QUANTUM THEORY OF RELATIVISTIC CHARGED PARTICLES IN EXTERNAL FIELDS

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PROEFSCHRIFT

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CONTENTS

INTRODUCTION

In many physical situations the behaviour of electrically charged particles interacting with the electromagnetic field can be adequately described by classical equations of motion for the particles only. The field caused by the particles is then neglected, and the electromagnetic field is described by fixed functions on space-time, determined by given charge and current distributions. This so-called external field approximation has important applications in quantum theory as well, especially in atomic and molecular physics, but also in the area of low energy nuclear physics.

In the domain of high energy collisions between charged elementary particles one can no longer treat the electromagnetic field as a classical, prescribed function on space-time: to explain the phenomena occurring at relativistic velocities the dynamics of both particles and electromagnetic field must be specified in terms of interacting quantized fields acting on a Fock space of many particles and photons. Unfortunately, such interacting quantum field theories are for various reasons purely formal from a mathematical point of view. One of the main obstacles for a rigorous treatment is the non-linearity of the formal equations for the quantized fields. In contrast, the equations describing the interaction of a quantized field with external fields are linear, which makes a mathematically unobjectionable formulation of these so-called external field theories much easier. Thus, several problems which also arise in the physically more important fully quantized theories can be analyzed with mathematical rigour in these theories.

In this thesis we study in a mathematically precise way external field theories in which the quantized field corresponds to relativistic elementary particles with non-zero rest mass. Furthermore, we assume that the particles are charged, i.e. that they have distinct antiparticles. Crudely speaking the thesis has two parts. The first part, viz. chapter 1 and section 2 of chapter 2, contains results of a general nature. Here, we have tried to accommodate the general features of theories of relativistic charged particles in external fields. In particular, spin and dynamics are not specified. In the second part these results are applied to charged spin-$\frac{1}{2}$ and spin-0 particles, the dynamics of which is given by the Dirac resp. Klein-Gordon equation. Most attention is paid to external fields which are rapidly decreasing infinitely differentiable functions on space-time, but we also consider time-independent fields. We shall not exclusively deal with electromagnetic fields, but shall also consider other external fields, e.g. scalar fields.

More in detail, chapter 1 contains a study, in a general context, of the field operator transformations which arise in a precise description of systems of charged particles in external fields. The main result is a simple expression for the normal form of the unitary operator on Fock space which implements such a transformation. We also rederive necessary and sufficient conditions for the transformation to be unitarily implementable.

Chapter 2 mainly deals with the classical (i.e. single particle) theory; although this theory is unphysical because of the negative energy solutions of classical relativistic wave equations, its properties largely determine those of the quantized (i.e. many particle) theory. First, some general perturbation-theoretic theorems are proved. These results are then applied to the classical Dirac and Klein-Gordon theories. The implementability in Fock space of the time evolution is investigated and several useful properties are established.

In chapter 3 the results of chapters 1 and 2 are applied to the quantized Dirac and Klein-Gordon theories. We introduce several field operators and establish their interrelationship and some of their properties. We then show that two different strategies which can be used to treat these external field theories are equivalent in the sense that they lead to the same unitary scattering operator. Subsequently, this operator is studied. We show that it has a divergence-free perturbation expansion. This expansion is then compared with the Feynman-Dyson series for the formal scattering operator. The formal vacuum-to-vacuum transition amplitude is divergent or, properly speaking, mathematically undefined. However, the formal relative amplitudes are shown to be equal to those from the rigorous theory. It follows from this that the modulus of the formal vacuum-to-vacuum transition amplitude ought to be defined as the modulus of its rigorous counterpart if the formal scattering operator is to correspond to a unitary operator on Fock space. Finally, precise analogues of the formal Furry theorem are proved.

From a physical point of view the established relation between the rigorous Fock space scattering operator and its formal counterpart is the main result of this thesis. It shows on one hand that the scattering amplitudes from the rigorous theory are the same as those from the formal one in so far as the latter are defined. This is a desirable feature since the formal external field theories can be regarded as approximations (albeit very crude) of their fully quantized counterparts, which lead to scattering amplitudes that agree with experiment. On the other hand it shows that, if the vacuum-to-vacuum transition amplitude is defined in the right way, the formal scattering operator corresponds in a natural way to a unitary operator on Fock space with the physically desirable properties of Lorentz covariance and causality, and with matrix elements having a convergent perturbation expansion.

CHAPTER 1

ON BOGOLIUBOV TRANSFORMATIONS FOR SYSTEMS OF RELATIVISTIC CHARGED PARTICLES

1. INTRODUCTION.

It is well-known that the interaction of relativistic particles with external fields should be considered as a many-particle problem. The classical (i.e. single-particle) theory leads to difficulties which are connected with the unphysical negative energy solutions of relativistic wave equations. In the many-particle framework the wave equation is looked upon as an equation for a quantized field, which is an operator-valued distribution acting on a Fock space. (For a more algebraic viewpoint see (1,2).) If the particle has a distinct antiparticle (which will be assumed in this paper) this space is the symmetric or antisymmetric Fock space over the direct sum of a one-particle and a one-antiparticle space, depending on whether the particle is a boson or a fermion. When the classical theory can be formulated in a Hilbert space it is convenient to smear field operators with vectors from this space instead of with test functions from a Schwartz space, since one can then easily use various operators from the classical theory, for instance the time-evolution operator. If these operators are pseudo-unitary resp. unitary (in the boson resp. the fermion case) they generate transformations of the field operators which amount to Bogoliubov transformations of the annihilation and creation operators, i.e. linear transformations which leave the canonical commutation relations (CCR) resp. canonical anticommutation relations (CAR) invariant. When these transformations are unitarily implementable the resulting unitary Fock space operator is assumed to be the physical operator corresponding to the unphysical operator from the classical theory.

More information on the connection between this type of Bogoliubov transformation and the external field problem can be found in (3, 4, 5). General Bogoliubov transformations are treated in the books by Friedrichs (6) and Berezin (7) and, for bosons, in (8).

The main result of this paper is a simple expression for the normal form of the unitary operator $\mathcal{U}$ which implements the field operator transformation generated by a (pseudo-)unitary operator U acting on the classical Hilbert space. We prove that on the dense subspace of "physical vectors", to be defined below, $\mathcal{U}$ equals a strongly convergent infinite series the terms of which contain creation and annihilation operators in the normal order. The coefficients of the terms are determined by an operator Λ which is closely related to U. In a forthcoming paper on the interaction of relativistic charged spin-0 and spin-$\frac{1}{2}$ particles with external fields (9) we will use this result to establish the connection between the formal Feynman-Dyson series for the Fock space S-operator and the unitary operator implementing the transformation generated by the classical S-operator. Our results might also be useful for higher spin theories.

Section 2 contains definitions and a summary of various equivalent requirements for the transformation to be unitarily implementable. In section 3 we introduce operators which are used in section 4 to obtain the normal form of $\mathcal{U}$. In the fermion case there is a restriction on U that is dropped in section 5, in which an expression for the normal form of $\mathcal{U}$ is obtained for the general fermion case. Section 6 contains a new proof that a certain well-known condition is necessary for our kind of Bogoliubov transformation to be unitarily implementable, and remarks about unbounded pseudo-unitary operators.

2. PRELIMINARIES.

The classical Hilbert space will be denoted by $\mathcal{H}$. It is the direct sum of 2 subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$, with corresponding projections P_+ and P_-. $\mathcal{H}_+$ will be the one-particle space, $\mathcal{H}_-$ the one-antiparticle space. This decomposition is closely connected with the occurrence of unphysical negative energies in the classical theory. For more details we refer to (10, 1, 4, 9). It is convenient to assume

$$\mathcal{H}_+, \mathcal{H}_- = L^2(R^3, dp)^M \qquad M < \infty . \tag{2.1}$$

This assumption has definite notational advantages and corresponds to physical applications (9). We will indicate at various points how one could proceed in a coordinate free way. It will also become clear that our results hold true as well if $\mathcal{H}_+$ or $\mathcal{H}_-$ are finite-dimensional.

We shall now summarize some results on second quantization, most of which are well-known. The elements of the (anti)symmetric Fock space $\mathcal{F}_\varepsilon$ over $\mathcal{H}$ (ε=a,s) can be written as

$$\{\psi^{n,r}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r)\}$$

where $n, r \in N$ and $\alpha_i,\beta_j = 1,\ldots,M$; $\psi^{n,r}$ is (anti)symmetric in particle and antiparticle variables separately. The inner product in $\mathcal{F}_\varepsilon$ is given by

$$(\psi_1,\psi_2) = \sum_{n,r=0}^{\infty} \sum_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_r=1}^{M} \int dp_1\ldots dp_n dq_1\ldots dq_r$$

$$\overline{\psi_1^{n,r}}(p_1,\alpha_1,\ldots,p_n\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r)\psi_2^{n,r}(p_1,\alpha_1,\ldots,p_n\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r). \tag{2.2}$$

An element $\psi \in \mathcal{F}_\varepsilon$ will be called a finite vector if there are $N,R < \infty$ such that $\psi^{n,r} = 0$ if $n > N$, $r > R$. The dense subspace of finite vectors will be denoted by D_f. In $\mathcal{F}_\varepsilon$ one has particle and antiparticle creation and annihilation operators $a^{(*)}(f)$ resp. $b^{(*)}(g)$, where $f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$. On a finite vector ψ they are defined by

$$(a(f)\psi)^{n,r}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r) = (n+1)^{\frac{1}{2}} \sum_{\alpha=1}^{M} \int dp\overline{f}_\alpha(p)$$

$$\cdot\,\psi^{n+1,r}(p,\alpha,p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r)$$

$$(b(g)\psi)^{n,r}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r) = (r+1)^{\frac{1}{2}}(\mp)^n \sum_{\beta=1}^{M} \int dq\overline{g}_\beta(q)$$

$$\cdot\,\psi^{n,r+1}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q,\beta,q_1,\beta_1,\ldots,q_r,\beta_r) \tag{2.3}$$

$$(a^*(f)\psi)^{n,r}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r) = n^{-\frac{1}{2}} \sum_{i=1}^{n} (\mp)^{i+1} f_{\alpha_i}(p_i)$$

$$\cdot\,\psi^{n-1,r}(p_1,\alpha_1,\ldots,\hat{p}_i,\hat{\alpha}_i,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r)$$

$$(b^*(g)\psi)^{n,r}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,q_r,\beta_r) = r^{-\frac{1}{2}} \sum_{j=1}^{r} (\mp)^{n+j+1} g_{\beta_j}(q_j)$$

$$\cdot\,\psi^{n,r-1}(p_1,\alpha_1,\ldots,p_n,\alpha_n;q_1,\beta_1,\ldots,\hat{q}_j,\hat{\beta}_j,\ldots,q_r,\beta_r)\,.$$

We will suppress the indices from now on. In (2.3) the upper sign refers to fermions, the lower to bosons. This convention will be used in the whole paper. One can easily show that these operators are bounded in the fermion case and unbounded, but closable, in the boson case. It is straightforward to verify that on D_f the well-known CAR (CCR) hold:

$$\begin{aligned}
&\left[a(f_1),a(f_2)\right]_\pm = \left[b(g_1),b(g_2)\right]_\pm = \left[a^{(*)}(f),b(g)\right]_\pm = 0 \\
&\left[a(f_1)a^*(f_2)\right]_\pm = (f_1,f_2) \qquad \left[b(g_1),b^*(g_2)\right]_\pm = (g_1,g_2) \ .
\end{aligned} \tag{2.4}$$

We will denote the spectral projection of the number operator $N \equiv \Omega(1)$ (for this notation and additional information see (11)) on the interval $[0,M]$ by P_M. One easily sees that the domain of the closure of $a^{(*)}(f)$ (which will be denoted by the same symbol) can be characterized as the set of vectors ψ for which $\underset{M\to\infty}{s\cdot\lim}\, a^{(*)}(f)P_M\psi$ exists, and that $D(a(f)) = D(a^*(f))$; this is also true for $b^{(*)}(g)$.

Hence

$$\begin{aligned}
a^{(*)}(f)\psi &= \underset{M\to\infty}{s\cdot\lim}\, a^{(*)}(f)P_M\psi \qquad \forall\, \psi\in D(a(f)) = D(a^*(f)) \\
b^{(*)}(g)\psi &= \underset{M\to\infty}{s\cdot\lim}\, b^{(*)}(g)P_M\psi \qquad \forall\, \psi\in D(b(g)) = D(b^*(g)) \ .
\end{aligned} \tag{2.5}$$

This implies that (2.3) holds true for any ψ in the domain of the respective operators. One also concludes, using relations like

$$a(f)a^*(f)P_M \leq \| f \|^2(N+1)P_M \ , \tag{2.6}$$

that the domain of $N^{\frac{1}{2}}$ belongs to the intersection of the domains of all creation and annihilation operators. The latter subspace will be denoted by $\tilde{D}$. (For fermions $\tilde{D} = \mathcal{F}_a$ of course.)

We will also have occasion to use the dense subspace D_∞ on which all powers of the number operator are defined:

$$D_\infty = \bigcap_{k=1}^{\infty} D(N^k). \tag{2.7}$$

From well-known (and easily proved) relations like (2.6) and like

$$a(f)N^kP_M = (N+1)^k a(f)P_M \tag{2.8}$$

one concludes that, for $\psi\in D_\infty$, $\underset{M\to\infty}{s\cdot\lim} \prod_{i=1}^{n} a^{(*)}(f_i) \prod_{j=1}^{r} b^{(*)}(g_j)P_M\psi$ exists and belongs to D_∞, i.e. the closure of any finite product of creation and/or annihilation operators (w.r.t. the subspace of finite vectors) is defined on D_∞ and leaves D_∞ invariant. One also verifies that on D_∞ the closure of the product equals the product of the closures. The relations (2.4) clearly hold true on D_∞.

We need one more subspace. Let Ω be the vacuum then we will call "physical vectors" the finite linear combinations of vectors of the form $\prod_{i=1}^{n} a^*(f_i) \prod_{j=1}^{r} b^*(g_j)\Omega$, where $n,r \geq 0$. (From a physical point of view these vectors are the relevant ones in describing initial states in a scattering theory.) The physical vectors form a dense subspace, denoted by D.

We define field operators on $\tilde{D}$ by

$$\Phi(v) = a(P_+v) + b^*(\overline{P_-v}) \qquad \forall\, v \in \mathcal{H} \tag{2.9}$$

where the bar denotes complex conjugation on $\mathcal{H}_-$; in a coordinate free approach one could take any conjugation K which maps $\mathcal{H}_-$ onto itself. (The connection between $\Phi(v)$ and the usual field operators from the Klein-Gordon and Dirac theories can be found in (9).) In the fermion case we consider transformations $\Phi(v) \to \hat{\Phi}(v)$, generated by unitary operators on $\mathcal{H}$ as follows:

$$\hat{\Phi}(v) = \Phi(U^*v) \qquad \forall\, v \in \mathcal{H}. \tag{2.10}$$

In the boson case we also have (2.10) but now U is pseudo-unitary, i.e.

$$UqU^* = U^*qU = q \tag{2.11}$$

where

$$q = P_+ - P_- . \tag{2.12}$$

(We will assume that U is bounded. At the end of the paper we shall comment on the case that U is unbounded.)

Defining

$$U_{\varepsilon\varepsilon'} = P_\varepsilon U P_{\varepsilon'} \qquad \varepsilon,\varepsilon' = +,- \tag{2.13}$$

we observe that

$${U_{\varepsilon\varepsilon'}}^* = U^*_{\varepsilon'\varepsilon} \tag{2.14}$$

and that the (pseudo-)unitarity of U is equivalent to the relations

$$\begin{array}{ll}
U^*_{++}U_{++} = 1_{++} \mp U^*_{+-}U_{-+} & U_{++}U^*_{++} = 1_{++} \mp U_{+-}U^*_{-+} \\
U^*_{--}U_{--} = 1_{--} \mp U^*_{-+}U_{+-} & U_{--}U^*_{--} = 1_{--} \mp U_{-+}U^*_{+-} \\
U^*_{++}U_{+-} = \mp U^*_{+-}U_{--} & U_{++}U^*_{+-} = \mp U_{+-}U^*_{--} \\
U^*_{--}U_{-+} = \mp U^*_{-+}U_{++} & U_{--}U^*_{-+} = \mp U_{-+}U^*_{++} \quad .
\end{array} \tag{2.15}$$

Decomposing $\hat{\Phi}(v)$ in new annihilation and creation operators as in (2.9), i.e. setting

$$\hat{\Phi}(v) \equiv \hat{a}(P_+v) + \hat{b}^*(\overline{P_-v}) \tag{2.16}$$

one easily sees that (2.10) is equivalent to the transformation $a(f) \to \hat{a}(f)$, $b(\overline{g}) \to \hat{b}(\overline{g})$, where

$$\begin{aligned} \hat{a}(f) &= a(U^*_{++}f) + b^*(\overline{U^*_{-+}f}) && \forall f \in \mathcal{H}_+ \\ \hat{b}(\overline{g}) &= b(\overline{U^*_{--}g}) + a^*(U^*_{+-}g) && \forall g \in \mathcal{H}_- \end{aligned} \tag{2.17}$$

Using (2.15) it is straightforward to verify that these operators also fulfil the CAR (CCR). The reader will have no difficulty in writing our transformation in terms of "one-body" annihilation and creation operators $c^{(*)}(v) \equiv$ $\equiv a^{(*)}(P_+v) + b^{(*)}(P_-v)$ and establishing the special character of the resulting Bogoliubov transformation.

The transformation (2.10) by definition is unitarily implementable if there exists a unitary operator $\mathcal{U}$, mapping $\tilde{D}$ onto $\tilde{D}$, such that

$$\hat{\Phi}(v) = \mathcal{U}^*\Phi(v)\mathcal{U} \qquad \forall v \in \mathcal{H} \tag{2.18}$$

or, equivalently, such that

$$\begin{aligned} \hat{a}(f) &= \mathcal{U}^* a(f)\mathcal{U} && \forall f \in \mathcal{H}_+ \\ \hat{b}(\overline{g}) &= \mathcal{U}^* b(\overline{g})\mathcal{U} && \forall g \in \mathcal{H}_- \, . \end{aligned} \tag{2.19}$$

It is well-known that this is equivalent to existence of a non-zero vector $\hat{\Omega} \in \tilde{D}$ such that

$$\hat{a}(f)\hat{\Omega} = \hat{b}(\overline{g})\hat{\Omega} = 0 \qquad \forall f \in \mathcal{H}_+ \quad \forall g \in \mathcal{H}_- ; \tag{2.20}$$

if such a vector exists it is a scalar multiple of $\mathcal{U}^*\Omega$.

Because we want to obtain an expression for $\mathcal{U}\Omega$ it is convenient to consider as well the transformation generated by the inverse of U, i.e. the transformation $\Phi(v) \to \Phi'(v)$ where

$$\Phi'(v) \equiv \Phi(Uv) \qquad \forall v \in \mathcal{H} \tag{2.21}$$

in the fermion case, and

$$\Phi'(v) \equiv \Phi(qUqv) \qquad \forall v \in \mathcal{H} \tag{2.22}$$

in the boson case. Existence of a unitary operator $\mathcal{U}$ satisfying (2.18) is obviously equivalent to existence of a unitary operator $\mathcal{U}$ satisfying

$$\Phi'(v) = \mathcal{U}\Phi(v)\mathcal{U}^* \qquad \forall v \in \mathcal{H} \tag{2.23}$$

or, equivalently, satisfying

$$\begin{aligned} a'(f) &= \mathcal{U}\, a(f)\, \mathcal{U}^* & \forall f \in \mathcal{H}_+ \\ b'(\bar{g}) &= \mathcal{U}\, b(\bar{g})\, \mathcal{U}^* & \forall g \in \mathcal{H}_- \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} a'(f) &\equiv a(U_{++}f) \pm b^*(\overline{U_{-+}f}) \\ b'(\bar{g}) &\equiv b(\overline{U_{--}g}) \pm a^*(U_{+-}g) \; . \end{aligned} \tag{2.25}$$

From (2.15) one again concludes that these operators fulfil the CAR (CCR), so implementability is also equivalent to existence of a non-zero vector $\Omega' \in \tilde{D}$ satisfying

$$a'(f)\Omega' = b'(\bar{g})\Omega' = 0 \qquad \forall f \in \mathcal{H}_+ \qquad \forall g \in \mathcal{H}_- \;; \tag{2.26}$$

if such a vector exists it is a scalar multiple of $\mathcal{U}\Omega$.

From work of several authors (12, 13, 14, 1) it follows that the transformation (2.10) or, equivalently, (2.21) - (2.22) is unitarily implementable if and only if

$$U_{+-},\ U_{-+} \in HS \tag{2.27}$$

where HS is the set of all Hilbert-Schmidt (H.S.) operators on $\mathcal{H}$. In sections 3, 4 and 5 it will be assumed that (2.27) holds true. We will denote the corresponding unitary operator on Fock space by $\mathcal{U}$. From the fact that the Fock-Cook representation of the Clifford algebra resp. the Weyl algebra over $\mathcal{H}$ is irreducible it follows that $\mathcal{U}$ is up to a phase factor uniquely determined.

We remark that the sufficiency of (2.27) will be a consequence of our results, while we will give a new proof of the necessity in section 6, so in this respect the paper is self-contained.

3. THE OPERATORS Λ AND EXP $(\Lambda_\pm a^* b^*)$.

In the boson case one easily concludes from (2.15) that U_{++} and U_{--} have bounded inverses (as operators on $\mathcal{H}_+$ resp. $\mathcal{H}_-$). In the fermion case this also follows from (2.15) and (2.27) if we make the additional assumption

$$\mathrm{Ker}\, U_{++} = \mathrm{Ker}\, U_{--} = 0. \tag{3.1}$$

We will assume (3.1) in this section and the next one and deal with the general case in section 5.

We now introduce a bounded operator Λ on $\mathcal{K}$ which will enable us to obtain a simple expression for the normal form of $\mathcal{U}$. Λ is defined by

$$
\begin{aligned}
\Lambda_{--} &= \pm(1_{--} - U_{--}^{-1}) \\
\Lambda_{-+} &= \pm U_{--}^{-1}U_{-+} \\
\Lambda_{+-} &= U_{+-}U_{--}^{-1} \\
\Lambda_{++} &= U_{++} - 1_{++} - U_{+-}U_{--}^{-1}U_{-+} \; .
\end{aligned}
\tag{3.2}
$$

From (2.15) it follows that this is equivalent to

$$
\begin{aligned}
\Lambda_{--} &= \pm(1_{--} - U^*_{--} + U^*_{-+}U^{*\,-1}_{++}U^*_{+-}) \\
\Lambda_{-+} &= - U^*_{-+}U^{*\,-1}_{++} \\
\Lambda_{+-} &= \mp U^{*\,-1}_{++}U^*_{+-} \\
\Lambda_{++} &= - 1_{++} + U^{*\,-1}_{++} \; .
\end{aligned}
\tag{3.3}
$$

One easily verifies that (3.2) is also equivalent to

$$\text{(fermions)} \quad (U-1)-\Lambda-(U-1)P_-\Lambda = (U-1)-\Lambda-\Lambda P_-(U-1) = 0 \tag{3.4}$$

$$\text{(bosons)} \quad (U-1)-q\Lambda+(U-1)P_-\Lambda = (U-1)-q\Lambda-q\Lambda P_-(U-1) = 0 \; . \tag{3.5}$$

We will need the following relations, which follow from (3.2) and (3.3):

$$
\begin{aligned}
U_{++} - \Lambda_{+-}U_{-+} &= 1_{++} + \Lambda_{++} \\
\Lambda_{-+} \mp U_{-+} + \Lambda_{--}U_{-+} &= 0
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
U_{--} \pm \Lambda^*_{-+}U_{+-} &= 1_{--} \mp \Lambda^*_{--} \\
U_{+-} + \Lambda^*_{+-} + \Lambda^*_{++}U_{+-} &= 0 \; .
\end{aligned}
\tag{3.7}
$$

From (2.27) and (3.2) we infer that Λ_{+-} and Λ_{-+} are H.S.; moreover one obtains from (2.15), in the boson case,

$$
\begin{aligned}
\|\Lambda_{+-}\|^2 &= \sup_{v\in\mathcal{K}} \frac{(U_{+-}U_{--}^{-1}v, U_{+-}U_{--}^{-1}v)}{(v,v)} = \sup_{w\in\mathcal{K}} \frac{(U_{+-}w, U_{+-}w)}{(U_{--}w, U_{--}w)} \\
&= \sup_{w\in\mathcal{K}} \frac{(U^*_{-+}U_{+-}w,w)}{(U^*_{-+}U_{+-}w,w) + (w,w)} < 1 \; .
\end{aligned}
\tag{3.8}
$$

Consequently

$$(\text{bosons})\ \Lambda_{+-} = \sum_{i=1}^{M_o} \lambda_i F_i(G_i,.) \qquad M_o \leq \infty \qquad 0 < \lambda_i \leq \theta < 1 \tag{3.9}$$

where $\{F_i\}$, $\{G_i\}$ are orthonormal sets in $\mathcal{H}_+$, $\mathcal{H}_-$ and where $\lambda_i \leq \lambda_j$ if $i > j$ (see (15)); furthermore

$$\sum_{i=1}^{M_o} \lambda_i^{\,2} < \infty \quad . \tag{3.10}$$

We set

$$\Lambda_{+-} a^* b^* \equiv \int dp dp' \Lambda_{+-}(p,p') a^*(p) b^*(p') \tag{3.11}$$

where $\Lambda_{+-}(p,p')$ is the kernel of Λ_{+-}. The operator $\Lambda_{+-} a^* b^*$ and its powers are clearly defined on D_f. The next lemma shows that the operator $\exp(\Lambda_{+-} a^* b^*)$ is defined on D.

<u>Lemma 3.1.</u> Let $\phi \in D$ and let

$$\phi_n = \frac{(\Lambda_{+-} a^* b^*)^n}{n!} \phi \quad . \tag{3.12}$$

Then $\operatorname{s\cdot lim}_{N\to\infty} \sum_{n=0}^{N} \phi_n$ exists and belongs to D_∞ .

<u>Proof</u>. A. <u>Bosons</u>. We assume first that $\phi = \Omega$. Existence of the limit is then obviously equivalent to existence of $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$, where

$$a_n = \left\| \frac{(\Lambda_{+-} a^* b^*)^n}{n!} \Omega \right\|^2 . \tag{3.13}$$

One easily obtains $(n \geq 1)$

$$a_n = \frac{1}{n!} \int dp_1 \ldots dp_n dq_1 \ldots dq_n \sum_{\sigma \in S_n} \prod_{i=1}^{n} \overline{\Lambda_{+-}}(q_i,p_i) \Lambda_{+-}(q_i,p_{\sigma(i)}) \tag{3.14}$$

where S_n is the symmetric group. We now define

$$a_{n,N} = \left\| \frac{(\Lambda_N a^* b^*)^n}{n!} \Omega \right\|^2 \tag{3.15}$$

where

$$\Lambda_N \equiv \sum_{i=1}^{N} \lambda_i F_i(G_i,.) \ . \tag{3.16}$$

The analogue of (3.14) for $a_{n,N}$ implies

$$\lim_{N\to\infty} a_{n,N} = a_n \ . \tag{3.17}$$

On the other hand

$$a_{n,N} = \left\| \frac{\left(\sum_{i=1}^{N} \lambda_i a^*(F_i) b^*(\overline{G_i})\right)^n}{n!} \Omega \right\|^2$$

$$= \left\| \sum_{\substack{k_1,\ldots,k_N=0 \\ k_1+\ldots+k_N=n}}^{n} \prod_{i=1}^{N} \frac{(\lambda_i a^*(F_i) b^*(\overline{G_i}))^{k_i}}{k_i!} \Omega \right\|^2$$

$$= \sum_{\substack{k_1,\ldots,k_N=0 \\ k_1+\ldots+k_N=n}}^{n} \prod_{i=1}^{N} \lambda_i^{2k_i} \ . \tag{3.18}$$

We now introduce a function

$$F_N(\alpha) = \prod_{i=1}^{N} (1-\alpha\lambda_i^2)^{-1} \tag{3.19}$$

which is clearly analytic in the disc O, defined by

$$O = \{\alpha\in C \,|\, |\alpha| < \theta^{-2}\} \ . \tag{3.20}$$

Using (3.18) one easily verifies

$$F_N(\alpha) = \sum_{n=0}^{\infty} a_{n,N}\alpha^n \ . \tag{3.21}$$

We shall prove that $\lim_{N\to\infty} F_N(\alpha)$ exists on O and that the limit function is analytic in O. Let r be such that

$$0 < r < \theta^{-2} . \tag{3.22}$$

If $|\alpha| \leq r$

$$|F_N(\alpha)| \leq \prod_{i=1}^{N} |1-\alpha\lambda_i^2|^{-1} \leq \prod_{i=1}^{N} (1-r\lambda_i^2)^{-1} = \exp(-\sum_{i=1}^{N} \ln(1-r\lambda_i^2))$$

$$\leq \exp(-N'\ln(1-r\theta^2) + 2r \sum_{i=N'+1}^{\infty} \lambda_i^2) \equiv C_r < \infty \tag{3.23}$$

where N' is such that

$$r\lambda_{N'+1}^2 \leq \tfrac{1}{2} . \tag{3.24}$$

From (3.23) ($|\alpha| \leq r$, $M > N$) :

$$|F_N(\alpha) - F_M(\alpha)| \leq C_r \left|1 - \prod_{i=N+1}^{M} (1-\alpha\lambda_i^2)^{-1}\right| = C_r \left|1 - \exp(-\sum_{i=N+1}^{M} \ln(1-\alpha\lambda_i^2))\right| . \tag{3.25}$$

We now observe ($N > N'$):

$$\left|\sum_{i=N+1}^{M} \ln(1-\alpha\lambda_i^2)\right| \leq \sum_{i=N+1}^{M} \big|\ln|1-\alpha\lambda_i^2|\big| + \sum_{i=N+1}^{M} |\mathrm{Arg}(1-\alpha\lambda_i^2)|$$

$$\leq -\sum_{i=N+1}^{\infty} \ln(1-r\lambda_i^2) + \frac{\pi}{2} \sum_{i=N+1}^{\infty} |\sin(\mathrm{Arg}(1-\alpha\lambda_i^2))| \leq (2r + \frac{\pi r}{2}) \sum_{i+N+1}^{\infty} \lambda_i^2 . \tag{3.26}$$

It evidently follows from (3.25) and (3.26) that

$$\lim_{N\to\infty} F_N(\alpha) = F(\alpha) \qquad \alpha \in O \tag{3.27}$$

where $F(\alpha)$ is analytic in O.

Using (3.17) and (3.21) we infer

$$F(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n . \tag{3.28}$$

Thus, since $\theta^{-2} > 1$,

$$\sum_{n=0}^{\infty} a_n = F(1) < \infty , \tag{3.29}$$

which proves that $s\cdot\lim_{N\to\infty} \sum_{n=0}^{N} \phi_n$ exists if $\phi = \Omega$. We notice that

$$F(1) \equiv \lim_{N\to\infty} F_N(1) = \prod_{i=1}^{M_o} (1-\lambda_i^2)^{-1} . \tag{3.30}$$

Hence (see (16)):

$$\text{(bosons)} \quad \| \exp(\Lambda_{+-} a^* b^*)\Omega \|^2 = \det(1_{--} - \Lambda_{+-}{}^* \Lambda_{+-})^{-1} . \tag{3.31}$$

We now introduce the functions

$$G_k(\alpha) = \sum_{n=0}^{\infty} (2n)^k a_n \alpha^n \qquad k \in N . \tag{3.32}$$

These functions are analytic in 0 because the power series on the right-hand side of (3.32) has the same convergence radius as the right-hand side of (3.28). Therefore

$$\sum_{n=0}^{\infty} (2n)^k a_n = G_k(1) < \infty . \tag{3.33}$$

From (3.13) and (3.33) it then follows that

$$\exp(\Lambda_{+-} a^* b^*)\Omega \in D_\infty . \tag{3.34}$$

One obviously has

$$\sum_{n=0}^{N} \frac{(\Lambda_{+-} a^* b^*)^n}{n!} \prod_{i=1}^{n} a^*(f_i) \prod_{j=1}^{r} b^*(g_j)\Omega$$

$$= \prod_{i=1}^{n} a^*(f_i) \prod_{j=1}^{r} b^*(g_j) P_{2N} \exp(\Lambda_{+-} a^* b^*)\Omega . \tag{3.35}$$

From (3.34) and (3.35) we finally conclude that the limit of the left-hand side of (3.35) exists and belongs to D_∞, which proves the lemma for bosons.

B. Fermions. Proceeding in the same way as for bosons one obtains instead of (3.14)

$$a_n = \frac{1}{n!} \int dp_1 \ldots dp_n dq_1 \ldots dq_n \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \prod_{i=1}^{n} \overline{\Lambda_{+-}(q_i, p_i)} \Lambda_{+-}(q_i, p_{\sigma(i)}). \tag{3.36}$$

Defining

$$T = \Lambda_{+-}{}^* \Lambda_{+-} \tag{3.37}$$

we have

$$a_n = \frac{1}{n!} T_n \tag{3.38}$$

where

$$T_n \equiv \int dp_1 \ldots dp_n T(p_1, \ldots, p_n) \tag{3.39}$$

and $T(p_1, \ldots, p_n)$ is the determinant the elements of which are $T(p_i, p_j)$ $(i,j = 1,\ldots,n)$. Introducing the entire function

$$\tilde{d}(\lambda) = \det(\underline{1} + \lambda T) \tag{3.40}$$

one has (see (16))

$$\tilde{d}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n \lambda^n \tag{3.41}$$

where

$$P_0 = 1 \qquad P_n = \begin{vmatrix} \sigma_1 & 1 & 0 & \ldots & 0 & 0 \\ \sigma_2 & \sigma_1 & 2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \ldots & \sigma_1 & n-1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \ldots & \sigma_2 & \sigma_1 \end{vmatrix} \tag{3.42}$$

and

$$\sigma_i = \mathrm{Tr}(T^i). \tag{3.43}$$

Expanding the determinant we obtain the recurrence relation

$$P_n = \sum_{k=1}^{n} (-)^{k-1} \frac{(n-1)!}{(n-k)!} \sigma_k P_{n-k} \qquad (n \geq 1). \tag{3.44}$$

Expanding $T(p_1, \ldots, p_n)$ in (3.39) one easily sees that T_n obeys the same recurrence relation. Thus, since $T_1 = P_1$,

$$T_n = P_n \qquad \forall\, n \in N^+ . \tag{3.45}$$

Therefore

$$\tilde{d}(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \tag{3.46}$$

so the limit exists if $\phi = \Omega$ and

$$\text{(fermions)} \quad \| \exp(\Lambda_{+-} a^* b^*) \Omega \|^2 = \det(\underline{1} + \Lambda_{+-}^* \Lambda_{+-}) . \tag{3.47}$$

Arguing in the same way as for bosons one concludes

$$\exp(\Lambda_{+-} a^* b^*)\Omega \in D_\infty \ . \tag{3.48}$$

The lemma now follows from (3.48) and (3.35). ∎

We remark that the proof of the lemma could be shortened in the boson case by using more results on infinite determinants (17, 18, 19).

