldots \mu_k}(k) = e^{-ik \cdot a} \Lambda_{\mu_1 \lambda_1} \ldots \Lambda_{\mu_s \lambda_s} a_{\lambda_1 \ldots \lambda_s}(\Lambda^{-1} k) \tag{3.24}$$

where Λ is an element of the proper Lorentz group and a is a translation parameter. According to the theory of induced representations [2,3] the irreducible unitary representations of the Poincaré group correspond uniquely to the irreducible representations of its little groups. For some fixed $\underline{k}$ the little group $L_{\underline{k}}$ associated with $\underline{k}$ is defined as the subgroup of elements of the Lorentz group leaving $\underline{k}$ invariant. Equivalent little groups give rise to equivalent representations if they are related to k-values on the same mass shell (light cone).

Written in a short-hand notation equation (3.24) reads

$$\tilde{a}(k) = D(a,\Lambda)a(k) \ . \tag{3.25}$$

Since $\mathcal{H}_1$ and $\mathcal{H}_0$ are Lorentz invariant subspaces of A, the representation (3.25) gives a representation of the Poincaré group on $\mathcal{H}_1/\mathcal{H}_0$ as follows. Let P represent the natural surjection from $\mathcal{H}_1$ onto $\mathcal{H}_1/\mathcal{H}_0$. Let a be a representative of an equivalence class $[a] \in (\mathcal{H}_1/\mathcal{H}_0)$, so $P(a) = [a]$, then

$$U(\Lambda)[a] = P\,D(\Lambda)\,a \tag{3.26}$$

defines a representation of the Poincaré group on $\mathcal{H}_1/\mathcal{H}_0$. This representation $U(\Lambda)$ is unitary with respect to the inner product on $\mathcal{H}_1/\mathcal{H}_0$.

According to the theory of induced representations, the representation (3.26) is irreducible if and only if its corresponding representation of the little group $L_{\underline{k}}$ for some fixed $\underline{k}$ is irreducible.

We choose $\underline{k}_\mu = (0,0,1,i)$. The restriction of all elements a(k) of $\mathcal{H}_1/\mathcal{H}_0$

to the k-value $\underline{k}$ forms a space $V_{\underline{k}}$ which can be proved to consist of elements $a_{i_1 \ldots i_s}(k)$, $i_1, i_2, \ldots, i_s = 1,2$

$$a_{11\, i_3 \ldots i_s} + a_{22\, i_3 \ldots i_s} = 0 \; . \tag{3.27}$$

Consequently $V_{\underline{k}}$ is spanned by the two independent elements $a_{11\ldots1}(\underline{k})$ and $a_{21\ldots1}(\underline{k})$. This agrees with the fact that the dimension of V_k and therefore of $V_{\underline{k}}$ is two. We consider the two independent combinations

$$\varepsilon(k,\pm) = \frac{1}{\sqrt{2}}\,(a_{11\ldots1} \mp i a_{21\ldots1}) \; . \tag{3.28}$$

Then it can be shown that the $\varepsilon(\underline{k},\pm)$ transform under elements Λ of the little group $L_{\underline{k}}$, according to

$$U(\Lambda)\varepsilon(\underline{k},\pm) = e^{\pm is\phi}\varepsilon(\underline{k},\pm) \; . \tag{3.29}$$

Here ϕ represents the angle of rotation around the k_3-axis. Clearly the representation of the little group (3.29) is reducible. However, in case one includes space reflections, the representation is not reducible, since space reflections transform different elements $\varepsilon(\underline{k},+)$ and $\varepsilon(\underline{k},-)$ into each other. Note that the elements $\varepsilon(\underline{k},\pm)$ are eigen vectors under infinitesimal elements of the little group at eigenvalues $\pm s$, Therefore only two helicity states $\pm s$ are described.

It has been shown that the original free field equation (3.12) with gauge invariance (3.9) describes massless particles with only helicities $\pm s$.

In the next chapter we shall show, at least for the case s=1,2,3, that field equations of the form (3.2) and (3.12) can be obtained by taking the $m \to 0$ limit in the massive theory.

We shall also study the propagator for zero mass. However, the definition of the propagator is more complicated than it was in the massive case, since inversion of the wave operator $\mathcal{O}$ is no longer possible. It has been shown in ref. [10], that a suitable propagator can be obtained. There it was proved that the propagator, corresponding to (2.32) sandwiched between two physical sources, equals the inverse of a regular submatrix of $\mathcal{O}$, sandwiched between two physical sources. Construction of such a propagator shall be done explicitly in the next chapter for the case s=1,2,3.

When we evaluate the propagator (still sandwiched between two physical sources) in the k-representation it should have a single pole in k^2. Due to the unitarity of the s-matrix, the corresponding residue should be positive definite.

Finally from the form of this residue it can also be seen whether or not only helicity states $\pm s$ propagate.

References

[1] N.N. Bogoliubov, D.V. Shirkov, Introduction to the theory of quantized fields, Interscience publishers, inc., New York (1959).
P. Roman, Introduction to Quantum Field Theory, John Wiley and Sons Inc. (1969).
C. Ytzikson, J.B. Zuber, Quantum Field Theory, Mc Graw Hill., Inc. (1980).

[2] A.D. Barut, R. Raczka, Theory of Group Representations and Applications, Polish Scientific Publishers, Warzawa (1977).

[3] G.W. Mackey, Induced Representations of Groups and Quantum Mechanics, W.A. Benjamin, Inc., New York, Amsterdam (1968).
D.J. Simms, Lie Groups and Quantum Mechanics, Lecture notes in Mathematics 52, Springer (1968).

[4] P.A.M. Dirac, Proc. R. Soc. A155, 447 (1936).

[5] M. Fierz, Helv. Phys. Acta, 12, 3 (1939).
M. Fierz, W. Pauli, Proc. R. Soc., A173, 211 (1939).

[6] S.J. Chang, Phys. Rev. 161, 1308 (1967).

[7] L.P.S. Singh, C.R. Hagen, Phys. Rev. D9, 898 (1974).
L.P.S. Singh, C.R. Hagen, Phys. Rev. D9, 910 (1974).

[8] C. Fronsdal, Phys. Rev. D18, 3624 (1978).

[9] V. Ogievetski, E. Sokatchev, J. of Phys. A10, 2021 (1977).

[10] F.A. Berends, J.W. van Holten, P. van Nieuwenhuizen and B. de Wit, Nucl. Phys. B154, 261 (1979).

[11] B. de Wit and D. Freedman, Phys. Rev. D21, 358 (1979).

CHAPTER VI

FIELD THEORIES FOR PARTICLES WITH SPIN 1, 2 AND 3

1. Introduction

In this chapter we shall discuss the construction of free field theories for the case of spin s=1, 2 and 3. First field equations representing the massive particles will be obtained using the root method, explained in the previous chapter. We shall also discuss the problem how to construct a Lagrangian from which this field equation can be derived. Then we study the properties of the propagator. When the propagator is given in the k-representation it should have a first order pole in k^2+m^2. Moreover, the corresponding residue should be positive definite.

Attention will be paid to the transition to the massless theory. We shall discuss the various gauge invariances and source constraints, which will arise. We have already shown that the gauge invariances must be used to verify that only highest helicities are present in the massless theory.

2. Spin-1 free field theory

In order to construct a field equation for massive particles with spin 1, we first introduce the spin projection operators

$$P^0_{\mu\nu} = \omega_{\mu\nu} = \frac{1}{\Box}\partial_\mu\partial_\nu ,$$
$$P^1_{\mu\nu} = \theta_{\mu\nu} = \delta_{\mu\nu} - \omega_{\mu\nu} . \qquad (2.1)$$

They satisfy

$$P^J P^L = \delta^{JL} P^J . \qquad (2.2)$$

For the description of spin 1 we choose a vector field $\phi_\mu(x)$.

The wave operator should be a homogeneous second order differential operator. Such an operator can be expressed in terms of the projection operators P^J. Therefore the most general wave operator O can be written as follows

$$O_{\mu\nu} = (aP^1_{\mu\nu} + bP^0_{\mu\nu})\Box . \qquad (2.3)$$

Clearly O does not contain negative powers of the d'Alembertian $\Box$. According to the root method we have to find the smallest N-value, for which

$$\mathcal{O}^N = P^1 \Box^N \tag{2.4}$$

holds. Since P^1 contains a term proportional to $\Box^{-1}$, whereas $\mathcal{O}$ does not, (2.4) might be satisfied by putting N=1. Clearly (2.4) holds for N=1, if we put a=1, b=0. Here we also used property (2.2). Thus the massive field equation for spin 1 particles reads

$$P^1_{\mu\nu}\Box\phi_\nu - m^2\phi_\mu = 0 \ , \tag{2.5A}$$

or using (2.1)

$$(\Box - m^2)\phi_\mu - \partial_\mu\partial_\nu\phi_\nu = 0 \ , \tag{2.5B}$$

which is the well-known Proca equation.

The Lagrangian from which this equation can be obtained is given by

$$\mathcal{L} = \tfrac{1}{2}\{(\partial_\nu\phi_\nu)(\partial_\mu\phi_\mu) - (\partial_\nu\phi_\mu)^2 - m^2\phi_\mu^2\} \ . \tag{2.6}$$

The propagator of a massive particle with spin 1 can be found by inverting (2.5A)

$$\pi(m) = \frac{1}{\Box - m^2} P^1 - \frac{1}{m^2} P^0 = \frac{1}{\Box - m^2}\left(\delta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu\right) \ . \tag{2.7}$$

When the propagator is studied in the k-representation it has only a single pole at k^2+m^2. Moreover, when the propagator is sandwiched between two external sources T the corresponding residue of the expression $T_\mu\pi(m)_{\mu\nu}T_\nu$ is positive definite as was shown for the general case in the previous chapter. This fact can directly be proved for the case of spin 1. According to (2.7) the residue of $T_\mu\pi_{\mu\nu}T_\nu$ is given by

$$(P^1T)^2 = J_\mu J_\mu \ , \quad \text{with } J = P^1T \ .$$

Since $P^0J = 0$, we have in k-representation

$$k_4J_4 = -k_\ell J_\ell \ .$$

In particular by choosing $k_1=k_2=0$; $k_3=\vec{k}$, we get

$$(P^1T)^2 = J_3^2+J_4^2 = J_3^2\left(\vec{k}^2 - \frac{\vec{k}^2}{\vec{k}^2+m^2}\right) \geq 0 \ .$$

The massless theory can be obtained by taking the $m \to 0$ limit in the massive field equation. In presence of an external source T the field equation then becomes

$$P^1_{\mu\nu} \Box \phi_\nu = T_\mu \ , \qquad (2.8A)$$

or

$$\Box \phi_\mu - \partial_\mu \partial_\nu \phi_\nu = T_\mu \ . \qquad (2.8B)$$

The resulting wave operator, $O = P^1 \Box$, has a left and a right null vector, both proportional to P^0. Consequently the field equation remains unchanged if one changes the vector field ϕ_μ by adding a right null vector:

$$\phi_\mu \to \phi_\mu + P^0_{\mu\nu} \Box \zeta_\nu$$

for some arbitrary vector field ζ_ν . Thus a gauge invariance of the massless theory is caused by the right null vectors of the wave operator.

The gauge transformation (2.9) can be written in the following form:

$$\phi_\mu \to \phi_\mu + \partial_\mu \chi \ , \qquad (2.10)$$

where χ is an arbitrary scalar field.

In a similar way left null vectors give rise to a source constraint. By multiplying eq. (2.8A) on both sides with a left null vector we get

$$P^0_{\mu\nu} \Box T_\nu = 0 \ , \qquad (2.11)$$

from which follows

$$\partial_\mu T_\mu = 0 \ . \qquad (2.12)$$

Thus in the massless spin-1 theory physical sources must be divergenceless.

Clearly the field equation (2.8B) with gauge invariance (2.10) is a special case of the general massless spin-s field equation discussed in the previous chapter (section 3). By using the gauge transformation (2.10) a class of field functions can be obtained, satisfying

$$\partial_\mu \phi_\mu(x) = 0 \qquad (2.13)$$

and the field equation (2.8B) turns into the massless Klein-Gordon equation

$$\Box \phi_\mu(x) = T_\mu(x) \ . \qquad (2.14)$$

As was shown generally in the previous chapter the solution of (2.14) describes only two helicity states $\pm s$.

In this section we shall also study the propagator. In order to define a propagator the wave operator should be invertible. However, since it has a null vector it cannot be inverted. The gauge condition (2.13), which in terms of projection operators reads

$$P^0_{\mu\nu}\Box\phi_\nu = 0 \ , \tag{2.15}$$

can be used to make the wave operator regular.

In fact we shall add equation (2.15) to the original field equation. Instead of equations (2.8) we shall study

$$(P^1 + P^0)_{\mu\nu}\Box\phi_\nu = T_\mu \tag{2.16}$$

together with condition (2.15).
Indeed, from (2.15) we obviously get (2.14).