If one does not assume (2.1) one should define

$$\text{(bosons)} \quad \Lambda_{+-} a^* b^* = \sum_{i=1}^{M_0} \lambda_i a^*(F_i) b^*(KG_i) \tag{3.49}$$

where K is the conjugation chosen in (2.9). One could then map $\mathcal{H}$ onto L^2 spaces as in (2.1) in such a way that K becomes complex conjugation on $\hat{\mathcal{H}}_-$, use the lemma, and transform back. Using the analogue of (3.49) for fermions one could prove the lemma in a similar way for fermions.

4. THE NORMAL FORM OF $\mathcal{U}$.

The result of the next lemma was obtained in different forms by several authors (7, 8, 3). It essentially dates back to the work of Friedrichs (6).

<u>Lemma 4.1</u>. The following relation holds true:

$$\mathcal{U}\Omega = e^{i\theta}\det(1_{--} \pm \Lambda_{+-}^*\Lambda_{+-})^{\mp\frac{1}{2}} \exp(\Lambda_{+-} a^* b^*)\Omega \qquad 0 \leq \theta < 2\pi . \tag{4.1}$$

<u>Proof.</u> By (3.31) and (3.47) the norm of the right-hand side equals 1, so by (2.26) and (2.25) it suffices to prove

$$\begin{aligned} &(a(U_{++}f) \pm b^*(\overline{U_{-+}f}))\exp(\Lambda_{+-} a^* b^*)\Omega = 0 \qquad \forall f \in \mathcal{H}_+ \\ &(b(\overline{U_{--}g}) \pm a^*(U_{+-}g))\exp(\Lambda_{+-} a^* b^*)\Omega = 0 \qquad \forall g \in \mathcal{H}_- \ . \end{aligned} \tag{4.2}$$

Notice that the left-hand sides of these equations are well-defined in virtue of (3.34). (In the fermion case this of course already follows from the relation $\exp(\Lambda_{+-} a^* b^*)\Omega \in \mathcal{F}_a$.) Using (2.5) and the CAR (CCR) we conclude that (4.2) is equivalent to

$$\begin{aligned} &(b^*(\overline{\Lambda_{+-}^* U_{++} f}) \pm b^*(\overline{U_{-+}f}))\exp(\Lambda_{+-} a^* b^*)\Omega = 0 \qquad \forall f \in \mathcal{H}_+ \\ &(\mp a^*(\Lambda_{+-} U_{--} g) \pm a^*(U_{+-}g))\exp(\Lambda_{+-} a^* b^*)\Omega = 0 \qquad \forall g \in \mathcal{H}_- \ . \end{aligned} \tag{4.3}$$

However, (4.3) follows immediately from (3.2) and (3.3). $\blacksquare$

We normalize $\mathcal{U}$ by setting $\theta = 0$ in (4.1):

$$\mathcal{U}\,\Omega = \det(1_{--} \pm \Lambda_{+-}{}^{*}\Lambda_{+-})^{\mp\frac{1}{2}}\exp(\Lambda_{+-}a^{*}b^{*})\Omega \quad . \tag{4.4}$$

We now introduce the operators which are needed for the normal form of $\mathcal{U}$. Let K, L, M be bounded operators on $\mathcal{H}$. We set

$$K^{j}_{++}L^{k}_{--}M^{\ell}_{-+}a^{*j}b^{*k}b^{k}b^{\ell}a^{\ell}a^{j} \equiv$$

$$\int dk_1\ldots dk_j' dp_1\ldots dp_k' dq_1\ldots dq_\ell' \prod_{\rho=1}^{j} (K_{++})(k_\rho,k_\rho') \prod_{\sigma=1}^{k} (L_{--})(p_\sigma,p_\sigma')$$

$$\cdot \prod_{\tau=1}^{\ell} (M_{-+})(q_\tau,q_\tau')a^{*}(k_1)\ldots a^{*}(k_j)b^{*}(p_1)\ldots b^{*}(p_k)b(p_k')\ldots b(p_1')$$

$$\cdot\, b(q_1)\ldots b(q_\ell)a(q_\ell')\ldots a(q_1')a(k_j')\ldots a(k_1') \tag{4.5}$$

where e.g. $(K_{++})(k,k')$ is the tempered distribution which corresponds to K_{++} by the nuclear theorem. The formal expression at the right-hand side of (4.5) is defined on D by writing

$$\begin{aligned} a^{*}(f) &= \int dp\,a^{*}(p)f(p) \\ b^{*}(g) &= \int dp\,b^{*}(p)g(p) \end{aligned} \tag{4.6}$$

and then using the formal CAR (CCR)

$$\begin{aligned} &\Big[a(p),a(p')\Big]_{\pm} = \Big[b(p),b(p')\Big]_{\pm} = \Big[a^{(*)}(p),b(p')\Big]_{\pm} = 0 \\ &\Big[a(p),a^{*}(p')\Big]_{\pm} = \Big[b(p),b^{*}(p')\Big]_{\pm} = \delta(p-p') \text{ and the relation} \end{aligned} \tag{4.7}$$

$$a(p)\Omega = b(p)\Omega = 0 \tag{4.8}$$

to get rid of all annihilation operators in (4.5). One should then set, e.g.,

$$\begin{aligned} &\int dk\,dk'(K_{++})(k,k')a^{*}(k)f(k') \equiv a^{*}(K_{++}f) \\ &\int dq\,dq'(M_{-+})(q,q')g(q)f(q') \equiv (\bar{g},M_{-+}f) \ . \end{aligned} \tag{4.9}$$

One easily convinces oneself that this gives rise to a well-defined linear operator mapping D into D. It is clear that one could define this operator on D without using (2.1) but this would obviously give rise to very unwieldy formulas. Denoting the operator by $O_{j,k,\ell}$ it is straightforward to verify relations like

$$O_{j,k,\ell}a^*(f) = a^*(f)O_{j,k,\ell} + ja^*(K_{++}f)O_{j-1,k,\ell} + \ell O_{j,k,\ell-1}b(\overline{M_{-+}f}) \quad (4.10)$$

which hold on D.

Defining the transpose N^T of a bounded operator N on $\mathcal{H}$ by

$$(N^T f)(p) = (\overline{N^* \bar{f}})(p) \quad (4.11)$$

we can now proceed to the first theorem. We define an operator $\tilde{\Gamma}(U)$ by

$$\tilde{\Gamma}(U) = \det(1_{--} \pm \Lambda_{+-}{}^*\Lambda_{+-})^{\mp\frac{1}{2}} \sum_{L=0}^{\infty} \mathcal{U}_L \quad (4.12)$$

where

$$\mathcal{U}_L = \sum_{\substack{i,j,k,\ell=0 \\ i+j+k+\ell=L}}^{L} \frac{1}{i!j!k!\ell!} \Lambda_{+-}^i \Lambda_{++}^j (\mp\Lambda_{--}^T)^k \Lambda_{-+}^\ell a^{*i} b^{*i} a^{*j} b^{*k} b^k b^\ell a^\ell a^j \; . \quad (4.13)$$

Notice that $\mathcal{U}_L$ is well-defined on D. We can abbreviate (4.12) and (4.13) as follows:

$$\tilde{\Gamma}(U) = \det(1_{--} \pm \Lambda_{+-}{}^*\Lambda_{+-})^{\mp\frac{1}{2}} :\exp(\Lambda_{+-}a^*b^* + \Lambda_{++}a^*a + \Lambda_{--}bb^* + \Lambda_{-+}ba):. (4.14)$$

Defining the domain of $\tilde{\Gamma}(U)$ by

$$D(\tilde{\Gamma}(U)) = \{\phi \in D \,|\, \text{s-}\lim_{N\to\infty} \sum_{L=0}^{N} \mathcal{U}_L \phi \text{ exists}\} \quad (4.15)$$

we have the following theorem.

<u>Theorem 4.1</u>. The domain of $\tilde{\Gamma}(U)$ equals D:

$$D(\tilde{\Gamma}(U)) = D \; , \quad (4.16)$$

and

$$\tilde{\Gamma}(U)D \subset D_\infty \; . \quad (4.17)$$

The operator $\mathcal{U}$ is equal to $\tilde{\Gamma}(U)$ on D:

$$\mathcal{U}\phi = \det(1_{--} \pm \Lambda_{+-}{}^*\Lambda_{+-})^{\mp\frac{1}{2}} : \exp(\Lambda_{+-}a^*b^* + \Lambda_{++}a^*a + \Lambda_{--}bb^* + \Lambda_{-+}ba): \phi \quad \forall\phi \in D. \quad (4.18)$$

Proof. From lemma 3.1 and (4.4) it evidently follows that $\Omega \in D(\tilde{\Gamma}(U))$, $\tilde{\Gamma}(U)\Omega = \mathcal{U}\Omega$ and $\mathcal{U}\Omega \in D_\infty$. To prove the existence of the limit in (4.15) for a vector ϕ of the form

$$\phi = \prod_{i=1}^{n} a^*(f_i) \prod_{j=1}^{r} b^*(g_j)\Omega \tag{4.19}$$

we observe that the individual term in (4.13) only contributes if

$$\ell + j \leq n \qquad \ell + k \leq r \, . \tag{4.20}$$

Since there is only a finite number of (j,k,ℓ) which fulfil (4.20) we conclude from lemma 3.1 that the limit in (4.15) exists and belongs to D_∞. It remains to prove (4.18).

In view of (2.24) and the relation $\tilde{\Gamma}(U)\Omega = \mathcal{U}\Omega$ it suffices to show that on D

$$\tilde{\Gamma}(U)a^*(f) = a'^*(f)\tilde{\Gamma}(U) \qquad \forall\, f \in \mathcal{H}_+ \tag{4.21}$$

$$\tilde{\Gamma}(U)b^*(\bar{g}) = b'^*(\bar{g})\tilde{\Gamma}(U) \qquad \forall\, g \in \mathcal{H}_- \, . \tag{4.22}$$

Using relations like (4.10) and the relations (3.6) and (2.25) we now have on D (observe that, e.g., $\lim_{N\to\infty} a^*(f) \sum_{L=0}^{N} \ldots = a^*(f) \lim_{N\to\infty} \sum_{L=0}^{\infty} \ldots$ on D according to lemma 3.1, (2.5) and the argument after (4.20))

$$\begin{aligned}
\tilde{\Gamma}(U)a^*(f) &= (a^*(f) + a^*(\Lambda_{++}f))\tilde{\Gamma}(U) + \tilde{\Gamma}(U)b(\overline{\Lambda_{-+}f}) \\
&= (a^*(U_{++}f) - a^*(\Lambda_{+-}U_{-+}f))\tilde{\Gamma}(U) + \tilde{\Gamma}(U)b(\overline{\Lambda_{-+}f}) \\
&= (a^*(U_{++}f) \pm b(\overline{U_{-+}f}))\tilde{\Gamma}(U) + \tilde{\Gamma}(U)b(\overline{\Lambda_{-+}f}) \\
&+ \det \ldots \Sigma\Sigma \ldots \Lambda^{i}_{+-} a^{*i} b^{*i} (\mp b(\overline{U_{-+}f}))\Lambda^{j}_{++} \ldots a^{*j} \ldots a^{j} \\
&= a'^*(f)\tilde{\Gamma}(U) + \tilde{\Gamma}(U)b(\overline{(\Lambda_{-+} \mp U_{-+} + \Lambda_{--}U_{-+})f}) = a'^*(f)\tilde{\Gamma}(U)
\end{aligned}$$

which proves (4.21). Similarly, using (3.7), we obtain

$$\begin{aligned}
\tilde{\Gamma}(U)b^*(\bar{g}) &= (b^*(\bar{g}) \mp b^*(\overline{\Lambda^*_{--}g}))\tilde{\Gamma}(U) + \tilde{\Gamma}(U)a(\mp\Lambda^*_{+-}g) \\
&= (b^*(\overline{U_{--}g}) \pm b^*(\overline{\Lambda^*_{-+}U_{+-}g}))\tilde{\Gamma}(U) + \tilde{\Gamma}(U)a(\mp\Lambda^*_{+-}g) \\
&= (b^*(\overline{U_{--}g}) \pm a(U_{+-}g))\tilde{\Gamma}(U) + \tilde{\Gamma}(U)a(\mp\Lambda^*_{+-}g) \\
&+ \det .. b^{*i}(\mp a(U_{+-}g)), .. a^{j} \\
&= b'^*(\bar{g})\tilde{\Gamma}(U) \mp \tilde{\Gamma}(U)a((\Lambda^*_{+-} + U_{+-} + \Lambda^*_{++}U_{+-})g) = b'^*(g)\tilde{\Gamma}(U)
\end{aligned}$$

which proves (4.22). $\blacksquare$

It should be noticed that as a consequence of this theorem one can write the "matrix element" $(\phi,\mathcal{U}\psi)$ for "physical vectors" ϕ,ψ as a finite sum of terms each of which is a finite product of the "matrix elements" of the operator Λ on $\mathcal{K}$ and the scalar $\det(\cdot\cdot)^{\mp\frac{1}{2}}$.

We further observe that a pseudo-unitary U is unitary if and only if

$$U_{+-} = U_{-+} = 0. \tag{4.23}$$

Assuming (4.23) for bosons and fermions one can define a unitary operator $\tilde{U}$ by

$$\tilde{U}_{+-} = \tilde{U}_{-+} = 0 \qquad \tilde{U}_{++} = U_{++} \qquad \tilde{U}_{--} = \bar{U}_{--} \tag{4.24}$$

where

$$(\bar{U}_{--}v)(p) \equiv \overline{(U_{--}\bar{v})(p)} \qquad \forall v\in\mathcal{K}. \tag{4.25}$$

Then

$$\tilde{\Gamma}(U)\subset\Gamma(\tilde{U}) \tag{4.26}$$

which motivates our notation (for a definition of $\Gamma(\tilde{U})$ see e.g. (15)).

Using Stone's theorem one can conclude from the Weyl algebra formulation of the CCR that $\mathcal{U}$ maps $\tilde{D}$ onto $\tilde{D}$. This also follows from our theorem. To show this, let $\psi\in D_f$. Then $P_N\psi = \psi$ if N is big enough. Now let $\psi_n\in D$ be such that $P_N\psi_n = \psi_n$ and $\psi_n \to \psi$ then, e.g.,

$$a(f)\mathcal{U}P_N\psi_n = \mathcal{U}a'(f)P_N\psi_n. \tag{4.27}$$

The l.h.s. is well-defined since by the theorem

$$\mathcal{U}D\subset D_\infty. \tag{4.28}$$

Now the limit $n \to \infty$ in (4.27) exists since $a'(f)$ is bounded on $P_N\mathcal{F}_\varepsilon$. Hence

$$\mathcal{U}\psi\in D(a(f)) \qquad a(f)\mathcal{U}\psi = \mathcal{U}a'(f)\psi. \tag{4.29}$$

Thus,

$$\mathcal{U}D_f\subset\tilde{D}. \tag{4.30}$$

If $\phi\in\tilde{D}$ then by (4.29)

$$a(f)\mathcal{U}P_M\phi = \mathcal{U}a'(f)P_M\phi. \tag{4.31}$$

The limit $M \to \infty$ in (4.31) exists in view of (2.5). Therefore

$$\mathcal{U}\phi\in D(a(f)) \qquad a(f)\mathcal{U}\phi = \mathcal{U}a'(f)\phi \tag{4.32}$$

so

$$\mathcal{U}\tilde{D}\subset\tilde{D}. \tag{4.33}$$

Repeating the argument for $\mathcal{U}^*$ we infer

$$\mathcal{U}\tilde{D} = \tilde{D} \tag{4.34}$$

as asserted.

In a coordinate free approach one should define the operator in (4.5) directly on D, replacing complex conjugation by the conjugation K. One could then proceed as indicated at the end of section 3.

5. THE GENERAL FERMION CASE.

We shall now treat the general fermion case, i.e. we drop the assumption (3.1). Let $\{g'_j\}_{j=1}^{M}$ and $\{f'_i\}_{i=1}^{L}$ be orthonormal bases for Ker U_{++} resp. Ker U_{--}. In view of our standing assumption (2.27) one has $M,L < \infty$. Defining

$$\begin{aligned} f_i &= Uf'_i \\ g_j &= Ug'_j \end{aligned} \tag{5.1}$$

one easily verifies that $\{f_i\}_{i=1}^{L}$ and $\{g_j\}_{j=1}^{M}$ are orthonormal bases for Ker U_{++}^* resp. Ker U_{--}^*. From (2.15) and (2.27) we now infer that U_{--}, as an operator from $(\text{Ker } U_{--})^{\perp}$ to $(\text{Ker } U_{--}^*)^{\perp}$, has a bounded inverse mapping $(\text{Ker } U_{--}^*)^{\perp}$ onto $(\text{Ker } U_{--})^{\perp}$. We extend this inverse to $\mathcal{H}_-$ by setting it equal to zero on Ker U_{--}^* and denote the resulting operator on $\mathcal{H}_-$ by U_{--}^{-1}. In an analogous fashion we define the bounded operator $U_{++}^{*\,-1}$.

Defining a bounded operator Λ by (3.2) it is straightforward to verify, using the unitarity relations (2.15), that (3.3), (3.6) and (3.7) again hold true. However, it should be noticed that (3.4) only holds if L=M=0 since it implies that U_{--}, as an operator from $\mathcal{H}_-$ to $\mathcal{H}_-$, has the inverse $1_{--} - \Lambda_{--}$.

The next lemma is the generalization of lemma 4.1. An analogous result has been obtained in (5).

Lemma 5.1. The following relation holds true:

$$\mathcal{U}\Omega = e^{i\theta}\det(1_{--}+\Lambda_{+-}^*\Lambda_{+-})^{-\frac{1}{2}}\prod_{i=1}^{L} a^*(f_i)\prod_{j=1}^{M} b^*(\overline{g_j})\exp(\Lambda_{+-}a^*b^*)\Omega \quad 0 \le \theta < 2\pi. \tag{5.2}$$

Proof. From

$$\Lambda_{+-}{}^{*}f_i = \Lambda_{+-}g_j = 0 \qquad i = 1,..,L \qquad j = 1,..,M \tag{5.3}$$

it follows that $a(f_i)$, $b(\bar{g}_j)$ commute with $\exp(\Lambda_{+-}a^*b^*)$. Consequently the norm of the r.h.s. of (5.2) equals 1. It remains to prove

$$(a(U_{++}f)+b^*(\overline{U_{-+}f}))\prod_{i=1}^{L} a^*(f_i)\prod_{j=1}^{M} b^*(\overline{g_j})\exp(\Lambda_{+-}a^*b^*)\Omega = 0 \qquad \forall f\in\mathcal{H}_+ \tag{5.4}$$

$$(b(\overline{U_{--}g})+a^*(U_{+-}g))\prod_{i=1}^{L} a^*(f_i)\prod_{j=1}^{M} b^*(\overline{g_j})\exp(\Lambda_{+-}a^*b^*)\Omega = 0 \qquad \forall g\in\mathcal{H}_-. \tag{5.5}$$

It follows from (5.1) that these relations hold if $f\in \text{Ker } U_{++}$ resp. $g\in \text{Ker } U_{--}$. If $f\in(\text{Ker } U_{++})^{\perp}$ then $a(U_{++}f)$ in (5.4) (anti)commutes with $\Pi..\Pi...$ Since

$$\Lambda_{+-}{}^{*}U_{++}f+U_{-+}f = 0 \qquad \forall f\in(\text{Ker } U_{++})^{\perp} \tag{5.6}$$

we conclude as in lemma 4.1 that (5.4) holds. Similarly, (5.5) follows from

$$-\Lambda_{+-}U_{--}g+U_{+-}g = 0 \qquad \forall g\in(\text{Ker } U_{--})^{\perp}. \quad \blacksquare \tag{5.7}$$

We normalize $\mathcal{U}$ by setting

$$\mathcal{U}\Omega = \det(1_{--}+\Lambda_{+-}{}^{*}\Lambda_{+-})^{-\frac{1}{2}}\prod_{i=1}^{L} a^*(f_i)\prod_{j=1}^{M} b^*(\overline{g_j})\exp(\Lambda_{+-}a^*b^*)\Omega \tag{5.8}$$

where the products are in the natural order of the indices. This convention will also be used in the sequel.

Defining the operator $\tilde{\Gamma}(U):D \to D_\infty$ by (4.14) one concludes in the same way as in the proof of Th. 4.1 that on D

$$\tilde{\Gamma}(U)a^*(f) = a'^*(f)\tilde{\Gamma}(U) \qquad \forall f\in\mathcal{H}_+ \tag{5.9}$$

$$\tilde{\Gamma}(U)b^*(\bar{g}) = b'^*(\bar{g})\tilde{\Gamma}(U) \qquad \forall g\in\mathcal{H}_-. \tag{5.10}$$

From this proof one also infers that on D

$$\tilde{\Gamma}(U)a^*(f) = \tilde{\Gamma}(U)b^*(\bar{g}) = 0 \qquad \forall f\in \text{Ker } U_{++} \qquad \forall g\in \text{Ker } U_{--}. \tag{5.11}$$

Hence, by (5.9) and (5.10),

$$b(\overline{f'})\tilde{\Gamma}(U) = a(g')\tilde{\Gamma}(U) = 0 \qquad \forall f'\in \text{Ker } U^*_{--} \qquad \forall g'\in \text{Ker } U^*_{++}. \tag{5.12}$$

We note that $\tilde{\Gamma}(-U)$ also satisfies (5.9-12) apart from a minus sign at the r.h.s. of (5.9) and (5.10).

Now let P be the set of all partitions of the index set $\{1,..,L\}\cup\{1,..,M\}$ into two subsets. P clearly contains 2^{L+M} elements. An element $(\rho,\tau)\in P$ is specified by two subsets $\{\rho_1,..,\rho_\ell\}\cup\{\tau_1,..,\tau_m\}$ and $\{\rho_{\ell+1},..,\rho_L\}\cup\{\tau_{m+1},..,\tau_M\}$ in which we take by convention the indices in the natural order. We now define a function on P by

$$\mathrm{sgn}(\rho,\tau) = \mathrm{sgn}(\rho_{\ell+1},..,\rho_L,\tau_{m+1}+L,..,\tau_M+L,\rho_1,..,\rho_\ell,\tau_1+L,..,\tau_m+L), \tag{5.13}$$

i.e. $\mathrm{sgn}(\rho,\tau)$ is the sign of the permutation of the indices $\{1,..,L+M\}$ which occur in the r.h.s. of (5.13).

Defining the operator $\mathcal{U}':D \to D_\infty$ by

$$\mathcal{U}' = \sum_{(\rho,\tau)\in P} \mathrm{sgn}(\rho,\tau)\prod_{i=1}^{\ell} a^*(f_{\rho_i})\prod_{j=1}^{m} b^*(\overline{g_{\tau_j}})\tilde{\Gamma}((-)^{L+M}U)\prod_{i=\ell+1}^{L} b(\overline{f'_{\rho_i}})$$

$$\cdot\prod_{j=m+1}^{M} a(g'_{\tau_j}) \tag{5.14}$$

we are in a position to state the following theorem.

<u>Theorem 5.1</u>. The operator $\mathcal{U}$ is equal to $\mathcal{U}'$ on D:

$$\mathcal{U}\phi = \sum_{(\rho,\tau)\in P}\mathrm{sgn}(\rho,\tau)\prod_{i=1}^{\ell} a^*(f_{\rho_i})\prod_{j=1}^{m} b^*(\overline{g_{\tau_j}})\tilde{\Gamma}((-)^{L+M}U)\prod_{i=\ell+1}^{L} b(\overline{f'_{\rho_i}})\prod_{j=m+1}^{M} a(g'_{\tau_j})\phi$$

$$\forall\phi\in D. \tag{5.15}$$

<u>Proof</u>. It follows from (5.8) that $\mathcal{U}'\Omega = \mathcal{U}\Omega$, so it suffices to prove

$$\mathcal{U}'a^*(f) = a'^*(f)\mathcal{U}' \qquad \forall f\in\mathcal{H}_+ \tag{5.16}$$

$$\mathcal{U}'b^*(\overline{g}) = b'^*(\overline{g})\mathcal{U}' \qquad \forall g\in\mathcal{H}_- \tag{5.17}$$

which should hold on D. To show this, first take $f\in(\mathrm{Ker}\ U_{++})^\perp$. Then $a^*(f)$ anticommutes with the a and b in (5.14) so we can use (5.9). Both $a^*(U_{++}f)$ and $b(\overline{U_{-+}f})$ now anticommute with all a^* and b^* in (5.14) since $U_{++}f\in(\mathrm{Ker}\ U_{++}^*)^\perp$ and $U_{-+}f\in(\mathrm{Ker}\ U_{--}^*)^\perp$. If L+M is odd the resulting minus sign is compensated by the extra minus sign from (5.9). We conclude that (5.16) and, similarly, (5.17) hold true if $f\in(\mathrm{Ker}\ U_{++})^\perp$ resp. $g\in(\mathrm{Ker}\ U_{--})^\perp$. It therefore suffices to show

$$\mathcal{U}'a^*(g'_{j_o}) = b(\overline{g_{j_o}})\mathcal{U}' \qquad \forall j_o\in\{1,..,M\} \tag{5.18}$$

$$\mathcal{U}'b^*(\overline{f'_{i_o}}) = a(f_{i_o})\mathcal{U}' \qquad \forall i_o\in\{1,..,L\}. \tag{5.19}$$

To prove (5.18) we observe that from (5.11) it follows that $\mathcal{U}'a^*(g'_{j_0})$ equals the sum of all terms in (5.14) in which the index j_0 is at the right of $\tilde{\Gamma}$, with the factor $a(g'_{j_0})$ suppressed, while from (5.12) it follows that $b(\overline{g_{j_0}})\mathcal{U}'$ is equal to the sum of all terms in which it is at the left, with the factor $b^*(\overline{g_{j_0}})$ suppressed; the terms get an extra minus sign if the number of transpositions required to pull the suppressed factor to the right resp. the left is odd. It is easily seen that the same terms occur in the l.h.s. and the r.h.s. of (5.18). To show that they have the same sign, let (ρ,τ) be a partition such that j_0 is at the right of $\tilde{\Gamma}$ and let (ρ,τ') be the corresponding partition, i.e. it equals (ρ,τ) except that j_0 is at the left. We should then prove that

$$\mathrm{sgn}(\rho,\tau')(-)^{\ell+j_1'-1} = \mathrm{sgn}(\rho,\tau)(-)^{M-j_1} \tag{5.20}$$

where j_1',j_1 are such that

$$j_0 = \tau_{j_1} = \tau'_{j_1'}. \tag{5.21}$$

However, this follows immediately from (5.13), so (5.18) is proved. The proof of (5.19) is similar. $\blacksquare$

6. <u>THE NECESSITY OF (2.27)</u>.

We observe that the sufficiency of (2.27) for implementability, i.e. for the existence of a non-zero vector $\Omega'\in\tilde{D}$ satisfying (2.26), follows from lemmas 3.1, 4.1 and 5.1. We now give a proof of the necessity of these conditions. For notational convenience (in the boson case) we again assume (2.1). One easily sees that the result does not depend on this choice.

<u>Theorem 6.1</u>. Let U be a (pseudo-)unitary operator on $\mathcal{H}$. If there exists a non-zero vector $\Omega'\in\tilde{D}$ such that

$$a'(f)\Omega' = b'(\bar{g})\Omega' = 0 \qquad \forall f\in\mathcal{H}_+ \qquad \forall g\in\mathcal{H}_- \tag{6.1}$$

where

$$\begin{aligned} a'(f) &= a(U_{++}f)\pm b^*(\overline{U_{-+}f}) \\ b'(\bar{g}) &= b(\overline{U_{--}g})\pm a^*(U_{+-}g) \end{aligned} \tag{6.2}$$

then U_{+-} and U_{-+} are H.S..

Proof. A. Bosons. We define projections $P^{n,r}(n,r\in Z)$ by (see(2.2))

$$(P^{n,r}\psi)^{n',r'} = \delta_{nn'}\,\delta_{rr'}\psi^{n,r} \tag{6.3}$$

where the notation should be clear. One easily verifies relations like

$$P^{n,r}a(f) = a(f)P^{n+1,r}, \tag{6.4}$$

which of course holds on $D(a(f))$. Since $P^{n,r}a'(f)\Omega' = 0$ we have $(n,r \geq 0)$

$$a(U_{++}f)P^{n+1,r}\Omega' = b^*(\overline{U_{-+}f})P^{n,r-1}\Omega'. \tag{6.5}$$

It evidently follows from (2.15) that Ran U_{++} equals $\mathcal{H}_+$ so from (6.5) we conclude that if $P^{n,r-1}\Omega' = 0$ then also $P^{n+1,r}\Omega' = 0$. This implies

$$P^{n+k,k}\Omega' = 0 \qquad \forall n > 0 \qquad \forall k \geq 0. \tag{6.6}$$

From $P^{n,r}b'(\bar{g})\Omega' = 0$ we infer analogously

$$P^{\ell,r+\ell}\Omega' = 0 \qquad \forall r > 0 \qquad \forall \ell \geq 0. \tag{6.7}$$

Since $||\Omega'|| \neq 0$ it follows from (6.6), (6.7) and the argument given above that we must have

$$P^{0,0}\Omega' \equiv \alpha \neq 0. \tag{6.8}$$

Defining

$$\psi(p,q) = \frac{1}{\alpha}(P^{1,1}\Omega')^{1,1}(p,q) \tag{6.9}$$

one has, using $P^{0,1}a'(f)\Omega' = 0$ resp. $P^{1,0}b'(\bar{g})\Omega' = 0$ and (2.3),

$$\int dp(\overline{U_{++}f})(p)\psi(p,q) = (\overline{U_{-+}f})(q) \qquad \forall f\in\mathcal{H}_+ \tag{6.10}$$

$$\int dq(U_{--}g)(q)\psi(p,q) = (U_{+-}g)(p) \qquad \forall g\in\mathcal{H}_-. \tag{6.11}$$

Introducing a H.S. operator $H_B:\mathcal{H}_- \to \mathcal{H}_+$ by

$$(H_Bg)(p) = \int dq\psi(p,q)g(q) \tag{6.12}$$

we can write (6.10) resp. (6.11) as

$$H_B^*U_{++} = U_{-+} \tag{6.13}$$

$$H_BU_{--} = U_{+-}. \tag{6.14}$$

We conclude that U_{-+} and U_{+-} are H.S..

B. Fermions. Let $\{f_i\}_{i=1}^{L}$ and $\{g_j\}_{j=1}^{M}$ be orthonormal (o.n.) bases for Ker U_{++}^{*} resp. Ker U_{--}^{*}. Let $\{f_i\}_{i=L+1}^{\infty}$ and $\{g_j\}_{j=M+1}^{\infty}$ be o.n. bases for $\overline{\text{Ran } U_{++}}$ resp. $\overline{\text{Ran } U_{--}}$. Then $\{f_i\}_{i=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ obviously are o.n. bases for $\mathcal{H}_+$ resp. $\mathcal{H}_-$. We now introduce an o.n. basis for $\mathcal{F}_a$ by setting

$$\phi_{\rho_1,\rho_2,..;\tau_1,\tau_2,..} \equiv \prod_{i=1}^{\infty} a^*(f_i)^{\rho_i} \prod_{j=1}^{\infty} b^*(\overline{g_j})^{\tau_j}\Omega \tag{6.15}$$

where

$$\rho_i,\tau_j = 0,1 \qquad\qquad \sum_{i=1}^{\infty}\rho_i + \sum_{j=1}^{\infty}\tau_j < \infty. \tag{6.16}$$

Then

$$\Omega' = \sum_{\rho_i,\tau_j} \alpha_{\rho_1,..;\tau_1,..}\,\phi_{\rho_1,..;\tau_1,..}\,. \tag{6.17}$$

From (6.1) and (6.2) (cf. section 5, esp. (5.1)):

$$a^*(f_i)\Omega' = b^*(\overline{g_j})\Omega' = 0 \qquad i = 1,..,L \qquad j = 1,..,M. \tag{6.18}$$

Using (6.17) one now concludes that $L,M < \infty$ (this was anticipated above for notational convenience) and that

$$\sum_{i=1}^{L} \rho_i < L \text{ or } \sum_{j=1}^{M} \tau_j < M \rightarrow \alpha_{\rho_1,..;\tau_1,..} = 0. \tag{6.19}$$

Thus,

$$P^{n,r}\Omega' = 0 \qquad \forall n < L \qquad\qquad \forall r < M \tag{6.20}$$

$$P^{L,M}\Omega' = \beta \prod_{i=1}^{L} a^*(f_i) \prod_{j=1}^{M} b^*(\overline{g_j})\Omega \tag{6.21}$$

$$P^{L+1,M+1}\Omega' = \sum_{k=L+1}^{\infty} \sum_{\ell=M+1}^{\infty} \gamma_{k\ell}\, a^*(f_k) b^*(\overline{g_\ell}) P^{L,M}\Omega'. \tag{6.22}$$

From $P^{L+n,M+r}a'(f)\Omega' = 0$ $(n,r \geq 0)$ it follows that

$$a(U_{++}f)P^{L+n+1,M+r}\Omega' = -b^*(\overline{U_{-+}f})P^{L+n,M+r-1}\Omega'. \tag{6.23}$$

Using (6.19) one easily concludes that (6.23) implies: if $P^{L+n,M+r-1}\Omega' = 0$ then $P^{L+n+1,M+r}\Omega' = 0$. Hence, from (6.20),

$$P^{L+n+k,M+k}\Omega' = 0 \qquad \forall n > 0 \qquad \forall k \geq 0. \tag{6.24}$$

Analogously,

$$P^{L+\ell,M+r+\ell}\Omega' = 0 \qquad \forall r > 0 \qquad \forall \ell \geq 0. \tag{6.25}$$

We therefore must have $\beta \neq 0$ in (6.21).

Defining a H.S. operator $H_F : \overline{\mathrm{Ran}\ U_{--}} \to \overline{\mathrm{Ran}\ U_{++}}$ by

$$H_F g = \sum_{k=L+1}^{\infty} \sum_{\ell=M+1}^{\infty} f_k \gamma_{k\ell} (g_\ell , g) \tag{6.26}$$

one infers from (6.21) and (6.22), using $P^{L,M+1} a'(f)\Omega' = 0$ resp. $P^{L+1,M} b'(\bar{g})\Omega' = 0$:

$$H_F^* U_{++} f = -U_{-+} f \qquad \forall f \in (\mathrm{Ker}\ U_{++})^{\perp} \tag{6.27}$$

$$H_F U_{--} g = U_{+-} g \qquad \forall g \in (\mathrm{Ker}\ U_{--})^{\perp}. \tag{6.28}$$

Thus, U_{-+} and U_{+-} are direct sums of a H.S. operator and a finite-rank operator. Therefore U_{-+} and U_{+-} are H.S.. $\blacksquare$

We finally make some remarks about unbounded pseudo-unitary operators. It seems reasonable to require that (2.11) hold on a dense subspace M belonging to the domains of U and U^* and invariant under $P_{\pm}$, U and U^*. If U_{+-} and U_{-+} are H.S. one concludes from (2.15), which holds on M, and from the relation

$$(U_{\varepsilon\varepsilon'} \restriction M)^* \supset U^*_{\varepsilon'\varepsilon} \restriction M \tag{6.29}$$

which follows from our assumptions, that U must be bounded.

On the other hand, if the conditions of Th.6.1 are met (for any $f, g \in M$), one is again led to (6.13) and (6.14) which now hold on M (use (2.15) and (6.29) to establish that $\mathrm{Ran}(U_{++} \restriction M)$ and $\mathrm{Ran}(U_{--} \restriction M)$ are dense in $\mathcal{H}_+$ resp. $\mathcal{H}_-$). Now from (6.29) and (2.15) it follows that $A \equiv U_{--} \restriction M$ has a bounded inverse A^{-1} and that

$$A^{-1*} A^{-1} = 1_{--} - H_B^* H_B \tag{6.30}$$

on $U_{--}M$, where (6.14) has been used. In virtue of the relation $\bar{A}^{-1} = \overline{A^{-1}}$ it follows from (6.30) that

$$\bar{A}^{*-1} \bar{A}^{-1} = 1_{--} - H_B^* H_B \tag{6.31}$$

on $\mathcal{H}_-$. Since H_B is compact,

$$\| A^{-1} g \| \geq \varepsilon \, \| g \| \qquad \varepsilon > 0 \qquad \forall g \in U_{--}M. \tag{6.32}$$

Thus, $U_{--} \restriction M$ is bounded. Similarly, $U_{++} \restriction M$ is bounded, so U must be bounded. We conclude that unbounded pseudo-unitary operators (as defined above) cannot give rise to implementable Bogoliubov transformations.

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CHAPTER 2

CHARGED PARTICLES IN EXTERNAL FIELDS
I CLASSICAL THEORY

1. INTRODUCTION.

This is the first of two papers on the external field problem, i.e. the problem of the interaction of relativistic particles with electromagnetic or other fields which are prescribed functions on space-time. The present paper is mainly concerned with classical aspects. The results of this paper and of a paper on the Bogoliubov transformations which occur in the (second-) quantized theory of charged particles (1) are used in (2) to study the quantized Dirac and Klein-Gordon theories.

The paper has two parts. The first part (section 2 and the appendix) contains mathematical results of a perturbation-theoretic character. A number of these can be regarded as generalizations of recent results of Bellissard (3, 4). Assuming that certain distributions, which are connected with the perturbed Green's functions, upon restriction to the mass shell give rise to bounded operators on the classical Hilbert space, he obtains relations between these operators which imply the (pseudo-)unitarity of closely related operators. This can be regarded as a special case of a quite general perturbation-theoretic structure which is presented in section 2.

In the second part (sections 3 and 4) these general results are applied to the classical Dirac and Klein-Gordon equations with external fields. For fields which are test functions on space-time two-sided tempered retarded and advanced fundamental solutions are shown to exist and their relation with the evolution operator is established. Furthermore, the S-operator is proved to be Lorentz covariant and causal. In the spin-$\frac{1}{2}$ case we show that the evolution is unitarily implementable in Fock space if only the timelike component of the vector field (i.e. the electric field) or the pseudovector field is non-zero. In the spin-0 case this holds true for electric or scalar fields. (Earlier results on implementability of the evolution can be found in (5-11).) It was recently proved by Hochstenbach (11) that in the spin-0 case a time-independent magnetic field gives rise to an evolution operator which is not implementable in Fock space. We show that in the spin-$\frac{1}{2}$ case the same Haag phenomenon occurs in general for the fourteen remaining kinds of fields, and rederive his result in a different way.

As a rule we do not state the weakest conditions under which our results apply. Thus, we usually assume that the fields are test functions although in particular the results on implementability hold true for more general functions. Similarly, the hypotheses of section 2 could be relaxed and generalized. Here, however, we did try to accommodate the general features of theories of relativistic particles. In view of the resulting generality the reader is advised to skip this section on first reading and refer back to it when needed.

2. CLASSICAL PERTURBATION THEORY.

In this section it is assumed that $\mathcal{H}$ is a separable Hilbert space which is the direct sum of two subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ with corresponding projections P_+ and P_-. A bounded operator U on $\mathcal{H}$ is defined to be pseudo-unitary if

$$UqU^* = U^*qU = q \tag{2.1}$$

where

$$q \equiv P_+ - P_-. \tag{2.2}$$

We define

$$U_{\varepsilon\varepsilon'} = P_\varepsilon U P_{\varepsilon'} \qquad \varepsilon, \varepsilon' = +, - \tag{2.3}$$

and note that

$$U_{\varepsilon\varepsilon'}{}^* = U^*_{\varepsilon'\varepsilon}. \tag{2.4}$$

The set $R \cup \{ -\infty, \infty \}$ is denoted by $\tilde{R}$; continuity on $\tilde{R}$ is defined in the obvious way. We also need $\tilde{R}^2$ which by definition has the product topology. Let O(t) be a function on R with values in the bounded operators on $\mathcal{H}$. We assume that O(t) is strongly continuous on R and that

$$|| O(\cdot) || \in L^1(R). \tag{2.5}$$

We define

$$K = \int dt \; || \; O(t) \; || \qquad\qquad \ell = K^{-1}. \tag{2.6}$$

We now introduce a number of operator-valued functions. The set of (λ, T_2, T_1) for which they are defined is delineated in theorems 1 and 2 below. Their arguments will be suppressed whenever this does not give rise to confusion. In this section any operator integration or differentiation is in the strong sense.

$$\left.\begin{array}{l} R_\lambda \; (T_2, \; T_1) \\ \\ A_\lambda \; (T_2, \; T_1) \end{array}\right\} = \sum_{n=1}^{\infty} (i\lambda)^n \int_{T_1}^{T_2} dt_1 .. \int_{T_1}^{T_2} dt_n O(t_1) \left[\pm\hat{\theta}(\pm(t_1 - t_2))\right] O(t_2) ..$$

$$\cdot \left[\pm\hat{\theta}(\pm(t_{n-1} - t_n)) \right] O(t_n) \tag{2.7}$$

$$\left.\begin{array}{l} F_\lambda \; (T_2, \; T_1) \\ \\ \bar{F}_\lambda \; (T_2, \; T_1) \end{array}\right\} = \sum_{n=1}^{\infty} (i\lambda)^n \sum_{\varepsilon_1, \ldots, \varepsilon_{n-1}} \int_{T_1}^{T_2} dt_1 .. \int_{T_1}^{T_2} dt_n O(t_1)$$

$$\cdot \left[\pm\varepsilon_1 P_{\varepsilon_1} \hat{\theta}(\pm\varepsilon_1 (t_1 - t_2))\right] .. \left[\pm\varepsilon_{n-1} P_{\varepsilon_{n-1}} \hat{\theta}(\pm\varepsilon_{n-1}(t_{n-1} - t_n))\right] O(t_n) \tag{2.8}$$

where

$$\hat{\theta}(t) \equiv \begin{cases} \theta(t) & T_2 > T_1 \\ \theta(-t) & T_2 < T_1 \end{cases} \tag{2.9}$$

$$\begin{array}{ll} U = 1 + R & Z = 1 + qF \\ V = 1 - A & \bar{Z} = 1 - q\bar{F} \end{array} \tag{2,10}$$

Finally,

$$R^{(o)} \; (T_2, \; T_1) = 1$$

$$R^{(n)} \; (T_2, \; T_1) = i^n \int_{T_1}^{T_2} dt_1 .. \int_{T_1}^{t_{n-1}} dt_n O(t_1) .. O(t_n) \qquad n \geq 1. \tag{2.11}$$

We have occasion to use the properties of these functions which are enumerated in the following theorems.

Th. 2.1. For any $(T_2, T_1) \in \tilde{R}^2$ R, A, U and V are $||.||$-entire functions of λ. For any $\lambda \in C$ they are $||.||$-continuous on $\tilde{R}^2$. On $C \times \tilde{R}^2$:

$$U_\lambda(T_2, T_1) = \sum_{n=0}^{\infty} \lambda^n R^{(n)}(T_2, T_1) \tag{2.12}$$

$$||U_\lambda(T_2, T_1)|| \leq \exp(|\lambda|K). \tag{2.13}$$

Th. 2.2. For any $(T_2, T_1) \in \tilde{R}^2$ F, $\bar{F}$, Z and $\bar{Z}$ are $||.||$-analytic functions of λ in the disc

$$D_\ell \equiv \left\{ \lambda \in C \,\middle|\, |\lambda| < \ell \right\}. \tag{2.14}$$

For any $\lambda \in D_\ell$ they are $||.||$-continuous on $\tilde{R}^2$.

Th. 2.3. On $C \times \tilde{R}^2$ ((2.15)) resp. $D_\ell \times \tilde{R}^2$ ((2.16-20)):

$$R - A - RA = R - A - AR = 0 \tag{2.15}$$
$$F - \bar{F} - Fq\bar{F} = F - \bar{F} - \bar{F}qF = 0 \tag{2.16}$$
$$R - F - RP_-F = R - F - FP_-R = 0 \tag{2.17}$$
$$R - \bar{F} - RP_+\bar{F} = R - \bar{F} - \bar{F}P_+R = 0 \tag{2.18}$$
$$A - F + AP_+F = A - F + FP_+A = 0 \tag{2.19}$$
$$A - \bar{F} + AP_-\bar{F} = A - \bar{F} + \bar{F}P_-A = 0\,. \tag{2.20}$$

Th. 2.4. (The interaction picture evolution operator) On $C \times \tilde{R}^2$:

$$U(T, T) = 1$$
$$U(T_3, T_2)U(T_2, T_1) = U(T_3, T_1). \tag{2.21}$$

For any $T_1(T_2) \in \tilde{R}$ and any $T_2(T_1) \in R$ $U(T_2, T_1)$ is differentiable w.r.t. $T_2(T_1)$. In these points

$$\partial_{T_2} U_\lambda(T_2, T_1) = i\lambda O(T_2)U_\lambda(T_2, T_1) \tag{2.22}$$

$$\partial_{T_1} U_\lambda(T_2, T_1) = -i\lambda U_\lambda(T_2, T_1)O(T_1). \tag{2.23}$$

If $\phi(T_2) \in \mathcal{H}$ is strongly differentiable on R and

$$\phi(T_1) = \phi \qquad T_1 \in R$$

$$i\dot{\phi}(T_2) = -\lambda O(T_2)\phi(T_2) \qquad \forall T_2 \in R \tag{2.24}$$

then

$$\phi(T_2) = U_\lambda(T_2, T_1)\phi. \tag{2.25}$$

<u>Th. 2.5.</u> (The Schrödinger picture evolution operator) Let

$$O(t) = \exp(iH_o t)V(t)\exp(-iH_o t) \tag{2.26}$$

where H_o is a self-adjoint operator on $\mathcal{H}$ and $V(t)$ is continuously differentiable. Let

$$U_\lambda^s(T_2, T_1) = \exp(-iH_o T_2)U_\lambda(T_2, T_1)\exp(iH_o T_1). \tag{2.27}$$

Then, on $C \times R^2$:

$$U^s(T, T) = 1$$

$$U^s(T_3, T_2)U^s(T_2, T_1) = U^s(T_3, T_1) \tag{2.28}$$

$$U^s D(H_o) = D(H_o). \tag{2.29}$$

For any $(T_2, T_1) \in R^2$ and $\phi \in D(H_o)$ $U^s(T_2, T_1)\phi$ is strongly differentiable w.r.t. T_2 and T_1, and

$$\partial_{T_2} U_\lambda^s(T_2, T_1)\phi = -i(H_o - \lambda V(T_2))\, U_\lambda^s(T_2, T_1)\phi \tag{2.30}$$

$$\partial_{T_1} U_\lambda^s(T_2, T_1)\phi = iU_\lambda^s(T_2, T_1)(H_o - \lambda V(T_1))\phi. \tag{2.31}$$

If $\phi(T_2) \in D(H_o)$ is strongly differentiable on R and

$$\phi(T_1) = \phi \qquad T_1 \in R$$

$$i\dot{\phi}(T_2) = (H_o - \lambda V(T_2))\phi(T_2) \qquad \forall T_2 \in R \tag{2.32}$$

then

$$\phi(T_2) = U_\lambda^s(T_2, T_1)\phi. \tag{2.33}$$

Th. 2.6. (Rearrangement) Let $\tilde{O}(t)$ satisfy the same conditions as $O(t)$ and let

$$\tilde{U}_\lambda(T_2, T_1) = \sum_{n=0}^{\infty} \tilde{R}^{(n)}_\lambda(T_2, T_1) \tag{2.34}$$

where

$$\tilde{R}^{(o)} \equiv 1 \tag{2.35}$$

$$\tilde{R}^{(1)}_\lambda(T_2, T_1) \equiv i \int_{T_1}^{T_2} dt(\lambda O(t)+\lambda^2\tilde{O}(t)) \tag{2.36}$$

$$\tilde{R}^{(n)}_\lambda(T_2, T_1) \equiv i \int_{T_1}^{T_2} dt(\lambda O(t)+\lambda^2\tilde{O}(t))\tilde{R}^{(n-1)}_\lambda(t, T_1)$$
$$-i \int_{T_1}^{T_2} dt\lambda^2\tilde{O}(t)\tilde{R}^{(n-2)}_\lambda(t, T_1) \qquad n \geq 2. \tag{2.37}$$

Then, on $C \times \tilde{R}^2$:

$$\tilde{U}_\lambda(T_2, T_1) = U_\lambda(T_2, T_1). \tag{2.38}$$

Th. 2.7. For any $(\lambda, T_2, T_1) \in C \times \tilde{R}^2$ $(D_\ell \times \tilde{R}^2)$ U and V (Z and $\bar{Z}$) are invertible, and

$$U^{-1} = V \tag{2.39}$$

$$Z^{-1} = \bar{Z}. \tag{2.40}$$

On $C \times \tilde{R}^2$:

$$U_{--}V_{--} = 1_{--} - U_{-+}V_{+-}$$
$$V_{--}U_{--} = 1_{--} - V_{-+}U_{+-}. \tag{2.41}$$

For any $(\lambda, T_2, T_1) \in D_\ell \times \tilde{R}^2$ U_{++}, V_{++} and U_{--}, V_{--} are invertible as operators on $\mathcal{H}_+$ resp. $\mathcal{H}_-$, and

$$F_{--} = 1_{--} - U_{--}^{-1} \qquad F_{+-} = U_{+-}U_{--}^{-1}$$
$$F_{-+} = U_{--}^{-1}\,U_{-+} \qquad F_{++} = U_{++} - 1_{++} - U_{+-}U_{--}^{-1}U_{-+}. \tag{2.42}$$

Th. 2.8. Let $R^{(1)}(T_2, T_1)_{\substack{+-\\-+}}$ be continuous on $\tilde{R}^2$ in the Hilbert-Schmidt (H.S.) norm and let

$$||R^{(1)}(T_2, T_1)_{\substack{+-\\-+}}||_2 \leq C_1 < \infty \tag{2.43}$$

on $\tilde{R}^2$. Then $U_{\substack{+-\\-+}}$ are $||.||_2$-entire functions of λ and $||.||_2$-continuous on $\tilde{R}^2$. Moreover, the set $E(T_2, T_1)\subset C$ where $U_\lambda(T_2, T_1)_{--}$ or $V_\lambda(T_2, T_1)_{--}$ are singular lies outside D_ℓ and has no limit points in C. $F_\lambda(T_2, T_1)$ has an $||.||$-analytic continuation to $C\backslash E(T_2, T_1)$. The operators $F_{\substack{+-\\-+}}$ are $||.||_2$-analytic functions of λ on $C\backslash E$ and they are $||.||_2$-continuous on $\tilde{R}^2$ for any $\lambda\in D_\ell$.

Th. 2.9. Let $\lambda_o\in C\backslash\{o\}$ and let

$$||U_{\lambda_o}(T_2, T_1)_{+-}||_2 \leq C < \infty \tag{2.44}$$

for any $(T_2, T_1)\in Q$, where

$$Q \equiv [a, b] \times [a, b] \qquad -\infty \leq a < b \leq \infty. \tag{2.45}$$

Then $R^{(1)}(T_2, T_1)_{+-}$ is H.S. on Q. Furthermore, U_{+-} is an $||.||_2$-entire function of λ for any $(T_2, T_1)\in Q$.

If, in addition, $U_{\lambda_o}(T_2, T_1)_{+-}$ is $||.||_2$-continuous on Q then U_{+-} is $||.||_2$-continuous on Q for any $\lambda\in C$.

The same statements hold true if +- is replaced by -+.

Th. 2.10. (Bosons) Let

$$O(t)^* = qO(t)q \qquad \forall t\in R. \tag{2.46}$$

Then

$$R^* = -qAq \qquad \forall\lambda\in R \tag{2.47}$$

$$F^* = -q\bar{F}q \qquad \forall\lambda\in(-\ell, \ell). \tag{2.48}$$

For any $\lambda\in R$ U and V are pseudo-unitary and for any $\lambda\in(-\ell, \ell)$ Z and $\bar{Z}$ are unitary.

Setting

$$\Lambda \equiv qF \tag{2.49}$$

$$U' \equiv Z \tag{2.50}$$

$$\Lambda' \equiv qR \tag{2.51}$$

one has for any $\lambda\in(-\ell,\ \ell)$

$$(U-1)-q\Lambda+(U-1)P_-\Lambda = (U-1)-q\Lambda-q\Lambda P_-(U-1) = 0 \tag{2.52}$$

$$(U'-1)-\Lambda'-(U'-1)P_-\Lambda' = (U'-1)-\Lambda'-\Lambda'P_-(U'-1) = 0. \tag{2.53}$$

<u>Th. 2.11.</u> (Fermions) Let

$$O(t)^* = O(t) \qquad \forall t\in R. \tag{2.54}$$

Then

$$R^* = -A \qquad \forall\lambda\in R \tag{2.55}$$

$$F^* = -\bar{F} \qquad \forall\lambda\in(-\ell,\ \ell). \tag{2.56}$$

For any $\lambda\in R$ U and V are unitary and for any $\lambda\in(-\ell,\ \ell)$ Z and $\bar{Z}$ are pseudo-unitary. Setting

$$\Lambda \equiv F \tag{2.57}$$

$$U' \equiv Z \tag{2.58}$$

$$\Lambda' \equiv R \tag{2.59}$$

one has for any $\lambda\in(-\ell,\ \ell)$

$$(U-1)-\Lambda-(U-1)P_-\Lambda = (U-1)-\Lambda-\Lambda P_-(U-1) = 0 \tag{2.60}$$

$$(U'-1)-q\Lambda'+(U'-1)P_-\Lambda' = (U'-1)-q\Lambda'-q\Lambda'P_-(U'-1) = 0. \tag{2.61}$$

<u>Proofs of Theorems 1-11.</u>

$\underline{\underline{1}}$. The relation (2.12) is obvious. For any $(T_2,\ T_1)\in\tilde{R}^2$ and $\phi\in\mathcal{H}$

$$||R^{(n)}(T_2,\ T_1)\phi|| \le \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n\ ||O(t_1)||\ldots||O(t_n)||\ ||\phi||$$

$$\le \frac{K^n}{n!}\ ||\phi|| \tag{2.62}$$

so

$$||R^{(n)}(T_2,\ T_1)|| \le \frac{K^n}{n!}. \tag{2.63}$$

Thus, (2.13) holds. Since U is the sum of a series of entire functions which converges uniformly on any bounded subset of C, U and R are entire functions of λ for any $(T_2, T_1)\in\tilde{R}^2$. To prove the second statement we observe that for any $(T_2, T_1)\in R^2$ and $\phi\in\mathcal{H}$

$$\|R^{(n)}(T_2+\delta_2, T_1+\delta_1)\phi - R^{(n)}(T_2, T_1)\phi\|$$

$$\leq \Big\| \Big[\Big(\int_{T_1+\delta_1}^{T_1} + \int_{T_1}^{T_2} + \int_{T_2}^{T_2+\delta_2}\Big) dt_1 \cdot\cdot \Big(\int_{T_1+\delta_1}^{T_1} + \int_{T_1}^{T_2} + \int_{T_2}^{T_2+\delta_2}\Big) dt_n$$

$$- \int_{T_1}^{T_2} dt_1 \cdot\cdot \int_{T_1}^{T_2} dt_n \Big] \cdot\cdot\ \phi \Big\| . \tag{2.64}$$

Multiplying out the first term and subtracting the second one, one obtains a finite sum of terms each of which contains at least one δ_1 or δ_2. Pulling the norm through the integral in all terms and using (2.5) one concludes that $R^{(n)}$ is norm continuous on R^2. Replacing either $T_2+\delta_2$ or $T_1+\delta_1$ or both by ∞ or $-\infty$ one similarly infers continuity on $\tilde{R}^2$. Thus, in view of the uniform convergence on $\tilde{R}^2$ of their perturbation series, U and R are norm continuous on $\tilde{R}^2$. The proofs for V and A are analogous. ∎

2. Two different choices for $\varepsilon_1, .., \varepsilon_{n-1}$ in (2.8) correspond to different integration regions in R^n. The first statement should therefore be clear from (2.6) and the proof that U is entire. The second one follows in the same way as for U. ∎

3. One easily sees that (2.18-20) follow from (2.15-17). Indeed, multiplying (2.17) with $q\bar{F}$ from the right resp. $\bar{F}q$ from the left and using (2.16) and (2.17) one e.g. obtains (2.18). It remains to prove (2.15-17). Since R and A are analytic, (2.15) holds if

$$\theta(t_1-t_2)..\theta(t_{N-1}-t_N)+(-)^N\theta(t_2-t_1)..\theta(t_N-t_{N-1})$$

$$= -\sum_{n=1}^{N-1}(-)^{N-n}\theta(t_1-t_2)..\theta(t_{n-1}-t_n)\theta(t_{n+2}-t_{n+1})..\theta(t_N-t_{N-1})$$

$$= -\sum_{n=1}^{N-1}(-)^{n}\theta(t_2-t_1)..\theta(t_n-t_{n-1})\theta(t_{n+1}-t_{n+2})..\theta(t_{N-1}-t_N). \tag{2.65}$$

The verification of (2.65) is straightforward. Choosing $\varepsilon_1,\dots,\varepsilon_{N-1}$ we note that (2.65) still holds if t_i-t_j is replaced by $\varepsilon_{\min(i,\,j)}(t_i-t_j)$, which proves (2.16). Finally, (2.17) follows from the easily verified relations

$$\theta(t_1-t_2)..\theta(t_{N-1}-t_N)O(t_1)..O(t_N) - \sum_{\varepsilon_1,\dots,\varepsilon_{N-1}} \theta(\varepsilon_1(t_1-t_2))..$$

$$.\theta(\varepsilon_{N-1}(t_{N-1}-t_N))O(t_1)\varepsilon_1 P_{\varepsilon_1}..\varepsilon_{N-1}P_{\varepsilon_{N-1}}O(t_N)$$

$$= \sum_{\varepsilon_1,\dots,\varepsilon_{N-1}}\ \sum_{n=1}^{N-1} \theta(t_1-t_2)..\theta(t_{n-1}-t_n)\theta(\varepsilon_{n+1}(t_{n+1}-t_{n+2}))..$$

$$.\theta(\varepsilon_{N-1}(t_{N-1}-t_N))O(t_1)P_{\varepsilon_1}..P_{\varepsilon_{n-1}}O(t_n)\tfrac{1}{2}(1-\varepsilon_n)P_{\varepsilon_n}O(t_{n+1})\varepsilon_{n+1}P_{\varepsilon_{n+1}}..$$

$$.\varepsilon_{N-1}P_{\varepsilon_{N-1}}O(t_N)$$

$$= \sum_{\varepsilon_1,\dots,\varepsilon_{N-1}}\ \sum_{n=1}^{N-1} \theta(\varepsilon_1(t_1-t_2))..\theta(\varepsilon_{n-1}(t_{n-1}-t_n))\theta(t_{n+1}-t_{n+2})..$$

$$.\theta(t_{N-1}-t_N)O(t_1)\varepsilon_1 P_{\varepsilon_1}..\varepsilon_{n-1}P_{\varepsilon_{n-1}}O(t_n)\tfrac{1}{2}(1-\varepsilon_n)P_{\varepsilon_n}O(t_{n+1})P_{\varepsilon_{n+1}}..$$

$$.P_{\varepsilon_{N-1}}O(t_N). \quad \blacksquare \tag{2.66}$$

4.To prove (2.21) it suffices to show that

$$\sum_{n=0}^{N} R^{(n)}(T_3,\,T_2)R^{(N-n)}(T_2,\,T_1) = R^{(N)}(T_3,\,T_1). \tag{2.67}$$

But (2.67) is equivalent to

$$\sum_{n=0}^{N}\int_{T_2}^{T_3}dt_1..\int_{T_2}^{t_{n-1}}dt_n\int_{T_1}^{T_2}dt_{n+1}..\int_{T_1}^{t_{N-1}}dt_N O(t_1)..O(t_N) = \int_{T_1}^{T_3}dt_1..\int_{T_1}^{t_{N-1}}dt_N O(t_1)..O(t_N) \tag{2.68}$$

which is easily verified. To prove (2.22) we note that

$$U_\lambda(T_2, T_1) = \sum_{n=0}^{\infty}(i\lambda)^n \int_{T_1}^{T_2} dt_1 \ldots \int_{T_1}^{t_{n-1}} dt_n O(t_1)\ldots O(t_n). \tag{2.69}$$

By the uniform boundedness principle each term is differentiable w.r.t. T_2. Since the resulting series strongly converges to $i\lambda O(T_2)U_\lambda(T_2, T_1)$, uniformly in T_2 on any bounded neighbourhood of any $T_2 \in R$, (2.22) follows. Transforming (2.69) into

$$U_\lambda(T_2, T_1) = \sum_{n=0}^{\infty}(i\lambda)^n \int_{T_1}^{T_2} dt_1 \ldots \int_{t_{n-1}}^{T_2} dt_n O(t_n)\ldots O(t_1) \tag{2.70}$$

one obtains (2.23) by a similar argument. To prove the uniqueness of the solution of (2.24) we define $\psi(T_2) = \phi(T_2) - U_\lambda(T_2, T_1)\phi$. Then $\psi(T_1) = 0$ and $i\dot{\psi}(T_2) = -\lambda O(T_2)\psi(T_2)$, so

$$\psi(T_2) = i\lambda \int_{T_1}^{T_2} dt O(t)\psi(t) = (i\lambda)^n \int_{T_1}^{T_2} dt_1 \ldots \int_{T_1}^{t_{n-1}} dt_n O(t_1)\ldots O(t_n)\psi(t_n). \tag{2.71}$$

Estimating in the obvious way it follows that $\psi(T_2) = 0$. $\blacksquare$

5. In virtue of Th.4 and Stone's theorem it suffices to prove that for any $(\lambda, T_2, T_1) \in C \times R^2$

$$U_\lambda(T_2, T_1)D(H_o) \subset D(H_o). \tag{2.72}$$

Defining

$$R_n^s(T_2, T_1) = \exp(-iH_oT_2)R^{(n)}(T_2, T_1)\exp(iH_oT_1) \tag{2.73}$$

we observe that

$$R_{n+1}^s(T_2, T_1) = i \int_{T_1}^{T_2} dt \exp(-iH_o(T_2 - t))V(t)R_n^s(t, T_1) \tag{2.74}$$

which can also be written as

$$R_{n+1}^s(T_2, T_1) = i \int_{T_1}^{T_2} dt \exp(-iH_o(t - T_1))V(T_2 + T_1 - t)R_n^s(T_2 + T_1 - t, T_1). \tag{2.75}$$

On $D(H_o)$ $V(T_2+T_1-t)R_o^s(T_2+T_1-t, T_1)$ is continuously differentiable w.r.t. T_2, so from (2.75) it follows, using dominated convergence and an induction argument, that on $D(H_o)$ $R_n^s(T_2, T_1)$ is continuously differentiable w.r.t. T_2 for any $n\in N$. It then follows from (2.74) by Stone's theorem that

$$R_n^s(T_2, T_1)D(H_o)\subset D(H_o) \qquad \forall n\in N. \tag{2.76}$$

Differentiating (2.74) and (2.75) one obtains after some straightforward manipulations:

$$H_oR^{(n+1)}(T_2, T_1) = R^{(n+1)}(T_2, T_1)H_o+O(T_2)R^{(n)}(T_2, T_1)-R^{(n)}(T_2, T_1)O(T_1)$$
$$-\sum_{k=o}^{n} i^k \int_{T_1}^{T_2}dt_o..\int_{T_1}^{t_{k-1}} dt_kO(t_o)..O(t_{k-1}) \exp (iH_ot_k)\dot{V}(t_k)$$
$$. \exp (-iH_ot_k)R^{(n-k)}(t_k, T_1) \tag{2.77}$$

which holds on $D(H_o)$. Using (2.63) we get for any $\phi\in D(H_o)$

$$||H_oR^{(n+1)}(T_2, T_1)\phi|| \le \frac{K^{n+1}}{(n+1)!} ||H_o\phi|| + \Big((||V(T_2)+||V(T_1)||) \frac{K^n}{n!}$$
$$+ \frac{L(2K)^n}{n!}\Big) ||\phi|| \tag{2.78}$$

where

$$L \equiv |T_2-T_1| \sup_{t\in[\min(T_1, T_2), \max (T_1, T_2)]} ||\dot{V}(t)||. \tag{2.79}$$

It evidently follows from (2.78) that the sequence $H_o\lambda^nR^{(n)}(T_2, T_1)\phi$ is absolutely summable. Thus, as H_o is closed, (2.72) holds true. ∎

6. If $n \ge 2$ and, e.g., $T_2 > T_1$:

$$||\tilde{R}_\lambda^{(n)}(T_2, T_1)|| \le \int_{T_1}^{T_2} dtF(t, |\lambda|)(||\tilde{R}_\lambda^{(n-1)}(t, T_1)||+||\tilde{R}_\lambda^{(n-2)}(t, T_1)||) \tag{2.80}$$

where

$$F(t, |\lambda|) \equiv |\lambda| \, ||O(t)|| + |\lambda|^2 \, ||\tilde{O}(t)||. \tag{2.81}$$

Furthermore,

$$||\tilde{R}_\lambda^{(1)}(T_2, T_1)|| \le \int_{T_1}^{T_2} dtF(t, |\lambda|). \tag{2.82}$$

Iterating (2.80) and then using (2.35) and (2.82) one obtains less than 2^n terms. For the generic term we write

$$M(m, T_2, T_1, |\lambda|) \equiv \int_{T_1}^{T_2} dt_1 .. \int_{T_1}^{t_{m-1}} dt_m \, F(t_1, |\lambda|)..F(t_m, |\lambda|) \tag{2.83}$$

where

$$\frac{n}{2} \leq m \leq n. \tag{2.84}$$

Clearly,

$$M(m, T_2, T_1, |\lambda|) \leq \frac{C(|\lambda|)^m}{m!} \tag{2.85}$$

where

$$C(|\lambda|) \equiv |\lambda| \, K + |\lambda|^2 \, \tilde{K}. \tag{2.86}$$

Thus,

$$||\tilde{R}^{(n)}_\lambda(T_2, T_1)|| \leq \frac{(2C'(|\lambda|))^n}{\left[\frac{n}{2}\right]!} \tag{2.87}$$

where

$$C'(|\lambda|) \equiv \max\,(1, C|\lambda|) \tag{2.88}$$

and where $\left[\frac{n}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n}{2}$. From (2.87) it follows as before that $\tilde{U}_\lambda(T_2, T_1)$ is $||.||$-entire in λ and $||.||$-continuous on $\tilde{R}^2$. Moreover,

$$\begin{aligned} \tilde{U}_\lambda(T_2, T_1) &= 1+\tilde{R}^{(1)}_\lambda(T_2, T_1)+\sum_{n=2}^{\infty} i \int_{T_1}^{T_2} dt(\lambda O(t)+\lambda^2\tilde{O}(t))\tilde{R}^{(n-1)}_\lambda(t, T_1) \\ &\quad -\sum_{n=1}^{\infty} i \int_{T_1}^{T_2} dt\lambda^2\tilde{O}(t)\tilde{R}^{(n-1)}_\lambda(t, T_1) \\ &= 1+i\lambda \int_{T_1}^{T_2} dt O(t)\tilde{U}_\lambda(t, T_1). \end{aligned} \tag{2.89}$$

In virtue of (2.22) $U_\lambda(T_2, T_1)$ satisfies the same Volterra type integral equation which is easily seen to have a unique norm continuous solution. Thus, (2.38) follows. $\blacksquare$

7. This theorem is an immediate consequence of (2.15-18). $\blacksquare$

$\underline{\underline{8}}$. We need the following lemma.