Since we can invert the wave operator in (2.16) the massless propagator $\pi_{\mu\nu}$ can be obtained

$$\pi_{\mu\nu} = \frac{1}{\Box}(P^1 + P^0)_{\mu\nu} = \frac{\delta\mu\nu}{\Box} \tag{2.17}$$

When this propagator is sandwiched between two physical sources it is possible to show that only helicity ± 1 modes propagate. According to eq. (2.17) we get

$$T_\mu \pi_{\mu\nu} T_\nu = \frac{1}{\Box} T_\mu T_\mu \quad . \tag{2.18}$$

We shall evaluate this expression in k-representation. In momentum representation the source constraint reads

$$k_\mu T_\mu = 0 \ . \tag{2.19}$$

The most general decomposition of the source T_μ on-shell (i.e. $k^2=0$) can be written in terms of the polarization vectors ε^i_μ (i=1,2) and the vectors $k_\mu = (\vec{k},k_4) = (\vec{k},ik_0)$ and $\bar{k}_\mu = (\vec{k},-ik_0)$ (see App. C)

$$T_\mu = \sum_{i=1,2} \varepsilon^i_\mu \ T^i + k_\mu A + \bar{k}_\mu B \ . \tag{2.20}$$

Due to the source constraint (2.19) we get, after contracting (2.19) with k_μ, the restriction B=0, or

$$T_\mu = \sum_i \varepsilon^i_\mu \ T^i + k_\mu A \ . \tag{2.21}$$

Since $\varepsilon_\mu^{(\pm)}$ terms give rise to helicities ± 1, where

$$\varepsilon_\mu^{(\pm)} = \mp \frac{1}{\sqrt{2}} \left(\varepsilon_\mu^1 \pm i\varepsilon_\mu^2\right) , \tag{2.22}$$

the ± 1 helicity source combinations are

$$T^{\pm} = \frac{1}{\sqrt{2}} (\mp T^1 + iT^2) . \tag{2.23}$$

With (2.21) and (2.23) the expression (2.18) becomes

$$T\pi T = \frac{1}{k^2} T_\mu T_\mu = \frac{1}{k^2} \left(|T^+|^2 + |T^-|^2\right) + |A|^2 . \tag{2.24}$$

Note the single pole in k^2. Further the residue of this pole is positive definite and its form guarantees propagation of only the higher helicity modes ± 1.

3. Spin-2 free field theory

In this section we shall first construct a field equation for massive particles with spin 2. Then we shall discuss the massless case. For the description of particles with spin 2 we choose a symmetric tensor field of second rank $\phi_{\mu\nu}(x)$. The field equation for massive particles can be constructed by using the root method. The projection and transition operators which we need are listed in appendix B. They satisfy the relation

$$P^J_{ij} P^L_{k\ell} = \delta^{JL} \delta_{jk} P^J_{i\ell} . \tag{3.1}$$

We start with the most general homogeneous second order differential operator $O_{\mu\nu,\rho\sigma}$. When we expand this operator in projection and transition operators, it takes the following form

$$O = (aP^2 + bP^1 + cP^0_{11} + dP_{22} + e\sqrt{3}\,P^0_{12} + f\sqrt{3}\,P^0_{21}) . \tag{3.2}$$

Terms proportional to $\Box^{-1}$ in (2.2) are eliminated if the following condition is satisfied

$$\frac{2}{3} a - 2b + \frac{1}{3} c + d - (e+f) = 0 . \tag{3.3}$$

The root method which guarantees description of spin 2 alone implies that

$$O^N = P^2 \Box^N \tag{3.4}$$

holds for some integer N. Since P^2 contains terms proportional to $\Box^{-2}$ and $\Box^{-1}$, whereas $\mathcal{O}$ does not, the lowest N-value for which (3.4) is satisfied must be N=2. For higher N-values it is then automatically satisfied.

The consequences of equation (3.4) can easily be understood if we use the matrix representation of $\mathcal{O}$ which follows from (3.2):

$$\mathcal{O} = \begin{pmatrix} a & & & \\ & b & & \\ & & c & f\sqrt{3} \\ & & e\sqrt{3} & d \end{pmatrix} . \tag{3.5}$$

Since, according to (3.4), $\mathcal{O}$ squared must be proportional to P^2, we find:

$$\begin{aligned} &a=1, \quad b=0, \\ &c+d=0, \\ &cd-3ef=0. \end{aligned} \tag{3.6}$$

The last two conditions correspond to the nilpotency of the spin-0 submatrix of $\mathcal{O}$.

Thus a massive spin-2 field equation has been constructed

$$\mathcal{O}_{\mu\nu,\rho\sigma}\phi_{\rho\sigma} - m^2\phi_{\mu\nu} = 0 \ , \tag{3.7}$$

where $\mathcal{O}$ is given by (3.2).
The coefficients in (3.2) have to satisfy the conditions (3.3) and (3.6). As was explained before (3.7) is equivalent to the Klein-Gordon equation

$$(\Box - m^2)\phi_{\mu\nu} = 0 \ , \tag{3.8}$$

and the subsidiary condition

$$P^2_{\mu\nu,\rho\sigma}\phi_{\rho\gamma} = \phi_{\mu\nu} \ , \tag{3.9}$$

which is equivalent to

$$\partial_\mu\phi_{\mu\nu} = \phi_{\mu\mu} = 0 \ . \tag{3.10}$$

However, the solution of (3.3) and (3.6) does not lead to a symmetric wave operator in (3.7). Therefore equation (3.7) cannot directly be obtained from a Lagrangian.

In order to symmetrize the wave operator we redefine the field by using a symmetric non singular transformation V:

$$\phi_{\mu\nu} = V_{\mu\nu,\rho\sigma}\phi'_{\rho\sigma} = \phi'_{\mu\nu} + p\delta_{\mu\nu}\delta_{\rho\sigma}\phi'_{\rho\sigma} \ . \tag{3.11}$$

In terms of the projection and transition operator V reads:

$$V = (P^2 + P^1 + \sum_{i,j=1,2} v^0_{ij} P^0_{ij}) \ . \tag{3.12}$$

In general a redefinition like (3.11) does not change the physical context of the theory. In particular it follows from (3.12) that the new field ϕ' still satisfies the subsidiary condition (3.9).

$$(\hat{O} - m^2 V)_{\mu\nu,\rho\sigma}\phi'_{\rho\sigma} = 0 \ , \tag{3.13}$$

where

$$\hat{O} = OV \tag{3.14}$$

is required to be symmetric in $(\mu\nu)$ and $(\rho\sigma)$. The coefficient matrix v^0 in (3.12) is given by

$$v^0 = \begin{pmatrix} 1+3p & p\sqrt{3} \\ p\sqrt{3} & 1+p \end{pmatrix} \ . \tag{3.15}$$

Note that for v^0 to be non singular we should have $1+4p \neq 0$.

Then the requirement that O is symmetric leads to the following condition

$$p(c-d) + (e-f) + p(e-3f) = 0 \ . \tag{3.16}$$

From the conditions (3,3), (3.6) and (3.16) all the coefficients can be calculated. Some of them depend on the variable p alone.

$$\begin{aligned} &a=1, \quad b=0 \ , \\ &c = -d = \tfrac{1}{4}\left(1 + 3(1+2p)s\right) \ , \\ &e = \tfrac{1}{4}\left(1 - (1+6p)s\right) \ , \\ &f = \tfrac{1}{4}\left(1 - (1-2p)s\right) \ , \end{aligned} \tag{3.17}$$

with

$$s^2 = -1/3(1+4p) \ .$$

Then the spin-0 sector of the symmetrized wave operator $\hat{O}$ reads

$$\hat{a}^0 = \tfrac{1}{4}\begin{pmatrix} \left((1+6p) \pm 3(1+4p)s\right) & \left((1+2p) \mp (1+4p)s\right)\sqrt{3} \\ \left((1+2p) \mp (1+4p)s\right)\sqrt{3} & \left((2p-1) \mp 3(1+4p)s\right) \end{pmatrix} \ . \tag{3.18}$$

By choosing the upper sign in (3.18) and by putting p=-1 a special solution is obtained

$$\hat{a}^0 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{3.19}$$

The general solution can be obtained by redefining the field in the Lagrangian, with general transformation V. The wave operator which corresponds to (3.19) reads

$$O = (P^2 + \sum_{ij} \hat{a}^0_{ij} P^0_{ij})\Box \ , \tag{3.20}$$

and it leads to the following field equation

$$\Box(\phi_{\mu\nu} - \delta_{\mu\nu}\phi_{\lambda\lambda}) - (\partial_\mu\partial_\lambda\phi_{\lambda\nu} + \partial_\nu\partial_\lambda\phi_{\lambda\mu}) + \partial_\mu\partial_\nu\phi_{\lambda\lambda} +$$

$$+ \delta_{\mu\nu}\partial_\lambda\partial_k\phi_{\lambda k} - m^2(\phi_{\mu\nu} - \delta_{\mu\nu}\phi_{\lambda\lambda}) = 0 \ . \tag{3.21}$$

This field equation can be obtained from the following Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(\phi^2_{\mu\nu,\lambda} - 2\phi_{\mu\nu,\nu}\phi_{\mu\lambda,\lambda} + 2\phi_{\mu\nu,\nu}\phi_{\lambda\lambda,\mu} - \phi_{\lambda\lambda,\mu}\phi_{\nu\nu,\mu}) -$$

$$- \tfrac{1}{4}m^2(\phi^2_{\mu\nu} - \phi^2_{\mu\mu}) \ . \tag{3.22}$$

The propagator of the massive theory can be found by inverting the field equation (3.20). Using the representation of $\hat{a}^0$, given by (3.19), we get

$$\pi(m) = \left(\frac{P^2}{\Box-m^2} - \frac{P^1}{m^2} + 2\,\frac{\Box-m^2}{3m^4}\,P^0_{22} + \frac{\sqrt{3}}{3m^2}\,(P^0_{12}+P^0_{21})\right) \ . \tag{3.23}$$

One immediately observes the first order pole in $(\Box-m^2)$. The P^2 projection operator in the pole part guarantees propagation of the highest spin modes only. As was already shown for the general case in the previous chapter, if the propagator is sandwiched between two sources, the corresponding residue $T_{\mu\nu}\pi_{\mu\nu,\rho\sigma}(m)T_{\rho\sigma}$ is positive definite.

Thus a massive spin-2 field theory satisfying the earlier mentioned conditions is obtained.

In the rest of this section we shall investigate whether the field equation (3.21) for vanishing mass will give a correct field equation describing massless spin 2-particles. By putting m=0, equation (3.21), in presence of a source $T_{\mu\nu}$, reads

$$O_{\mu\nu,\rho\sigma}\phi_{\rho\sigma} = T_{\mu\nu} \tag{3.24}$$

where

$$O_{\mu\nu,\rho\sigma} = \Big[\tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - 2\delta_{\mu\nu}\delta_{\rho\sigma}) + \omega_{\mu\nu}\delta_{\rho\sigma} - \omega_{\rho\sigma}\delta_{\mu\nu}$$

$$- \tfrac{1}{2}(\omega_{\mu\rho}\delta_{\nu\sigma} + \omega_{\mu\sigma}\delta_{\nu\rho} + \omega_{\nu\rho}\delta_{\mu\sigma} + \omega_{\nu\sigma}\delta_{\mu\rho})\Big]\Box \ . \tag{3.25}$$

Explicitly the field equation (3.24) reads

$$\Box\phi_{\mu\nu} - \partial_\mu(\partial\cdot\phi)_\nu - \partial_\nu(\partial\cdot\phi)_\mu + \partial_\mu\partial_\nu\phi_{\lambda\lambda} - \delta_{\mu\nu}(\Box\phi_{\lambda\lambda} - \partial\cdot\partial\cdot\phi) = T_{\mu\nu} \ . \tag{3.26}$$

The operator O can also be expressed in terms of spin projection operators:

$$O = (P^2 - 2P^0_{11})\Box \tag{3.27}$$

and consequently it takes the following matrix form with respect to these spin projection operators:

$$O = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -2 & 0 \\ & & 0 & 0 \end{pmatrix} . \tag{3.28}$$

From (3.28) it is easily seen that O possesses two left and two right null vectors:

in the spin-1 sector $x^1 = \alpha$, and
in the spin-0 sector $x^2 = (0,\beta)$, α and β real. (3.29)

The right null vectors expressed in terms of projection operators of O defines the following gauge transformation

$$\phi_{\mu\nu} \to \phi_{\mu\nu} + (\textstyle\sum_\ell x^J_\ell P^J_{\ell m})_{\mu\nu\rho\sigma}\,\Box\chi_{\rho\sigma} \ , \tag{3.30}$$

with J=0, and $\chi_{\rho\sigma}$ arbitrary. Clearly the field equation (3.24) is invariant under transformation (3.30).

The various gauge transformations for different J and m can be combined in order to eliminate singular terms, proportional to $\Box^{-1}$. Thus the transformations (3.30) turn into

$$\phi_{\mu\nu} \to \phi_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu \ , \tag{3.31}$$

where ζ_ν denotes an arbitrary vector field.

On the other hand constraints on the source $T_{\mu\nu}$ are defined by the left null vectors of O.

For J=0,1

$$(\textstyle\sum_\ell x^J P^J_{m\ell}\Box)_{\mu\nu,\rho\sigma}T_{\rho\sigma} = 0 \ . \tag{3.32}$$

Again a convenient combination of these source constraints leads to a form of the source constraint that does not contain negative powers of $\Box$

$$\partial_\nu T_{\mu\nu} = 0 \ . \tag{3.33}$$

We shall consider now the field equation, given by (3.24) and (3.27) or by (3.26) in absence of external sources.