Lemma 2.1. Let $X(t)$ be a function on R with values in the H.S. operators on $\mathcal{H}$, which is strongly continuous on R and such that

$$||X(t)||_2 \leq \alpha(t) \qquad \alpha(\cdot)\in L^1(R). \tag{2.90}$$

If $Y \equiv \int dtX(t)$ then

$$||Y||_2 \leq \int dt||X(t)||_2 < \infty. \tag{2.91}$$

Proof. Y is well-defined since $||X(t)\phi|| \leq \alpha(t)||\phi||$. If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ then $||X(t)||_2^2 = \sum_{n=1}^{\infty}(X(t)e_n,X(t)e_n)$ so $||X(t)||_2$ is measurable and, by (2.90), integrable on R.

Moreover, $\sum_{n=1}^{N}(Ye_n,Ye_n) \leq \int dt_1dt_2 \sum_{n=1}^{N}||X(t_1)e_n||\ ||X(t_2)e_n||<(\int dt||X(t)||_2)^2$, so (2.91) holds true. $\blacksquare$

We have $(n \geq 1)$

$$-iR^{(n+1)}(T_2,T_1)_{+-} = \int_{T_1}^{T_2} dt\Big(O(t)_{++}R^{(n)}(t,T_1)_{+-}-i\big(\partial_t R^{(1)}(t,T_1)_{+-}\big)R^{(n)}(t,T_1)_{--}\Big)$$

$$= \int_{T_1}^{T_2} dt\Big(O(t)_{++}R^{(n)}(t,T_1)_{+-}-R^{(1)}(t,T_1)_{+-}\big(O(t)R^{(n-1)}(t,T_1)\big)_{--}\Big)$$
$$-i\,R^{(1)}(T_2,T_1)_{+-}R^{(n)}(T_2,T_1)_{--}. \tag{2.92}$$

An induction argument using (2.92), (2.43) and the lemma now shows that

$$||R^{(n)}(T_2,T_1)_{+-}||_2 \leq C_n < \infty \tag{2.93}$$

on $\tilde{R}^2$. In fact we can choose

$$C_n = C_1K^{n-1}\left[\frac{2^{n-1}}{(n-1)!} + \frac{2^{n-2}}{(n-2)!}\right] \qquad n > 1 \tag{2.94}$$

since by (2.92) and (2.63), if e.g. $T_2 > T_1$,

$$||R^{(n+1)}(T_2,T_1)_{+-}||_2 \le C_1K^n\left(\frac{1}{n!}+\frac{1}{(n-1)!}\right)+\int_{T_1}^{T_2} dt_1\, ||O(t_1)||\ \ ||R^{(n)}(t_1,T_1)_{+-}||_2$$

$$\le C_1K^n\left(\frac{1}{n!}+\frac{1}{(n-1)!}\right)+\int_{T_1}^{T_2} dt_1\, ||O(t_1)||\Big[C_1K^{n-1}\left(\frac{1}{(n-1)!}+\frac{1}{(n-2)!}\right)$$

$$+\int_{T_1}^{t_1} dt_2\, ||O(t_2)||\Big[C_1K^{n-2}\left(\frac{1}{(n-2)!}+\frac{1}{(n-3)!}\right)+\,..$$

$$+\int_{T_1}^{t_{n-1}} dt_n\, ||O(t_n)||\ \ ||R^{(1)}(t_n,T_1)||_2\Big]..\Big]$$

$$\le C_1K^n\left(\frac{1}{o!n!}+\frac{1}{o!(n-1)!}+\cdots+\frac{1}{(n-1)!1!}+\frac{1}{(n-1)!o!}+\frac{1}{n!o!}\right)$$

$$= C_1K^n\left[\frac{2^n}{n!}+\frac{2^{n-1}}{(n-1)!}\right]. \tag{2.95}$$

Thus, U_{+-} is an $||.||_2$-entire function of λ for any $(T_2,T_1)\in\tilde{R}^2$. Since $\tilde{R}^2$ is a compact topological space the topology of which is generated by the metric

$$d((x_1,x_2),(y_1,y_2)) \equiv \sum_{i=1}^{2} |\mathrm{Arctg}\ x_i - \mathrm{Arctg}\ y_i| \tag{2.96}$$

we can conclude from (2.92), using induction and uniform continuity w.r.t. d, that $R^{(n)}{}_{+-}$ is $||.||_2$-continuous on $\tilde{R}^2$. Hence, in view of (2.93 -94), U_{+-} is $||.||_2$-continous on $\tilde{R}^2$. The statements regarding U_{-+} follow on replacing +,- by -,+ in (2.92-95). The remaining statements follow from (2.41) and the analytic Fredholm theorem (12), and from (2.42). $\blacksquare$

9. Since $U_{\lambda_o}(T_2,T_1)_{+-} = \int_{T_1}^{T_2} dt\,\partial_t U_{\lambda_o}(t,T_1)_{+-} = i\lambda_o\int_{T_1}^{T_2} dt\,O(t)_{++}U_{\lambda_o}(t,T_1)_{+-}$

$$-\lambda_o\int_{T_1}^{T_2} dt\left(\partial_t R^{(1)}(T_2,t)_{+-}\right)U_{\lambda_o}(t,T_1)_{--}$$

$$= i\lambda_o\int_{T_1}^{T_2} dt\,O(t)_{++}U_{\lambda_o}(t,T_1)_{+-}+\lambda_o R^{(1)}(T_2,T_1)_{+-}+i\lambda_o^2\int_{T_1}^{T_2} dt\,R^{(1)}(T_2,t)_{+-}$$

$$\left(O(t)U_{\lambda_o}(t,T_1)\right)_{--}$$

(we used (2.22)):

$$R^{(1)}(T_2,T_1)_{+-} = \lambda_0^{-1}U_{\lambda_0}(T_2,T_1)_{+-}-i\int_{T_1}^{T_2}dt\,O(t)_{++}U_{\lambda_0}(t,T_1)_{+-}$$
$$-i\lambda_0\int_{T_1}^{T_2}dt\,R^{(1)}(T_2,t)_{+-}\Big(O(t)U_{\lambda_0}(t,T_1)\Big)_{--} \tag{2.97}$$

i.e. $R^{(1)}(T_2,T_1)_{+-}$ is a solution of the Volterra type integral equation

$$x(T_2,T_1) = f_1(T_2,T_1)+\int_{T_1}^{T_2}dt\,x(T_2,t)f_2(t,T_1) \tag{2.98}$$

where

$$f_1(T_2,T_1) \equiv \lambda_0^{-1}U_{\lambda_0}(T_2,T_1)_{+-}-i\int_{T_1}^{T_2}dt\,O(t)_{++}U_{\lambda_0}(t,T_1)_{+-} \tag{2.99}$$

$$f_2(t,T_1) \equiv -i\lambda_0\Big(O(t)U_{\lambda_0}(t,T_1)\Big)_{--}. \tag{2.100}$$

Using by now familiar estimates one concludes that the iteration solution of (2.98) is the unique solution which is $||.||$-continuous on $\tilde{R}^2$. Thus,

$$R^{(1)}(T_2,T_1)_{+-} = f_1(T_2,T_1)+\sum_{n=1}^{\infty}\int_{T_1}^{T_2}dt_1\ldots\int_{t_{n-1}}^{T_2}dt_n\,f_1(T_2,t_n)f_2(t_n,T_1)\ldots f_2(t_1,T_1). \tag{2.101}$$

In view of (2.44), (2.99) and lemma 2.1 it follows from (2.101) that

$$||R^{(1)}(T_2,T_1)_{+-}||_2 \leq C_1 < \infty \tag{2.102}$$

for any $(T_2,T_1)\in Q$. The remaining statements follow from (2.102), the proof of Th.8 and (2.101). $\blacksquare$

<u>10., 11.</u> These theorems immediately follow from (2.7-8) and (2.15-17). $\blacksquare$

The Dyson expansion is of course well-known (cf. e.g. (13)). Th.5 was inspired by results of Phillips (14). The method of proof of part of Th.8, in particular the lemma and (2.92), are taken from a paper by Bongaarts (7). The idea of using an integral equation to derive, as in the proof of Th.9, from properties of the evolution operator properties of its "Born approximation" is due to Hochstenbach (11).

It should be noted that if one only assumes (cf.(2.5))

$$||O(\cdot)||\in L^1([\alpha,\beta]) \qquad -\infty < \alpha < \beta < \infty \tag{2.103}$$

then all theorems still hold true with some obvious changes (e.g. $\tilde{R}^2 \to [\alpha,\beta]\times[\alpha,\beta]$, $K = \int_\alpha^\beta dt||O(t)||$). This observation permits us to use the results for time-independent external fields as well. We conclude this section with some results which are useful in this connection.

We assume that H_o and V are self-adjoint resp. bounded operators on $\mathcal{H}$, and that $P_\pm$ are spectral projections of H_o. Defining the closed operator

$$H(\lambda) = H_o - \lambda V \qquad \lambda\in C \tag{2.104}$$

one concludes, using the second Neumann series for its resolvent and the Hille-Yosida-Phillips theorem (13), that it generates a strongly continuous group $\exp(-iH(\lambda)t)$. Defining

$$O(t) = \exp(iH_o t)V\exp(-iH_o t) \tag{2.105}$$

we are ready for the following theorems.

<u>Th.2.12</u>. The following relations hold:

$$\exp(-iH_o t)U_\lambda(t,o) = \exp(-iH(\lambda)t) \tag{2.106}$$

$$||\exp(-iH(\lambda)t)|| \leq \exp(|\lambda|\;||V||\;|t|). \tag{2.107}$$

<u>Proof</u>. On $D(H_o)$:

$$\frac{d}{dt}\exp(iH_o t)\exp(-iH(\lambda)t) = i\lambda O(t)\exp(iH_o t)\exp(-iH(\lambda)t). \tag{2.108}$$

Thus,

$$\exp(iH_o t)\exp(-iH(\lambda)t) = 1+i\lambda\int_o^t dt' O(t')\exp(iH_o t')\exp(-iH(\lambda)t') \tag{2.109}$$

which holds on $\mathcal{H}$ by continuity. Since $U_\lambda(t,o)$ is the unique norm continuous solution of this integral equation, (2.106) must hold. The inequality (2.107) follows from estimates analogous to (2.62) or alternatively from semigroup theory. ∎

<u>Th.2.13</u>. Let

$$||\exp(-iH(\lambda_o)t)_{+-}||_2 < \infty \qquad \forall t\in R \tag{2.110}$$

where $\lambda_o \in C\setminus\{o\}$. Then $R^{(1)}(T_2,T_1)_{+-}$ is H.S. for any $(T_2,T_1)\in R^2$. Moreover, $\exp(-iH(\lambda)t)_{+-}$ is an $||\cdot||_2$-entire function of λ for any $t\in R$. The same statements hold true if +- is replaced by -+.

Proof. From (2.21) and (2.106) it is obvious that

$$U_{\lambda_o}(T_2,T_1) = \exp(iH_oT_2)\exp(-iH(\lambda_o)(T_2-T_1))\exp(-iH_oT_1). \tag{2.111}$$

Thus,

$$||U_{\lambda_o}(T_2,T_1)_{+-}||_2 = ||\exp(-iH(\lambda_o)(T_2-T_1))_{+-}||_2. \tag{2.112}$$

In view of Th.9 it therefore remains to show that

$$f(t) \equiv ||\exp(-iH(\lambda_o)t)_{+-}||_2 \tag{2.113}$$

is bounded on bounded sets of R.

From the group property and (2.107) it follows that

$$f(t_1+t_2) \leq \exp(a|t_1|)f(t_2)+\exp(a|t_2|)f(t_1) \tag{2.114}$$

where

$$a \equiv |\lambda_o|\ ||V||. \tag{2.115}$$

Thus, by induction,

$$f(nt) \leq n\exp(a(n-1)|t|)f(t), \tag{2.116}$$

which implies that it suffices to prove that $f(t)$ is bounded on a neighbourhood of the origin. We will derive a contradiction from the asumption that this is not true.

Indeed, assume there exists a sequence $t_i \to o$ such that $f(t_i) \to \infty$. We assert that this implies that for any $n\in N$

$$V_n \equiv \{t\in R|f(t) > n\} \tag{2.117}$$

is dense in R. To see this, let $t_o\in R$. Then, by (2.114),

$$\exp(a|t_o|)f(t_o+t_i) \geq f(t_i)-\exp(a|t_o+t_i|)f(-t_o) \tag{2.118}$$

so $f(t_o+t_i) \to \infty$, from which our assertion follows. On the other hand, $f_N(t) \uparrow f(t)$, where

$$f_N(t) \equiv \left(\sum_{i=1}^{N} ||\exp(-iH(\lambda_o)t)_{+-}e_i||^2\right)^{\frac{1}{2}} \tag{2.119}$$

and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Thus,

$$V_n = \bigcup_{N=1}^{\infty} f_N^{-1}(n,\infty). \tag{2.120}$$

Since f_N is continuous it follows from (2.120) that V_n is open. As V_n is open and dense, its complement V_n^c is nowhere dense. Thus, by the Baire category theorem, $\bigcup_{n=1}^{\infty} V_n^c \neq R$. Therefore, there exists a $t_o\in R$ for which $f(t_o) > n$, $\forall n\in N$. This contradicts (2.110) so the theorem is proved. ∎

3. THE SPIN-$\frac{1}{2}$ CASE.

Using section 2 and the appendix we will now obtain various results in the context of the classical Dirac theory. It is assumed that the reader is familiar with the Dirac theory as treated in (15, 16). We choose the Dirac representation of the γ-algebra (16, Ch.2). It is easily seen that the results hold true in any representation.

A. Preliminaries.

The free Dirac Hamiltonian $\check{H}_o$ is defined as an operator on the Hilbert space $\check{\mathcal{H}} \equiv L^2(R^3,d\vec{x})^4$ by

$$\check{H}_o = \frac{1}{i}\vec{\alpha}.\vec{\nabla}+\beta m \qquad D(\check{H}_o) = W_1(R^3)^4. \tag{3.1}$$

(For a definition of the Sobolev spaces W_m see (13).) $\check{H}_o$ is easily seen to be self-adjoint. We have occasion to use a simultaneous spectral representation of $\check{H}_o$ and the momentum operator, generated by the unitary operator

$$W : \mathcal{H} \equiv L^2(R^3,d\vec{p})^4 \to \check{\mathcal{H}} \tag{3.2}$$

which satisfies

$$(Wg)(\vec{x}) \equiv \sum_{i,\varepsilon} \int d\vec{p}W_{i,\varepsilon}(\vec{x},\vec{p})g^i_\varepsilon(\vec{p}) \qquad \varepsilon = +,- \qquad i = 1,2 \tag{3.3}$$

$$(W^{-1}f)^i_\varepsilon(\vec{p}) = \int d\vec{x}W^{-1}_{i,\varepsilon}(\vec{x},\vec{p}).f(\vec{x}). \tag{3.4}$$

The integrals are limits in the mean and

$$\begin{aligned} W_{i,\varepsilon}(\vec{x},\vec{p}) &\equiv (2\pi)^{-3/2}\left(\frac{m}{E_p}\right)^{\frac{1}{2}} w^i_\varepsilon(\vec{p}) \exp(i\varepsilon\vec{p}.\vec{x}) \\ W^{-1}_{i,\varepsilon}(\vec{x},\vec{p}) &\equiv \bar{W}_{i,\varepsilon}(\vec{x},\vec{p}) \\ E_p &\equiv (p^2+m^2)^{\frac{1}{2}} \qquad w^i_+(\vec{p}) \equiv u_i(\vec{p}) \qquad w^i_-(\vec{p}) \equiv v_i(\vec{p}). \end{aligned} \tag{3.5}$$

In (3.5) $u_i(\vec{p})$ and $v_i(\vec{p})$ are defined by

$$\begin{aligned} u_1(\vec{p}) &= \left(\frac{E_p+m}{2m}\right)^{\frac{1}{2}} \left(1,0,\frac{p_3}{E_p+m}, \frac{p_1+ip_2}{E_p+m}\right) \\ u_2(\vec{p}) &= \left(\frac{E_p+m}{2m}\right)^{\frac{1}{2}} \left(0,1, \frac{p_1-ip_2}{E_p+m}, \frac{-p_3}{E_p+m}\right) \end{aligned} \tag{3.6}$$

and $$v_i(\vec{p}) = \check{C}u_i(\vec{p}) \tag{3.7}$$

where $\check{C}$ is the charge conjugation operator on $\check{\mathcal{H}}$:

$$(\check{C}f)(\vec{x}) \equiv i\gamma^2\bar{f}(\vec{x}). \tag{3.8}$$

In the sequel we will use the convention

$$O \equiv W^{-1}\check{O}W \tag{3.9}$$

if $\check{O}$ is an operator on $\check{\mathcal{H}}$. With this convention,

$$(H_o f)^i_\varepsilon(\vec{p}) = \varepsilon E_p f^i_\varepsilon(\vec{p}) \qquad \forall f \in D(H_o) \tag{3.10}$$

$$(Cf)^i_\varepsilon(\vec{p}) = \bar{f}^i_{-\varepsilon}(\vec{p}) \qquad \forall f \in \mathcal{H}. \tag{3.11}$$

We omit the easy proofs of the unitarity of W and of (3.4), (3.10-11).

We will frequently need the well-known commutator and Green's functions associated with the free Klein-Gordon and Dirac equations (15). Our conventions for these distributions are:

$$\begin{aligned} \Delta_\varepsilon(x) &= -i\varepsilon(2\pi)^{-3}\int d\vec{p}\, \exp(i\vec{p}.\vec{x})\frac{\exp(-i\varepsilon E_p t)}{2E_p} \\ \Delta(x) &= \Delta_+(x)+\Delta_-(x) \\ S_{(\varepsilon)}(x) &= (i\not{\partial}+m)\Delta_{(\varepsilon)}(x). \end{aligned} \tag{3.12}$$

Of course, the integral is a distributional Fourier transform.

$$\begin{aligned} &\Delta_R(x) = -\theta(t)\Delta(x) && \Delta_F(x) = -\theta(t)\Delta_+(x)+\theta(-t)\Delta_-(x) \\ &\Delta_A(x) = \theta(-t)\Delta(x) && \Delta_{\bar{F}}(x) = -\theta(t)\Delta_-(x)+\theta(-t)\Delta_+(x) \\ &S_I(x) = (i\not{\partial}+m)\Delta_I(x) && I = R,A,F,\bar{F}. \end{aligned} \tag{3.13}$$

As a consequence the propagators fulfil the equations

$$\begin{aligned} (\vec{\Box}_x+m^2)\Delta_I(x-y) &= \Delta_I(x-y)(\overleftarrow{\Box}_y+m^2) = \delta(x-y) \\ (-i\vec{\not{\partial}}_x+m)S_I(x-y) &= S_I(x-y)(i\overleftarrow{\not{\partial}}_y+m) = \delta(x-y). \end{aligned} \tag{3.14}$$

(Notice that the Feynman propagators differ by a constant from the usual ones.)

We define the (partial) Fourier transforms of these distributions by

$$\begin{aligned} \hat{D}(t,\vec{p}) &= \int d\vec{x}\, \exp(-i\vec{p}.\vec{x})D(t,\vec{x}) \\ \tilde{D}(p) &= \int dx\, \exp(ipx)D(x). \end{aligned} \tag{3.15}$$

One then has, e.g.,

$$\hat{\Delta}_R(t,\vec{p}) = \theta(t)\,\frac{\sin E_p t}{E_p} \tag{3.16}$$

$$\tilde{\Delta}_\varepsilon(p) = -2\pi i\varepsilon\theta(\varepsilon p^o)\delta(p^2-m^2) \tag{3.17}$$

$$\begin{cases} \check{\Delta}_I(p) = \lim_{\delta\downarrow 0} \Delta_I^\delta(p) \\ \Delta^\delta_{\substack{R\\A}}(p) \equiv (E_p^2-(p_o\pm i\delta)^2)^{-1} \qquad \Delta^\delta_{\substack{F\\\bar{F}}}(p) \equiv (m^2-p^2\mp i\delta)^{-1} \end{cases} \tag{3.18}$$

$$\tilde{S}_I(p) = \lim_{\delta\downarrow 0} S_I^\delta(p) \qquad S_I^\delta(p) \equiv (\not{p}+m)\Delta_I^\delta(p). \tag{3.19}$$

It is convenient to define the (partial) Fourier transform of a function F in $S(R^4)$ by

$$\begin{aligned} \hat{F}(t,\vec{p}) &= (2\pi)^{-3}\int d\vec{x}\, \exp(-i\vec{p}.\vec{x})F(t,\vec{x}) \\ \tilde{F}(p) &= (2\pi)^{-4}\int dx\, \exp(ipx)F(x). \end{aligned} \tag{3.20}$$

Let $K(T_2,T_1)$ be a function from R^2 to the bounded operators on $\mathcal{H}$ which is strongly continuous on R^2, except possibly on the set $T_2 = T_1$, and such that

$$\sup_{(T_1,T_2)\in R^2} ||K(T_2,T_1)|| < \infty. \tag{3.21}$$

We define for any $f,g\in S(R^3)^4$ resp. $F, G\in S(R^4)^4$:

$$[K]_{T_2,T_1}(f,g) = (\bar{\check{f}}, \check{K}(T_2,T_1)\check{g}) \tag{3.22}$$

$$[K](F,G) = \int dt dt'(\bar{\check{F}}(t,\cdot),\check{K}(t,t')\check{G}(t',\cdot)) \tag{3.23}$$

where, e.g., $\check{f}$ denotes the injection of f into $\check{\mathcal{H}}$. In virtue of the nuclear theorem $[K]_{T_2,T_1}$ extends to a distribution in $S'(R^6)^{16}$ for any $(T_2,T_1)\in R^2$, while $[K]$ extends to a distribution in $S'(R^8)^{16}$.

The free Schrödinger picture evolution operator is defined by

$$U_o^s(T_2,T_1) = \exp(-iH_o(T_2-T_1)). \tag{3.24}$$

We denote the spectral projections of H_o on $[m,\infty)$ and $(-\infty,-m]$ by P_+ resp. P_-. Using the operator W, the well-known relations

$$\sum_{i=1}^{2} w_\varepsilon^i(\vec{p})\bar{w}_\varepsilon^i(\vec{p}) = \frac{(\not{p}+\varepsilon m)\gamma^o}{2m} \qquad p \equiv (E_p,\vec{p}), \tag{3.25}$$

(3.10) and (3.12) one obtains

$$\left[P_\varepsilon U_o^s\right]_{T_2,T_1}(\vec{x},\vec{y}) = iS_\varepsilon(T_2-T_1, \vec{x}-\vec{y})\gamma^o \tag{3.26}$$

$$\left[P_\varepsilon U_o^s\right](x,y) = iS_\varepsilon(x-y)\gamma^o \tag{3.27}$$

$$(\check{P}_\varepsilon \exp(-i\check{H}_o t)\check{f})(\vec{x}) = i\int d\vec{y}S_\varepsilon(t,\vec{x}-\vec{y})\gamma^o f(\vec{y}) \tag{3.28}$$

$$(\int dt'\check{P}_\varepsilon \exp(-i\check{H}_o(t-t'))\check{F}(t',\cdot))(\vec{x}) = i\int dy S_\varepsilon(x-y)\gamma^o F(y) \tag{3.29}$$

where f and F belong to $S(R^3)^4$ resp. $S(R^4)^4$. The integrals at the r.h.s. stand for convolutions and the integral at the l.h.s. is a strong Riemann integral in $\check{\mathcal{K}}$. One easily sees that the l.h.s. of (3.28) and (3.29) belong to $S(R^3)^4$ for any $t\in R$, and belong to $O_M(R^4)^4$ as functions of x.

B. The operators R, A, F, and $\bar{F}$.

We now introduce the interaction with external fields. We shall consider functions $V(x)$ from R^4 to the Hermitean 4x4 matrices, the matrix elements of which belong to $S(R^4)$. At any time t such a function defines a bounded self-adjoint multiplication operator $\check{V}(t)$ on $\check{\mathcal{K}}$. We define

$$H(t) = H_o - V(t) \tag{3.30}$$

$$O(t) = \exp(iH_o t)V(t)\exp(-iH_o t). \tag{3.31}$$

It is obvious that $O(t)$ satisfies the assumptions of section 2 so we can use its results. (The coupling constant is absorbed in $V(t)$ since analyticity properties will only be used in the quantized theory.) In particular, by Ths. 2.4, 2.5 and 2.11, $\check{U}(T_2,T_1)$ $(\equiv\check{U}_1(T_2,T_1))$ and $\check{U}^s(T_2,T_1)$ solve the interaction picture resp. Schrödinger picture Cauchy problems and are unitary.

We will now show that

$$(\check{I}(\infty,-\infty)_{\varepsilon\varepsilon'}\check{f})(\vec{x}) = -i\sum_{n=1}^{\infty}\int dx_1..dx_n d\vec{x}' S_\varepsilon(x-x_1)B(x_1)S_I(x_1-x_2)B(x_2)$$

$$..S_I(x_{n-1}-x_n)B(x_n)S_{\varepsilon'}(x_n-x')\gamma^o f(\vec{x}') \quad f\in S(R^3)^4 \tag{3.32}$$

where

$$I = R,A,F,\bar{F} \qquad x^{(\prime)} \equiv (0,\vec{x}^{(\prime)}) \tag{3.33}$$

and

$$B(x) \equiv \gamma^o V(x). \tag{3.34}$$

(For F and $\bar{F}$ one should require $\int dt||V(t)|| < 1$, cf.Th.2.2.)

The integrals stand for convolutions which should be performed in the indicated order. The limit is a limit in the mean in $\check{\mathcal{K}}$. It is easily seen that each term in the sum belongs to $S(R^3)^4$.

To prove that (3.32) holds, we observe that, e.g. $(g\in S(R^3)^4$:

$$(\check{g},\check{R}^{(n)}(\infty,-\infty)_{\varepsilon\varepsilon'},\check{f}) = i^n\int dt_1..dt_n(W^{-1}\check{g},P_\varepsilon\exp(iH_ot_1)V(t_1)..$$

$$.V(t_n)\exp(-iH_ot_n)P_{\varepsilon'}W^{-1}\check{f})$$

$$= i^n\int dt_1..dt_n d\vec{p}d\vec{k}_1..d\vec{k}_{n-1}d\vec{q}\,\Sigma_{i,i'}\overline{(W^{-1}\check{g})^i_\varepsilon(\varepsilon\vec{p})}\left(\frac{m}{E_p}\right)^{\frac{1}{2}}\overline{w}^i_\varepsilon(\varepsilon\vec{p})\exp(i\varepsilon E_pt_1)$$

$$.\hat{V}(t_1,\vec{p}-\vec{k}_1)\left[\Sigma_{\varepsilon_1}(E_{k_1}\gamma^o-\varepsilon_1\vec{k}_1.\vec{\gamma}+\varepsilon_1 m)\frac{\exp(-i\varepsilon_1E_{k_1}(t_1-t_2))}{2E_{k_1}}\theta(t_1-t_2)\right]\gamma^o$$

$$.\hat{V}(t_2,\vec{k}_1-\vec{k}_2)..\gamma^o\hat{V}(t_n,\vec{k}_{n-1}-\vec{q})\exp(-i\varepsilon' E_qt_n)w^{i'}_{\varepsilon'}(\varepsilon'\vec{q})\left(\frac{m}{E_q}\right)^{\frac{1}{2}}(W^{-1}\check{f})^{i'}_{\varepsilon'}(\varepsilon'\vec{q}). \tag{3.35}$$

Notice that the factor in the brackets equals $-i\hat{S}_R(t_1-t_2,\vec{k}_1)$ and that the integral is absolutely convergent. We now choose a sequence T_N, the matrix elements of which belong to $S(R^4)$ and satisfy

$$(\hat{T}_N)_{ij}(t,\vec{k}) \to (\hat{S}_R)_{ij}(t,\vec{k}) \qquad \forall i,j\in\{1,..,4\} \qquad \forall(t,\vec{k})\in R^4$$
$$\sup_{t,\vec{k},i,j,N}|(\hat{T}_N)_{ij}(t,\vec{k})| < \infty. \tag{3.36}$$

Then, using dominated convergence and Fubini's theorem,

$$(\check{g},\check{R}^{(n)}(\infty,-\infty)_{\varepsilon\varepsilon'},\check{f}) = i\lim_{N_{n-1}}..\lim_{N_1}\int dt_1..d\vec{q}..\hat{T}_{N_{n-1}}(t_1-t_2,\vec{k}_1)..$$

$$.\hat{T}_{N_1}(t_{n-1}-t_n,\vec{k}_{n-1})..$$

$$= -i\lim_{N_{n-1}}..\lim_{N_1}\int dx_1..dx_n(\int d\vec{x}\bar{g}(\vec{x})S_\varepsilon(x-x_1))B(x_1)T_{N_{n-1}}(x_1-x_2)..T_{N_1}(x_{n-1}-x_n)$$

$$.B(x_n)\,(\int d\vec{x}'S_{\varepsilon'}(x_n-x')\gamma^of(\vec{x}'))$$

$$\equiv \lim_{N_{n-1}}..\lim_{N_1}\int dx_{n-1}dx_nG_{N_{n-1},..,N_2}(x_{n-1})T_{N_1}(x_{n-1}-x_n)F(x_n) \tag{3.37}$$

where G and F are in $S(R^4)^4$. Since, by (3.36), $T_N \to S_R$ weakly, $\lim_{N_1}\int..T_{N_1}..$ equals $\int..S_R..$ in virtue of the weak continuity of the convolution (13). In the same fashion one successively removes the other limits. Further details are left to the reader.

Similarly, one concludes that

$$(I(\infty,-\infty)f)^i_\varepsilon(\vec{p}) = 2\pi i\sum_{n=1}^{\infty}\Sigma_{i',\varepsilon'}\int dk_1..dk_{n-1}d\vec{q}\left(\frac{m}{E_p}\right)^{\frac{1}{2}}\tilde{\overline{w}}^i_\varepsilon(\vec{p})\tilde{B}(\varepsilon p-k_1)\tilde{S}_I(k_1)$$

$$.\tilde{B}(k_1-k_2)..\tilde{S}_I(k_{n-1})\tilde{B}(k_{n-1}-\varepsilon' q)w^{i'}_{\varepsilon'}(\vec{q})\left(\frac{m}{E_q}\right)^{\frac{1}{2}}f^{i'}_{\varepsilon'}(\vec{q}) \qquad f\in S(R^3)^4 \tag{3.38}$$

where

$$p \equiv (E_p,\vec{p}) \quad , \qquad q \equiv (E_q,\vec{q}) \qquad \tilde{w} \equiv \bar{w}\gamma^o. \tag{3.39}$$

The injection of f into $\mathcal{H}$ is denoted by f. The k_i-integrals stand again for convolutions and the limit is a limit in the mean in $\mathcal{H}$. Note that each term in the sum belongs to $S(R^3)^4$ since W and W^{-1} map $S(R^3)^4$ onto $S(R^3)^4$.

We write (3.38) as

$$(I(\infty,-\infty)f)^i_\varepsilon(\vec{p}) \equiv \sum_{n=1}^{\infty} \sum_{i',\varepsilon'} \int d\vec{q}\, I^{(n)ii'}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) f^{i'}_{\varepsilon'}(\vec{q}). \tag{3.40}$$

Using properties of O'_c (17) it is easily seen that $I^{(n)ii'}_{\varepsilon\varepsilon'}(\vec{p},\vec{q})$ is in $S(R^3)$ in $\vec{p}$ and $\vec{q}$ separately. We further observe that

$$I^{(n)ii'}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) = 2\pi i \lim_{\delta_{n-1}\downarrow o} \,..\lim_{\delta_1\downarrow o} \int dk_1..dk_{n-1}\left(\frac{m}{E_p}\right)^{\frac{1}{2}} \tilde{w}^i_\varepsilon(\vec{p})\tilde{B}(\varepsilon p-k_1)S_I^{\delta_{n-1}}(k_1)$$

$$.\tilde{B}(k_1-k_2)..S_I^{\delta_1}(k_{n-1})\tilde{B}(k_{n-1}-\varepsilon' q)w^{i'}_{\varepsilon'}(\vec{q})\left(\frac{m}{E_q}\right)^{\frac{1}{2}}. \tag{3.41}$$

Evidently, the integral in (3.41) is absolutely convergent.

C. Lorentz covariance, causality.

We will now show that the operators $I(\infty,-\infty)$ are Lorentz covariant in the sense that for, e.g., an electromagnetic field:

$$U(a,\Lambda)I(A_\mu)U^*(a,\Lambda) = I(A_\mu{}^{a,\Lambda}) \tag{3.42}$$

where

$$A_\mu{}^{a,\Lambda}(x) \equiv \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}(x-a)). \tag{3.43}$$

In (3.42) $U(a,\Lambda)$ is the representation of iSL(2,C) in $\mathcal{H}$ (we do not consider inversions):

$$(U(a,\Lambda)g)^i_\varepsilon(\vec{p}) \equiv \exp(i\varepsilon pa)\left(\frac{(\Lambda^{-1}p)_0}{p_0}\right)^{\frac{1}{2}} \sum_j \overline{\left[\left(\frac{\tilde{p}}{m}\right)^{\frac{1}{2}} A\left(\frac{\widetilde{\Lambda^{-1}p}}{m}\right)^{\frac{1}{2}}\right]}^{\varepsilon}_{ij} g^j_\varepsilon(\overrightarrow{\Lambda^{-1}p}) \tag{3.44}$$

where

$$\overline{f(\vec{p})}^{\varepsilon} \equiv \begin{cases} f(\vec{p}) & \varepsilon = + \\ \overline{f(\vec{p})} & \varepsilon = - \,. \end{cases} \tag{3.45}$$

In (3.44) the notation of (18) is used. The validity of (3.42) follows from (3.38), (3.44), the relation

$$\sum_i \overline{\left[\left(\frac{\widetilde{\Lambda^{-1}p}}{m}\right)^{\frac{1}{2}} A^{-1}\left(\frac{\tilde{p}}{m}\right)^{\frac{1}{2}}\right]}^{\varepsilon}_{ij} w^i_\varepsilon(\overrightarrow{\Lambda^{-1}p}) = S(A^{-1})w^j_\varepsilon(\vec{p}) \tag{3.46}$$

and the well-known relations

$$S(A)^*\gamma^o = \gamma^o S(A)^{-1} \tag{3.47}$$

$$S(A)^{-1}\gamma^\mu S(A) = \Lambda^\mu{}_\nu\gamma^\nu.$$

(To see that (3.46) holds for u_i, use

$$(u_i(\vec{p}))_j = S\left(\left(\frac{p}{m}\right)^{\frac{1}{2}}\right)_{ji} \tag{3.48}$$

and

$$S(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \qquad \forall A \in SU(2). \tag{3.49}$$

To prove it for v_i, use

$$\bar{A} = \zeta A \zeta^{-1} \qquad \forall A \in SU(2) \tag{3.50}$$

where

$$\zeta \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.51}$$

and

$$(v_i'(\vec{p}))_j = S\left(\left(\frac{p}{m}\right)^{\frac{1}{2}}\right)_{j,i+2} \tag{3.52}$$

where

$$v_i'(\vec{p}) \equiv \sum_j \zeta^{-1}{}_{ij} v_j(\vec{p}). \tag{3.53}$$

Of course, (3.48), (3.49) and (3.52) only hold true in the Dirac representation of the γ-algebra. However, (3.46) holds in any representation.)

Introducing the classical S-operator

$$S(V) \equiv U(\infty,-\infty;V) \tag{3.54}$$

we now have the following theorem.

__Theorem 3.1__. $S(V)$ is Lorentz covariant and causal in the sense that

$$S(V_1+V_2) = S(V_1)S(V_2) \tag{3.55}$$

if the supports of V_1 and V_2 are spacelike separated.

__Proof__. The Lorentz covariance is obvious from (3.42).

Defining

$$R(V) = R(\infty,-\infty;V) \tag{3.56}$$

we have (cf. (3.32) and its proof)

$$(\check{R}(V_1)\check{R}(V_2)\check{f})(\vec{x}) = \sum_{n,m=1}^{\infty}\int \ldots B_1(x_n)S(x_n-x_{n+1})B_2(x_{n+1})\ldots f(\vec{x}') = 0 \qquad f \in S(R^3)^4 \tag{3.57}$$

so

$$R(V_1)R(V_2) = 0. \tag{3.58}$$

Similarly, from (3.32),

$$R(V_1+V_2) = R(V_1) + R(V_2). \tag{3.59}$$

(3.55) now follows from (3.58) and (3.59). $\blacksquare$

We now define

$$G_{\substack{R\\A}}(T_2,T_1) = \pm i U^s(T_2,T_1)\gamma^o\theta(\pm(T_2-T_1)). \tag{3.60}$$

Then

$$\left[G_I\right](F,G) = \sum_{n=o}^{\infty}\int dx dx_1 \ldots dx_n dy F(x) S_I(x-x_1)B(x_1)S_I(x_1-x_2)\ldots B(x_n)S_I(x_n-y)G(y)$$

$$I = R,A. \tag{3.61}$$

To see this, note that, e.g.,

$$\int dt dt'(\bar{\check{F}}(t,\cdot), i\check{U}^s(t,t')\gamma^o\theta(t-t')\check{G}(t',\cdot)) = \sum_{n=o}^{\infty} i^{n+1}\int_{-\infty}^{\infty} dt_o\int_{-\infty}^{t_o} dt_{n+1}\int_{t_{n+1}}^{t_o} dt_1 \ldots \int_{t_{n+1}}^{t_{n-1}} dt_n$$

$$(W^{-1}\bar{\check{F}}(t_o,\cdot), \exp(-iH_o(t_o-t_1))V(t_1)\ldots V(t_n)\exp(-iH_o(t_n-t_{n+1}))W^{-1}\check{G}(t_{n+1},\cdot))$$

$$= \sum_{n=o}^{\infty} i^{n+1}\int_{-\infty}^{\infty} dt_o \ldots \int_{-\infty}^{t_n} dt_{n+1}(\ldots) \tag{3.62}$$

and then proceed as in the proof of (3.32). Since the arguments of the $S_R(S_A)$ in (3.61) must be all in the closed forward (backward) light cone, denoted by $\bar{V}_+(\bar{V}_-)$, to give a non-zero contribution, their sum, x-y, must be in $\bar{V}_+(\bar{V}_-)$. Hence,

$$\operatorname{supp}\left[G_{\substack{R\\A}}\right](x,y) \subset \{(x,y)/x-y\in\bar{V}_\pm\}. \tag{3.63}$$

Furthermore, using (3.14),

$$(-i\overrightarrow{\not\partial}_x+m-B(x))\left[G_I\right](x,y) = \left[G_I\right](x,y)(i\overleftarrow{\not\partial}_y+m-B(y)) = \delta(x-y) \quad I = R,A. \tag{3.64}$$

Thus, $\left[G_{R,A}\right]$ are two-sided retarded resp. advanced fundamental solutions of the perturbed Dirac equation. We note that the existence of retarded and advanced solutions, given by (3.61), also follows from Th.A.1. This theorem moreover implies that they are regular in the sense defined by Wightman (19), i.e. they define continuous maps from $S(R^4)^4$ into $O_M(R^4)^4$.

It should be noted that all results obtained thus far also hold for $V(x)\in S(R^4)^{16}$ since we did not use the self-adjointness of V(t). In particular, if one defines G_I by (3.60) for these V, then (3.61), (3.63) and (3.64) again hold true. Thus, we have proved:

Theorem 3.2. For any $V \in S(R^4)^{16}$ the perturbed Dirac equation admits two-sided tempered retarded and advanced fundamental solutions, given by (3.61). They are connected with the evolution operator through (3.60) and (3.23), and are Wightman regular. $\blacksquare$

D. Implementability of the evolution in $\mathcal{F}_a(\mathcal{H})$.

It is well-known that V can be decomposed in 16 fields of 5 different tensor types: scalar, vector, tensor, pseudovector and pseudoscalar. We will now show that the hypothesis of Th.2.8 is satisfied if only the timelike component of the vector field or the pseudovector field is non-zero. In the former case (only electric field):

$$(\check{V}_1(t)\phi)(\vec{x}) \equiv F(t,\vec{x})\phi(\vec{x}) \qquad \forall\phi\in\mathcal{H} \tag{3.65}$$

where F is a real-valued function in $S(R^4)$. In the latter case:

$$(\check{V}_2(t)\phi)(\vec{x}) \equiv F(t,\vec{x})(\gamma^5\phi)(\vec{x}) \tag{3.66}$$

where

$$\gamma^5 \equiv i\gamma^o\gamma^1\gamma^2\gamma^3. \tag{3.67}$$

Theorem 3.3. Let

$$O_k(t) \equiv \exp(iH_o t)V_k(t)\exp(-iH_o t) \qquad k = 1,2 \tag{3.68}$$

Then the operators $\int_{T_1}^{T_2} dt O_k(t)_{\pm\mp}$, k = 1,2, are $||.||_2$-continuous on $\tilde{R}^2$ and

$$\left|\left|\int_{T_1}^{T_2} dt O_k(t)_{\pm\mp}\right|\right|_2 \leq C_1 < \infty \qquad \forall(T_2,T_1)\in\tilde{R}^2. \tag{3.69}$$

Proof. It is easily seen that it suffices to prove our statements for the integral operators on $L^2(R^3)^4$ with the kernels

$$P_\pm(\vec{p})\int_{T_1}^{T_2} dt\ \exp(\pm i(E_p+E_q)t)\hat{F}(t,\vec{p}-\vec{q})(\gamma^5)^{k-1}P_\mp(\vec{q}) \qquad k = 1,2 \tag{3.70}$$

where

$$P_\pm(\vec{k}) \equiv \frac{1}{2}\left(1\pm\frac{\vec{\alpha}.\vec{k}+\beta m}{E_k}\right). \tag{3.71}$$

Since, for any $a\in R$ and $(T_2,T_1)\in R^2$,

$$a\int_{T_1}^{T_2} dt\, \exp(iat)\hat{F}(t,\vec{k}) = -i\,\exp(iaT_2)\hat{F}(T_2,\vec{k}) + i\,\exp(iaT_1)\hat{F}(T_1,\vec{k}) + i\int_{T_1}^{T_2} dt\, \exp(iat)\partial_t\hat{F}(t,\vec{k}), \qquad (3.72)$$

there exists a $C > 0$ such that for any $(T_2,T_1)\in \tilde{R}^2$, $a\in R$, $\vec{k}\in R^3$,

$$|a|\left|\int_{T_1}^{T_2} dt\, \exp(iat)\hat{F}(t,\vec{k})\right| \le \frac{C}{(1+k^2)^2}\,. \qquad (3.73)$$

From well-known trace relations it follows that

$$\mathrm{Tr}P_{\pm}(\vec{p})(\gamma^5)^{k-1}P_{\pm}(\vec{q})(\gamma^5)^{k-1} = 1-\frac{\vec{p}.\vec{q}\pm m^2}{E_pE_q} \qquad k = \frac{1}{2}\,. \qquad (3.74)$$

Thus, (3.69) holds if

$$\int d\vec{p}d\vec{q}\left(1-\frac{\vec{p}.\vec{q}\pm m^2}{E_pE_q}\right)(E_p+E_q)^{-2}(1+|\vec{p}-\vec{q}|^2)^{-4} < \infty. \qquad (3.75)$$

Taking the upper sign, the integral belongs to a class of integrals investigated in detail and proved to be convergent by Bongaarts(7). Using his estimates one easily sees that the lower sign leads to a convergent integral as well. The $||.||_2$-continuity follows from dominated convergence. $\blacksquare$

Corollary 3.4. The evolution operator $U(T_2,T_1)$, corresponding to an electric field in $S(R^4)$, or to the timelike component (in $S(R^4)$) of a pseudovector field, or to both together, induces a unitarily implementable Bogoliubov transformation in $\mathcal{F}_a(\mathcal{H})$ for any $(T_2,T_1)\in\tilde{R}^2$.

Proof. This follows from Ths. 3.3, 2.8 and (1). $\blacksquare$

We now define, for any $\phi\in\check{\mathcal{H}}$,

$$(\check{V}_k\phi)(\vec{x}) = f(\vec{x})(\Gamma_k\phi)(\vec{x}) \qquad k = 1,..,16 \qquad (3.76)$$

where f is a real-valued function in $S(R^3)$ and where $\Gamma_1,..,\Gamma_{16}$ are the 16 Hermitean matrices $1,\gamma^5,(i)\gamma^\mu,(i)\gamma^\mu\gamma^\nu,(i)\gamma^\mu\gamma^\nu\gamma^\rho$. Defining

$$O^k(t) = \exp(iH_ot)V_k\,\exp(-iH_ot) \qquad k = 1,..,16 \qquad (3.77)$$

one concludes as in the proof of Th.3.3 that

$$||\int_{T_1}^{T_2} dt O^k(t)_{\substack{+-\\-+}}||_2^2 = 4\int d\vec{p}d\vec{q}\,\mathrm{Tr}\Big(P_\pm(\vec{p})\Gamma_k P_\mp(\vec{q})\Gamma_k\Big)\sin^2(\alpha(E_p+E_q))(E_p+E_q)^{-2}$$

$$|\hat{f}(\vec{p}-\vec{q})|^2 \qquad k = 1,..,16 \quad (T_2,T_1)\in R^2 \quad (3.78)$$

where

$$\alpha \equiv (T_2-T_1)/2. \tag{3.79}$$

It clearly follows from (3.75) that the integral converges for k=1,2. Defining

$$H_k = H_o - V_k \qquad k = 1,..,16 \tag{3.80}$$

it then follows from Th.2.8, (2.106) and (1) that $\exp(-iH_k t)$ is implementable in $\mathcal{F}_a(\mathcal{K})$ for any $t\in R$ and $k = 1,2$. However, for $k > 2$ we have

<u>Theorem 3.5</u>. Let $(T_2,T_1)\in R^2$ and $T_2\neq T_1$. Then the operators $\int_{T_1}^{T_2} dt O^k(t)_{\substack{+-\\-+}}$ are not H.S. if $V_k\neq 0$, $k = 3,..,16$.

<u>Proof</u>. It suffices to show that the integral in (3.78) diverges for $k = 3,..,16$ if $\alpha\neq 0$. Since, if $k > 2$, Γ_k anticommutes with at least one α_i, $i = 1,2,3$, one obtains for the trace one of the two expressions at the r.h.s. of (3.74), but with at least one of the terms containing $p_i q_i$ having the opposite sign. Thus, subtracting a convergent integral the r.h.s. of (3.78) becomes

$$8\int d\vec{p}d\vec{q}\sum_{i=1}^{3}\delta_i(k)p_iq_i\sin^2(\alpha(E_p+E_q))(E_pE_q)^{-1}(E_p+E_q)^{-2}\,|\hat{f}(\vec{p}-\vec{q})|^2 \tag{3.81}$$

where $\delta_i(k) = 0$ or 1; for fixed $k > 2$ at least one of the $\delta_i(k)$ is 1. Introducing variables $\vec{s} = \vec{p}+\vec{q}$, $\vec{t} = \vec{p}-\vec{q}$ and adding a convergent integral we obtain

$$\tfrac{1}{4}\sum_{i=1}^{3}\delta_i(k)\int d\vec{s}d\vec{t}\,s_i^2\sin^2(\alpha(E_++E_-))(E_+E_-)^{-1}(E_++E_-)^{-2}\,|\hat{f}(\vec{t})|^2 \tag{3.82}$$

where

$$E_\pm \equiv \left(\tfrac{1}{4}|\vec{s}\pm\vec{t}|^2+m^2\right)^{\frac{1}{2}}. \tag{3.83}$$

We now assume that the integral

$$\int d\vec{s}d\vec{t}\,s_{i_o}^2\sin^2(\alpha(E_++E_-))(E_+E_-)^{-1}(E_++E_-)^{-2}|\hat{f}(\vec{t})|^2 \qquad i_o\in\{1,2,3\} \tag{3.84}$$

converges. Introducing polar variables (s,θ,ϕ) it then follows from Fubini's theorem that there exist a $\vec{t}_o$ with $\hat{f}(\vec{t}_o)\neq 0$ and $\theta_o,\phi_o\neq 0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$ such that the integral over s converges. This integral is a non-zero multiple of

$$\int_0^\infty ds\,s^4\sin^2(\alpha(E_+^o+E_-^o))(E_+^oE_-^o)^{-1}(E_+^o+E_-^o)^{-2} \tag{3.85}$$

where

$$E^o_\pm = (\tfrac14 s^2+\tfrac14 t_o^2\pm\tfrac12 s t_o c_o+m^2)^{\frac12} \qquad (3.86)$$

in which c_o is the cosine of the angle between (θ_o,ϕ_o) and $\vec{t}_o$. However, it is easily seen that (3.85) diverges. Thus, (3.84) diverges. Therefore (3.78) diverges for $k > 2$. $\blacksquare$

<u>Corollary 3.6</u>. Assume that $\exp(-iH_{k_o}t)$, $k_o\in\{3,..,16\}$, is implementable in $\mathcal{F}_a(\mathcal{H})$ for any $t\in R$. Then V_{k_o} is zero.

<u>Proof</u>. According to (1,Th.6.1) the assumption implies that the hypothesis of Th.2.13 is satisfied. Thus,

$\int_{T_1}^{T_2} dt O^{k_o}(t)_{+-}$ is H.S. for any $(T_2,T_1)\in R^2$. In view of Th.3.5 this implies that V_{k_o} is zero. $\blacksquare$

Finally, we define

$$(\check{V}\phi)(\vec{x}) = \sum_{k=1}^{16} g_k f_k(\vec{x})(\Gamma_k\phi)(\vec{x}) \qquad (3.87)$$

$$O'(t) = \exp(iH_o t)V\exp(-iH_o t) \qquad (3.88)$$

$$H = H_o - V \qquad (3.89)$$

where g_k are real coupling constants and f_k are non-zero real-valued functions in $S(R^3)$. It is of course quite plausible that the operators

$\int_{T_1}^{T_2} dt O'(t)_{\pm\mp}$ are only H.S. for all $(T_2,T_1)\in R^2$ if $g_k = 0$ for $k > 2$. If this is true, then the assumption that $\exp(-iHt)$ is implementable for any $t\in R$ clearly implies that $g_k = 0$ for $k > 2$. However, it does not seem easy to prove this, as cancellations may occur. In any case, these operators are in general not H.S., in the sense that if they are H.S. for some set of $\{g_k\}$, then they are not H.S. if one changes one of the g_k with $k > 2$. Thus, if some g_k with $k > 2$ are non-zero, then $\exp(-iHt)$ is in general not implementable. Clearly, if one replaces the f_k in (3.87) by real-valued F_k in $S(R^4)$, similar remarks can be made regarding $U(T_2,T_1)$. In particular, we conjecture that Th.3.3 will not hold for any $V(x)$ having non-zero components for $k > 2$.

4. THE SPIN-0 CASE.

In this section we treat the Klein-Gordon case along the same lines as the Dirac case. To stress the analogy the same symbols are used for corresponding operators and distributions.

A. Preliminaries.

We write the free Klein-Gordon equation

$$(\Box+m^2)\phi(x) = 0 \tag{4.1}$$

in the two-component form

$$\check{H}_o\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} = i\partial_t\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \tag{4.2}$$

where

$$\check{H}_o \equiv \begin{pmatrix}0 & 1\\ -\Delta+m^2 & 0\end{pmatrix}. \tag{4.3}$$

Obviously, the upper component of a weak solution of (4.2) satisfies (4.1).
We define a multiplication operator $\hat{H}_o$ on the Hilbert space
$\hat{\mathcal{H}} \equiv L^2(R^3,E_p d\vec{p})\oplus L^2(R^3,E_p^{-1}d\vec{p})$ by

$$\hat{H}_o = \begin{pmatrix}0 & 1\\ E_p^2 & 0\end{pmatrix}. \tag{4.4}$$

$\hat{H}_o$ is clearly self-adjoint on

$$D(\hat{H}_o) \equiv L^2(R^3,E_p^3 d\vec{p})\oplus L^2(R^3,E_p d\vec{p}). \tag{4.5}$$

Since the functions in $\hat{\mathcal{H}}$ are tempered distributions we can introduce the space $\check{\mathcal{H}}$ of distributional Fourier transforms, which becomes a Hilbert space by defining as inner product

$$(\check{f},\check{g}) = (\hat{f},\hat{g}) \qquad \forall \check{f},\check{g}\in\check{\mathcal{H}}. \tag{4.6}$$

Evidently,

$$\check{\mathcal{H}} = W_{\frac{1}{2}}(R^3)\oplus W_{-\frac{1}{2}}(R^3). \tag{4.7}$$

We can now regard (4.2) as an equation in $\check{\mathcal{H}}$: The action of $\check{H}_o$ is defined by

$$\check{H}_o\check{f} = \hat{H}_o\hat{f} \qquad \forall\hat{f}\in D(\hat{H}_o); \tag{4.8}$$

$\check{H}_o$ is then self-adjoint on

$$D(\check{H}_o) = W_{\frac{3}{2}}(R^3)\oplus W_{\frac{1}{2}}(R^3). \tag{4.9}$$

Observe that the distributions in $D(\check{H}_o)$ are square-integrable functions.

We introduce a unitary operator

$$W: \mathcal{H} \equiv L^2(R^3,d\vec{p})^2 \to \check{\mathcal{H}} \tag{4.10}$$

which gives rise to a simultaneous spectral representation of $\check{H}_o$ and the momentum operator (defined on $\hat{\mathcal{H}}$ as multiplication by $\vec{p}$):

$$(Wg)(\vec{x}) \equiv \sum_{\varepsilon}\int d\vec{p}W_{\varepsilon}(\vec{x},\vec{p})g_{\varepsilon}(\vec{p}) \qquad \varepsilon = +,- \tag{4.11}$$

$$(W^{-1}f)_{\varepsilon}(\vec{p}) = \int d\vec{x}W_{\varepsilon}^{-1}(\vec{x},\vec{p}).f(\vec{x}) \tag{4.12}$$

where

$$\begin{aligned} W_{\varepsilon}(\vec{x},\vec{p}) &\equiv (2\pi)^{-\frac{3}{2}} w_{\varepsilon}(\vec{p})\exp(i\varepsilon\vec{p}.\vec{x}) \\ W_{\varepsilon}^{-1}(\vec{x},\vec{p}) &\equiv (2\pi)^{-\frac{3}{2}} w_{\varepsilon}^{*}(\vec{p})\exp(-i\varepsilon\vec{p}.\vec{x}) \end{aligned} \tag{4.13}$$

in which

$$\begin{aligned} w_{+}(\vec{p}) &\equiv 2^{-\frac{1}{2}}\begin{pmatrix} E_p^{-\frac{1}{2}} \\ E_p^{\frac{1}{2}} \end{pmatrix} & w_{+}^{*}(\vec{p}) &\equiv 2^{-\frac{1}{2}}\begin{pmatrix} E_p^{\frac{1}{2}} \\ E_p^{-\frac{1}{2}} \end{pmatrix} \\ w_{-}(\vec{p}) &\equiv 2^{-\frac{1}{2}}\begin{pmatrix} E_p^{-\frac{1}{2}} \\ -E_p^{\frac{1}{2}} \end{pmatrix} & w_{-}^{*}(\vec{p}) &\equiv 2^{-\frac{1}{2}}\begin{pmatrix} E_p^{\frac{1}{2}} \\ -E_p^{-\frac{1}{2}} \end{pmatrix} . \end{aligned} \tag{4.14}$$

The integrals are distributional Fourier transforms. We define the charge conjugation operator $\check{C}$ on $\check{\mathcal{H}}$ by

$$(\check{C}f)_i(\vec{x}) = (-)^{i+1}\bar{f}_i(\vec{x}) \qquad i = 1,2. \tag{4.15}$$

Then the following relations hold (cf. (3.9)):

$$(H_of)_{\varepsilon}(\vec{p}) = \varepsilon E_p f_{\varepsilon}(\vec{p}) \qquad \forall f\in D(H_o) \tag{4.16}$$

$$(Cf)_{\varepsilon}(\vec{p}) = \bar{f}_{-\varepsilon}(\vec{p}) \qquad \forall f\in\mathcal{H}. \tag{4.17}$$

The straightforward proofs of the unitarity of W and of the relations (4.12), (4.16-17) are omitted.

Denoting the spectral projections of H_o on $[m,\infty)$ and $(-\infty,-m]$ by P_+ resp. P_- we have (cf.(2.2))

$$(qf)_{\varepsilon}(\vec{p}) = \varepsilon f_{\varepsilon}(\vec{p}) \qquad \forall f\in\mathcal{H}. \tag{4.18}$$

It easily follows that for vectors $\phi,\psi\in\check{\mathcal{H}}$ which are square-integrable functions:

$$(\psi,\check{q}\phi) = \int d\vec{x}(\bar{\psi}_1(\vec{x})\phi_2(\vec{x})+\bar{\psi}_2(\vec{x})\phi_1(\vec{x})). \tag{4.19}$$

Thus, $\check{q}$ corresponds to the well-known indefinite metric of the Klein-Gordon theory.

Proceeding now as in section 3, but replacing (3.22-23) by

$$[K]_{T_2,T_1}(f,g) \equiv (\check{L}\check{\bar{f}},\check{q}\check{K}(T_2,T_1)\check{g}) \tag{4.20}$$

$$[K](F,G) \equiv \int dt\,dt'(\check{L}\check{\bar{F}}(t,\cdot),\check{q}\check{K}(t,t')\check{G}(t',\cdot)) \tag{4.21}$$

where

$$\check{L} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D(L) \equiv D(H_o) \tag{4.22}$$

one obtains (3.26-29) with $S_\varepsilon \gamma^o \to S_\varepsilon$, where

$$S_\varepsilon(x) \equiv \begin{pmatrix} i\partial_t & 1 \\ -\Delta+m^2 & i\partial_t \end{pmatrix} \Delta_\varepsilon(x). \tag{4.23}$$

In particular, for $f \in S(R^3)^2$,

$$(\check{P}_\varepsilon \exp(-i\check{H}_o t)\check{f})(\vec{x}) = i\int d\vec{y}\, S_\varepsilon(t,\vec{x}-\vec{y}).f(\vec{y}). \tag{4.24}$$

Again one concludes that the l.h.s. of (4.24) belongs to $S(R^3)^2$ for any $t \in R$, and belongs to $O_M(R^4)^2$ as a function of x. We further note that

$$\overrightarrow{D^o_x} S_I(x-y) = S_I(x-y)\overleftarrow{D^o_y} = \delta(x-y) \qquad I = R,A,F,\bar{F} \tag{4.25}$$

where

$$D^o \equiv \begin{pmatrix} -i\partial_t & 1 \\ -\Delta+m^2 & -i\partial_t \end{pmatrix}. \tag{4.26}$$

B. The operators R,A,F and $\bar{F}$.

If electromagnetic and scalar fields are present (which we assume to be real-valued functions in $S(R^4)$) the Klein-Gordon equation reads:

$$[(\partial_\mu - iA_\mu(x))(\partial^\mu - iA^\mu(x)) + m^2 - A_4(x)]\phi(x) = 0. \tag{4.27}$$

We write it in the following two-component form:

$$\check{H}(t)\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = i\partial_t \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \tag{4.28}$$

where

$$\check{H}(t) \equiv \begin{pmatrix} -A_o(x) & 1 \\ -\Delta+m^2-i\vec{\nabla}.\vec{A}(x)-i\vec{A}(x).\vec{\nabla}+A^2(x)-A_4(x) & -A_o(x) \end{pmatrix}. \tag{4.29}$$

Clearly, the upper component of a weak solution of (4.28) satisfies (4.27).

One easily sees that, formally,

$$(H(t)g)_\varepsilon(\vec{p}) = \varepsilon E_p g_\varepsilon(\vec{p}) - \varepsilon\sum_{\varepsilon'}\int d\vec{q}(4E_pE_q)^{-\frac{1}{2}}\Big[\hat{A}_o(t,\varepsilon\vec{p}-\varepsilon'\vec{q})\ (\varepsilon E_p+\varepsilon' E_q)$$
$$-\hat{\vec{A}}(t,\varepsilon\vec{p}-\varepsilon'\vec{q}).(\varepsilon\vec{p}+\varepsilon'\vec{q})-\widehat{A^2}(t,\varepsilon\vec{p}-\varepsilon'\vec{q})+\hat{A}_4(t,\varepsilon\vec{p}-\varepsilon'\vec{q})\Big]g_{\varepsilon'}(\vec{q}). \tag{4.30}$$

Since convolution with an L^1 function is a bounded operation on L^2, the interaction term defines a bounded operator on $\mathcal{H}$ if

$$E_p^{\frac{1}{2}}\hat{A}_\mu(t,\vec{p}), \hat{A}_4(t,\vec{p})\in L^1(R^3,d\vec{p}). \tag{4.31}$$

Thus, $H(t)$ is actually a well-defined closed operator on $D(H_o)$ for any $t\in\mathbb{R}$. We write (4.30), with obvious notation, as

$$H(t) = H_o - V(t) \tag{4.32}$$

$$V(t) = \sum_{k=e,m,s} V_k(t). \tag{4.33}$$

From (4.29) and (4.15) it follows that if $A_4(x) = 0$ and the dependence onthe electric charge e (which is absorbed in A_μ) is explicitly indicated,

$$CH_e(t) = -H_{-e}(t)C, \tag{4.34}$$

which motivates the name charge conjugation operator for C.

Defining

$$O(t) = \exp(iH_ot)V(t)\exp(-iH_ot) \tag{4.35}$$

it is clear that $O(t)$ satisfies the conditions of section 2. However, λ can only be interpreted as the coupling constant if there is no magnetic field present. If the magnetic field is non-zero one should replace $\lambda\check{O}(t)$ by

$$\check{O}(t,\lambda) \equiv \lambda\exp(i\check{H}_ot)\begin{pmatrix} A_o(t,\cdot) & 0 \\ i\vec{\nabla}.\vec{A}(t,\cdot)+i\vec{A}(t,\cdot).\vec{\nabla}+A_4(t,\cdot) & A_o(t,\cdot)\end{pmatrix}\exp(-i\check{H}_ot)$$
$$-\lambda^2\exp(i\check{H}_ot)\begin{pmatrix} 0 & 0 \\ A^2(t,\cdot) & 0\end{pmatrix}\exp(-i\check{H}_ot). \tag{4.36}$$

The results of section 2 have straightforward analogues for this case. In particular, one should now define $\dot{\ell}$ (cf.(2.6)) as the supremum of the numbers $\alpha > 0$ such that

$$\int dt\,||O(t,\lambda)|| < 1 \qquad \forall\lambda\in D_\alpha \tag{4.37}$$

(cf.(2.14)).

It follows from (4.18) and (4.30) that

$$V^*(t) = qV(t)q \tag{4.38}$$

i.e. $V(t)$ is pseudo-self-adjoint (w.r.t. q). Th.2.10 then implies that U and U^s are pseudo-unitary. In view of Ths. 2.4 and 2.5 these operators solve the interaction picture resp. Schrödinger picture Cauchy problems.

Before we obtain the analogue of (3.32) we make some remarks. Clearly, (4.38) implies that $qH(t)$ is self-adjoint. Moreover, using (4.19) and (4.29) one infers that for $f\in D(\check{H}_o)$

$$(f,\check{q}\check{H}(t)f) = \int d\vec{x}\Big[|(\vec{\nabla}+i\vec{A}(t,\vec{x}))f_1(\vec{x})|^2+(m^2-A_o^2(t,\vec{x})-A_4(t,\vec{x}))|f_1(\vec{x})|^2 + |f_2(\vec{x})-A_o(t,\vec{x})f_1(\vec{x})|^2\Big]. \tag{4.39}$$

Thus, $qH(t)$ is strictly positive if

$$\sup_{\vec{x}\in R^3} |A_o^2(t,\vec{x})+A_4(t,\vec{x})| < m^2. \tag{4.40}$$

Assuming now that the external fields are time-independent real-valued functions satisfying (4.31) it follows from the pseudo-unitarity of $\exp(-iHt)$ (cf. Ths. 2.10,2.12) that on $D(\check{H})$

$$(\exp(-i\check{H}t)f,\check{q}\check{H}\exp(-i\check{H}t)f) = (f,\check{q}\check{H}f) \qquad \forall t\in R \tag{4.41}$$

i.e. the "energy" of solutions is time-independent. If the fields moreover satisfy (4.40) one can make $W_1(R^3)\oplus W_o(R^3)$ $(=D((\check{q}\check{H})^{\frac{1}{2}}))$ into a Hilbert space $\mathcal{K}_v$ by setting

$$(f,g)_{\mathcal{K}_v} \equiv ((\check{q}\check{H})^{\frac{1}{2}}f,(\check{q}\check{H})^{\frac{1}{2}}g) \qquad \forall f,g\in\mathcal{K}_v. \tag{4.42}$$

In view of (4.41) $\exp(-i\check{H}t)\upharpoonright D(\check{H})$ extends to a $||\cdot||_{\mathcal{K}_v}$-strongly continuous one-parameter unitary group in $\mathcal{K}_v$, the generator of which is a restriction of $\check{H}$.

In (20), Lundberg presents a scattering theory for the time-independent case, which is based on a different two-component equation (if $A_o(\vec{x})\neq 0$). It would be worthwile to investigate to what extent the two approaches are equivalent. Of course, on the level of solutions to the partial differential equation (4.27) one obtains the same results, like the time-independence of the "charge" and the "energy". However, it is e.g. not clear whether the spectrum of the operator $\check{H}$ on $\mathcal{K}_v$ and the spectrum of his operator A on $\mathcal{K}$ are equal, or whether the two approaches lead to the same quantized theory in the time-dependent case (cf.(2) and (10)). Similar remarks can be made regarding the Pétiau-Duffin-Kemmer approach to the Klein-Gordon equation (cf.(19)).

Proceeding now again as in section 3 one obtains, using (4.24) (cf.(3.32) and its proof):

$$(\check{I}(\infty,-\infty)_{\varepsilon\varepsilon'}\check{f})(\vec{x}) = -i\sum_{n=1}^{\infty}\int dx_1..dx_n d\vec{x}' S_\varepsilon(x-x_1)B(x_1)S_I(x_1-x_2)\,B(x_2)..S_I(x_{n-1}-x_n)B(x_n)S_{\varepsilon'}(x_n-x')f(\vec{x}') \qquad f\in S(R^3)^2 \tag{4.43}$$

where

$$B(x) \equiv \begin{pmatrix} A_o(x) & o \\ i\vec{\nabla}.\vec{A}(x)+i\vec{A}(x).\vec{\nabla}-A^2(x)+A_4(x) & A_o(x) \end{pmatrix} \tag{4.44}$$

For F and $\bar{F}$ it is assumed that $\ell > 1$. As in section 3, each term belongs to $S(R^3)^2$.

The analogue of (3.38) is

$$(I(\infty,-\infty)f)_\varepsilon(\vec{p}) = 2\pi i\varepsilon \sum_{n=1}^{\infty} \sum_{\varepsilon'} \int dk_1..dk_{n-1} d\vec{q}(2E_p)^{-\frac{1}{2}} \Big[\tilde{V}_1(\varepsilon p,k_1)\tilde{\Delta}_I(k_1)\,\tilde{V}_1(k_1,k_2)$$

$$..\tilde{\Delta}_I(k_{n-1})\tilde{V}_1(k_{n-1},\varepsilon' q)+\text{all}\tilde{A}_o\tilde{A}^o\text{-contractions}\Big](2E_q)^{-\frac{1}{2}} f_{\varepsilon'}(\vec{q})$$

$$f \in S(R^3)^2 \tag{4.45}$$

where

$$\tilde{V}_1(k,k') \equiv \tilde{A}_\mu(k-k')(k^\mu+k'^\mu)-\widetilde{A^2}(k-k')+\tilde{A}_4(k-k'). \tag{4.46}$$

An $\tilde{A}_o\tilde{A}^o$-contraction of the term in brackets is by definition the same term where one or several different triplets $\tilde{V}_1(k_i,k_{i+1})\tilde{\Delta}_I(k_{i+1})\tilde{V}_1(k_{i+1},k_{i+2})$ are replaced by $\tilde{A}_o(k_i-k_{i+1})\tilde{A}^o(k_{i+1}-k_{i+2})$ $(i= 0,..,n-2;\ k_o \equiv \varepsilon p, k_n \equiv \varepsilon' q)$.

To see that (4.45) holds, obtain the analogue of (3.35) (with, of course, $\check{f} \to Wf$, $\check{g} \to Wg$) and note that the factor $\varepsilon_i E_{k_i}$ (cf.(4.30)) amounts to a distributional time derivative of $\hat{\Delta}_R(t_i-t_{i+1},\vec{k}_i)$ in those terms where it occurs once. Choose then uniformly bounded sequences $\hat{F}_N,\hat{G}_N \in S(R^4)$ such that $\lim_{N\to\infty} \hat{F}_N,\hat{G}_N = \hat{\Delta}_R,\dot{\hat{\Delta}}_R$ pointwise and proceed as in section 3. Finally, replace $E_{k_i}^2$ by $m^2-k_i^2+k_i^{o2}$ in those terms where $\varepsilon_i E_{k_i}$ occurred twice and use (3.18).