After contraction of the indices μ and ν we have

$$(P^2 - 2P^0_{11})_{\mu\mu,\rho\sigma}\phi_{\rho\sigma} = 0 \ ,$$

or (3.34)

$$\Box\phi_{\lambda\lambda} - \partial\partial\cdot\phi = 0 \ .$$

Therefore the field equation (3.26) becomes

$$\Box\phi_{\mu\nu} - \partial_\nu(\partial\cdot\phi)_\nu - \partial_\nu(\partial\cdot\phi)_\mu + \partial_\mu\partial_\nu\phi_{\lambda\lambda} = 0 \tag{3.35}$$

Like the spin-1 case the wave operator in (3.35) is not invertible since it has null vectors. In order to construct a propagator we shall use a gauge condition. We choose a gauge condition such that the field equation turns into the massless Klein-Gordon equation

$$\Box\phi_{\mu\nu}(x) = 0. \tag{3.36}$$

In the last chapter we noticed that gauge transformations (3.31) can be used, in order to obtain a class of field functions $\phi_{\mu\nu}(x)$, which satisfy such a gauge condition.

Here we choose the gauge condition to be

$$\left(P^1 + \frac{3}{2}\,P^0_{11} + \frac{1}{2}\,P^0_{22} - \frac{1}{2}\,\sqrt{3}(P^0_{12}+P^0_{21})\right)_{\mu\nu,\rho\sigma}\phi_{\rho\sigma} = 0,$$

or equivalently (3.37)

$$\partial_\mu(\partial\cdot\phi)_\nu + \partial_\nu(\partial\cdot\phi)_\mu - \partial_\mu\partial_\nu\phi'_{\lambda\lambda} - \delta_{\mu\nu}(\partial\cdot\partial\cdot\phi - \tfrac{1}{2}\Box\phi_{\lambda\lambda}) = 0$$

Clearly from (3.37) one obtains a simpler form of the covariant gauge condition (usually called the harmonic gauge)

$$\partial_\nu\phi_{\nu\mu} - \frac{1}{2}\,\partial_\mu\phi_{\lambda\lambda} = 0 \ . \tag{3.38}$$

Inserting (3.38) in the free field equation (3.35) leads indeed to the Maxwell equation (3.36).

Obviously the free field equation (3.35) together with the gauge invariance (3.31) and the gauge condition (3.38) forms a special case of the set of equations that was discussed in general in section 3 of the previous chapter. From this discussion it follows that the solution of (3.35) describes only the highest helicity modes ± 2.

The fact that this theory describes only the highest helicity modes can also be seen by studying the propagator. The propagator cannot be obtained by inverting the wave operator in (3.27) since the latter has null vectors. However, by using the gauge condition the wave operator can be brought into the form

$$\mathcal{O} = \left(P^2 + P^1 - \frac{1}{2} P^0_{11} + \frac{1}{2} P^0_{22} - \frac{1}{2} \sqrt{3}(P^0_{12}+P^0_{21})\right)\Box \quad . \tag{3.39}$$

The inverse of this expression defines a propagator

$$\pi = \frac{1}{\Box} \left(P^2 + P^1 - \frac{1}{2} P^0_{11} + \frac{1}{2} P^0_{22} - \frac{1}{2} \sqrt{3}(P^0_{12}+P^0_{21})\right) \tag{3.40}$$

which can be written as

$$\pi_{\mu\nu,\rho\sigma} = \frac{1}{\Box} \left[\frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}) - \frac{1}{2} \delta_{\mu\nu}\delta_{\rho\sigma}\right] . \tag{3.41}$$

More generally a propagator is obtained by choosing a gauge condition. With this gauge condition the wave operator $\mathcal{O}$ can be made invertible [1]. Its inverse gives a propagator, which consists of two parts

$$\pi = \pi_1 + \pi_2 \tag{3.42}$$

π_1 corresponds to the inverse of the regular part of $\mathcal{O}$. For the case of spin-2

$$\pi_1 = \frac{1}{\Box} (P^2 - \frac{1}{2} P^0_{11}) \ . \tag{3.43}$$

The second part, depending on the choice of the gauge condition, is proportional to the null vectors of $\mathcal{O}$:

$$\pi_2 = \frac{1}{\Box} \ \Big(\sum_{J=0,1} \alpha^J x^J_i x^J_j P^J_{ij} \Big) \ . \tag{3.44}$$

It should be noted that, due to the source constraint, the expression $\bar{T}\pi_2 T$ always vanishes. Consequently the expression $\bar{T}\pi T = \bar{T}\pi_1 T$ is only determined by the regular part of $\mathcal{O}$.

In case the gauge condition (3.38) is chosen π_2 turns out to be

$$\pi_2 = \frac{1}{\Box} \left(P^1 + \frac{1}{2} P^0_{22} - \frac{1}{2} \sqrt{3}(P^0_{12}+P^0_{21})\right) \ , \tag{3.45}$$

and we find the massless propagator as given by (3.41).

We shall study the expression $\bar{T}\pi T$, where T is a physical source. In order to evaluate this expression in momentum space we shall give a decomposition of the source in terms of the polarization vectors ε^i_μ (i=1,2) and the vectors $k_\mu = (\vec{k},k_4) = (\vec{k},ik_0)$ and $\bar{k}_\mu = (\vec{k},-ik_0)$. (See also App. C). On-shell these vectors satisfy the following relations

$$\begin{aligned} &k^2 = \bar{k}^2 = 0 \ , \\ &k_\mu \varepsilon^i_\mu = \bar{k}_\mu \varepsilon^i_\mu = 0 \ , \\ &\varepsilon^i_\mu \varepsilon^j_\mu = \delta_{ij} \ , \\ &\varepsilon^i_\mu \varepsilon^i_\nu = \delta_{\mu\nu} - \frac{k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu}{(k,\bar{k})} \ . \end{aligned} \tag{3.46}$$

The most general decomposition of $T_{\mu\nu}$ in terms of these vectors:

$$\begin{aligned} T_{\mu\nu} = {} & \varepsilon^i_\mu \varepsilon^j_\nu T^{ij} + (\varepsilon^i_\mu k_\nu + \varepsilon^i_\nu k_\mu) A^i + (\varepsilon^i_\mu \bar{k}_\nu + \varepsilon^i_\nu \bar{k}_\mu) B^i + \\ & + k_\mu k_\nu A + (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu) B + \bar{k}_\mu \bar{k}_\nu C \ . \end{aligned} \tag{3.47}$$

Since physical sources satisfy source constraints, which in momentum representation are given by

$$k_\mu T_{\mu\nu} = 0 \ , \tag{3.48}$$

we get

$$B^i = B = C = 0.$$

Thus the source $T_{\mu\nu}$ can be written as

$$T_{\mu\nu} = \varepsilon^i_\mu \varepsilon^j_\nu T^{ij} + (\varepsilon^i_\mu k_\nu + \varepsilon^i_\nu k_\mu) A^i + k_\mu k_\nu A \ . \tag{3.49}$$

We can evaluate the expression $\bar{T}\pi T$ now.

$$\bar{T}\pi T = \frac{1}{k^2} \left(\bar{T}_{\mu\nu} T_{\mu\nu} - \frac{1}{2} \bar{T}_{\mu\mu} T_{\nu\nu} \right) \ . \tag{3.50}$$

Using the fact that terms with $\varepsilon^{(\pm)}_\mu \varepsilon^{(\pm)}_\nu$ correspond to helicity modes ± 2, where

$$\varepsilon^{(\pm)}_\mu = \mp \frac{1}{\sqrt{2}} (\varepsilon^1_\mu \pm \varepsilon^2_\mu) \ , \tag{3.51}$$

we find as ± 2 helicity parts in the source decomposition

$$T^{\pm} := \frac{1}{2}(T^{11} - T^{22}) \pm iT^{12} \; . \qquad (3.52)$$

Then one easily shows that the residue of the pole $\frac{1}{k^2}$ in (3.50) is given by

$$\bar{T}^{ij}T^{ij} - \frac{1}{2}\bar{T}^{ii}T^{jj} = |T^{+}|^2 + |T^{-}|^2 \; . \qquad (3.53)$$

Thus the following has been shown:

i) The expression $\bar{T}\pi T$ has a single pole in k^2 .

ii) The corresponding residue is positive definite.

iii) The form of the residue shows propagation of only the helicities ± 2.

A massive field theory for spin-2 particles has been constructed by using the root method. The corresponding massless theory followed from the massive theory by putting m=0. It was verified that this theory describes only two helicity states ± 2.

The massive and massless free field theories constructed here were earlier obtained by van Nieuwenhuizen [2]. He also proved the uniqueness of both the massive and massless Lagrangian for the case of spin 2. In particular the massless spin-2 theory is equivalent to the linearized version of Einstein's theory of gravity. This so-called linearized theory of gravitation forms a starting point for the construction of a quantum theory of gravitation. The particles associated with the field $\phi_{\mu\nu}$ are, after quantization, called gravitons.

4. Spin-3 free field theory

In this section we shall first discuss the construction of a field theory for massive spin-3 particles. Then we shall pay attention to the transition to the massless case.

For the description of massive particles with spin 3 we use a symmetric tensor field of rank 3: $\phi_{\lambda\mu\nu}(x)$. Again we shall make use of spin projection operators and transition operators P^J_{ij}. For the spin-3 case these operators are listed in appendix B. The operators given there form a complete set and moreover they satisfy

$$P^J_{ij}P^L_{k\ell} = \delta^{JL}\delta_{jk}P^J_{i\ell} \; . \qquad (4.1)$$

The field equation can be written as

$$O_{\omega,\omega'}\phi_{\omega'}(x) = m^2\phi_\omega \ , \qquad (4.2)$$

where ω is a shorthand notation for the three 4-indices $\lambda\mu\nu$, and where O is a second order differential operator. The most general second order differential operator can be expanded in the operators P^J_{ij} as follows:

$$O = \left(a^3P^3 + a^2P^2 + \sum_{J=0,1} \sum_{i,j=1,2} a^J_{ij}P^J_{ij}\right)\square \ . \qquad (4.3)$$

The coefficients a^J_{ij} have to be chosen in such a way that O does not contain negative powers of $\square$.

Clearly the matrices a^J of the spin-J sector have dimension 1, if J=2,3, and dimension 2 if J=0,1. According the root method the differential operator O has to satisfy

$$O^N = P^3\square^N \qquad (4.4)$$

for some integer N.
Since O does not contain negative powers of $\square$, whereas P^3 contains terms proportional to $\square^{-3}$, the smallest N-value, for which (4.4) can be satisfied, is N=3. Then it automatically holds for greater N-values.

Because the matrices a^0 and a^1 both are 2×2 matrices, eq. (4.4) is satisfied for N=2, which is smaller than the minimal value N=3, mentioned above. Consequently, the dimension of at least one of the matrices a^0 or a^1 should be enlarged. This can be done by introducing an auxiliary field. In this case the only possibility is addition of an auxiliary vector field $A_\mu(x)$ (see the previous chapter, section 2). Accordingly, the set of spin projection and transition operators is extended with the set of projection and transition operators associated with the vector field $A_\mu(x)$. These operators can also be found in appendix B. Therefore, eq. (4.3) should be modified as follows

$$O = \left(a^3P^3 + a^2P^2 + \sum_{J=0,1} \sum_{i,j=1,2,3} a^J_{ij}P^J_{ij}\right)\square \ . \qquad (4.5)$$

In contrast to (4.3) the i,j-summation now runs over 1,2 and 3, because of the enlargement of the spin-0 and spin-1 sectors.