Since I is analytic in the coupling constant we can rearrange (4.45):

$$(I(\infty,-\infty)f)_\varepsilon(\vec{p}) = 2\pi i\varepsilon \sum_{n=1}^{\infty} \sum_{\varepsilon'} \int dk_1..dk_{n-1} d\vec{q}(2E_p)^{-\frac{1}{2}} \Big[\tilde{V}_2(\varepsilon p,k_1)\tilde{\Delta}_I(k_1)\,\tilde{V}_2(k_1,k_2)$$

$$..\tilde{\Delta}_I(k_{n-1})\tilde{V}_2(k_{n-1},\varepsilon' q)+\text{all}\tilde{A}_\mu\tilde{A}^\mu\text{-contractions}\Big](2E_q)^{-\frac{1}{2}} f_{\varepsilon'}(\vec{q}) \tag{4.47}$$

where

$$\tilde{V}_2(k,k') \equiv \tilde{A}_\mu(k-k')(k^\mu+k'^\mu)+\tilde{A}_4(k-k'). \tag{4.48}$$

We write (4.47) as

$$(I(\infty,-\infty)f)_\varepsilon(\vec{p}) \equiv \sum_{n=1}^{\infty} \sum_{\varepsilon'} \int d\vec{q}\, I^{(n)}{}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) f_{\varepsilon'}(\vec{q}) \tag{4.49}$$

and observe that $I^{(n)}{}_{\varepsilon\varepsilon'}(\vec{p},\vec{q})$ is in $S(R^3)$ in $\vec{p}$ and $\vec{q}$ separately. Furthermore:

$$I^{(n)}{}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) = 2\pi i\varepsilon \lim_{\delta_{n-1}\downarrow o} .. \lim_{\delta_1\downarrow o} \int dk_1..dk_{n-1}(2E_p)^{-\frac{1}{2}} \Big[\tilde{V}_2(\varepsilon p,k_1)\Delta_I^{\delta_{n-1}}(k_1)$$

$$.\tilde{V}_2(k_1,k_2)..\Delta_I^{\delta_1}(k_{n-1})\tilde{V}_2(k_{n-1},\varepsilon' q)+\text{all}\tilde{A}_\mu\tilde{A}^\mu\text{-contractions}\Big](2E_q)^{-\frac{1}{2}}. \tag{4.50}$$

Formally rearranging (4.45) we can also write

$$(I(\infty,-\infty)f)_\varepsilon(\vec{p}) = 2\pi i\varepsilon \sum_{n=1}^{\infty} \sum_{\varepsilon'} \int dk_1..dk_{n-1} d\vec{q}(2E_p)^{-\frac{1}{2}} \tilde{V}_3(\varepsilon p,k_1)\tilde{\Delta}_I(k_1)\, \tilde{V}_3(k_1,k_2)$$
$$..\tilde{\Delta}_I(k_{n-1})\tilde{V}_3(k_{n-1},\varepsilon' q)(2E_q)^{-\frac{1}{2}} f_{\varepsilon'}(\vec{q}) \tag{4.51}$$

where

$$\tilde{V}_3(k,k') \equiv \tilde{A}_\mu(k-k')(k^\mu+k'^\mu) + \widetilde{A_\mu A^\mu}(k-k') + \tilde{A}_4(k-k'). \tag{4.52}$$

In fact, this rearrangement is allowed if I = R,A. Indeed, defining

$$\check{\mathfrak{V}}(t) = \exp(i\check{H}_o t)\begin{pmatrix} 0 & 0 \\ A_0^2(t,\cdot) & 0 \end{pmatrix} \exp(-i\check{H}_o t) \tag{4.53}$$

it follows from Th.2.6 that

$$R(\infty,-\infty) = \sum_{n=1}^{\infty} \tilde{R}_1^{(n)}(\infty,-\infty). \tag{4.54}$$

However, it can be seen that $(\tilde{R}_1^{(n)}(\infty,-\infty)f)_\varepsilon(\vec{p})$ is equal to the n^{th} term in (4.51). Thus, (4.51) holds for I = R, and, similarly, for I = A.

(Provided the coupling constant is small enough, the rearrangement is also permitted if $I = F,\bar{F}$. This is a consequence of work of Bellissard (3,4). It follows from his results (in particular (3, Lemmas A 5.4, A 5.6) and (4,Th.II1.1)) that the r.h.s. of (4.51) defines a bounded operator on $\mathcal{H}$ if the coupling constant is small enough. Since, moreover, this operator is analytic in a neighbourhood of the origin, we can rearrange (4.51) and obtain (4.47).)

C. Lorentz covariance, causality.

The representation of the Poincaré group in $\mathcal{H}$ is defined by

$$(U(a,\Lambda)g)_\varepsilon(\vec{p}) = \exp(i\varepsilon pa)\left(\frac{(\Lambda^{-1}p)_o}{p_o}\right)^{\frac{1}{2}} g_\varepsilon(\Lambda^{-1}p). \tag{4.55}$$

It easily follows from (4.47) and (4.55) that

$$U(a,\Lambda)I(A_\ell)U^*(a,\Lambda) = I(A_\ell{}^{a,\Lambda}) \tag{4.56}$$

where $A^{a,\Lambda}$ is defined by (3.43), and

$$A_4{}^{a,\Lambda}(x) \equiv A_4(\Lambda^{-1}(x-a)). \tag{4.57}$$

Thus, the operators $I(\infty,-\infty)$ are Lorentz covariant. Defining the classical S-operator by (3.54) we have

Theorem 4.1. S(V) is Lorentz covariant and causal.

Proof. This follows as in section 3. $\blacksquare$

Defining

$$G_{\substack{R\\A}}(T_2,T_1) = \pm iU^S(T_2,T_1)\theta(\pm(T_2-T_1)) \tag{4.58}$$

it follows as in section 3 from (4.21) that (3.61) and (3.63) hold true. Hence, using (4.25),

$$\vec{D}_x\left[G_I\right](x,y) = \left[G_I\right](x,y)\overleftarrow{\bar{D}}_y = \delta(x-y) \qquad I = R,A \tag{4.59}$$

where

$$D_x \equiv D^o_x - B(x). \tag{4.60}$$

Thus, $\left[G_{R,A}\right]$ are retarded resp. advanced fundamental solutions of the perturbed two-component Klein-Gordon equation (4.28). It should be noted that their existence does not follow from Th.A.1 if the electromagnetic field is non-zero, since in this case second order derivatives occur in $B{*}S_R$.

As in section 3 we did not use the assumption that the fields are real-valued in obtaining (3.61) and (3.63). Consequently, these relations also hold if the fields are only assumed to be in $S(R^4)$. Clearly, (4.59) holds as well if one does not complex conjugate the fields in the expression $\overleftarrow{\bar{D}}_y$. Thus, we have proved:

Theorem 4.2. For any $A_\ell \in S(R^4)$, $\ell = 0,..,4$, the perturbed two-component Klein-Gordon equation (4.28) admits two-sided tempered retarded and advanced fundamental solutions given by (3.61). They are connected with the evolution operator through (4.58) and (4.21), and are Wightman regular if the electromagnetic field vanishes. $\blacksquare$

Setting, for $F,G,A_o,..,A_4 \in S(R^4)$,

$$\overset{o}{G}_I(F,G) \equiv \sum_{n=o}^{\infty}\int dx dx_1..dx_n dy F(x)\Delta_I(x-x_1)K(x_1)\Delta_I(x_1-x_2)..K(x_n)\Delta_I(x_n-y)G(y)$$

where $\qquad I = R,A \qquad$ (4.61)

$$K \equiv iA_\mu\partial^\mu + i\partial_\mu A^\mu + A_\mu A^\mu + A_4 \tag{4.62}$$

it is clear that $\overset{o}{G}_{R,A}$ are, formally, two-sided retarded resp. advanced fundamental solutions of the perturbed Klein-Gordon equation (4.27):

$$(\Box_x + m^2 - \vec{K}(x))\overset{o}{G}_I(x,y) = \overset{o}{G}_I(x,y)(\overleftarrow{\Box}_y + m^2 - \overleftarrow{K}(y)) = \delta(x-y). \tag{4.63}$$

(Again, the complex conjugation in $\overset{\leftarrow}{\bar{K}}$ is supposed not to act on the fields.) In fact, it easily follows from Th.A.1 that $\overset{\circ}{G}_{R,A}$ are Wightman regular distributions in $S'(R^8)$.

Defining now

$$[G_I] = \begin{pmatrix} G_I^{11} & G_I^{12} \\ G_I^{21} & G_I^{22} \end{pmatrix} \tag{4.64}$$

it follows from (4.59) that $G_{R,A}^{12}$ are retarded resp. advanced fundamental solutions of (4.27). Using (4.23) and (4.44) it moreover follows from (3.61) that, if $A_o(x) = 0$,

$$G_I^{12} = \overset{\circ}{G}_I. \tag{4.65}$$

Actually, (4.65) holds as well if $A_o(x) \neq 0$, which can be seen if one uses Th.2.6. Therefore we have proved:

<u>Theorem 4.3</u>. For any $A_\ell \in S(R^4)$, $\ell = 0,..,4$, the perturbed Klein-Gordon equation (4.27) admits two-sided tempered retarded and advanced fundamental solutions, given by (4.61). They are connected with the evolution operator through (4.65) and are Wightman regular. $\blacksquare$

D. Implementability of the evolution in $\mathcal{F}_s(\mathcal{H})$.

In this subsection we again assume that the fields are real-valued since we need the pseudo-unitarity of $U(T_2,T_1)$. We first show that the hypothesis of Th.2.8 is satisfied if the magnetic field vanishes.

<u>Theorem 4.4</u>. Let

$$O_k(t) \equiv \exp(iH_o t)V_k(t)\exp(-iH_o t) \qquad k = e,s. \tag{4.66}$$

Then the operators $\int_{T_1}^{T_2} dt O_k(t)_{\substack{+- \\ -+}}$, $k = e,s$, are $||.||_2$-continuous on $\tilde{R}^2$ and

$$\left|\left|\int_{T_1}^{T_2} dt O_k(t)_{\substack{+- \\ -+}}\right|\right|_2 \leq C_1 < \infty \qquad \forall\, (T_2,T_1) \in \tilde{R}^2. \tag{4.67}$$

Proof. It follows from (4.30) that it suffices to prove these statements for the integral operators on $L^2(R^3)^2$ with the kernels

$$\int_{T_1}^{T_2} dt\, \exp(\pm i(E_p+E_q)t)\hat{A}_o(t,\vec{p}-\vec{q})(E_p-E_q)(E_pE_q)^{-\frac{1}{2}} \tag{4.68}$$

$$\int_{T_1}^{T_2} dt\, \exp(\pm i(E_p+E_q)t)\hat{A}_4(t,\vec{p}-\vec{q})(E_pE_q)^{-\frac{1}{2}}. \tag{4.69}$$

Using (3.73) we conclude that it is sufficient to show:

$$\int d\vec{p}d\vec{q}(E_p-E_q)^2(E_pE_q)^{-1}(E_p+E_q)^{-2}(1+|\vec{p}-\vec{q}|^2)^{-4} < \infty \tag{4.70}$$

$$\int d\vec{p}d\vec{q}(E_pE_q)^{-1}(E_p+E_q)^{-2}(1+|\vec{p}-\vec{q}|^2)^{-4} < \infty. \tag{4.71}$$

However, (4.70) and (4.71) are easily seen to hold true (cf.(7)). $\blacksquare$

Corollary 4.5. The evolution operator $U(T_2,T_1)$, corresponding to an electric or scalar field in $S(R^4)$, or to both together, gives rise to a unitarily implementable Bogoliubov transformation in $\mathcal{F}_s(\mathcal{H})$ for any $(T_2,T_1)\in\tilde{R}^2$.

Proof. This follows as in section 3. $\blacksquare$

We now assume that the fields are real-valued functions in $S(R^3)$. Defining

$$O^k(t) = \exp(iH_ot)V_k\exp(-iH_ot) \qquad k = e,m,s \tag{4.72}$$

one obtains

$$\left\|\int_{T_1}^{T_2} dt O^k(t)_{\substack{+-\\-+}}\right\|_2^2 = \int d\vec{p}d\vec{q}|F_k(\vec{p},\vec{q})|^2\sin^2(\alpha(E_p+E_q))(E_pE_q)^{-1}(E_p+E_q)^{-2} \tag{4.73}$$

where

$$F_e(\vec{p},\vec{q}) \equiv \hat{A}_o(\vec{p}-\vec{q})(E_p-E_q) \tag{4.74}$$

$$F_m(\vec{p},\vec{q}) \equiv \hat{\vec{A}}(\vec{p}-\vec{q}).(\vec{p}+\vec{q}) \tag{4.75}$$

$$F_s(\vec{p},\vec{q}) \equiv \hat{A}_4(\vec{p}-\vec{q}). \tag{4.76}$$

Defining

$$H = H_o - \sum_{k=e,m,s} V_k \tag{4.77}$$

it follows as in section 3 that $\exp(-iHt)$ is implementable in $\mathcal{F}_s(\mathcal{H})$ for any $t\in R$ if $V_m = 0$. However, if a magnetic field is present, the situation is different:

Theorem 4.6. Let $(T_2,T_1)\in R^2$ and $T_2\neq T_1$. Then the operators $\int_{T_1}^{T_2} dt O^m(t)_{\substack{+-\\-+}}$ are not H.S. if $V_m\neq 0$.

Proof. It suffices to show that the integral at the r.h.s. of (4.73) diverges for $k = m$ if $\alpha\vec{A}\neq 0$. We introduce variables s,θ,ϕ and $\vec{t}$ as in the proof of Th3.5 and assume the integral converges. It then follows from Fubini's theorem that there exist a $\vec{t}_o$ and θ_o,ϕ_o with $\vec{n}_{\theta_o,\phi_o}\cdot\hat{\vec{A}}(\vec{t}_o)\neq 0$ such that the integral over s converges ($\vec{n}_{\theta,\phi}$ is the unit vector in the direction θ,ϕ). However, the latter integral is a non-zero multiple of (3.85) so it actually diverges. This contradiction completes the proof. $\blacksquare$

Corollary 4.7. Assume that $\exp(-iHt)$ is implementable in $\mathcal{F}_s(\mathcal{H})$ for any $t\in R$. Then the magnetic field vanishes.

Proof. This follows as in section 3. $\blacksquare$

This corollary was first proved by Hochstenbach (11) using results of Shale (21). Again, we expect that an analogous result holds for time-dependent fields.

APPENDIX.

We consider two transformations T_R and T_A of $S(R^4)^K$. Let $V_\ell^{\sigma\sigma'}(x)$ $(\sigma,\sigma' = 1,..,K;\ \ell = 0,..,4)$ be functions in $S(R^4)$ and let

$$A_{\sigma\sigma'}(x) \equiv V_o^{\sigma\sigma'}(x)\partial_t + \sum_{j=1}^{3} V_j^{\sigma\sigma'}(x)\partial_{x_j} + V_4^{\sigma\sigma'}(x). \qquad (A.1)$$

For any $F \in S(R^4)^K$ we define

$$T_{\substack{A\\R}} F = (1 - J_{\substack{R\\A}})F \tag{A.2}$$

where

$$(J_I F)_\sigma(x) \equiv A_{\sigma\sigma'}(x)\int dy \Delta_I(x-y)F_{\sigma'}(y) \qquad I = R,A. \tag{A.3}$$

In (A.3), as in the sequel, the summation convention is used. The following theorem is a generalization of a result of Bellissard (4).

Theorem A.1. The transformations T_R and T_A are linear bicontinuous bijections of $S(R^4)^K$ onto $S(R^4)^K$ with inverses

$$T_{\substack{R\\A}}^{-1} = \sum_{N=0}^{\infty} J_{\substack{A\\R}}^N . \tag{A.4}$$

Proof. It is easily seen that T_R and T_A are linear continuous injections. Since $S(R^4)^K$ is a Fréchet space they are bicontinuous if they are also surjections (12). Thus, in view of (A.2), it suffices to prove that $\lim_{M\to\infty} \sum_{N=0}^{M} J_I^N F$ exists in $S(R^4)^K$. We will show this for I = R. The proof for I = A is similar. We first note that (cf.(3.16)):

$$\hat{\Delta}_R(t,\vec{p}) = \theta(t)\frac{\sin E_p t}{E_p}$$

$$\widehat{(\partial_t \Delta_R)}(t,\vec{p}) = \theta(t)\cos E_p t \tag{A.5}$$

$$\widehat{(\partial_{x_j} \Delta_R)}(t,\vec{p}) = \theta(t)\frac{ip_j \sin E_p t}{E_p} \qquad j = 1,2,3.$$

We now define for any $F \in S(R^4)^K$

$$||F||_{\tau,\alpha,\sigma} = \sup_{(t,\vec{p})\in R^4} |(1+t^2+p^2)^\tau D^\alpha \hat{F}_\sigma(t,\vec{p})| \tag{A.6}$$

where $\alpha \equiv (\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ is a multi-index, $\tau = 0,1..$ and $\sigma = 1,..,K$. Clearly, $\lim_{M\to\infty} \sum_{N=0}^{M} J_R^N F$ exists if the sequence $J_R^N F$ is absolutely summable in all $||.||_{\tau,\alpha,\sigma}$. In order to show this we shall majorize the function

$$H_{N\tau\alpha\sigma}(t,\vec{p}) \equiv (1+t^2+p^2)^\tau D^\alpha \widehat{(J_R^N F)}_\sigma(t,\vec{p}). \tag{A.7}$$

It is easy to see that

$$(\widehat{J_R^N F})_\sigma(t,\vec{p}) = \int_{-\infty}^{t} dt_1 .. \int_{-\infty}^{t_{N-1}} dt_N \int d\vec{q}_1 .. d\vec{q}_N \hat{V}_{\ell_1}^{\sigma\sigma_1}(t,\vec{p}-\vec{q}_1) G_{\ell_1}(t-t_1,\vec{q}_1)$$

$$.. \hat{V}_{\ell_N}^{\sigma_{N-1}\sigma_N}(t_{N-1},\vec{q}_{N-1}-\vec{q}_N) G_{\ell_N}(t_{N-1}-t_N,\vec{q}_N) \hat{F}_{\sigma_N}(t_N,\vec{q}_N) \tag{A.8}$$

where

$$G_o(t,\vec{p}) \equiv \cos E_p t$$

$$G_j(t,\vec{p}) \equiv ip_j \frac{\sin E_p t}{E_p} \qquad j = 1,2,3 \tag{A.9}$$

$$G_4(t,\vec{p}) \equiv \frac{\sin E_p t}{E_p} .$$

Obviously, for any $\ell = 0,..4$; $(t,\vec{p}) \in R^4$; $\beta = 1,..,\alpha_o$:

$$|G_\ell(t,\vec{p})| \leq \delta \tag{A.10}$$

and

$$|\partial_t^\beta G_\ell(t,\vec{p})| \leq \delta^{\alpha_o} E_p^{2\alpha_o} \tag{A.11}$$

where

$$\delta \equiv \max(m^{-2},1). \tag{A.12}$$

Assuming from now on that $N > \alpha_o$ we conclude that $H_{N\tau\alpha\sigma}$ is equal to a sum of terms the number of which is bounded by $(5K)^N L$, where L only depends on α_o. Denoting the generic term by H_i we have, suppressing the indices,

$$H_i(t,\vec{p}) = \int_{-\infty}^{t} dt_n .. \int_{-\infty}^{t_{N-1}} dt_N \int d\vec{q}_1 .. d\vec{q}_N (1+t^2+p^2)^\tau V_{\beta_1,\vec{\alpha}}(t,\vec{p}-\vec{q}_1)(\partial_t^{\beta_2} G)(0,\vec{q}_1)$$

$$..(\partial_t^{\beta_{2n-1}}\hat{V})(t,\vec{q}_{n-1}-\vec{q}_n)(\partial_t^{\beta_{2n}} G)(t-t_n,\vec{q}_n)\hat{V}(t_n,\vec{q}_n-\vec{q}_{n+1})..\hat{F}(t_N,\vec{q}_N) \tag{A.13}$$

where

$$\hat{V}_{\beta_1,\vec{\alpha}}(t,\vec{k}) \equiv (\partial_t^{\beta_1} \partial_{k_1}^{\alpha_1} \partial_{k_2}^{\alpha_2} \partial_{k_3}^{\alpha_3} \hat{V})(t,\vec{k}) \tag{A.14}$$

and

$$0 \leq n-1, \beta_1,..,\beta_{2n} \leq \alpha_o . \tag{A.15}$$

Thus, for any H_i with a fixed n:

$$|H_i(t,\vec{p})| \leq \delta^{\alpha_o^2+\alpha_o+N} \int_{-\infty}^{\infty} dt_n .. \int_{-\infty}^{t_{N-1}} dt_N \int d\vec{k}_1 .. d\vec{k}_N (1+t^2)^\tau (1+p^2)^\tau$$

$$.|\hat{V}_{\beta_1,\vec{\alpha}}(t,\vec{p}-\sum_{j=1}^{N}\vec{k}_j)| (|\sum_{j=1}^{N}\vec{k}_j|^2+m^2)^{\alpha_o} .. |(\partial_t^{\beta_{2n-1}}\hat{V})(t,\vec{k}_{n-1})| (|\sum_{j=n}^{N}\vec{k}_j|^2+m^2)^{\alpha_o}$$

$$. \prod_{j=n}^{N-1} |\hat{V}(t_j,\vec{k}_j)| \; |\hat{F}(t_N,\vec{k}_N)| . \tag{A.16}$$

Using the estimate

$$|\sum_{j=r}^{N} \vec{k}_j|^2+m^2 \le \gamma N \prod_{j=r}^{N} (1+k_j^2) \qquad 1 \le r \le N \tag{A.17}$$

$$\gamma \equiv \max(m^2,1)$$

we conclude:

$$|H_i(t,\vec{p})| \le (\delta\gamma N)^{\alpha_o^2+\alpha_o}\delta^N\int_{-\infty}^{\infty} dt_1 .. \int_{-\infty}^{t_{N-1}} dt_N \int d\vec{k}_1..d\vec{k}_N (1+t^2)^\tau(1+p^2)^\tau$$

$$\cdot|\hat{V}_{\beta_1,\vec{\alpha}}(t,\vec{p}-\sum_{j=1}^{N}\vec{k}_j)|\prod_{j=1}^{n-1}(1+k_j^2)^{\alpha_o^2+\alpha_o}|(\partial_t^{\beta_{2j+1}}\hat{V})(t,\vec{k}_j)|\prod_{j=n}^{N-1}(1+k_j^2)^{\alpha_o^2+\alpha_o}|\hat{V}(t_j,\vec{k}_j)|$$

$$\cdot(1+k_N^2)^{\alpha_o^2+\alpha_o}|\hat{F}(t_N,\vec{k}_N)|. \tag{A.18}$$

Since $(\hat{V}_\ell^{\sigma\sigma'})_{\beta,\vec{\alpha}}, \partial_t^\beta\hat{V}_\ell^{\sigma\sigma'}, \hat{F}_\sigma$ are test functions ($\beta = 0,..,\alpha_o$; $\ell = 0,..,4$; $\sigma,\sigma' = 1,..,K$) there exists a $C > 0$, only depending on V,F,α,τ, such that the sum of all H_i with the same n, denoted by S_n, satisfies

$$|S_n(t,\vec{p})| \le (5K)^N L(\delta\gamma N)^{\alpha_o^2+\alpha_o}\delta^N C^{N+1}\pi^{N-n+1}((N-n+1)!)^{-1}$$

$$\cdot\int d\vec{k}_1..d\vec{k}_N(1+p^2)^\tau(1+|\vec{p}-\sum_{j=1}^{N}\vec{k}_j|^2)^{-\tau}\prod_{j=1}^{N}(1+k_j^2)^{-\tau-2}. \tag{A.19}$$

Using the estimate

$$(1+p^2)(1+q^2)^{-1} \le 2(1+|\vec{p}-\vec{q}|^2) \tag{A.20}$$

and (A.17) we obtain:

$$|S_n(t,\vec{p})| \le \frac{C_1C_2^N N^{\alpha_o^2+\alpha_o+\tau}}{(N-n+1)!} \tag{A.21}$$

where C_1,C_2 do not depend on N and n. Thus,

$$|H_{N\tau\alpha\sigma}(t,\vec{p})| \le \frac{(\alpha_o+1)C_1C_2^N N^{\alpha_o^2+\alpha_o+\tau}}{(N-\alpha_o)!}. \tag{A.22}$$

It easily follows from (A.22) that the sequence $J_R^N F$ is absolutely summable in all $||.||_{\tau,\alpha,\sigma}$, so the theorem is proved. ∎

It can be seen that the theorem still holds true if R^4 is replaced by R^{1+n}, with n odd, and Δ_R,Δ_A by the (unique) retarded and advanced fundamental solutions of the partial differential operator $\partial_t^2-\sum_{j=1}^{n}\partial_{x_j}^2+m^2$ (the retarded one satisfies (A.5) with $j = 1,..,n$). We note further that we have made essential use of the boundedness of the G_ℓ, i.e. of the fact that the derivatives of Δ_R in T_A are only first order. The theorem can therefore in general not be applied to higher spin external field theories (see, however, (4)). This is in agreement with the non-existence of retarded solutions in most of these theories (cf. (19)).

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CHAPTER 3

CHARGED PARTICLES IN EXTERNAL FIELDS
II THE QUANTIZED DIRAC AND KLEIN-GORDON THEORIES

1. INTRODUCTION.

In a previous paper (1) (which we shall refer to as I) a number of perturbation-theoretic results were obtained which were applied to the classical Dirac and Klein-Gordon equations with external fields. The quantized theories will now be considered.

As detailed in (2) (referred to as B), one can treat the external field problem in a rigorous way by using the operators from the classical theory to generate transformations of the field operators on Fock space which amount to Bogoliubov transformations. If such a transformation is implementable the resulting Fock space operator is regarded as the physical operator corresponding to the unphysical operator on the classical Hilbert space. This approach, which goes back to Friedrichs (3), was further developed in (4-8,2). A closely related but more algebraic approach, inspired by the ideas of Segal (9), was used in (10-17).

A rather different strategy, based on the Yang-Feldman equations, was initiated by Capri (18) and further developed by Wightman (19); it can be used for any generalized Dirac equation. Yet another treatment, using ideas from renormalization theory, was recently given by Bellissard (20, 21).

On a formal level the scattering of (especially spin-$\frac{1}{2}$) particles at external fields has been considered some time ago. Some references are (22-24). Detailed accounts of the formal theory can be found in the books by Schweber (25) and Thirring (26).

One of the main results of this paper is that for (massive, relativistic, charged) spin-0 and spin-$\frac{1}{2}$ particles in external fields which are test functions on space-time the Friedrichs-Segal and Capri-Wightman approaches lead to the same S-operator, and that the divergence-free perturbation expansion of this unitary S-operator corresponds in a quite natural way to the formal Feynman-Dyson series. (Earlier results going in this direction can be found in (20).) In particular, the relative (cf.(22,25)) S-matrix elements are the same in the rigorous and the formal theory, while the vacuum-to-vacuum transition amplitude resp. its modulus are equal in a formal sense to be specified below.

In section 2 the Dirac theory is treated. Subsection A contains definitions of various field operators, the equations they satisfy and their interrelationship. It is proved that the interpolating field is local and satisfies the Yang-Feldman equations, and the equivalence of the Capri-Wightman and Friedrichs-Segal approaches is established. In subsection B the evolution operator and S-operator are studied. Explicit expressions are derived and various continuity and analyticity properties are proved. The S-operator is shown to be Lorentz covariant and causal up to a phase factor. In subsection C perturbation expansions are derived, the connection with the Feynman-Dyson series is established, and the analogue of Furry's theorem is proved. In section 3 the Klein-Gordon theory is treated along the same lines as the Dirac theory. The paper ends with section 4, which contains concluding remarks.

In sections 2 and 3 it is assumed that the external fields are real-valued test functions on space time. As in I, several results could easily be extended to more general functions, but we shall not consider this.

2. THE QUANTIZED DIRAC THEORY.

A. Field operators.

In this section and the next one we shall make extensive use of the notation and results of I and B. Thus (cf.B(2.9)), we have field operators on $\mathcal{F}_a(\mathcal{H})$, defined by

$$\Phi(v) = a(P_+v) + b^*(\overline{P_-v}) \qquad \forall v\in\mathcal{H} \tag{2.1}$$

where $P_+(P_-)$ is the projection corresponding to the positive (negative) part of the Dirac Hamiltonian H_0 acting on the classical Hilbert space $\mathcal{H}$ (cf.I §3A). Clearly,

$$[\Phi(u),\Phi^*(v)]_+ = (u,v) \qquad \forall u,v\in\mathcal{H} \tag{2.2}$$

where

$$\Phi^*(v) \equiv \Phi(v)^*. \tag{2.3}$$

Defining

$$\psi_t^o(f) = \Phi(\exp(iH_o t)W^{-1}f) \qquad \forall f\in\mathcal{H} \tag{2.4}$$

one has the formal relation

$$\psi_t^o(f) = \int d\vec{x}\ \bar{f}(\vec{x}).\psi^o(t,\vec{x}) \tag{2.5}$$

where $\psi^o(x)$ is the usual formal free Dirac field

$$\psi^o(x) = (2\pi)^{-3/2} \sum_i \int d\vec{p}(\frac{m}{E_p})^{\frac{1}{2}}\ (a_i(\vec{p})u_i(\vec{p})\exp(-ipx)+b_i^*(\vec{p})v_i(\vec{p})\exp(ipx)) \tag{2.6}$$

with our conventions for the u_i and v_i (cf.I(3.6-7)). To see this, use I(3.4), (3.10) and set, e.g. (cf.B(4.6)),

$$\sum_i \int d\vec{p}\ a_i(\vec{p})\overline{g_i}(\vec{p}) \equiv a(g) \qquad g\in\mathcal{H}_+. \tag{2.7}$$

If $f\in D(\check{H}_o)$, then

$$\frac{d}{dt}\psi_t^o(f) = \psi_t^o(i\check{H}_o f) \tag{2.8}$$

where the differentiation is in the norm topology. Note that if one smears the formal relation

$$\beta(-i\not{\partial}+m)\psi^o(x) = 0 \tag{2.9}$$

with $\bar{f}(\vec{x})$ and uses partial integration and (2.5) one obtains (2.8). One can therefore regard (2.8) as a rigorous analogue of (2.9). Similarly, the relation

$$[\psi_t^o(f),\psi_t^o(g)^*]_+ = (f,g) \qquad \forall f,g\in\mathcal{H} \tag{2.10}$$

is the analogue of

$$[\psi_\alpha^o(t,\vec{x}),\psi_\beta^{o*}(t,\vec{x}')]_+ = \delta_{\alpha\beta}\ \delta(\vec{x}-\vec{x}'). \tag{2.11}$$

Since $\exp(iH_o t)$ acts as multiplication by $\exp(iE_p t)$ on $\mathcal{H}_+$ and by $\exp(-iE_p t)$ on $\mathcal{H}_-$ it follows from (2.1) and (2.4) that

$$\psi_t^o(f) = \exp(iB_o t)\psi_o^o(f)\exp(-iB_o t) \tag{2.12}$$

where

$$B_o \equiv \Omega(qH_o). \tag{2.13}$$

B_o is by definition the free Fock space Hamiltonian. It is the sum operator derived from the operator qH_o, which acts as multiplication by E_p both on the one-particle space $\mathcal{H}_+$ and on the one-antiparticle space $\mathcal{H}_-$. Thus, B_o is a positive self-adjoint operator on $\mathcal{F}_a(\mathcal{H})$ with continuous spectrum in $[m,\infty)$ and eigenvalue 0 on Ω, the vacuum. On physical vectors (cf.B § 2) built up from vectors in $D(H_o)$ one has (cf. B § 4)

$$B_o = \int d\vec{p}\, E_p (a^*(\vec{p})a(\vec{p}) + b^*(\vec{p})b(\vec{p})) \tag{2.14}$$

which is of course the usual expression (the spin indices are suppressed).

More generally, the transformation

$$\phi(v) \to \phi(U^*(a,\Lambda)v) \tag{2.15}$$

where $U(a,\Lambda)$ is the representation of iSL(2,C) in $\mathcal{H}$ (see I (3.44)) is implemented by unitary operators

$$\mathcal{U}(a,\Lambda) = \Gamma(\tilde{U}(a,\Lambda)) \tag{2.16}$$

in which

$$(\tilde{U}(a,\Lambda)v)^i_\varepsilon(\vec{p}) \equiv \exp(ipa)\left(\frac{(\Lambda^{-1}p)_o}{p_o}\right)^{\frac{1}{2}} \sum_j \left[\left(\frac{\tilde{p}}{m}\right)^{\frac{1}{2}} A\left(\frac{\widetilde{\Lambda^{-1}p}}{m}\right)^{\frac{1}{2}}\right]_{ij} v^j_\varepsilon(\overrightarrow{\Lambda^{-1}p})$$

$$\forall v \in \mathcal{H}. \tag{2.17}$$

Thus, $\mathcal{U}(a,\Lambda)$ acts in the same fashion on particle and antiparticle states.

One also verifies that the gauge transformation

$$\phi(v) \to \phi(\exp(i\alpha)v) \tag{2.18}$$

is implemented by the unitary operator $\exp(iQ\alpha)$, where Q is the charge operator

$$Q \equiv \Omega(q). \tag{2.19}$$

Moreover (cf. I (3.11)),

$$\phi^*(Cv) = \mathfrak{C}\phi(v)\mathfrak{C}^*, \tag{2.20}$$

where $\mathfrak{C}$ is the Fock space charge conjugation operator, given by

$$\mathfrak{C} = \Gamma(C') \tag{2.21}$$

where

$$(C'v)^i_\varepsilon(\vec{p}) \equiv v^i_{-\varepsilon}(\vec{p}) \qquad \forall v \in \mathcal{H}. \tag{2.22}$$

It should be noted that $\mathfrak{C}$ is unitary, whereas C is anti-unitary.

The formal perturbed Dirac field should satisfy

$$\beta(-i\not{\partial}+m-B(x))\psi(x) = 0 \tag{2.23}$$

$$[\psi_\alpha(t,\vec{x}),\psi^*_\beta(t,\vec{x}')]_+ = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}'). \tag{2.24}$$

In (2.23) $V(x) \equiv \beta B(x)$ is by definition a function from R^4 to the Hermitean 4x4 matrices, the matrix elements of which belong to $S(R^4)$. Smearing with

$\bar{f}(\vec{x})\in\check{\mathcal{H}}$ and partially integrating one obtains as rigorous analogues:

$$\frac{d}{dt}\psi_t(f) = \psi_t(i\check{H}(t)f) \qquad \forall f\in D(\check{H}_o) \tag{2.25}$$

$$\left[\psi_t(f),\psi_t(g)^*\right]_+ = (f,g) \qquad \forall f,g\in\check{\mathcal{H}}. \tag{2.26}$$

We assert that for any $a\in\tilde{\mathbb{R}}$

$$\psi_{t,a}(f) \equiv \Phi(U^*(t,a)\exp(iH_o t)W^{-1}f) \tag{2.27}$$

is a solution to (2.25) and (2.26), where U is the interaction picture evolution operator corresponding to V(x) (cf. I § 3B). Indeed, the verification of (2.26) is trivial and (2.25) follows from the relation

$$U^*(t,a) = U(a,t) \tag{2.28}$$

and I (2.23).

Evidently,

$$\lim_{t\to-\infty} \psi_{t,-\infty}(\exp(-i\check{H}_o t)f) = \Phi(W^{-1}f) \tag{2.29}$$

$$\lim_{t\to\infty} \psi_{t,-\infty}(\exp(-i\check{H}_o t)f) = \Phi(S^*W^{-1}f) \tag{2.30}$$

where the limits are norm limits and S is the classical S-operator on $\mathcal{H}$.

Hence, we set

$$\psi_{in}(f) \equiv \Phi(W^{-1}f) \tag{2.31}$$

$$\psi_{int,t}(f) \equiv \Phi(U^*(t,-\infty)W^{-1}f) = \psi_{t,-\infty}(\exp(-i\check{H}_o t)f) \tag{2.32}$$

$$\psi_{out}(f) \equiv \Phi(S^*W^{-1}f) \tag{2.33}$$

where int stands for interpolating. In a formal sense, $\psi_{int,t}$ is the interaction picture analogue of the Heisenberg picture field $\psi_{t,-\infty}$.

We now define field operators,smeared with test functions $F\in S(R^4)^4$ by

$$\psi^{ex}(F) = \int dt\ \psi_{ex}(\exp(i\check{H}_o t)\check{\bar{F}}(t,\cdot)) \tag{2.34}$$

$$\psi^{int}(F) = \int dt\ \psi_{t,-\infty}(\check{\bar{F}}(t,\cdot)) \tag{2.35}$$

where the integrals are Riemann integrals in $\mathfrak{L}(\mathcal{F}_a)$ and ex = in,out. To conform with common usage these field operators depend linearly on F. Clearly, we can also write,

$$\psi^{ex}(F) = \psi_{ex}\left(\int dt\ \exp(i\check{H}_o t)\check{\bar{F}}(t,\cdot)\right) \tag{2.36}$$

$$\psi^{int}(F) = \Phi\left(\int dt\ U^*(t,-\infty)\exp(iH_o t)W^{-1}\check{\bar{F}}(t,\cdot)\right) \tag{2.37}$$

where the integrals are strong Riemann integrals in $\check{\mathcal{H}}$ resp. $\mathcal{H}$. The relation

$$\psi^{in}(F) = \int dx F(x)\psi^{o}(x) \tag{2.38}$$

now follows in the same way as (2.5). One also verifies that

$$\psi^{ex}((i\not{\partial}^{T}+m)F) = 0 \tag{2.39}$$

$$\psi^{int}((i\not{\partial}^{T}+m-B^{T})F) = 0. \tag{2.40}$$

Thus, $\psi^{ex}(x)$ and $\psi^{int}(x)$ satisfy the free resp. perturbed Dirac equation in the sense of operator-valued distributions. Moreover, we have

Theorem 2.1. The interpolating field ψ^{int} is local and satisfies the Yang-Feldman equations:

$$\psi^{int}(T_R F) = \psi^{in}(F) \tag{2.41}$$

$$\psi^{int}(T_A F) = \psi^{out}(F) \tag{2.42}$$

where

$$(T_I F)(x) \equiv F(x) - \int dy F(y) S_I(y-x) B(x) \qquad I = R, A. \tag{2.43}$$

Proof. If supp F and supp G are spacelike separated:

$$\begin{aligned}\left[\psi^{int}(F),\psi^{int}(G)^{*}\right]_{+} &= \left(\int dt\, U^{*}(t,-\infty)\exp(iH_{o}t)W^{-1}\check{\bar{F}}(t,\cdot),\int dt'\, U^{*}(t',-\infty)\right.\\ &\qquad \left.\exp(iH_{o}t')W^{-1}\check{\bar{G}}(t',\cdot)\right)\\ &= \int dt\,dt'\left(\check{\bar{F}}(t,\cdot),\check{U}^{s}(t,t')\check{\bar{G}}(t',\cdot)\right)\\ &= \left(-i\left[G_R\right]+i\left[G_A\right]\right)(F,\gamma^{o}\bar{G}) = 0\end{aligned} \tag{2.44}$$

where we used (2.28) and I (2.21), (2.27), Th.3.2.

To prove (2.41) we note that

$$(\overline{T_R F})(x) = \bar{F}(x) - \gamma^{o}B(x)\int dy S_A(x-y)\gamma^{o}\bar{F}(y) \tag{2.45}$$

$$= \bar{F}(x) + iV(x)\left[\int_{t}^{\infty} dt' \exp(-i\check{H}_{o}(t-t'))\check{\bar{F}}(t',\cdot)\right](\vec{x}) \tag{2.46}$$

where we used the well-known relation

$$S_R^{*}(y-x) = \gamma^{o}S_A(x-y)\gamma^{o} \tag{2.47}$$

and I Th.3.2 (for $V(x) = 0$). Thus,

$$\int dt\check{U}^*(t,-\infty)\exp(i\check{H}_o t)(\overline{T_R\check{F}})(t,\cdot) = \int dt\check{U}^*(t,-\infty)\exp(i\check{H}_o t)\check{\bar{F}}(t,\cdot)$$

$$+i\sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty}dt\int_{-\infty}^{t}dt_1..\int_{-\infty}^{t_{n-1}}dt_n\check{O}(t_n)..\check{O}(t_1)\check{O}(t)\int_t^{\infty}dt'\exp(i\check{H}_o t')\check{\bar{F}}(t',\cdot)$$

$$= \int dt'\check{U}^*(t',-\infty)\exp(i\check{H}_o t')\check{\bar{F}}(t',\cdot)+i\sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty}dt'\int_{-\infty}^{t'}dt\int_{-\infty}^{t}dt_1..$$

$$\int_{-\infty}^{t_{n-1}}dt_n\check{O}(t_n)..\check{O}(t_1)\check{O}(t)\exp(i\check{H}_o t')\check{\bar{F}}(t',\cdot)$$

$$= \int dt\ \exp(i\check{H}_o t)\check{\bar{F}}(t,\cdot). \tag{2.48}$$

Hence, (2.41) holds. Similarly,

$$\int dt\check{U}^*(t,-\infty)\exp(i\check{H}_o t)(\overline{T_A\check{F}})(t,\cdot) = \int dt'\check{U}^*(t',-\infty)\exp(i\check{H}_o t')\check{\bar{F}}(t',\cdot)$$

$$-i\sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty}dt'\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{t'}\right)dt\int_{-\infty}^{t}dt_1..\int_{-\infty}^{t_{n-1}}dt_n\check{O}(t_n)..\check{O}(t_1)\check{O}(t)\exp(i\check{H}_o t')\check{\bar{F}}(t',\cdot)$$

$$= \int dt\check{U}^*(\infty,-\infty)\exp(i\check{H}_o t)\check{\bar{F}}(t,\cdot). \tag{2.49}$$

Thus, (2.42) holds. ∎

We note that, if $V(x) \neq 0$, ψ^{in} and ψ^{int} can not be unitarily equivalent. (If they were, one would have

$$[\psi^{int}(F),\psi^{int}(G)^*]_+ = [\psi^{in}(F),\psi^{in}(G)^*]_+ \qquad \forall F,G\in S(R^4)^4 \tag{2.50}$$

so

$$U(t,t') = 1 \qquad \forall t,t'\in R^2 \tag{2.51}$$

from which it follows by I (2.22) that $V(x) = 0$.) However, this does not imply that ψ_{in} and $\psi_{int,t}$ are unitarily inequivalent.

In contrast, if $\mathcal{S}$ is a unitary operator on $\mathcal{F}_a$ such that

$$\psi^{out}(F) = \mathcal{S}^*\psi^{in}(F)\mathcal{S} \qquad \forall F\in S(R^4)^4 \tag{2.52}$$

then also

$$\psi_{out}(f) = \mathcal{S}^*\psi_{in}(f)\mathcal{S} \qquad \forall f\in\check{\mathcal{H}} \tag{2.53}$$

and vice versa. This is an easy consequence of (2.36), (2.31) and (2.33). Evidently, existence of the Fock space S-operator is in turn equivalent to the implementability of the field operator transformation

$$\Phi(v) \to \Phi(S^*v). \tag{2.54}$$

Following Capri (18) and Wightman (19) one can define, provided that T_R,T_A are bijections of $S(R^4)^4$, an out field by

$$\psi_{cw}^{out}(F) = \psi^{in}(T_R^{-1}T_A F) \qquad \forall F \in S(R^4)^4. \tag{2.55}$$

It easily follows from I Th.A.1 that this condition is satisfied (use (2.45) and its analogue for T_A and observe that $\bar{\Delta}_I = \Delta_I$, $I = R,A$). Using (2.41-42) we now conclude

$$\psi_{cw}^{out}(F) = \psi^{out}(F). \tag{2.56}$$

Thus,

<u>Theorem 2.2</u>. The Capri-Wightman approach and the Friedrichs-Segal approach lead to the same S-operator. $\blacksquare$

Using (2.16) and I (3.46) one verifies that

$$\mathcal{U}(a,\Lambda)\psi^{in}(F)\mathcal{U}^*(a,\Lambda) = \psi^{in}(F^{a,\Lambda}) \tag{2.57}$$

where

$$F^{a,\Lambda}(x) \equiv S(\Lambda^{-1})^T F(\Lambda^{-1}(x-a)). \tag{2.58}$$

Hence, ψ^{in} satisfies the Wightman axioms (27).

We finally observe that if one chooses a different representation of the γ-algebra, $\{\gamma^{\mu'}\}$, and proceeds in the same way as we have done (defining, e.g.,

$$W' = MW \tag{2.59}$$

where M is the unitary matrix which connects the representations), then the resulting field $\psi^{in'}$ is not unitarily equivalent with ψ^{in}. Indeed, if it were, (2.50) should hold with $\psi^{int} \to \psi^{in'}$. However, $S(x-y)\gamma^o$ and $S'(x-y)\gamma^{o'}$ are different distributions. Even the Wightman axioms do not determine the free field up to unitary equivalence. Indeed, there exist different representations of the γ-algebra having the same $\{S(A)\}$, since the $\{S(A)\}$ have a non-trivial commutant.

B. <u>The evolution operator and S-operator in $\mathcal{F}_a(\mathcal{K})$.</u>

We now assume that only the timelike component of the vector field (i.e. the electric field) and/or the pseudovector field are non-zero. It then follows from ITh.3.3 that the hypothesis of ITh.2.8 is satisfied, and from

ICor.3.4 that $U_\lambda(T_2,T_1)$ is implementable in $\mathcal{F}_a$ for any $(\lambda,T_2,T_1)\in R\times\tilde{R}^2$. Denoting the resulting three-parameter family of unitary operators by $\mathcal{U}_\lambda(T_2,T_1)$, it follows in particular that

$$\psi_{int,t}(f) = \mathcal{U}_1^*(t,-\infty)\psi_{in}(f)\mathcal{U}_1(t,-\infty) \qquad \forall f\in\check{\mathcal{H}} \qquad \forall t\in\tilde{R}. \tag{2.60}$$

If (λ,T_2,T_1) is such that

$$(\Omega,\mathcal{U}_\lambda(T_2,T_1)\Omega) \neq 0 \tag{2.61}$$

then we normalize $\mathcal{U}$ by requiring

$$(\Omega,\mathcal{U}\Omega) > 0. \tag{2.62}$$

In the next theorem the set $E(T_2,T_1)$ is defined in ITh.2.8, and the operator Λ by I(2.57).

<u>Theorem 2.3</u>. (i) For any $(\lambda,T_2,T_1)\in R\setminus E(T_2,T_1)\times\tilde{R}^2$ (2.61) holds true. For these values of the arguments one has for any $\phi\in D$:

$$\mathcal{U}\phi = \det(1_{--}+\Lambda_{+-}{}^*\Lambda_{+-})^{-\frac{1}{2}}:\exp(\Lambda_{+-}a^*b^*+\Lambda_{++}a^*a+\Lambda_{--}bb^*+\Lambda_{-+}ba):\phi. \tag{2.63}$$

(ii) $\mathcal{U}_\lambda(T_2,T_1)$ is strongly continuous on $\tilde{R}^2$ for any $\lambda\in(-\ell,\ell)$, and on $R\setminus E(T_2,T_1)$ for any $(T_2,T_1)\in\tilde{R}^2$.

(iii) On $(-\ell,\ell)\times\tilde{R}^2$:

$$\mathcal{U}_\lambda(T,T) = 1$$
$$\mathcal{U}_\lambda(T_3,T_2)\mathcal{U}_\lambda(T_2,T_1) = \exp(i\chi(\lambda,T_3,T_2,T_1))\mathcal{U}_\lambda(T_3,T_1) \tag{2.64}$$

where χ is a real-valued function.

(iv) For any $(T_2,T_1)\in\tilde{R}^2$ and $\phi\in D$ the vector-valued function $\mathcal{U}_\lambda(T_2,T_1)\phi$ on $(-\ell,\ell)$ has a unique analytic continuation to $D_{\ell_{E(T_2,T_1)}}$ where

$$\ell_E \equiv \mathrm{dist}(E,\{0\}) \tag{2.65}$$

and a, possibly two-valued, analytic continuation to $\mathbb{C}\setminus E(T_2,T_1)$.

<u>Proof</u>. It follows from B that (2.61) is equivalent to existence of a uniquely determined bounded operator Λ on $\mathcal{H}$, satisfying

$$(U-1)-\Lambda-(U-1)P_-\Lambda = (U-1)-\Lambda-\Lambda P_-(U-1) = 0. \tag{2.66}$$

Comparing (2.66) and I(2.60) we conclude that the first statement of (i) holds true. The second one then follows from (2.62) and B(4.18).

Since $\mathcal{U}_\lambda(T_2,T_1)$ is unitary its strong continuity will follow if we prove that the function $M(\lambda,T_2,T_1)$, defined by

$$M(\lambda,T_2,T_1) = \Big(\prod_{i=1}^{n} a^*(f_i)\prod_{j=1}^{r} b^*(\overline{g_j})\Omega, \mathcal{U}_\lambda(T_2,T_1)\prod_{i=1}^{n'} a^*(f_i')\prod_{j=1}^{r'} b^*(\overline{g_j'})\Omega\Big)$$

$$n,r,n',r'\in\mathbb{N} \qquad f_i,f_i'\in\mathcal{K}_+ \qquad g_j,g_j'\in\mathcal{K}_- \tag{2.67}$$

is continuous on $\tilde{\mathbb{R}}^2$ for any $\lambda\in(-\ell,\ell)$ and on $\mathbb{R}\setminus E(T_2,T_1)$ for any $(T_2,T_1)\in\tilde{\mathbb{R}}^2$. Using (2.63) and the CAR one easily sees that $M(\lambda,T_2,T_1)$ is equal to a finite sum of terms, each of which is the product of (+ or −)$\det(..)^{-\frac{1}{2}}$ and a finite number of terms of the form $(f_i,\Lambda_{+-}g_j)$, $(f_i,\Lambda_{++}f_j')$, $(g_i',\Lambda_{--}g_j)$, $(g_i',\Lambda_{-+}f_j')$, (f_i,f_j') or (g_i',g_j). Since $\det(1+\cdot)$ is a continuous function on the trace class (28) we conclude from ITh.2.8 that $\det(\cdot\cdot)^{-\frac{1}{2}}$ has the required continuity properties. The same conclusion for the remaining terms follows from the norm continuity of $\Lambda_\lambda(T_2,T_1)$ in λ and (T_2,T_1). Thus, M has the required properties.

(2.64) is an easy consequence of I(2.21) and the irreducibility of the field operators.

To prove (iv) we first observe that

$$E(T_2,T_1) = \bar{E}(T_2,T_1) \qquad \forall(T_2,T_1)\in\tilde{\mathbb{R}}^2. \tag{2.68}$$

Indeed, $U_{\lambda--}$ is singular if and only if $V_{\bar{\lambda}--}$ is, since

$$V_{\bar{\lambda}--} = U_{\lambda--}^* \qquad \forall\lambda\in\mathbb{C}. \tag{2.69}$$

(To see this, use I(2.39) and the unitarity of U_λ for $\lambda\in\mathbb{R}$.) We then note that on $(-\ell,\ell)\times\tilde{\mathbb{R}}^2$ (cf.B(3.2))

$$1_{--}+\Lambda_{+-}^*\Lambda_{+-} = U_{--}^{*\,-1}U_{--}^{-1} \tag{2.70}$$

so

$$(1_{--}+\Lambda_{+-}^*\Lambda_{+-})(1_{--}-U_{-+}U_{-+}^*) = 1_{--}. \tag{2.71}$$

It follows from ITh.2.8 that (2.70-71) can be continued to $\mathbb{C}\setminus E$ and that $\Lambda_{\bar{\lambda}+-}^*\Lambda_{\lambda+-}$ and $U_{\lambda-+}U_{\bar{\lambda}-+}^*$ are $||\cdot||_1$-analytic functions in $\mathbb{C}\setminus E$ resp. $\mathbb{C}$. Hence (28),

$$g(\lambda) \equiv \det(1_{--}+\Lambda_{\bar{\lambda}+-}^*\Lambda_{\lambda+-})^{-1} = \det(1_{--}-U_{\lambda-+}U_{\bar{\lambda}-+}^*) \tag{2.72}$$

is an entire function which only vanishes in the points of E. Denoting its positive square root on $(-\ell,\ell)$ by $v(\lambda)$ it follows from the monodromy theorem that $v(\lambda)$ has a unique analytic continuation to D_{ℓ_E}. Clearly, $v(\lambda)$ can be analytically continued to an, in general two-valued, function on $\mathbb{C}\setminus E$. We assume first that $v(\lambda)$ can be continued to an entire function.

We define for any $(T_2,T_1)\in\tilde{\mathbb{R}}^2$, $\lambda\in\mathbb{C}\setminus E(T_2,T_1)$ and ϕ of the form

$$\phi = \prod_{i=1}^{n} a^*(f_i) \prod_{j=1}^{r} b^*(\overline{g_j})\Omega \qquad n,r \in N \qquad f_i \in \mathcal{H}_+ \qquad g_j \in \mathcal{H}_-, \tag{2.73}$$

$$\mathcal{U}_\lambda \phi \equiv v(\lambda) \prod_{i=1}^{n} (a^*(U_{\lambda++} f_i) + b(\overline{U_{\lambda-+} f_i})) \prod_{j=1}^{r} (b^*(\overline{U_{\lambda--} g_j}) + a(U_{\lambda+-} g_j)) \\ . \exp(\Lambda_{\lambda+-} a^* b^*)\Omega. \tag{2.74}$$

The products in (2.73-74) are by definition in the natural order of the indices. It follows from B that the r.h.s. of (2.74) belongs to $\mathcal{F}_a$ and that it is equal to $\mathcal{U}_\lambda \phi$ if $\lambda \in (-\ell, \ell)$. Choosing $\lambda_o \in \mathbb{C} \setminus E$ and an open ball O_{λ_o} with center λ_o such that $\overline{O_{\lambda_o}} \cap E = \emptyset$, it follows from I(2.13) and B(3.47) that $||\mathcal{U}_\lambda \phi||$ is bounded on $\overline{O_{\lambda_o}}$. Hence, $\mathcal{U}_\lambda \phi$ is analytic in λ_o if $(\psi, \mathcal{U}_\lambda \phi)$ is analytic in O_{λ_o} for any $\psi \in D$. However, this follows from the analyticity of $v(\lambda)$, U_λ and $\Lambda_{\lambda+-}$ in $\mathbb{C} \setminus E$ by the same argument that we used to prove the continuity properties of $\mathcal{U}$.

It is clear from the above that (iv) holds as well if some points of E are branch points of $v(\lambda)$. In this case the continuation to $\mathbb{C} \setminus E$ is two-valued. ∎

Using results recently obtained by Bellissard (20,21) and Palmer (17) we shall now study the Fock space S-operator, which corresponds to a function V(x) as considered in subsection A, multiplied by a real coupling constant λ. It is by definition the unitary operator $\mathcal{S}_\lambda$ satisfying

$$\phi(S_\lambda^* v) = \mathcal{S}_\lambda^* \phi(v) \mathcal{S}_\lambda \qquad \forall v \in \mathcal{H} \tag{2.75}$$

(if such an operator exists, cf.B). We denote by ℓ_s the supremum of the numbers $\alpha > 0$ such that $S_{\lambda\pm\mp}$ are $||.||_2$-analytic in D_α. It follows from (20, Lemmas A 5.4, A 5.6), using the Neumann series argument of (21, Th.II 1.1), that $\ell_s > 0$. Since each term of their perturbation series is H.S. and the series converges in norm, $S_{\lambda\pm\mp}$ are compact for any $\lambda \in \mathbb{C}$. Thus, arguing as in the proof of ITh.2.8, it follows that $E \equiv E(\infty, -\infty)$ is a discrete set outside D_ℓ, and that $\Lambda_\lambda \equiv F_\lambda(\infty, -\infty)$ can be continued to $\mathbb{C} \setminus E$. It moreover follows from (17) that $S_{\lambda\pm\mp}$ are H.S. for any $\lambda \in \mathbb{R}$, so $\mathcal{S}_\lambda$ exists for any $\lambda \in \mathbb{R}$ and $\Lambda_{\lambda\pm\mp}$ are H.S. for any $\lambda \in \mathbb{R} \setminus E$. If $\lambda \in \mathbb{R}$ is such that

$$(\Omega, \mathcal{S}_\lambda \Omega) \neq 0 \tag{2.76}$$

then we require

$$(\Omega, \mathcal{S}_\lambda \Omega) > 0. \tag{2.77}$$

Setting

$$\ell_c \equiv \min(\ell_E,\ell_s) \tag{2.78}$$

we have

<u>Theorem 2.4</u>. (i) For any $\lambda\in R\backslash E$ (2.76) holds true. For these λ one has for any $\phi\in D$:

$$\mathcal{S}\phi = \det(1_{--}+\Lambda_{+-}{}^{*}\Lambda_{+-})^{-\frac{1}{2}}:\exp(\Lambda_{+-}a^{*}b^{*}+\Lambda_{++}a^{*}a+\Lambda_{--}bb^{*}+\Lambda_{-+}ba):\phi. \tag{2.79}$$

(ii) $\mathcal{S}_\lambda$ is strongly continuous on $R\backslash E$.

(iii) For any $\phi\in D$ the vector-valued function $\mathcal{S}_\lambda\phi$ on $(-\ell_c,\ell_c)$ has a unique analytic continuation to D_{ℓ_c}; if $\ell_s > \ell_E$ it has a, possibly two-valued, analytic continuation to $D_{\ell_s}\backslash E^c$.

(iv) For any $\lambda\in R\backslash E$ $\mathcal{S}_\lambda$ is causal, up to a phase factor, and Lorentz covariant; for any $\lambda\in R\cap E$ $\mathcal{S}_\lambda$ is causal and Lorentz covariant, up to a phase factor.

<u>Proof</u>. In view of the above it suffices to prove (iv). However, this statement is an easy consequence of ITh.3.1. $\blacksquare$

We remark that, by IThs.3.3,3.1, $\ell_s = \infty$ for any $V(x)$ which, in some inertial frame, is equal to the sum of an electric and a "pseudo-electric" field. Thus, $\ell_c = \ell_E$ for these V. We further note that if the vacuum-to-vacuum transition amplitude $(\Omega,\mathcal{S}_\lambda\Omega) = 0$, i.e. if $\lambda\in R\cap E$, then $\mathcal{S}_\lambda\phi$ is given by the r.h.s. of B(5.15), with $U \to S_\lambda$ (up to a phase factor). Finally, we mention that Schwinger (24) formally obtained the expression (2.79) for $\mathcal{S}_\lambda$ in the case of an electromagnetic field.

C. <u>The connection with the Feynman-Dyson series</u>.

According to Th.2.4 $\mathcal{S}_\lambda\phi$ can, for any $\phi\in D$, be expanded in a power series, the convergence radius of which is greater than of equal to ℓ_c. In this subsection we will derive explicit expressions for the expansions of

$$v(\lambda) \equiv (\Omega,\mathcal{S}_\lambda\Omega) \qquad\qquad \lambda\in D_{\ell_c} \tag{2.80}$$

and of

$$\mathcal{R}_\lambda\phi \equiv S_\lambda\phi/v(\lambda) \qquad \phi\in D \qquad \lambda\in D_{\ell_c} \tag{2.81}$$

and compare the result with the expressions which one obtains from the formal Feynman-Dyson (F.D.) series for the Fock space S-operator (25,26).

We set for any $\lambda\in D_{\ell_c}$ (cf.B § 4)

$$M_\lambda \equiv \Lambda_{\lambda+-}a^*b^*+\Lambda_{\lambda++}a^*a+\Lambda_{\lambda--}bb^*+\Lambda_{\lambda-+}ba. \tag{2.82}$$

From I § 3B we have, explicitly exhibiting the spin indices,

$$\Lambda^{ii'}_{\lambda\varepsilon\varepsilon'}(\vec{p},\vec{q}) = \sum_{n=1}^{\infty}\lambda^n\Lambda^{(n)ii'}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) \tag{2.83}$$

where

$$\Lambda^{(n)ii'}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) = 2\pi i\int dk_1..dk_{n-1}\left(\frac{m}{E_p}\right)^{\frac{1}{2}}\tilde{w}^i_\varepsilon(\vec{p})\tilde{B}(\varepsilon p-k_1)\tilde{S}_F(k_1)\tilde{B}(k_1-k_2)$$
$$..\tilde{S}_F(k_{n-1})\tilde{B}(k_{n-1}-\varepsilon'q)w^{i'}_{\varepsilon'}(\vec{q})\left(\frac{m}{E_q}\right)^{\frac{1}{2}}. \tag{2.84}$$

Clearly,

$$:M_\lambda:\phi = \underset{N\to\infty}{\mathrm{s.lim}}\sum_{n=1}^{N}\lambda^n:M^{(n)}:\phi \qquad \forall\phi\in D \tag{2.85}$$

where

$$M^{(n)} \equiv \Lambda^{(n)}_{+-}a^*b^*+\Lambda^{(n)}_{++}a^*a+\Lambda^{(n)}_{--}bb^*+\Lambda^{(n)}_{-+}ba. \tag{2.86}$$

In (2.86) we again suppressed the spin indices. It follows from (2.79) (cf. B § 4) that

$$\mathcal{R}_\lambda\phi = \underset{N\to\infty}{\mathrm{s.lim}}\sum_{L=0}^{N}\frac{1}{L!}:M_\lambda{}^L:\phi \qquad \forall\phi\in D \tag{2.87}$$

where, explicitly,

$$:M_\lambda{}^L: = \sum_{\substack{i,j,k,\ell=0\\ i+j+k+\ell=L}}^{L}\frac{L!}{i!j!k!\ell!}\Lambda^i_{\lambda+-}\Lambda^j_{\lambda++}(-\Lambda^T_{\lambda--})^k\Lambda^\ell_{\lambda-+}$$
$$.a^{*i}b^{*i}a^{*j}b^{*k}b^kb^\ell a^\ell a^j. \tag{2.88}$$

Using arguments familiar by now we conclude that $:M_\lambda{}^L:\phi$ is analytic in D_{ℓ_c} for any $\phi\in D$. In fact, one easily sees that

$$:M_\lambda{}^L:\phi = \underset{N\to\infty}{\mathrm{s.lim}}\sum_{n=L}^{L}\lambda^n:M^{(n,L)}:\phi \qquad L\geq 1 \tag{2.89}$$

where

$$M^{(n,L)} \equiv \sum_{\substack{j_1,\dots,j_L=1\\ j_1+..+j_L=n}}^{n-L+1}\prod_{\ell=1}^{L}M^{(j_\ell)} \qquad n\geq L\geq 1. \tag{2.90}$$

Thus,

$$\mathcal{R}_\lambda\phi = \phi+\sum_{L=1}^{\infty}\frac{1}{L!}\sum_{n=L}^{\infty}\lambda^n:M^{(n,L)}:\phi \qquad \forall\phi\in D. \tag{2.91}$$

We now have

Theorem 2.5. For any $\phi \in D$ and $\lambda \in D_{\ell_c}$:

$$\mathcal{R}_\lambda \phi = \underset{N\to\infty}{\text{s.lim}} \sum_{n=o}^{N} \lambda^n \mathcal{R}^{(n)} \phi \tag{2.92}$$

where

$$\begin{aligned} \mathcal{R}^{(o)} &\equiv 1 \\ \mathcal{R}^{(n)} &\equiv \sum_{L=1}^{n} \frac{1}{L!} :M^{(n,L)}: \qquad\qquad n \geq 1. \end{aligned} \tag{2.93}$$

Proof. Since $\mathcal{R}_\lambda \phi$ is analytic it suffices to show that

$$\sum_{n=o}^{\infty} \lambda^n (\psi, \mathcal{R}^{(n)} \phi) = (\psi, \mathcal{R}_\lambda \phi) \qquad\qquad \forall \psi \in D. \tag{2.94}$$

However, as $\phi, \psi \in D$, there exists a $K < \infty$ such that

$$(\psi, \mathcal{R}_\lambda \phi) = (\psi, \phi) + \sum_{L=1}^{K} \frac{1}{L!} \sum_{n=L}^{\infty} \lambda^n (\psi, :M^{(n,L)}: \phi) \tag{2.95}$$

where we used (2.91). Hence, (2.94) follows. $\blacksquare$

With our conventions for the Dirac equation and the field operators (cf. (2.23) and (2.6)) the F.D. S-operator is given by

$$S_\lambda^{F.D.} = T(\exp(i \int dx \, \mathcal{L}_I(x))) \tag{2.96}$$

where

$$\mathcal{L}_I(x) \equiv \lambda :\tilde{\psi}^o(x) B(x) \psi^o(x):. \tag{2.97}$$

Expanding the exponential and using Wick's theorem the fully contracted factors sum up to the well-known multiplicative divergent factor $(\Omega, S_\lambda^{F.D.} \Omega)$ by the usual arguments (25); omitting this factor one obtains $\mathcal{R}_\lambda^{F.D.}$. We denote the term of the coefficient of λ^n in its expansion ($n \geq 1$) which has L uncontracted ψ and $\tilde{\psi}$ ($1 \leq L \leq n$) by $(L!)^{-1} M_{F.D.}^{(n,L)}$. Using the relation (cf. (2.6) and I § 3A)

$$(\Omega, T(\psi_\alpha^o(x) \tilde{\psi}_\beta^o(y))\Omega) = -iS_F(x-y)_{\alpha\beta} \tag{2.98}$$

and a combinatorial argument it then follows that

$$\begin{aligned} M_{F.D.}^{(n,L)} = i^L \sum_{\substack{j_1,\ldots,j_L=1 \\ j_1+\ldots+j_L=n}}^{n-L+1} \int dx_1 \ldots dx_n : &\left[\tilde{\psi}^o(x_1) B(x_1) S_F(x_1-x_2) B(x_2) \ldots S_F(x_{j_1-1}-x_{j_1})\right. \\ &\left. B(x_{j_1}) \psi^o(x_{j_1})\right] \ldots \left[\tilde{\psi}^o(x_{j_1+\ldots+j_{L-1}+1}) B(x_{j_1+\ldots+j_{L-1}+1}) \ldots B(x_n) \psi^o(x_n)\right] :. \end{aligned} \tag{2.99}$$

However, from (2.6), (2.84) and (2.86) it easily follows that

$$i\int dx_1 \ldots dx_\ell \tilde{\psi}^o(x_1) B(x_1) S_F(x_1 - x_2) \ldots B(x_\ell) \psi^o(x_\ell) = M^{(\ell)}, \tag{2.100}$$

Thus, using (2.90),

$$M^{(n,L)}_{F.D.} = :M^{(n,L)}: \tag{2.101}$$

so

$$\mathcal{R}_\lambda^{F.D.} = \mathcal{R}_\lambda . \tag{2.102}$$

Of course, the r.h.s. of (2.99) is a priori not defined in a rigorous mathematical sense. What we have shown by the formal calculations leading from (2.96) to (2.102) is that expressions like (2.99) can be associated in a quite natural way with well-defined operators mapping the subspace of physical vectors D into D_f and that, with this association, the relative F.D. S-operator acts in the same way on physical vectors as $\mathcal{R}_\lambda$. Thus, the formal relative S-matrix elements, given by

$$\begin{aligned}
&\mathcal{R}_\lambda^{F.D.}(\vec{p}_1,\ldots,\vec{p}_n,\vec{q}_1,\ldots,\vec{q}_r;\vec{p}_1{}',\ldots,\vec{p}_{n'}{}',\vec{q}_1{}',\ldots,\vec{q}_{r'}{}') \\
&=\Big(\prod_{i=1}^{n} a^*(\vec{p}_i)\prod_{j=1}^{r} b^*(\vec{q}_j)\Omega, \mathcal{R}_\lambda^{F.D.}\prod_{i=1}^{n'} a^*(\vec{p}_i{}')\prod_{j=1}^{r'} b^*(\vec{q}_j{}')\Omega\Big)
\end{aligned} \tag{2.103}$$

$(n,r,n',r' = 0,1,\ldots)$, are equal to their rigorous counterparts, viz. the tempered distributions $\mathcal{R}_\lambda(\vec{p}_1,\ldots,\vec{p}_n,\vec{q}_1,\ldots,\vec{q}_r;\vec{p}_1{}',\ldots,\vec{p}_{n'}{}',\vec{q}_1{}',\ldots,\vec{q}_{r'}{}')$ which are defined by the requirement

$$\begin{aligned}
&\mathcal{R}_\lambda(f_1,\ldots,f_n,g_1,\ldots,g_r;f_1',\ldots,f_{n'}',g_1',\ldots,g_{r'}') \\
&= \Big(\prod_{i=1}^{n} a^*(\overline{f_i})\prod_{j=1}^{r} b^*(\overline{g_j})\Omega, \mathcal{R}_\lambda\prod_{i=1}^{n'} a^*(f_i')\prod_{j=1}^{r'} b^*(g_j')\Omega\Big) \qquad \forall f_i,g_j,f_i',g_j' \in S(R^3)^2
\end{aligned} \tag{2.104}$$

We further note that the $\Lambda^{(n)ii'}_{\varepsilon\varepsilon'}(\vec{p},\vec{q})$ (which together with the δ-function are the constituents of the terms of the perturbation series of any relative S-matrix element) are functions in $S(R^3)$ in $\vec{p}$ and $\vec{q}$ separately (cf. I § 3B) whereas the complete amplitudes $\Lambda^{ii'}_{\lambda\varepsilon\varepsilon'}(\vec{p},\vec{q})$ are tempered distributions which are not necessarily functions.