Our field configuration can be represented symbolically as

$$\phi_\omega = \begin{pmatrix} \phi_{\lambda\mu\nu} \\ A_\rho \end{pmatrix} . \qquad (4.6)$$

Now the necessity of an auxiliary field has been demonstrated, we can give a wave operator in which negative powers of $\Box$ do not appear. With respect to the original tensor field $\phi_{\lambda\mu\nu}$ six, in this sense regular, second order differential operators can be found, and we directly expand them in spin projection and transition operators P^J_{ij} :

$$\frac{1}{6}\sum_{\alpha\beta\gamma} \Box\delta_{\alpha\mu}(\delta_{\beta\nu}\delta_{\gamma\lambda} + \delta_{\beta\lambda}\delta_{\gamma\nu}) = (P^3 + P^2 + P^1_{11} + P^1_{22} + P^0_{11} + P^0_{22})\,\Box\,,$$

$$\frac{1}{6}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \omega_{\alpha\mu}(\delta_{\beta\nu}\delta_{\gamma\lambda} + \delta_{\beta\lambda}\delta_{\gamma\nu})\Box = (P^2 + 2P^1_{22} + P^0_{22} + 3P^0_{11})\,\Box\,,$$

$$\frac{1}{3}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \delta_{\alpha\beta}\delta_{\gamma\lambda}\delta_{\mu\nu}\Box = \left(5P^1_{22} + \sqrt{5}(P^1_{21}+P^1_{12}) + P^1_{11} + 3(P^0_{11}+P^0_{22}+P^0_{12}+P^0_{21})\right)\Box\,,$$

$$\frac{1}{9}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \delta_{\alpha\beta}\omega_{\gamma\lambda}\delta_{\mu\nu}\Box = (P^0_{11} + P^0_{22} + P^0_{12} + P^0_{21})\,\Box\,, \tag{4.7}$$

$$\frac{1}{3}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \omega_{\alpha\beta}\delta_{\gamma\lambda}\delta_{\mu\nu}\Box = (3P^0_{11} + P^1_{11} + 3P^0_{12} + \sqrt{5}P^1_{12})\,\Box\,,$$

$$\frac{1}{3}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \delta_{\alpha\beta}\delta_{\gamma\lambda}\omega_{\mu\nu}\Box = (3P^0_{11} + P^1_{11} + 3P^0_{21} + \sqrt{5}P^1_{21})\,\Box\,.$$

Associated with the auxiliary vector field A_ρ we have eight other regular second order differential operators, not containing negative powers of $\Box$.

$$(\delta_{\rho\sigma} - \omega_{\rho\sigma})\Box = P^1_{33}\Box\,; \quad \omega_{\rho\sigma}\Box = P^0_{33}\Box\,,$$

$$\frac{1}{3}\sum_{\alpha\beta\gamma} \delta_{\alpha\beta}\omega_{\gamma\sigma}\Box = (P^0_{13} + P^0_{23})\Box\,,$$

$$\frac{1}{3}\sum_{\lambda\mu\nu} \omega_{\rho\lambda}\delta_{\mu\nu}\Box = (P^0_{31} + P^0_{32})\Box\,,$$

$$\sum_{\alpha\beta\gamma} \omega_{\alpha\beta}\delta_{\gamma\sigma}\Box = (\sqrt{3}P^1_{13} + 3P^0_{13})\,\Box\,, \tag{4.8}$$

$$\sum_{\lambda\mu\nu} \delta_{\rho\lambda}\omega_{\mu\nu}\Box = (\sqrt{3}P^1_{31} + 3P^0_{31})\,\Box\,,$$

$$\sum_{\alpha\beta\gamma} (\delta_{\alpha\beta}-\omega_{\alpha\beta})\delta_{\gamma\sigma}\,\Box = (\sqrt{15}P^1_{23} + 3P^0_{13})\,\Box\,,$$

$$\sum_{\lambda\mu\nu} \delta_{\rho\lambda}(\delta_{\mu\nu}-\omega_{\mu\nu}) = (\sqrt{15}P^1_{32} + 3P^0_{31})\,\Box\,.$$

Therefore, the most general wave operator, homogeneous in second order wave operator, homogeneous in second order partial derivatives and not containing negative powers of $\Box$, will become a linear combination of the differential operators, listed in (4.7). With respect to the set of spin projection operators O takes the following form

$$O = \Big[A(P^3 + P^2 + P^1_{11} + P^1_{22} + P^0_{11} + P^0_{22}) +$$

$$+ B(P^2 + 2P^1_{11} + P^0_{22} + 3P^0_{11}) + C\big(5P^1_{22} + P^1_{11} + \sqrt{5}(P^1_{12} + P^1_{21}) +$$

$$+ 3(P^0_{11} + P^0_{22} + P^0_{21} + P^0_{12})\big)$$

$$+ D(P^0_{11} + P^0_{22} + P^0_{12} + P^0_{21}) + E(P^1_{11} + \sqrt{5}P^1_{12} + 3P^0_{11} + 3P^0_{12}) +$$

$$+ F(P^1_{11} + \sqrt{5}P^1_{21} + 3P^0_{11} + 3P^0_{21}) +$$

$$+ GP^1_{33} + H(\sqrt{3}P^1_{13} + 3P^0_{13}) + I(\sqrt{3}P^1_{31} + 3P^0_{31}) + P(\sqrt{15}P^1_{23} + 3P^0_{23}) +$$

$$+ Q(\sqrt{15}P^1_{32} + 3P^0_{32}) + L(P^0_{13} + P^0_{23}) + M(P^0_{31} + P^0_{32}) + KP^0_{33}\Big]\Box \; . \qquad (4.9)$$

Thus, O has become a 14 parameter dependent differential operator.

Note that in eq. (4.9) the part with coefficients A-F is related to the original tensor field $\phi_{\lambda\mu\nu}$, the other part being related to the auxiliary fields. Equivalently O can be represented by its coefficient matrices a^J_{ij} from (4.5). From the last eq. (4.9) it follows that these matrices are given by

$$a^3 = A$$

$$a^2 = A+B$$

$$a^1 = \begin{pmatrix} A+2B+C+E+F & (C+E)\sqrt{5} & H\sqrt{3} \\ (C+F)\sqrt{5} & A+5C & P\sqrt{15} \\ I\sqrt{3} & Q\sqrt{15} & G \end{pmatrix}$$

$$a^0 = \begin{pmatrix} A+3(B+C+E+F)+D & D+3(C+E) & L+3H \\ D+3(C+F) & A+B+3C+D & L+3P \\ M+3I & M+3Q & K \end{pmatrix} \qquad (4.10)$$

According to the root method the condition $(a^J)^3 = 0$ has to be satisfied for all J, but for at least one J value

$$(a^J)^2 \neq 0 \; . \qquad (4.11)$$

This leads to a number of restrictions on the coefficients a^J_{ij}.

$$A = 1, \quad B = -1 . \tag{4.12}$$

And for J = 0,1 :

$$\left.\begin{aligned} &\mathrm{Tr}(a^J) = 0, \\ &\begin{vmatrix} a^J_{11} a^J_{12} \\ a^J_{21} a^J_{22} \end{vmatrix} + \begin{vmatrix} a^J_{11} a^J_{13} \\ a^J_{31} a^J_{33} \end{vmatrix} + \begin{vmatrix} a^J_{22} a^J_{23} \\ a^J_{32} a^J_{33} \end{vmatrix} = 0 , \\ &\mathrm{Det}(a^J) = 0 , \end{aligned}\right\} \tag{4.13}$$

where the matrix elements a^J_{ij} are given by (4.11).

In general the field equation

$$\mathcal{O}_{\omega,\omega'}\phi_{\omega'} - m^2\phi_\omega = 0 , \tag{4.14}$$

obtained by using the root method, does not follow directly from a Lagrangian, since the wave operator $\mathcal{O}$ is not symmetric in ω and ω'. It is possible, however, to redefine the field ϕ_ω, by means of a local transformation V with non vanishing determinant

$$\phi_\omega = V_{\omega,\omega'}\phi'_{\omega'} , \tag{4.15}$$

or explicitly

$$\left.\begin{aligned} \phi_{\lambda\mu\nu} &= \phi'_{\lambda\mu\nu} + \frac{p}{9}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu}\delta_{\lambda\mu}\delta_{\nu\alpha}\delta_{\beta\gamma}\phi'_{\alpha\beta\gamma} + q\sum_{\lambda\mu\nu}\delta_{\lambda\mu}\delta_{\gamma\sigma}A'_\sigma \\ A_\rho &= (1+s)\delta_{\rho\sigma}A'_\sigma + r\sum_{\lambda\mu\nu}\delta_{\rho\lambda}\delta_{\mu\nu}\phi'_{\lambda\mu\nu} . \end{aligned}\right. \tag{4.16}$$

If V is expressed in terms of projection and transition operators, V turns out to act as an identity in the spin-3 and spin-2 sector. Therefore, condition (4.4) is still satisfied.

In the lower spin sectors the matrix representation of V with respect to the projection and transition operators is given by

$$V^1 = \begin{pmatrix} 1+\frac{1}{3}p & \frac{1}{3}p\sqrt{5} & q\sqrt{3} \\ \frac{1}{3}p\sqrt{5} & 1+\frac{5}{3}p & q\sqrt{15} \\ r\sqrt{3} & r\sqrt{15} & 1+s \end{pmatrix}$$

$$V = \begin{pmatrix} 1+p & p & 3q \\ p & 1+p & 3q \\ 3r & 3r & 1+s \end{pmatrix} \tag{4.17}$$

In order V to be invertible we must have

$$18qr - (1+s)(1+2p) \neq 0 \; . \tag{4.18}$$

By (4.15) or (4.16) the original field equation (4.14) turns into

$$(\mathcal{O}V) - m^2V)_{\omega,\omega'}\phi'_{\omega'} = 0 \; . \tag{4.19}$$

The operator $\mathcal{O}V-m^2V$ is symmetric if

i) V is symmetric; this implies

$$q=r \; , \tag{4.20}$$

ii) $\hat{\mathcal{O}} = \mathcal{O}V$ is symmetric, or

$$\hat{\mathcal{O}}_{\omega,\omega'} = \hat{\mathcal{O}}_{\omega',\omega} \; . \tag{4.21}$$

As a consequence of (4.21) the submatrices $\hat{a}^J = a^J v^J$ (J=0,1) of $\hat{\mathcal{O}}$ have to be symmetric, i.e.

$$\hat{a}^J_{\omega,\omega'} = \hat{a}^J_{\omega',\omega} \; , \tag{4.22}$$

Further, by imposing the additional requirements

$$\hat{a}^J_{13} = \hat{a}^J_{23} = 0 \; , \quad \text{for } J=0,1 \; , \tag{4.23}$$

a convenient transition to the massless theory becomes possible.

The conditions (4.12), (4.13), (4.20), (4.22) and (4.23) lead to a number of equations for the original parameters A, B, etc. These equations are solvable, as can be shown by giving one particular solution:

$$\begin{aligned} &A=1, \quad B=-1 \\ &p=-3, \quad s=-(18r^2+1), \quad r\neq 0 \; . \end{aligned} \tag{4.24}$$

The matrices a^J and $\hat{a}^J$ are then given by:

$$a^0 = \frac{1}{4}\begin{pmatrix} 3 & -1 & \frac{1}{3}r^{-1} \\ 9 & -3 & r^{-1} \\ 0 & 0 & 0 \end{pmatrix} \;;\; \hat{a}^0 = -\frac{1}{2}\begin{pmatrix} 1 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a^1 = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 5\sqrt{5} & 1 & \frac{1}{3}r^{-1}\sqrt{15} \\ -\frac{1}{5}r\sqrt{3} & -\frac{1}{5}r\sqrt{15} & -1 \end{pmatrix} \;;\; \hat{a}^1 = -4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{5}r^2 \end{pmatrix} \tag{4.25}$$

It is easily verified that $(a^1)^3 = 0$, whereas $(a^1)^2 \neq 0$, which was required in (4.11). Furthermore, the matrices $\hat{a}^J$ are symmetric.

The solution given above corresponds to the following field equation

$$(\Box - m^2)(\phi_{\lambda\mu\nu} - \sum_{\lambda\mu\nu} \delta_{\lambda\mu}\phi'_\nu) + \sum_{\lambda\mu\nu} [\partial_\lambda\partial_\mu\phi_\nu - \partial_\lambda(\partial\cdot\phi)_{\mu\nu}] +$$

$$\sum_{\lambda\mu\nu} \delta_{\lambda\mu}\left(\partial\cdot\partial\cdot\phi_\nu - \frac{1}{2}\partial_\nu(\partial\cdot\phi')\right) - m^2 r \sum_{\lambda\mu\nu} \delta_{\lambda\mu}A_\nu = 0 \,, \tag{4.26}$$

$$\frac{12}{5} r^2\left(\Box A_\rho - \partial_\rho(\partial\cdot A)\right) - m^2(-18A_\rho + 3r\phi'_\rho) = 0 \,.$$

Here again $\partial\cdot A$ and ϕ'_μ denote contractions.

The Lagrangian, from which this field equation follows is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\rho\phi_{\lambda\mu\nu})^2 + \frac{3}{2}(\partial\cdot\phi)_{\mu\nu}(\partial\cdot\phi)_{\mu\nu} + \frac{3}{2}(\partial_\mu\phi'_\nu)^2 + \frac{3}{4}(\partial\cdot\phi')^2 -$$

$$- 3(\partial_\mu\phi'_\nu)(\partial\cdot\phi)_{\mu\nu} - \frac{6}{5}r^2\left[(\partial_\mu A_\nu)^2 - (\partial\cdot A)^2\right] -$$

$$- \frac{1}{2}m^2\left[\phi^2_{\lambda\mu\nu} - 3(\phi'_\mu)^2 + 6r\phi'_\mu A_\mu - 18r^2(A_\mu)^2\right] \,. \tag{4.27}$$

As indicated by the parameter r, we have obtained a family of Lagrangians. In fact this family of Lagrangians is larger, since it is possible to redefine the field in the Lagrangian by using a symmetric transformation V: $\phi_\omega = V_{\omega,\omega'}\phi'_{\omega'}$.

The original Lagrangian

$$\mathcal{L} = \phi^T_\omega(\hat{O}-m^2)_{\omega,\omega'}\phi_{\omega'} \tag{4.28}$$

then turns into

$$\mathcal{L} = \phi'_\omega(V\hat{O}V - V^2m^2)_{\omega\omega'}\phi'_{\omega'} \quad . \tag{4.29}$$

Clearly VOV is again symmetric for arbitrary p', r' and s' (p', r' and s' being the parameters which determine V). Moreover, for arbitrary p' and s' but r'=0, the new field operator $V\hat{O}V$ still satisfies condition (4.23).

The propagator of the massive theory is given by

$$\pi(m) = \left(\frac{P^3\Box^3}{m^6(\Box^3-m^6)} - \frac{1}{m^6}\right)(O^2+m^2O+m^4) \ . \tag{4.30}$$

As we proved for the general case it has the following properties:

i) $\pi(m)(O-m^2) = \mathbb{1}$

ii) $\pi(m)$ has a first order pole at $\Box$-m^2 .
F rther its residue is positive definite, if $\pi(m)$ is sandwiched between two sources.

A massless field equation can be obtained by taking m=0 in the massive field equation (4.26). Due to the special choice of the coefficients $\hat{a}^J_{ij}$ (see (4.23)) we obtain two equations: one for the original field $\phi_{\lambda\mu\nu}$ and one for the auxiliary field A_ρ. The fields decouple, such that we can take the auxiliary field to be zero.

We shall show that the field equation for the original field describes massless spin-3 particles. For this field equation we use the short-hand notation

$$O_{\lambda\mu\nu,\alpha\beta\gamma}\phi_{\alpha\beta\gamma} = 0 \ . \tag{4.32}$$

Here the wave operator O is still expressed in terms of projection and transition operators

$$O = \left(P^3 - 4P^1_{22} - \frac{1}{2}P^0_{11} - \frac{9}{2}P^0_{22} - \frac{3}{2}(P^0_{12}+P^0_{21})\right)\Box \ . \tag{4.33}$$

The wave equation takes the explicit form:

$$\Box\phi_{\lambda\mu\nu} + \sum\left(\partial_\lambda\partial_\mu\phi'_\nu - \partial_\lambda(\partial\cdot\phi)_{\mu\nu}\right) + \sum\delta_{\lambda\mu}\left((\partial\cdot\partial\cdot\phi)_\nu - \Box\phi'_\nu - \tfrac{1}{2}\partial_\nu(\partial\cdot\phi')\right) = 0\ , \tag{4.34}$$

where the sum is taken over all independent permutations of the indices λ, μ and ν.

The spin-1 and spin-0 sector of the wave operator O can be represented by coefficient matrices O^1 and O^0 , as was done before (see eq. (4.25))

$$O^1 = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \quad ; \qquad O^0 = -\tfrac{1}{2}\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \ . \tag{4.35}$$

Obviously O possesses a set of right and left null vectors

i) in the spin-2 sector $\quad x^2 = \alpha$,
ii) in the spin-1 sector $\quad x^1 = \beta(1,0)$, (4.36)
iii) in the spin-0 sector $\quad x^2 = \gamma(3,-1)$,
with α, β and γ real.

One can redefine the field by the following gauge transformation for J=0,1 and 2

$$\phi_\omega \to \phi_\omega + \sum_\ell x^J_\ell (P^J_{\ell,m})_{\omega,\omega'} \Box\chi_{\omega'}\ , \tag{4.37}$$

with χ_ω an arbitrary rank 3 tensor.

Here the x^J represent the right null vectors of O and consequently the field equation remains invariant under (4.37). Thus a gauge invariance of the theory is caused by the right null vector of the corresponding wave operator O. Again it is possible to combine the gauge transformations (4.37) such that terms with $\Box^{-2}$, $\Box^{-1}$ cancel and the gauge transformation takes the form:

$$\phi_{\lambda\mu\nu} \to \phi_{\lambda\mu\nu} + \sum_{\lambda\mu\nu} \partial_\lambda \varepsilon_{\mu\nu}\ . \tag{4.38}$$

where $\varepsilon_{\lambda\mu}$ is an arbitrary traceless rank 2 tensor, i.e.

$$\varepsilon_{\lambda\lambda} = 0\ . \tag{4.39}$$

In case the field $\phi_{\lambda\mu\nu}$ is coupled to an external source $T_{\lambda\mu\nu}$ the field equation takes the form:

$$O_{\lambda\mu\nu,\alpha\beta\gamma}\phi_{\alpha\beta\gamma} = T_{\lambda\mu\nu}\ . \tag{4.40}$$

Multiplying both sides with a combination of left null vectors of O and projection operators gives

$$\sum_{\ell} x_{\ell}^{J} (P_{m\ell}^{J})_{\omega,\omega'} \Box T_{\omega'} = 0 \tag{4.41}$$

for J=0,1,2.

Clearly left null vectors of O cause source constraints. Again the various source constraints (4.41) can be combined such that terms proportional to $\Box^{-2}$, $\Box^{-1}$ cancel. The source constraint can then be written as

$$\sum_{\lambda\mu\nu} \partial_{\lambda}\left((\partial\cdot T)_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}(\partial\cdot T')\right) = 0 \ , \tag{4.42}$$

which is equivalent to

$$\partial_{\lambda} T_{\lambda\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}\partial_{\lambda} T_{\lambda\rho\rho} = 0 \ . \tag{4.43}$$

Like the spin-1 and spin-2 case it should be noticed that the field equation (4.34) and the gauge transformation (4.38) are a special case of the set of equations discussed in the previous chapter, section 3. From the general consideration given there it follows that this system of equations (4.34) and (4.38) describes massless particles with only two helicity modes ± 3. There it was also shown that the field equation could be brought into the simpler form of the massless Klein-Gordon equation

$$\Box\phi_{\lambda\mu\nu}(x) = 0 \ , \tag{4.44}$$

by using the gauge condition

$$(\partial\cdot\phi)_{\mu\nu} = \tfrac{1}{2} \sum_{\lambda\mu} \partial_{\lambda}\phi'_{\mu} \ . \tag{4.45}$$

Instead of the free field equation, we can consider the field equation in presence of a physical source (4.40). In this case we shall study the propagator π and in particular we shall show that the pole part of $\bar{T}\pi T$ (the propagator sandwiched between two physical sources) gives rise to propagation of only helicity modes ± 3. However, due to its null vector, it is no longer possible to define the massless propagator as the inverse of the wave operator O.

By using a gauge condition, like for instance (4.45), it is possible to make the operator O regular. The propagator is defined by the inverse of the wave operstor:

$$\pi = \pi_1 + \pi_2 \ . \tag{4.46}$$

Here π_1 is the inverse of the regular part of O. In particular inverting the regular submatrices O^1 and O^0 of O gives

$$\pi_1 = \frac{1}{\Box}\,(P^3 - \tfrac{1}{4}P^1_{22} - 2P^0_{11})\,\Box \quad . \tag{4.47}$$

The second term in (4.46) depends on the choice of the gauge condition. But in any case it turns out to be proportional to a number of left null vectors of O:

$$\pi_2 = \frac{1}{\Box}\,\Big(\alpha P^2 + \beta P^1_{11} + \gamma\sqrt{5}(P^1_{21} + P^1_{12}) + \delta\,(6P^0_{11} - P^0_{12} - P^0_{21}) + \\ + \varepsilon(3P^0_{21} + 3P^0_{12} + 2P^0_{22})\Big). \tag{4.48}$$

Therefore this last term, sandwiched between two physical sources, always vanishes due to the source constraints (4.41) and we get

$$\bar{T}\pi T = \bar{T}\pi_1 T \quad . \tag{4.49}$$

Thus the expression $\bar{T}\pi T$ appears to be uniquely determined by the regular part of the wave operator O.

If we put $\alpha=1$, $\beta=\frac{3}{4}$, $\gamma=-\frac{1}{4}$, $\delta=\frac{3}{8}$ and $\varepsilon=-\frac{1}{8}$ in eq. (4.48), the massless propagator (4.46) can be written as follows:

$$\pi = \frac{1}{\Box}\,\Big(\frac{1}{6}\sum_{\lambda\mu\nu}\delta_{\alpha\lambda}(\delta_{\beta\mu}\delta_{\gamma\nu} + \delta_{\beta\nu}\delta_{\gamma\mu}) - \frac{1}{12}\sum_{\lambda\mu\nu}\sum_{\alpha\beta\gamma}\delta_{\alpha\lambda}\delta_{\beta\mu}\delta_{\gamma\nu}\Big). \tag{4.50}$$

It can thus be shown that this form of the propagator corresponds to the choice of the gauge condition given by (4.45).

We shall evaluate now the expression $\bar{T}\pi T$. in order to show that only helicity modes ± 3 propagate between the sources $T_{\lambda\mu\nu}$. From (4.50) we find

$$\bar{T}\pi T = \frac{1}{\Box}\,\Big(\bar{T}_{\lambda\mu\nu}T_{\lambda\mu\nu} - \frac{3}{4}\,\bar{T}_{\lambda\mu\mu}T_{\lambda\nu\nu}\Big) \; . \tag{4.51}$$

Like we did in the spin-1 and spin-2 case we shall decompose the source in terms of the polarization vectors ε^i_μ , i=1,2, and the vectors $k_\mu = (\vec{k}, ik_0)$ and $\bar{k}_\mu = (\vec{k}, -ik_0)$ (see also app. C).

The decomposition of T is done in momentum space. Furthermore we use the on-shell relations which are satisfied by ε^i_μ, k_μ and $\bar{k}_\mu$.

The most general decomposition of the source

$$T_{\lambda\mu\nu} = \varepsilon^i_\lambda \varepsilon^j_\mu \varepsilon^k_\nu T^{ijk} + \sum_{\lambda\mu\nu} (k_\lambda A^{ij} + \bar{k}_\lambda B)\varepsilon^i_\mu \varepsilon^j_\nu +$$

$$+ \sum_{\lambda\mu\nu} \left(\varepsilon^i_\lambda (k_\mu k_\nu A^i + (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu) B^i + \bar{k}_\mu \bar{k}_\nu C^i)\right) +$$

$$+ k_\lambda k_\mu k_\nu A + \sum_{\lambda\mu\nu} (k_\lambda k_\mu \bar{k}_\nu B + k_\lambda \bar{k}_\mu \bar{k}_\nu C) + \bar{k}_\lambda \bar{k}_\mu \bar{k}_\nu D \tag{4.52}$$

will be restricted by the source constraint.

In momentum space the source constraint reads (see (4.43))

$$k_\lambda \left(T_{\lambda\mu\nu} - \frac{1}{4}\,\delta_{\mu\nu} T_{\lambda\alpha\alpha}\right) = 0\ . \tag{4.53}$$

It is rather easy to show that due to this source constraint the following identities hold

$$\begin{aligned} &B^{ij} = 0, \ \forall i \neq j, \\ &B^i = C^i = B = D = 0\ , \\ &(k,\bar{k})C = \frac{1}{2}\, B^{ii}; \quad B^{11} = B^{22} = C\cdot(k,\bar{k})\ . \end{aligned} \tag{4.54}$$

$T_{\lambda\mu\nu}$ turns into

$$T_{\lambda\mu\nu} = \varepsilon^i_\lambda \varepsilon^j_\mu \varepsilon^k_\nu T^{ijk} + \sum_{\lambda\mu\nu} \left(\varepsilon^i_\lambda \varepsilon^j_\mu k_\nu A^{ij} + \varepsilon^i_\lambda k_\mu k_\nu A^i\right) +$$

$$+ C \sum_{\lambda\mu\nu} \left(\bar{k}_\lambda \varepsilon^i_\mu \varepsilon^i_\nu (k,\bar{k}) + k_\lambda \bar{k}_\mu \bar{k}_\nu\right) + k_\lambda k_\mu k_\nu A\ . \tag{4.55}$$

Now we find as ± 3 helicity source combinations

$$T^\pm = \frac{1}{2\sqrt{2}} \left(\mp T^{111} \pm 3T^{122} + i(3T^{112} - T^{222})\right)\ . \tag{4.56}$$

Here we used the fact that $\varepsilon^{(\pm)}_\lambda \varepsilon^{(\pm)}_\mu \varepsilon^{(\pm)}_\nu$ terms give rise to helicity ± 3, where

$$\varepsilon^{(\pm)}_\lambda = \mp \frac{1}{\sqrt{2}} (\varepsilon^1_\lambda \pm i\varepsilon^2_\lambda)\ . \tag{4.57}$$

We already found that $\bar{T}\pi T$ is given by (see (4.51)):

$$\bar{T}\pi T = \frac{1}{\Box}\left(\bar{T}_{\lambda\mu\nu} T_{\lambda\mu\nu} - \frac{3}{4}\,\bar{T}_{\lambda\mu\mu} T_{\lambda\nu\nu}\right)\ .$$

Using the source decomposition (4.55) and the properties of the polarization vectors ε^i_λ and k_μ, $\bar{k}_\mu$ (see app. C) the pole part of $\bar{T}\pi T$ takes the following form

$$\frac{1}{k^2}\left(T^{ijk}T^{ijk} - \frac{3}{4}T^{iik}T^{jjk}\right) = \frac{1}{4k^2}\left((T^{111} - 3T^{122})^2 + (3T^{112} - T^{222})^2\right) =$$

$$= \frac{1}{k^2}\left(|T^+|^2 + |T^-|^2\right) . \qquad (4.58)$$

From eq. (4.58) follows immediately that

(a) the residue of the $\frac{1}{k^2}$ pole is positive definite and

(b) only ± 3 helicity modes do propagate, since the residue equals the sum of the moduli of the ± 3 helicity combinations T^+ and T^-.

In this section we have constructed field equations and Lagrangians for massive and massless particles with spin-3. In both cases the field equations are homogeneous second order differential equations. The corresponding propagators turned out to have the required properties.

Lagrangians for massive spin-3 particles have also been constructed by Hagen and Singh [3]. However, their field equations are not homogeneous, since they contain also first order partial derivatives.

By using the root method for the case of spin-1, 2 and 3, we were able to eliminate all superfluous spin components in a systematic way in order to obtain the correct massive field equation. The massless field equation could be obtained by putting m=0 in the corresponding massive equation. In the cases considered here they agreed with the equations found by Fronsdal [4]. The massless theories exhibit gauge invariances, leading to source constraints. An advantage of the root method is that the origin of this phenomenon can be clearly understood, at least after performing the $m\to 0$ limit. However, it seems rather difficult to generalize the root method to the case of arbitrary spin. This is due in particular to the fact that a systematic way to choose the field and auxiliary fields has not yet been found.

However, this approach has general features which are useful in the next chapter. There we shall study the problem of the zero mass limit in more detail.

Rerefences

[1] F.A. Berends, J.W. van Holten, P. van Nieuwenhuizen and B. de Wit, Nucl. Phys, B154, 261 (1979).

[2] P. van Nieuwenhuizen, Nucl. Phys. B60, 478 (1973).

[3] L.P.S. Singh, C.R. Hagen, Phys. Rev. D9, 898 (1974).

[4] C.Fronsdal, Phys. Rev. D18, 3624 (1978).

CHAPTER VII

ON THE ZERO MASS LIMIT OF HIGHER SPIN THEORIES

1. Introduction

In this chapter the higher spin field equations, constructed according to the root method will be considered from a different point of view. We shall first repeat the main features of the way in which these equations have been derived. Then it is possible to sketch the problems which we shall discuss here in more detail.

A massive spin-s particle can be described by a symmetric rank s tensor field $\phi_{\mu_1 \ldots \mu_s}(x)$, or $\phi^{(s)}$, and a number of auxiliary fields $\phi^{(s-2)}$, $\phi^{(s-4)}$, etc. The complete field configuration is denoted by ϕ_ω. In the following we assume that a massive field equation can be derived by using the root method, i.e.

$$\mathcal{O}_{\omega,\omega'}\phi_{\omega'} - m^2\phi_\omega = 0 \; , \tag{1.1}$$

where the wave operator $\mathcal{O}$ satisfies

$$\mathcal{O}^n = P^s\Box^n, \quad \text{for some } n\geq s \; . \tag{1.2}$$

In general (1.1) does not immediately follow from a Lagrangian. Therefore it is also assumed that the field ϕ_ω can be redefined by a symmetric, nonsingular transformation V

$$\phi_\omega = V_{\omega,\omega'}\phi'_{\omega'} \; , \tag{1.3}$$

such, that the new wave operator

$$\hat{\mathcal{O}} = \mathcal{O}V \tag{1.4}$$

satisfies certain symmetry conditions, necessary for a derivation from a Lagrangian. Then the massive field equation turns into

$$(\hat{\mathcal{O}}_{\omega,\omega'} - m^2V_{\omega,\omega'})\phi'_{\omega'} = 0 \; . \tag{1.5}$$

As was discussed before, the massless theory can be obtained from (1.5) by putting m=0. In particular, if one makes a suitable choice for V, one obtains a convenient transition to the m=0 theory: then the auxiliary fields decouple directly and can be set equal to zero.

In both the massive and the massless case the fields ϕ_ω are now supposed to be coupled to a similar external source T_ω. The simplest sources are considered, i.e. it is assumed that they do not depend on the mass of the particle.

We shall study the exchange of a spin-s particle between two sources. In lowest order, the amplitude A(m) for the exchange of a massive particle is given by

$$A(m) = \bar{T}\pi(m)T, \tag{1.6}$$

where $\pi(m)$ is the propagator obtained from (1.5).

A similar expression can be given for the exchange of a massless particle between the same sources

$$B = \bar{T}\pi_0 T \ , \tag{1.7}$$

where π_0 represents the massless propagator.

The aim of this chapter is to compare A(m) with B for arbitrary small mass. In other words, we shall study the existence of $\lim_{m\to 0} A(m)$, and we shall investigate whether or not this limit equals B.

This problem has already been studied by van Dam and Veltman for the case of spin 1 and 2. Whereas for the spin-1 case the $m\to 0$ limit does not pose a problem for the particle exchange, they showed that for spin 2 the zero mass limit (1.6) is different from (1.7). We shall show that for spin 3 one cannot even take the $m\to 0$ limit of A(m).

In the next section we shall make some general statements concerning the amplitude A(m). In the third section the special cases of spin 1, 2 and 3 will be considered. Throughout both sections we shall again make use of projection and transition operators. In the final section the results of section 3 will be briefly discussed.

2. The amplitude A(m)

In this section we shall consider the amplitude A(m) from a rather general point of view. The question whether lim A(m) exists and whether it equals the massless amplitude B can then be answered in a systematic way.

We should stress again here, that in both amplitudes the same sources are used. Consequently the sources in the massive and in the massless case both satisfy the constraints which follow from the gauge invariances of the massless theory. It is also assumed that these sources are independent of the mass of the particle.

Our starting point is the massive free field equation (1.1). For the non-symmetric wave operator O, which occurs in (1.1), we give the following decomposition

$$O_{\omega,\omega'} = P^{s}_{\omega,\omega'}\Box + \tilde{O}_{\omega,\omega'} \quad , \tag{2.1}$$

where $\tilde{O}$ is related to the lower spin sectors, i.e.

$$P^{s}\tilde{O} = 0 \ . \tag{2.2}$$

As a consequence of condition (1.2), $\tilde{O}$ satisfies

$$\tilde{O}^{s} = 0, \quad \text{but} \quad \tilde{O}^{s-1} \neq 0 \ . \tag{2.3}$$

For at least one spin sector (2.3) must be satisfied.

In order to evaluate the amplitude A(m) we give the general expression for the propagator in terms of

$$\pi(m) = \frac{P^{s}}{\Box-m^{2}} - \frac{1}{m^{2s}} \ V^{-1} \ \left(\tilde{O}^{s-1} + m^{2}\tilde{O}^{s-2} + \ldots + m^{2(s-1)}(\mathbf{1} - P^{s})\right) \ . \tag{2.4}$$

This propagator is obtained from the symmetric field equation (1.5). Then the amplitude A(m) is given by

$$A(m) = \frac{1}{\Box-m^{2}} \bar{T}P^{s}T - A_{c} \ ,$$

with

$$A_{c} = \frac{1}{m^{2s}} \bar{T}V^{-1} \ \left(\tilde{O}^{s-1} + \tilde{O}^{s-2}m^{2} + \ldots + m^{2(s-1)}(\mathbf{1} - P^{s})\right)T \ . \tag{2.5}$$

Clearly, $\lim_{m\to 0} A(m)$ exists if A_c vanishes for any value of the mass m.

In the following we shall only consider the particular spin-J sector for which (2.3) is satisfied. The corresponding part of $\tilde{O}$ in this sector is given by

$$\tilde{O}^{J}_{\omega,\omega'} = \sum_{i,j=1}^{N} a_{ij}P^{J}_{ij;\omega,\omega'} \quad . \tag{2.6}$$

N denotes the dimension of the matrix a ($N \geq s$).

According to (2.3) we must have

$$a^s = 0 \text{ and } a^{s-1} \neq 0. \tag{2.7}$$

The spin-J part of the sources can be represented as follows

$$T_\omega = \sum_{j=1}^{N} \tau_j e_{j,\omega} . \tag{2.8}$$

Here $\{e_j = e_{j,\omega}\}$ represents an orthonormal set of vectors defined by

$$P^J_{jj} e_j = e_j ; \quad P^J_{ii} e_j = 0, \quad \text{if } i \neq j . \tag{2.9}$$

By using (2.6) and (2.8) we get for A_c in momentum representation

$$A_c = \frac{1}{m^{2s}} \sum_{k,\ell,n} \tau_k (V^{-1})_{k\ell} [\, a^{s-1}_{\ell n} (-q^2)^{s-1} + \ldots + m^{2(s-1)} \delta_{\ell n} \,] \tau_n . \tag{2.10}$$

From (2.10) it is clear that A_c vanishes, only if the numbers τ_k, which determine the source T_ω, satisfy the following n conditions

$$\tau_k (V^{-1})_{k\ell} (a^p)_{\ell n} \tau_n = 0, \quad \text{for } p=0,1,\ldots,s-1. \tag{2.11}$$

Or, in other words, $\lim_{m \to 0} A(m)$ exists, if the numbers τ_k satisfy (2.11).

On the other hand, as already was stressed, we used the same sources in the massive and massless theory. Therefore the massive source must obey the constraints of the massless theory. Such source constraints are caused by left null vectors x_L of a, as was demonstrated in the previous chapter.

By using the representation (2.8) for the massive source, it is easy to see that source constraints must have the following form

$$\sum_{k=1}^{N} x_{L,k} \tau_k = 0 . \tag{2.12}$$

In case of higher spin auxiliary fields have to be introduced. In the massless limit these auxiliary fields decouple. Therefore the source components, associated with these auxiliary fields, vanish. Suppose that in the spin-J sector k components correspond to the original field i.e. $\tau_1, \ldots, \tau_k$, then the remaining N-k components are zero in the massless theory

$$\tau_{k+1} = \tau_{k+2} = \ldots = \tau_N = 0 . \tag{2.13}$$

We summarize this discussion as follows.

In both the massive and the massless case we used the same, mass independent, sources in order to evaluate the amplitudes A(m) and B. Therefore in both cases the sources must satisfy (2.12) and (2.13).

On the other hand, the existence of $\lim_{m\to 0} A(m)$ leads to an additional set of requirements (2.11). Thus is can be understood that $\lim_{m\to 0} A(m)$ - evaluated with sources from the massless theory - exists, if the conditions (2.11) follow from (2.12) and (2.13).

In the next section we shall investigate whether massless sources give rise to existence of this limit in case of spin 1, 2 and 3.

3. The massless limit in case of spin 1, 2 and 3

A. *Spin-1*

The spin-1 case is the simplest case. Condition (2.7) is satisfied in the spin-0 sector which has dimension N=1. With the results of chapter 6, section 2 we find for the massive and massless amplitude A(m) and B

$$A(m) = \frac{1}{\square - m^2}\,\bar{T}P^1T - \frac{1}{m^2}\,\bar{T}P^0T \qquad (3.1)$$

$$B = \frac{1}{\square}\,\bar{T}(P^1+P^0)T\ . \qquad (3.2)$$

In both cases the source satisfies the massless source constraint

$$P^0T = 0\ . \qquad (3.3)$$

Since there are no auxiliary fields in the spin-1 case we do not have conditions of the type (2.13). Clearly $\lim_{m\to 0} A(m)$ exists, and moreover we get

$$\lim_{m\to 0} A(m) = B = \frac{1}{\square}\,\bar{T}P^1T \qquad (3.4)$$

B. *Spin-2*

For the main features of spin-2 field theory we refer to chapter 6, section 3. There it can be checked that condition (2.7) is satisfies in the spin-0 sector, which has dimension N=2.

Firstly we evaluate the source in the spin-0 sector, which in this case is determined by two real numbers τ_1 and τ_2 (see (2.8)). The spin-0 part of the wave operator is given by

$$\hat{a} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.5}$$

Its left null vector $x_L = (0, \lambda)$ leads to the following source constraint (see (2.12)):

$$P^0_{22}T = 0, \quad \text{or} \quad \tau_2 = 0 . \tag{3.6}$$

This is the only condition on the massless source since there are no auxiliary fields in case of spin-2.

Secondly we investigate whether $\lim_{m\to 0} A(m)$ exists. If this limit exists, the requirements (2.11) must be satisfied. Since $\tau_2 = 0$ we get from (2.11):

$$\tau_1(V^{-1})_{11}\tau_1 = 0 ,$$

and (3.7)

$$\sum_{\ell=1}^{2} \tau_1(V^{-1})_{1\ell}a_{\ell 1}\tau_1 = 0 .$$

Here a represents the spin-0 component of the massive wave operator

$$a = \frac{2}{3}\begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix} , \tag{3.8}$$

and V^{-1} is given by

$$V^{-1} = \frac{1}{3}\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix} . \tag{3.9}$$

The expression for V^{-1} can be obtained directly by inverting V, which has been determined in the previous chapter. With (3.8) and (3.9) it can easily be verified that (3.7) is satisfied. Consequently $\lim_{m\to 0} A(m)$ exists.

Finally we check whether $\lim_{m\to 0} A(m)$ equals B.

By using the massive and massless propagator we find for the amplitudes A(m) and B

$$A(m) = \frac{\bar{T}P^2T}{\Box - m^2} - \frac{1}{m^2}\bar{T}P^1T - \frac{1}{m^4}\bar{T}\left(\frac{2}{3}(q^2+m^2)P^0_{22} - \frac{1}{3}\sqrt{3}\,m^2(P^0_{12}+P^0_{21})\right)T , \tag{3.10}$$

$$B = \frac{1}{\Box}\bar{T}(P^2 - \frac{1}{2}P^0_{11})T . \tag{3.11}$$

Note that one can add combinations of P^1, P^0_{12} and P^0_{22} to the massless propagator, without modifying B. This fact is a consequence of the source constraint (3.6). Of course another consequence of (3.6) is that A(m) reduces to

$$A(m) = \frac{1}{\Box - m^2} \bar{T} P^2 T \ , \tag{3.12}$$

from which one can understand that $\lim_{m \to 0} A(m)$ never equals B.

C. *Spin-3*

In this part we investigate the zero mass limit for the spin-3 case. Here we shall use some of the results which were found in chapter 6, section 4. One of these results for instance is the fact that condition (2.7) is satisfied in the spin-1 sector which has dimension N=3.

Consequently the spin-1 component of the source, given by (2.8), is determined by 3 real numbers τ_1, τ_2 and τ_3.

First we evaluate this source in the massless case. The spin-1 part of the wave operator is given by

$$\hat{a}^1 = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} . \tag{3.13}$$

The left null vector $x_L = (\lambda, 0)$ leads to the following source constraint (see (2.12)):

$$P^1_{11} T = 0, \quad \text{or } \tau_1 = 0 \ . \tag{3.14}$$

In the massive spin-3 case we introduced a spin-1 auxiliary field. As a consequence of the decoupling of this field in the massless case, the corresponding source component must be set equal to zero, i.e.

$$\tau_3 = 0 \ . \tag{3.15}$$

Thus the spin-1 part of the source T is determined by the vector $(\tau_1, \tau_2, \tau_3) = (0, \tau, 0)$.

Then we can answer the question whether this source leads to a vanishing A_c. As was explained in the previous chapter this leads to new conditions (2.11).

Since we use the same source $(0, \tau_2, 0)$ in the massive case, the conditions (2.11) reduce to the following set

$$\begin{aligned} &\tau_2 (V^{-1})_{22} \tau_2 = 0 \ , \\ &\tau_2 (V^{-1})_{2\ell} (a^p)_{\ell 2} \tau_2 = 0 \ ; \quad p = 1,2 \ . \end{aligned} \tag{3.16}$$

Here a represents the spin-1 part of the wave operator which, according to the previous chapter, is given by

$$a = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 5\sqrt{5} & 1 & \frac{1}{3}\,r^{-1}\sqrt{15} \\ -\frac{r}{5}\sqrt{3} & -\frac{r}{5}\sqrt{15} & -1 \end{pmatrix}. \tag{3.17}$$

Then

$$a^2 = \frac{2}{15}\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{5} & 0 & 0 \\ -r\sqrt{3} & 0 & 0 \end{pmatrix}. \tag{3.18}$$

From (3.17) and (3.18) one immediately sees that the second requirement of (3.16) is satisfied. According to chapter 6, V is given by

$$V = \begin{pmatrix} 0 & -\sqrt{5} & r\sqrt{3} \\ -\sqrt{5} & -4 & r\sqrt{15} \\ r\sqrt{3} & r\sqrt{15} & -18r^2 \end{pmatrix}. \tag{3.19}$$

Then

$$V^{-1} = \frac{-1}{24}\begin{pmatrix} -19 & 5\sqrt{5} & \frac{1}{3r}\sqrt{3} \\ 5\sqrt{5} & 1 & \frac{1}{3r}\sqrt{15} \\ \frac{1}{3r}\sqrt{3} & \frac{1}{3r}\sqrt{15} & -\frac{5}{3r^2} \end{pmatrix} \tag{3.20}$$

Since $(V^{-1})_{22} \neq 0$, the first condition of (3.16) is not satisfied and $\lim_{m\to 0} A(m)$ does not exist.

The only possibility for lim A(m) to exist is to take $\tau_1 = \tau_2 = \tau_3 = 0$, which implies that the spin-1 components of T are zero. Clearly a source satisfying only the constraints of the massless theory does a priori not lead to a well defined limit of A(m) for vanishing mass. A source which only has a contribution in the highest spin sector makes the zero mass limit possible. For this limit, however, one is again faced with the inequality $\lim_{m\to 0} A(m) \neq B$, which arises in the same way as in the spin-2 case.

4. Conclusions and summary

In the previous two chapters we showed the construction of field theories for massive particles with arbitrary spin by using the root method and the formalism of projection operators. This has been done explicitly for the case of spin 1, 2 and 3.

If one compares the massive and massless field theory for a particle with spin s, some discontinuities between both theories can be observed.

First if the spin value is greater than two, there is a discontinuity on the level of the Lagrangian. In this case a massive particle is associated with an s rank symmetric tensor field $\phi^{(s)}$ and a set of auxiliary fields $\phi^{(s-2)}$, $\phi^{(s-4)}$ etc. These fields have been transformed in such a way that the coupling between this field and the auxiliary fields takes place only in the mass term. By putting m=0 in this Lagrangian, the massless Lagrangian is obtained, which consists of two terms, one Lagrangian for the field $\phi^{(s)}$ and one for the auxiliary fields.

By taking the auxiliary fields to be zero a suitable massless Lagrangian in terms of the original field $\phi^{(s)}$ has been obtained. This phenomenon has been explicitly demonstrated for the case of spin 3.

Another type of discontinuity between the massive and the massless theory arises if one compares the amplitudes describing exchange of a spin-s particle between two external sources. In both the massive and the massless case the same sources, obeying only the constraints from the massless theory, are used. It was first noticed by van Dam and Veltman [1] that the m=0 limit of the amplitude in the massive case does not lead to a corresponding amplitude for a massless particle. They have demonstrated this discontinuity for the case of spin 2. The same discontinuity, however, arises also in the case of spin $\frac{3}{2}$ [2]. As was shown in the previous section, the origin of these discontinuities can be easily understood by using the formalism of projection operators.

For higher spin one can in general not even take the zero mass limit of the amplitude in the massive theory. In the previous section we showed that for the case of spin 3 the massive amplitude becomes infinite for vanishing mass, if a general source of the massless theory is taken. However, if this source satisfies more restrictions than only the massless source constraint, the zero mass limit becomes possible. This can easily be verified for sources containing only the highest spin.

Throughout this chapter it was assumed that the external sources were independent of the mass of the particle. However, others [3] have studied the same problem, making use of sources which do depend on the particle's mass.

References

[1] H. van Dam and M. Veltman, Nucl.Phys. B22 (1970) 446.

[2] F.A. Berends and J.C.J.M. van Reisen, Nucl.Phys. B164 (1980) 286.

[3] C. Fronsdal, Nucl.Phys. B167 (1980) 237.

Appendix A. Conventions and metric

Units

Throughout this thesis we use units in which $\hbar = c = 1$.

Metric

A four-vector a is given by

$$a = (\vec{a},a_4) = (\vec{a},ia_0) \tag{A.1}$$

In particular: $x = (\vec{x},x_4) = (\vec{x},it)$,
$k = (\vec{k},k_4) = (\vec{k},iE)$.

We use a metric which is given by

$$\delta_{\mu\nu} = \text{diag}\ (+1,+1,+1,+1)\ . \tag{A.2}$$

Also the Einstein summation convention is used, i.e. instead of $\sum_{\mu=1}^{4} a_\mu b_\mu$ we write $a_\mu b_\mu$.

Thus the scalar product of two four-vectors a and b is given by

$$a\cdot b = a_\mu b_\mu = (\vec{a},\vec{b}) - a_0 b_0\ . \tag{A.3}$$

It should be stressed that the factor i in the fourth component is only useful for ease of notation and it should not be reversed when one takes the complex conjugate of a four vector i.e.

$$\text{if}\quad a = (\vec{a},ia_0)\ ,$$

$$\text{then}\ a^* = (\vec{a}^*,ia_0^*) = (\vec{a}^*,-a_4^*) \tag{A.4}$$

Appendix B. Projection and transition operators

In this appendix we list the projection and transition operators for spin 1, 2 and 3. In all these cases the operators are combinations of the following basic quantities

$$\omega_{\mu\nu} = \frac{1}{\square}\,\partial_\mu\partial_\nu\ ,$$

and (B.1)

$$\theta_{\mu\nu} = \delta_{\mu\nu} - \omega_{\mu\nu}\ .$$

Spin-1

The spin-1 projection operators are given by

$$P^1_{\mu\nu} = \theta_{\mu\nu} \;; \quad P^0_{\mu\nu} = \omega_{\mu\nu} \;. \tag{B.2}$$

Spin-2

The spin-2 projection and transition operators are given by

$$\begin{aligned}
P^2_{\mu\nu,\rho\sigma} &= \frac{1}{2} \sum_{\mu\nu} \theta_{\mu\rho}\theta_{\nu\sigma} - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \\
P^1_{\mu\nu,\rho\sigma} &= \frac{1}{2} \sum_{\mu\nu}\sum_{\rho\sigma} \theta_{\mu\rho}\omega_{\nu\sigma} \;, \\
P^0_{11,\mu\nu\;\rho\sigma} &= \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \qquad P^0_{22\;\mu\nu,\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma} \;, \\
P^0_{12\;\mu\nu,\rho\sigma} &= P^0_{21\;\rho\sigma,\mu\nu} = \frac{1}{3}\sqrt{3}\,\theta_{\mu\nu}\omega_{\rho\sigma} \;.
\end{aligned} \tag{B.3}$$

Here the summations denote summations over all independent permutations of μ,ν,ρ and σ. For instance $P^1_{\mu\nu\rho\sigma}$ reads explicitly

$$P^1_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\sigma}\omega_{\mu\rho}\right) \;. \tag{B.4}$$

Spin-3

Here we give the projection and transition operators for a field configuration ϕ consisting of the original field $\phi_{\lambda\mu\nu}$ and an auxiliary field A_σ.

From the number of indices one can see whether the operator works on the $\phi_{\lambda\mu\nu}$ or the A_σ part of ϕ.

Again the summations below are performed over all independent permutations of the indices, which in this case are cyclic permutations.

$$\begin{aligned}
P^3_{\alpha\beta\gamma,\lambda\mu\nu} &= \frac{1}{6}\sum_{\alpha\beta\gamma}\left[\theta_{\alpha\lambda}(\theta_{\beta\mu}\theta_{\gamma\nu} + \theta_{\beta\nu}\theta_{\gamma\mu})\right] - P^1_{22\;\alpha\beta\gamma,\lambda\mu\nu} \;, \\
P^2_{\alpha\beta\gamma,\lambda\mu\nu} &= \frac{1}{6}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \omega_{\alpha\lambda}(\theta_{\beta\mu}\theta_{\gamma\nu} + \theta_{\beta\nu}\theta_{\gamma\mu}) - P^0_{22\;\alpha\beta\gamma,\lambda\mu\nu} \;, \\
P^1_{11\;\alpha\beta\gamma,\lambda\mu\nu} &= \frac{1}{3}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \omega_{\alpha\beta}\theta_{\gamma\lambda}\omega_{\mu\nu} \;, \\
P^1_{22\;\alpha\beta\gamma,\lambda\mu\nu} &= \frac{1}{15}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \theta_{\alpha\beta}\theta_{\gamma\lambda}\theta_{\mu\nu} \;, \\
P^1_{21\;\alpha\beta\gamma,\lambda\mu\nu} &= \frac{1}{15}\sqrt{5}\sum_{\alpha\beta\gamma}\sum_{\lambda\mu\nu} \theta_{\alpha\beta}\theta_{\gamma\lambda}\omega_{\mu\nu} \;,
\end{aligned}$$

$$P^0_{11\ \alpha\beta\gamma,\lambda\mu\nu} = \omega_{\alpha\beta}\omega_{\gamma\lambda}\omega_{\mu\nu} ,$$

$$P^0_{22\ \alpha\beta\gamma,\lambda\mu\nu} = \frac{1}{9} \sum_{\alpha\beta\gamma} \sum_{\lambda\mu\nu} \omega_{\alpha\lambda}\theta_{\beta\gamma}\theta_{\mu\nu} ,$$

$$P^0_{21\ \alpha\beta\gamma,\lambda\mu\nu} = \frac{1}{3} \sum_{\alpha\beta\gamma} \theta_{\alpha\beta}\omega_{\gamma\lambda}\omega_{\mu\nu} ,$$

$$P^1_{33\ \rho,\sigma} = \theta_{\rho\sigma} ,$$

$$P^0_{33\ \rho,\sigma} = \omega_{\rho\sigma} ,$$

$$P^1_{31\ \sigma,\lambda\mu\nu} = \frac{1}{3}\sqrt{3} \sum_{\lambda\mu\nu} \theta_{\sigma\lambda}\omega_{\mu\nu} ,$$

$$P^1_{32\ \sigma,\lambda\mu\nu} = \frac{1}{15}\sqrt{15} \sum_{\lambda\mu\nu} \theta_{\sigma\lambda}\theta_{\mu\nu} ,$$

$$P^0_{31\ \sigma,\lambda\mu\nu} = \omega_{\sigma\lambda}\omega_{\mu\nu} ,$$

$$P^0_{32\ \sigma,\lambda\mu\nu} = \frac{1}{3} \sum_{\lambda\mu\nu} \omega_{\sigma\lambda}\theta_{\mu\nu} . \tag{B.5}$$

As an example P^0_{32} is also given explicitly

$$P^0_{32} = \frac{1}{3}\left(\omega_{\sigma\lambda}\theta_{\mu\nu} + \omega_{\sigma\mu}\theta_{\lambda\nu} + \omega_{\sigma\nu}\theta_{\lambda\mu}\right) . \tag{B.6}$$

The remaining transition operators follow from

$$P^J_{ij\ \omega,\omega'} = P^J_{ji\ \omega',\omega} , \tag{B.7}$$

where ω and ω' denote the 3- or 1-index set in eq. (B.5).
In all these cases it can be shown that the set of projection and transition operators have the following properties:

$$P^{L'}_{ij}P^L_{k\ell} = \delta^{L'L}\delta_{jk}P^L_{i\ell} , \tag{B.8}$$

and

$$\sum_{L=0}^{S} \sum_{j} P^L_{jj} = \quad , \tag{B.9}$$

where S is the highest spin present.

Appendix C. Polarization vectors

In this appendix we summarize the definition and the main properties of polarizaiton vectors $\varepsilon_\mu^\lambda(\vec{k})$ in the massless case. To a four-momentum vector $k = (\vec{k}, ik_0)$ are assigned four polarization vectors ε_μ^λ, $\lambda = 1,2,3,4$.

$$\begin{aligned} \varepsilon_\mu^1(k) &= \left(\vec{e}^1(\vec{k}),0\right) \\ \varepsilon_\mu^2(k) &= \left(\vec{e}^2(\vec{k}),0\right) \\ \varepsilon_\mu^3(k) &= \left(\vec{e}^3(\vec{k}),0\right) \\ e_\mu^4(k) &= \left(\vec{0},\ 1\right) \quad . \end{aligned} \tag{C.1}$$

The three-vectors $\vec{e}_i$ (i=1,2,3) form an orthonormal set with $\vec{e}^3 = \vec{k}/k_0$. The polarization vectors ε_μ^λ satisfy the following relations

$$\varepsilon_\mu^\lambda(k)\varepsilon_\mu^{\lambda'}(k) = \delta_{\lambda\lambda'}, \tag{C.2}$$

$$\sum_{\lambda=1}^{4} \varepsilon_\mu^\lambda(k)\varepsilon_\nu^\lambda(k) = \delta_{\mu\nu} \quad . \tag{C.3}$$

Instead of ε_μ^1 and ε_μ^2 one may use $\varepsilon_\mu^{(\pm 1)}$ corresponding to states with helicity eigenvalues ±1

$$\varepsilon_\mu^{(\pm)} = \mp \frac{1}{\sqrt{2}} (\varepsilon_\mu^1 \pm i\varepsilon_\mu^2) \quad . \tag{C.4}$$

Products of s times $\varepsilon_\mu^{(\pm 1)}$ i.e.

$$\varepsilon^{(\pm s)}_{\mu_1 \dots \mu_s} \sim \varepsilon^{(\pm 1)}_{\mu_1} \varepsilon^{(\pm 1)}_{\mu_2} \dots \varepsilon^{(\pm 1)}_{\mu_s}$$

correspond to states with helicity eigenvalue ±s .

SAMENVATTING

In de elementaire deeltjes fysica houdt men zich onder meer bezig met de volgende twee problemen.

1) het langs experimentele weg verkrijgen van inzicht in het bestaan van diverse deeltjes en hun eigenschappen,
2) het construeren van theorieën - bekend onder de naam quantumveldentheorieën - waarmee bekende deeltjes kunnen worden beschreven en waarmee mogelijkerwijs ook het bestaan van nieuwe deeltjes kan worden voorspeld.

In dit proefschrift worden enige aspecten, die op beide problemen betrekking hebben, behandeld.

Veel informatie in de elementaire deeltjes fysica wordt verkregen uit experimenten, waarbij men bundels deeltjes op elkaar laat botsen. Bij dergelijke verstrooiingsexperimenten meet men hoe grootheden als differentiële werkzame doorsnede en polarisatie afhangen van de energie en de verstrooiingshoek van deze deeltjes. Anderzijds staan deze grootheden in betrekking tot theoretisch belangrijke grootheden zoals de botsingsamplitude. De botsingsamplitude is een complexe functie en de differentiële werkzame doorsnede en de polarisatie zijn te schrijven als kwadratische combinaties ervan. Daar kennis van de amplitude informatie levert omtrent het bestaan van deeltjes en hun onderlinge wisselwerking, tracht men deze amplitude te bepalen uit de experimenteel bepaalde grootheden.

Een belangrijke vraag bij deze procedure is, in hoeverre de amplitude eenduidig is bepaald door differentiële werkzame doorsnede, polarisatie en unitariteitseisen. Zelfs als men hierbij uitgaat van de ideale situatie, dat experimentele fouten kunnen worden verwaarloosd, blijkt dat de amplitude niet altijd uniek is bepaald. In een dergelijk geval spreekt men van een faseverschuivingsmeerduidigheid.

Nadat in hoofdstuk II de voornaamste resultaten met betrekking tot deze problematiek zijn samengevat, worden in de daaropvolgende twee hoofdstukken dergelijke faseverschuivingsmeerduidigheden geconstrueerd. In hoofdstuk III wordt dit gedaan voor elastische verstrooiing van deeltjes met spin 0 en in hoofdstuk IV wordt het geval behandeld van elastische verstrooiing van deeltjes met spin 0 aan deeltjes met spin $\frac{1}{2}$. Het uitgangspunt in beide hoofdstukken is dat de amplitude afhangt van een willekeurig, maar eindig aantal partiële golven.

Het tweede hierboven genoemde probleem is, om voor de nu bekende deeltjes

theorieën te construeren, die de eigenschappen van deze deeltjes en hun onderlinge wisselwerkingen bevredigend beschrijven. In dergelijke, zo geheten quantumveldentheorieën, wordt een deeltje beschreven door een veld, dat voldoet aan een zekere differentiaalvergelijking of veldvergelijking.

In het geval dat men deeltjes zonder wisselwerking beschouwt spreekt men van een vrije veldentheorie en van vrije veldvergelijkingen. In dit proefschrift wordt de constructie van dergelijke vrije veldvergelijkingen besproken voor deeltjes met hogere spin en met massa ongelijk aan nul. Het is bekend dat deze deeltjes worden beschreven door een tensorveld, dat voldoet aan de Klein-Gordon vergelijking.

Aangezien dit tensorveld meer vrijheidsgraden bevat dan nodig zijn voor de beschrijving van een deeltje met spin s, voldoet het veld aan een aantal aanvullende vergelijkingen, waardoor de overtollige vrijheidsgraden kunnen worden geëlimineerd.

Op theoretische gronden is het gewenst de genoemde vergelijkingen af te leiden uitgaande van het actie principe. Een centrale rol wordt hierbij gespeeld door de Lagrangiaan, een Lorentz covariante functie, die afhankelijk is van het veld en zijn eerste orde partiële afgeleiden. De Lagrangiaan moet zó geconstrueerd zijn, dat de vergelijkingen die daar volgens het actie principe uit volgen, gelijkwaardig zijn met de eerder genoemde Klein-Gordon vergelijking en de aanvullende eisen.

Hoewel dezelfde vergelijkingen worden verkregen, kan kennis van de Lagrangiaan waardevol zijn om diverse redenen, zoals het uitvoeren van quantisatieprocedures en het invoeren van interactie.

Na een inleiding in deze problematiek in hoofdstuk 5 worden in hoofdstuk 6 vrije veldvergelijkingen en de bijbehorende Lagrangiaan geconstrueerd voor deeltjes met spin 1, 2 en 3. Bovendien wordt aangetoond hoe uit de verkregen veldvergelijking voor massieve deeltjes de veldvergelijking voor massaloze deeltjes volgt. In hoofdstuk 7 worden de theorieën voor massieve en massaloze deeltjes in meer detail bekeken. In het bijzonder wordt de uitdrukking vergeleken voor de amplitude die uitwisseling van een deeltje tussen twee uitwendige bronnen beschrijft. Aangetoond wordt dat het nemen van de m=0 limiet in de uitdrukking zoals die verkregen is in de massieve theorie, aanleiding geeft tot problemen: in bepaalde gevallen kunnen discontinuïteiten optreden en in andere gevallen blijkt de limiet niet te bestaan.

CURRICULUM VITAE

De schrijver van dit proefschrift is geboren te Sassenheim op 27 november 1947. In 1967 behaalde hij het eindexamen HBS-B.

Na het vervullen van de dienstplicht studeerde hij vanaf 1968 natuurkunde aan de Rijksuniversiteit te Leiden, hetgeen resulteerde in het behalen van het kandidaatsexamen met bijvak sterrenkunde in 1971 en het doctoraalexamen theoretische natuurkunde met bijvak wiskunde en de onderwijsbevoegdheid in de wis- en natuurkunde, in 1974. Eveneens tijdens deze studie verrichtte hij experimenteel werk in de groep van Prof.dr. J.H. van der Waals.

Sinds augustus 1974 is hij als leraar natuurkunde verbonden aan het Rijnlands Lyceum te Sassenheim. In deze periode verrichtte hij onder leiding van Prof.dr. F.A. Berends onderzoek op het gebied van elementaire deeltjes fysica en veldentheorie.

LIJST VAN PUBLIKATIES

1) F.A. Berends and J.C.J.M. van Reisen. On the existence of phase-shift ambiguities in elastic spin 0 - spin ½ scattering. Phys.Lett. 64B (1976).
2) F.A. Berends and J.C.J.M. van Reisen. On the existence of phase-shift ambiguities in spinless elastic scattering. Nucl.Phys. B115 (1976).
3) F.A. Berends and J.C.J.M. van Reisen. Chains of phase-shift ambiguities in elastic spin 0 - spin ½ scattering. Nucl.Phys. B118 (1977).
4) F.A. Berends and J.C.J.M. van Reisen. On spin 3 field theory and the zero-mass limit of higher spin theories. Nucl.Phys. B164 (1980).

STELLINGEN

behorende bij het proefschrift van

J.C.J.M. van Reisen

Leiden
11 mei 1983

1. Met een methode, die verschilt van de in dit proefschrift behandelde 'root method', kan een acceptabele Lagrangiaan worden geconstrueerd voor een deeltje met spin 4 en massa ongelijk nul.

2. Het uitgangspunt dat veldvergelijkingen voor massieve deeltjes met spin s in de limiet $m \to 0$ overgaan in de door Fronsdal gevonden veldvergelijkingen voor overeenkomstige massaloze deeltjes, suggereert de mogelijkheid om voor massieve deeltjes met heeltallige spin een Lagrangiaan te construeren in termen van symmetrische tensorvelden, waarvan het 2-voudige spoor nul is. Een dergelijke Lagrangiaan zal echter niet het resultaat kunnen zijn van de in dit proefschrift behandelde 'root method'.

C. Fronsdal, Phys.Rev. D18 (1978).

3. Voor een ferromagnetisch Ising model, gedefiniëerd op een planaire graaf, geldt voor een randrij (v_1, v_2, v_3, v_4) de volgende correlatiefunctie-ongelijkheid:

$$(\sigma_{v_1}\sigma_{v_2})(\sigma_{v_3}\sigma_{v_4}) \geq (\sigma_{v_1}\sigma_{v_3})(\sigma_{v_2}\sigma_{v_4}) \ .$$

4. In de hieronder vermelde referentie is voor een Ising model op een planaire graaf een correlatiefunctie-identiteit afgeleid voor een randrij $(v_1, \ldots, v_n)$ met n even.
Deze identiteit is als volgt uit te breiden voor een willekeurige deelverzameling A van de vertexverzameling van de graaf:

$$\sum_{k=1}^{n} (-1)^k (\sigma_{v_1}\sigma_{v_k}\sigma_A)(\sigma_{v_1}\sigma_{v_k}\prod_{i=1}^{n} \sigma_{v_i}\sigma_A) = 0 \ .$$

J. Groeneveld, R.J. Boel and P.W. Kasteleyn, Physica 93A (1978).

5. Bij systemen, bestaande uit alleen bosonen of alleen fermionen, bestaat een natuurlijk verband tussen de representaties van een Lie algebra in de 1-deeltjes ruimte en in de Fock-ruimte. Bij gemengde systemen komt daarvoor in de plaats een dergelijk natuurlijk verband van 'graded' representaties van 'graded' Lie algebras.

6. Stel $a \in \mathbb{N}$ en $b \in \mathbb{Z}$, dan is bekend, dat voor ieder polynoom $P(x)$ met gehele coëfficiënten geldt, dat $P(b+a) - P(b)$ deelbaar is door a.
Voor het Legendre polynoom $P_n(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n$, $n \in \mathbb{N}$, geldt zelfs dat $P_n(b+a) - P_n(b)$ deelbaar is door 2a.

7. Met de huidige kennis van de werkzame doorsnede voor het proces $e^+e^- \rightarrow$ hadronen, is het niet mogelijk om massa's van quarks met acceptabele nauwkeurigheid te bepalen.

8. Met behulp van het door Dirac ontwikkelde formalisme voor mechanische systemen met restricties, kan een expliciete reductie worden verkregen van het O(N) niet-lineaire σ-model, waarbij het aantal vrijheidsgraden met 1 is teruggebracht.

9. Het manipuleren met Grassmann-variabelen in een theorie met supersymmetrie krijgt een duidelijke wiskundige betekenis, wanneer men de fysische Hilbert-ruimte uitbreidt tot een moduul over een Grassmann algebra. Dit kan worden geïllustreerd aan de hand van een door van Hove behandeld eenvoudig supersymmetrisch model.

L. van Hove, Nucl. Phys. B207 (1982).

10. Het door N.A. Dyson gepubliceerde boek over toepassing van kernfysica in de geneeskunde, dient over de diagnostische methoden met radio-actieve nucliden informatie te verschaffen overeenkomstig de meest recente inzichten en technische verworvenheden op dit terrein.

N.A. Dyson, Nuclear Physics with applications of radioisotopes in Medicine and Biology (Ellis Horwood Ltd. Chichester 1981).

11. In het onderwijs is het plan ontwikkeld om in de toekomst leerlingen langer in heterogeen klasseverband bijeen te houden. Een maatregel, waardoor het aantal leerlingen per klas wordt vergroot heeft echter een negatief effect op de gewenste succesvolle uitwerking van dit plan.