We shall now show that the coefficients of the expansion of $v(\lambda)$ easily follow from the $L^2(R^6)$ functions $\Lambda^{(n)ii'}_{+-}(\vec{p},\vec{q})$. Indeed, denoting by $[x]$ the greatest integer less than or equal to x, we have

<u>Theorem 2.6</u>. For any $\lambda \in D_{\ell_{\ddot{c}}}$:

$$v(\lambda) = \sum_{n=o}^{\infty} d_n \lambda^n \tag{2.105}$$

where

$$d_o \equiv 1 \qquad\qquad d_1 \equiv o$$
$$nd_n \equiv \sum_{k=2}^{n} ka_k d_{n-k} \qquad\qquad n \geq 2 \tag{2.106}$$

and

$$a_k \equiv \sum_{n=1}^{[\frac{1}{2}k]} \frac{(-)^n}{2n} \sum_{\substack{i_1,\ldots,j_n=1 \\ i_1+\ldots+j_n=k}}^{k-2n+1} \mathrm{Tr}\Lambda^{(i_1)}_{+-}\Lambda^{*(j_1)}_{+-}\ldots\Lambda^{(i_n)}_{+-}\Lambda^{*(j_n)}_{+-} \qquad k\geq 2. \tag{2.107}$$

<u>Proof</u>. Since $\Lambda_{\lambda+-}$ is $||.||_2$-analytic in D_{ℓ_c} and $\Lambda_{o+-}=0$ we have, if $|\lambda|$ is small enough (28),

$$v(\lambda) = \det(1_{--}+\Lambda^{*}_{\bar\lambda+-}\Lambda_{\lambda+-})^{-\frac{1}{2}} = \exp(\tfrac{1}{2}\sum_{n=1}^{\infty}(-)^n n^{-1}\sigma_n(\lambda)) \equiv \exp(f(\lambda)) \tag{2.108}$$

where

$$\sigma_n(\lambda) \equiv \mathrm{Tr}(\Lambda^{*}_{\bar\lambda+-}\Lambda_{\lambda+-})^n. \tag{2.109}$$

Clearly, $f(\lambda)$ is analytic in D_{ℓ_c} and

$$f(\lambda) = \sum_{k=2}^{\infty} a_k\lambda^k. \tag{2.110}$$

Thus, differentiating the identity

$$v(\lambda) = \exp(\sum_{k=2}^{\infty} a_k\lambda^k) \tag{2.111}$$

at both sides and equating coefficients, (2.106) follows. ∎

The F.D. analogue of a_k is given by

$$a_k^{F.D.} = -k^{-1}\int dx_1\ldots dx_k \mathrm{Tr}B(x_1)S_F(x_1-x_2)B(x_2)\ldots S_F(x_k-x_1) \qquad k\geq 2. \tag{2.112}$$

Formally Fourier transforming to time-momentum variables the integrand becomes a measurable function, since $\hat{S}_F(t,\vec{p})$ is. However, one easily sees that the integral is not absolutely convergent for k=2. For higher k it presumably does not converge either, but this is difficult to prove. Transforming to energy-momentum variables and replacing $\tilde{S}_F$ by S_F^{δ} with $\delta>o$ (cf.I(3.19)) it might be convergent for $k\geq 5$, but it is not clear whether the limit $\delta\downarrow o$ then exists. If it does, one could probably obtain any other number by choosing a different sequence of functions approximating $\tilde{S}_F$ (in the sense of distributions).We also note that two renormalizations (in the sense of Hepp (29)) of the undefined product of the S_F in (2.112) differ in general by a finite renormalization which gives a non-zero contribution.

The same remarks apply to the real part of $a_k^{F.D.}$, given by

$$\mathrm{Re}\,a_k^{F.D.} = -(2k)^{-1}\int dx_1..dx_k \mathrm{Tr}\Big[B(x_1)S_F(x_1-x_2)..S_F(x_k-x_1)+B(x_1)S_{\bar F}(x_1-x_2)$$
$$..S_{\bar F}(x_k-x_1)\Big], \tag{2.113}$$

which of course is the "observable" part, since the imaginary part only gives rise to a phase factor. In particular, we have not been able to show that one obtains a_k if one evaluates Re $a_k^{F.D.}$ by the usual Feynman techniques.

However, it should be realized that in view of (2.102) one ought to require

$$\left|(\Omega,\mathcal{S}_\lambda^{F.D.}\Omega)\right| = v(\lambda) \qquad \forall\lambda\in(-\ell_c,\ell_c) \tag{2.114}$$

if $\mathcal{S}_\lambda^{F.D.}$ is to correspond to a unitary operator on Fock space. Since $v(\lambda)$ is analytic this requirement can only be satisfied if

$$\mathrm{Re}\;a_k^{F.D.} = a_k \qquad \forall k \geq 2. \tag{2.115}$$

As noted above, we could not obtain a satisfactory definition of the r.h.s. of (2.113) which implies (2.115) or, equivalently, (2.114). We shall now show that, nevertheless, (2.114) can be formally derived.

Substituting

$$S_F = S_R + S_- \tag{2.116}$$

in (2.112) and multiplying through, the term without S_- drops out since the integrand is zero a.e.. The sum of the remaining terms can be written as

$$a_k^{F.D.} = -i\sum_{n=1}^{k} n^{-1} \sum_{\substack{i_1,..,i_n=1\\ i_1+..+i_n=k}}^{k-n+1} \int d\vec{x}dx_1..dx_k \mathrm{Tr}S_-(\vec{x}-x_1)B(x_1)S_R(x_1-x_2)$$
$$..B(x_{i_1})S_-(x_{i_1}-x_{i_1+1})B(x_{i_1+1})S_R(x_{i_1+1}-x_{i_1+2})..B(x_{k-i_n})S_-(x_{k-i_n}-x_{k-i_n+1})$$
$$B(x_{k-i_n+1})S_R(x_{k-i_n+1}-x_{k-i_n+2})..B(x_k)S_-(x_k-\vec{x})\gamma^0. \tag{2.117}$$

(To see this, first use (cf.I(3.28))

$$i\int d\vec{x}S_-(x_k-\vec{x})\gamma^0 S_-(\vec{x}-x_1) = S_-(x_k-x_1). \tag{2.118}$$

Observe then that two terms with the same n but with $\{i_1,..,i_n\}$ which differ by a cyclic permutation give the same contribution. A moment's reflection now shows that the number of such terms, multiplied by n^{-1}, equals the number of such terms in the expansion of (2.112), multiplied by k^{-1}. Thus, (2.117) follows.) We now note (cf.I § 3B) that the r.h.s. of (2.117) is formally equal to the coefficient of λ^k in the expansion of

$$\sum_{n=1}^{\infty}(-)^{n+1}n^{-1}\mathrm{Tr}\check{R}_{\lambda--}{}^{n}. \tag{2.119}$$

(This would be rigorously true if $R_{\lambda--}$ were $||.||_1$-analytic in a neighbourhood of the origin, which it is not in the present case.) Thus,

$$(\Omega, \mathfrak{S}_\lambda^{F.D.}\Omega) = \det(1_{--}+R_{\lambda--}) \tag{2.120}$$

so

$$\left|(\Omega, \mathfrak{S}_\lambda^{F.D.}\Omega)\right| = (\det(1_{--}+R_{\lambda--})\det(1_{--}-A_{\lambda--}))^{\frac{1}{2}} = \det(U_{\lambda--}U_{\bar{\lambda}--}^{*})^{\frac{1}{2}} = v(\lambda) \tag{2.121}$$

where we used I(2.55), some properties of infinite determinants (28) and (2.70).

Of course, this derivation is purely formal. Nevertheless, we have the following analogue of Furry's theorem, which closes this section.

<u>Theorem.2.7</u>. Let $V(t)$ be such that there exists a conjugation C, satisfying

$$CV(t) = V(t)C \qquad \forall t\in R \tag{2.122}$$

$$CH_o = -H_oC. \tag{2.123}$$

Then

$$a_{2n+1} = 0 \qquad \forall n\in N^{+}. \tag{2.124}$$

<u>Proof</u>. It follows from (2.123) by the functional calculus that

$$CP_\varepsilon = P_{-\varepsilon}C. \tag{2.125}$$

Thus, using I(2.8),

$$C\Lambda_\lambda C = -\Lambda_{-\lambda}^{*} \qquad \forall\lambda\in(-\ell_c,\ell_c). \tag{2.126}$$

Hence (cf.(2.109))

$$\sigma_n(\lambda) = \mathrm{Tr}(C(\Lambda_{\lambda+-}^{*}\Lambda_{\lambda+-})^nC) = \mathrm{Tr}(\Lambda_{-\lambda+-}\Lambda_{-\lambda+-}^{*})^n = \sigma_n(-\lambda)$$

$$\forall\lambda\in(-\ell_c,\ell_c). \tag{2.127}$$

In the last step we used the fact that $\Lambda_{+-}\Lambda_{+-}^{*}$ has the same eigenvalues, including multiplicities, as $\Lambda_{+-}^{*}\Lambda_{+-}$. From (2.108) and (2.127) it clearly follows that $f(\lambda)$ is even, so (2.124) holds true. $\blacksquare$

• The theorem holds in particular for electromagnetic fields, since the charge conjugation operator then satisfies (2.122-123).

3. THE QUANTIZED KLEIN-GORDON THEORY.

A. Field operators.

The basic field operators on $\mathcal{F}_s(\mathcal{H})$ are defined on $\tilde{D}$ by (2.1) (cf. B§2). The analogue of (2.2) is

$$[\phi(u),\phi^*(v)]_- = (u,qv) \qquad \forall u,v\in\mathcal{H} \tag{3.1}$$

where $\phi^*(v)$ is the restriction of $\phi(v)^*$ to $\tilde{D}$; (3.1) holds on D_∞.

One could now, in analogy to the spin-$\frac{1}{2}$ case, introduce a formal two-component Klein-Gordon field

$$\psi^o(x) \equiv (2\pi)^{-3/2}\int d\vec{p}(a(\vec{p})w_+(\vec{p})\exp(-ipx)+b^*(\vec{p})w_-(\vec{p})\exp(ipx)) \tag{3.2}$$

and an adjoint field

$$\tilde{\psi}^o(x) \equiv (2\pi)^{-3/2}\int d\vec{p}(a^*(\vec{p})w_+^*(\vec{p})\exp(ipx)-b(\vec{p})w_-^*(\vec{p})\exp(-ipx)) \tag{3.3}$$

(cf. I§4A). It can be seen that the interaction Lagrangean

$$\mathcal{L}_I(x) \equiv :\tilde{\psi}^o(x)B(x)\psi^o(x): \tag{3.4}$$

(cf. I(4.44)) leads to the same Feynman-Dyson S-matrix as the one obtained from the usual theory. However, since one of our main goals is to establish the relation of the rigorous theory with the customary formal theory, we shall not consider these fields any further.

The usual free fields are (25):

$$\begin{cases}\phi^o(x) \equiv (2\pi)^{-3/2}\int d\vec{p}(2E_p)^{-\frac{1}{2}}(a(\vec{p})\exp(-ipx)+b^*(\vec{p})\exp(ipx)) \\ \phi^{o*}(x) \equiv \phi^o(x)^*\end{cases} \tag{3.5}$$

$$\pi^o(x) \equiv \partial_t\phi^{o*}(x) \qquad \pi^{o*}(x) \equiv \partial_t\phi^o(x). \tag{3.6}$$

They satisfy the relations

$$(\Box+m^2)\phi^o(x) = 0 \tag{3.7}$$

$$\begin{cases}[\phi^o(t,\vec{x}),\pi^o(t,\vec{x}')]_- = i\delta(\vec{x}-\vec{x}') \\ [\phi^o(t,\vec{x}),\phi^{o*}(t,\vec{x}')]_- = [\phi^o(t,\vec{x}),\pi^{o*}(t,\vec{x}')]_- = [\pi^o(t,\vec{x}),\pi^{o*}(t,\vec{x}')]_- = 0\end{cases} \tag{3.8}$$

and the adjoint relations (i.e. the relations obtained by taking formal adjoints.

We define for any $f\in W_{\frac{1}{2}}(R^3)$

$$\phi_t^o(f) = \phi(\cdot\exp(iH_ot)|H_o|^{-1}W^{-1}f) \tag{3.9}$$

$$\pi_t^o(f) = \phi^*(i\exp(iH_o t)qW^{-1}f) \tag{3.10}$$

where, at the r.h.s.,

$$f \equiv \binom{f}{o}. \tag{3.11}$$

The operators $\phi_t^{o*}(f), \pi_t^{o*}(f)$ are defined in the obvious way. Clearly,

$$\phi_t^{o(*)}(f) = \left(\int d\vec{x}\bar{f}(\vec{x})\phi^o(t,\vec{x})\right)^{(*)} \tag{3.12}$$

$$\pi_t^{o(*)}(f) = \left(\int d\vec{x}f(\vec{x})\pi^o(t,\vec{x})\right)^{(*)}. \tag{3.13}$$

Moreover, for any $\psi \in D_f$,

$$\pi_t^o(f)\psi = \frac{d}{dt}\phi_t^{o*}(f)\psi \qquad \forall f\in W_{\frac{1}{2}}(R^3) \tag{3.14}$$

$$\frac{d^2}{dt^2}\phi_t^o(f)\psi = \phi_t^o((\Delta - m^2)f)\psi \qquad \forall f\in W_{\frac{5}{2}}(R^3). \tag{3.15}$$

The time differentiations are in the strong sense, and Δ acts in the sense of distributions. One also verifies, using (3.1), that on D_∞

$$\begin{aligned} &\left[\phi_t^o(f), \pi_t^o(g)\right]_- = i\int d\vec{x}\bar{f}(\vec{x})g(\vec{x}) \\ &\left[\phi_t^o(f), \phi_t^{o*}(g)\right]_- = \left[\phi_t^o(f), \pi_t^{o*}(g)\right]_- = \left[\pi_t^o(f), \pi_t^{o*}(g)\right]_- = 0 \end{aligned} \tag{3.16}$$

for any $f,g \in W_{\frac{1}{2}}(R^3)$. The relations (3.14-16) and their adjoints can be regarded as rigorous analogues of the relations (3.6-8) and their adjoints.

The analogue of (2.12) is

$$\psi_t^o(f) = \exp(iB_o t)\psi_o^o(f)\exp(-iB_o t) \qquad \psi = \pi, \phi \tag{3.17}$$

where B_o is the free Fock space Hamiltonian, defined by (2.13). It clearly has the same properties as in the Dirac case (in particular (2.14) holds true in the same sense). Similarly, the field operator transformation (2.15), where $U(a,\Lambda)$ is the representation of the Poincaré group in $\mathcal{H}$, given by I(4.55), is implemented by unitary operators $\mathcal{U}(a,\Lambda)$ given by (2.16), in which

$$(\hat{U}(a,\Lambda)v)_\varepsilon(\vec{p}) \equiv \exp(ipa)\left(\frac{(\Lambda^{-1}p)_o}{p_o}\right)^{\frac{1}{2}} v_\varepsilon(\overrightarrow{\Lambda^{-1}p}) \qquad \forall v\in\mathcal{H}. \tag{3.18}$$

Also, the gauge transformation (2.18) is implemented by $\exp(iQ\alpha)$, where Q is the charge operator (2.19). It follows from I(4.17) that (2.20-21) hold true as well, with

$$(C'v)_\varepsilon(\vec{p}) \equiv v_{-\varepsilon}(\vec{p}) \qquad \forall v\in\mathcal{H}. \tag{3.19}$$

Again, $\mathcal{C}$ is unitary while C is anti-unitary.

The perturbed Klein-Gordon fields should satisfy

$$(\Box + m^2 - K(x))\phi(x) = 0 \tag{3.20}$$

$$\pi(x) \equiv \partial_t\phi^*(x)+iA_o(x)\phi^*(x) \tag{3.21}$$

and (3.8) with the o's omitted, and the adjoint relations. (In (3.20)

$$K \equiv iA_\mu\partial^\mu+i\partial_\mu A^\mu+A_\mu A^\mu+A_4 \tag{3.22}$$

where A_ℓ $(\ell=0,..,4)$ are real-valued functions in $S(R^4)$.) Smearing with $\bar{f}(\vec{x})$ resp. $f(\vec{x})$ and partially integrating one obtains

$$\frac{d^2}{dt^2}\phi_t(f)+\frac{d}{dt}\phi_t(2iA_of)+\phi_t((-\Delta+m^2-K)f) = 0 \tag{3.23}$$

$$\pi_t(f) = \frac{d}{dt}\phi_t^*(f)+\phi_t^*(iA_of) \tag{3.24}$$

and (3.16) with the o's omitted, and the adjoint relations. We assert that for any $a\in\hat{R}$

$$\phi_{t,a}(f) \equiv \phi(U^*(t,a)\exp(iH_ot)|H_o|^{-1}\, W^{-1}f) \tag{3.25}$$

$$\pi_{t,a}(f) \equiv \phi^*(iU^*(t,a)\exp(iH_ot)qW^{-1}f) \tag{3.26}$$

and their adjoints are solutions to (3.23-24), commutation relations and adjoint relations in the sense specified before (cf. (3.14-16)). (In (3.25-26) U is the interaction picture evolution operator corresponding to $\{A_\ell\}_{\ell=o}^4$ (cf. I§4B).) Indeed, the commutation relations follow from (3.1) and the pseudo-unitarity of U while on $D(H_o)$, by I(2.23),

$$\frac{d}{dt}U^*(t,a)\exp(iH_ot) = \frac{d}{dt}qU(a,t)q\, \exp(iH_ot) = iU^*(t,a)\exp(iH_ot)qH(t)q. \tag{3.27}$$

It therefore remains to show

$$(-\check{q}\check{H}(t)^2\check{q}-i\check{q}\dot{\check{V}}(t)\check{q})|\check{H}_o|^{-1}\begin{pmatrix} f \\ 0\end{pmatrix}+2i|\check{H}_o|^{-1}\begin{pmatrix} \dot{A}_of \\ 0\end{pmatrix}-2\check{q}\check{H}(t)\check{q}|\check{H}_o|^{-1}\begin{pmatrix} A_of \\ 0\end{pmatrix}$$
$$+|\check{H}_o|^{-1}\begin{pmatrix} (-\Delta+m^2-K)f \\ 0\end{pmatrix} = 0 \tag{3.28}$$

$$i\check{q}\begin{pmatrix} f \\ 0\end{pmatrix} = i\check{q}\check{H}(t)\check{q}|\check{H}_o|^{-1}\begin{pmatrix} f \\ 0\end{pmatrix}+i|\check{H}_o|^{-1}\begin{pmatrix} A_of \\ 0\end{pmatrix}. \tag{3.29}$$

The verification of (3.28-29) is straightforward.

Clearly, on D_f,

$$\underset{t\to-\infty}{\text{s.lim}}\ \phi_{t,-\infty}(\exp(-i\check{H}_ot)f) = \phi(|H_o|^{-1}W^{-1}f) \tag{3.30}$$

$$\underset{t\to\infty}{\text{s.lim}}\ \phi_{t,-\infty}(\exp(-i\check{H}_ot)f) = \phi(S^*|H_o|^{-1}W^{-1}f). \tag{3.31}$$

Similar relations hold for $\pi_{t,-\infty}(f)$. Therefore we set

$$\begin{cases} \phi_{in}(f) \equiv \phi(|H_o|^{-1}W^{-1}f) \\ \pi_{in}(f) \equiv \phi^*(iqW^{-1}f) \end{cases} \tag{3.32}$$

$$\begin{cases} \phi_{int,t}(f) \equiv \phi(U^*(t,-\infty)|H_o|^{-1}W^{-1}f) \\ \pi_{int,t}(f) \equiv \phi^*(iU^*(t,-\infty)qW^{-1}f) \end{cases} \tag{3.33}$$

$$\begin{cases} \phi_{out}(f) \equiv \phi(S^*|H_o|^{-1}W^{-1}f) \\ \pi_{out}(f) \equiv \phi^*(iS^*qW^{-1}f). \end{cases} \tag{3.34}$$

The adjoints are defined in the obvious way. We note that, if f ranges over $W_{\frac{1}{2}}(R^3)$, the arguments of the ϕ and ϕ^* in e.g. (3.32) and its adjoint range over a dense subspace of $\mathcal{H}$, which ensures the irreducibility of the field operators. (Properly speaking, of e.g. the set of unitary operators $\exp(i(\phi_{in}(f)+\phi_{in}^*(f)))$, $\exp(i(\pi_{in}(f)+\pi_{in}^*(f)))$.)

We now define field operators smeared with test functions $F\in S(R^4)$ by, e.g. (cf. (3.32-34)),

$$\begin{cases} \phi^{in}(F) = \phi(\int dt\ \exp(iH_o t)|H_o|^{-1}W^{-1}\check{\bar{F}}(t,\cdot)) \\ \pi^{in}(F) = \phi^*(i\int dt\ \exp(iH_o t)qW^{-1}\check{\bar{F}}(t,\cdot)) \end{cases} \tag{3.35}$$

$$\phi^{int}(F) = \phi(\int dt\ U^*(t,-\infty)\exp(iH_o t)|H_o|^{-1}W^{-1}\check{\bar{F}}(t,\cdot)) \tag{3.36}$$

etc.; adjoints are defined by, e.g.,

$$\phi^{in*}(F) = \phi^{in}(\bar{F})^*\restriction\tilde{D}. \tag{3.37}$$

Thus these field operators depend linearly on F. Clearly, (2.34-35) hold with $\psi = \pi,\phi$ if the integral is interpreted as a strong Riemann integral on D_f. One easily verifies the relations

$$\phi^{in}(F) = \int dx F(x)\phi^o(x) \tag{3.38}$$

$$\phi^{ex}((\Box+m^2)F) = 0 \tag{3.39}$$

$$\pi^{ex}(F) = \phi^{ex*}(-\partial_t F) \tag{3.40}$$

$$\phi^{int}((\Box+m^2-\bar{K})F) = 0 \tag{3.41}$$

$$\pi^{int}(F) = \phi^{int*}((-\partial_t+iA_o)F). \tag{3.42}$$

(Use (3.27-29) to obtain (3.41-42).) Thus ϕ^{ex}, π^{ex}, ϕ^{int}, π^{int} satisfy (3.7), (3.6), (3.20), (3.21) in the sense of operator-valued distributions. Notice that the $\phi^{ex(*)}(F)$, $\phi^{int(*)}(F)$ form an irreducible set of operators, in contrast to the sharp time fields $\phi_{ex}^{(*)}(f)$, $\phi_{int,t}^{(*)}(f)$. Furthermore:

Theorem 3.1. The interpolating field ϕ^{int} is local and satisfies the Yang-Feldman equations:

$$\phi^{int}(T_R F) = \phi^{in}(F) \tag{3.43}$$

$$\phi^{int}(T_A F) = \phi^{out}(F) \tag{3.44}$$

where

$$(T_I F)(x) \equiv F(x) - \int dy F(y)\Delta_I(y-x)\overleftarrow{K}(x) \qquad I = R,A. \tag{3.45}$$

Proof. If supp F and supp G are spacelike separated one has on $\dot{D}_\infty$:

$$\begin{aligned}
[\phi^{int}(F),\phi^{int*}(G)]_- &= \Big(\int dt U^*(t,-\infty)\exp(iH_o t)|H_o|^{-1}W^{-1}\check{\bar F}(t,\cdot), \\
&\quad q\int dt' U^*(t',-\infty)\exp(iH_o t)|H_o|^{-1}W^{-1}\check G(t',\cdot)\Big) \\
&= \int dt\,dt'\Big(\check L\begin{pmatrix}\bar F(t,\cdot)\\ 0\end{pmatrix}, \acute{q}\check U^s(t,t')\begin{pmatrix}0\\ G(t',\cdot)\end{pmatrix}\Big) \\
&= (-i\mathring{G}_R + i\mathring{G}_A)(F,G) = 0
\end{aligned} \tag{3.46}$$

where we used I(2.21), (2.27), Th.4.3.

To prove (3.43) we observe that

$$(\overline{T_R F})(x) = \bar F(x) - K(x)\int dy\Delta_A(x-y)\bar F(y). \tag{3.47}$$

Using (3.27) and ITh.4.2 (for K(x) = 0) it then follows that

$$\begin{aligned}
&\int dt\check U^*(t,-\infty)\exp(i\check H_o t)|\check H_o|^{-1}(\check{\overline{T_R F}})(t,\cdot) = \int dt\check U^*(t,-\infty)\exp(i\check H_o t)|\check H_o|^{-1}\check{\bar F}(t,\cdot) \\
&-\sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty} dt\int_{-\infty}^{t} dt_1\cdots\int_{-\infty}^{t_{n-1}} dt_n\,\check q\check O(t_n)\cdots\check O(t_1)\exp(i\check H_o t)\check H_o^{-1} \\
&\cdot\Bigg[\begin{pmatrix}(A_\mu A^\mu(t,\cdot)+i\vec\nabla\cdot\vec A(t,\cdot)+i\vec A(t,\cdot)\cdot\vec\nabla+A_4(t,\cdot))\int dx'\Delta_A(t-t',\cdot-\vec x')\bar F(x')\\ 0\end{pmatrix} \\
&+\begin{pmatrix}iA_o(t,\cdot)\int dx'\dot\Delta_A(t-t',\cdot-\vec x')\bar F(x')\\ 0\end{pmatrix}+\check H_o\check H(t)\check H_o^{-1}\begin{pmatrix}A_o(t,\cdot)\int dx'\Delta_A(t-t',\cdot-\vec x')\bar F(x')\\ 0\end{pmatrix}\Bigg] \\
&= \cdots-\Sigma\cdots\check O(t_1)\exp(i\check H_o t)\begin{pmatrix}A_o(t,\cdot)\int\cdots\Delta_A\cdots\\ iA_o(t,\cdot)\int\cdots\dot\Delta_A\cdots+(-A^2(t,\cdot)+\cdots+A_4(t,\cdot))\int\cdots\Delta_A\cdots\end{pmatrix} \\
&= \cdots+i\Sigma\cdots\check O(t_1)\exp(i\check H_o t)\check V(t)\check H_o^{-1}\int_t^{\infty} dt'\exp(-i\check H_o(t-t'))\check{\bar F}(t',\cdot) \\
&= \int dt\,\exp(i\check H_o t)|\check H_o|^{-1}\check{\bar F}(t,\cdot).
\end{aligned} \tag{3.48}$$

The proof of (3.44) is similar (cf. the proof of (2.42)). ∎

It is easily seen that ϕ^{in} and ϕ^{int} cannot be unitarily equivalent if $K(x) \neq 0$, and that ϕ^{out} and ϕ^{in} are unitarily equivalent if and only if ψ_{out} and ψ_{in} are ($\psi = \pi,\phi$), and if and only if the transformation (2.54) is unitarily implementable.

Since, by ITh. A.1, $T_{R,A}$ are bijections of $S(R^4)$, one can define an out field by

$$\phi^{out}_{cw}(F) = \phi^{in}(T_R^{-1}T_A F) \qquad \forall F\in S(R^4). \tag{3.49}$$

From (3.43-44) it then follows that

$$\phi^{out}_{cw}(F) = \phi^{out}(F). \tag{3.50}$$

Hence,

<u>Theorem 3.2</u>. The Capri-Wigthman approach and the Friedrichs-Segal approach lead to the same S-operator. ∎

We finally note that on $\tilde{D}$

$$\mathcal{U}(a,\Lambda)\phi^{in}(F)\mathcal{U}^*(a,\Lambda) = \phi^{in}(F^{a,\Lambda}) \tag{3.51}$$

where

$$F^{a,\Lambda}(x) \equiv F(\Lambda^{-1}(x-a)). \tag{3.52}$$

Thus ϕ^{in} satisfies the Wightman axioms. Note that in view of (3.40) π^{in} is Lorentz non-covariant.

<u>B. The evolution operator and S-operator in $\mathcal{F}_s(\mathcal{H})$.</u>

We now assume that the magnetic field vanishes. It then follows from ITh.4.4 that the hypothesis of ITh.2.8 is satisfied, and from ICor 4.5 that $U_\lambda(T_2,T_1)$ is implementable in $\mathcal{F}_s(\mathcal{H})$ for any $(\lambda,T_2,T_1)\in R\times\tilde{R}^2$. Denoting the resulting 3-parameter family of unitary operators by $\mathcal{U}_\lambda(T_2,T_1)$ it follows in particular that on $\tilde{D}$, for any $f\in W_{\frac{1}{2}}(R^3)$ and $t\in\tilde{R}$,

$$\psi_{int,t}(f) = \mathcal{U}_1^*(t,-\infty)\psi_{in}(f)\mathcal{U}_1(t,-\infty) \qquad \psi = \pi,\phi. \tag{3.53}$$

If (λ,T_2,T_1) is such that (2.61) holds then we require that (2.62) hold. The operator Λ in the next theorem is defined by I(2.49).

<u>Theorem 3.3</u>. (i) For any $(\lambda,T_2,T_1)\in R\times\tilde{R}^2$ (2.61) holds true. For these values of the arguments one has for any $\phi\in D$:

$$\mathcal{U}\phi = \det(1_{--}-\Lambda_{+-}{}^*\Lambda_{+-})^{\frac{1}{2}}:\exp(\Lambda_{+-}a^*b^*+\Lambda_{++}a^*a+\Lambda_{--}bb^*+\Lambda_{-+}ba):\phi. \tag{3.54}$$

(ii) $\mathcal{U}_\lambda(T_2,T_1)$ is strongly continuous on $\tilde{R}^2$ for any $\lambda\in R$ and on R for any $(T_2,T_1)\in\tilde{R}^2$.

(iii) On $R\times\tilde{R}^2$ (2.64) holds true.

(iv) For any $(T_2,T_1)\in\tilde{R}^2$ and $\psi,\phi\in D$ the function $(\psi,\mathcal{U}_\lambda(T_2,T_1)\phi)$ on $(-\frac{1}{2}\ell,\frac{1}{2}\ell)$ has an analytic continuation to $D_{\frac{1}{2}\ell}$.

<u>Proof</u>. The statements (i)-(iii) follow from I and B by the arguments used in the proof of Th.2.3. To prove (iv) we first note that on $R\times\tilde{R}^2$

$$1_{--}-\Lambda_{+-}{}^*\Lambda_{+-} = U_{--}^{*\ -1}U_{--}^{\ -1} \tag{3.55}$$

so

$$(1_{--}-\Lambda_{+-}{}^*\Lambda_{+-})(1_{--}+U_{-+}U_{-+}{}^*) = 1_{--}. \tag{3.56}$$

Continuing (3.55-56) to D_ℓ we conclude that

$$g(\lambda) \equiv \det(1_{--}-\Lambda_{\bar{\lambda}+-}{}^*\Lambda_{\lambda+-}) = \det(1_{--}+U_{\lambda-+}U_{\bar{\lambda}-+}{}^*)^{-1} \tag{3.57}$$

is a non-vanishing analytic function in D_ℓ. Thus, its positive square root on $(-\ell,\ell)$ has a (unique) analytic continuation to D_ℓ, which we denote by $v(\lambda)$.

We now observe, using I(2.8), that for any $\lambda\in D_{\frac{1}{2}\ell}$,

$$||\Lambda_{\lambda+-}|| < 1. \tag{3.58}$$

Thus, defining for any $(\lambda,T_2,T_1)\in D_{\frac{1}{2}\ell}\times\tilde{R}^2$ and ϕ of the form (2.73)

$$\mathcal{U}_\lambda\phi \equiv v(\lambda)\prod_{i=1}^{n}(a^*(U_{\lambda++}f_i)-b(\overline{U_{\lambda-+}f_i}))\prod_{j=1}^{r}(b^*(\overline{U_{\lambda--}g_j})-a(U_{\lambda+-}g_j))$$
$$.\exp(\Lambda_{\lambda+-}a^*b^*)\Omega, \tag{3.59}$$

it follows from B that the r.h.s. of (3.59) belongs to $\mathcal{F}_s$ and equals $\mathcal{U}_\lambda\phi$ if $\lambda\in(-\frac{1}{2}\ell,\frac{1}{2}\ell)$. The statement now easily follows. ∎

Of course, $(\psi,\mathcal{U}_\lambda(T_2,T_1)\phi)$ can be analytically continued to a larger set which is determined both by $E(T_2,T_1)$ and by the requirement (3.58), but we shall not pursue this.

Using the properties of D_∞ mentioned in B and relations like B (2.6), (2.8), it can be seen that $\mathcal{U}_\lambda\phi$ is analytic in $D_{\frac{1}{2}\ell}$ if for any $k\in N^+$ and $\alpha < \frac{1}{2}\ell$

$$\sup_{|\lambda|=\alpha} ||N^k \exp(\Lambda_{\lambda+-}a^*b^*)\Omega|| < \infty. \tag{3.60}$$

However, we do not know whether (3.60) holds true.

We further observe that $\mathcal{U}_\lambda\Omega$ is analytic in $D_{\frac{1}{2}\ell}$, but that it has no analytic continuation to C unless

$$U_{\lambda{+-\atop -+}} = 0 \qquad \forall\lambda\in C. \tag{3.61}$$

(Indeed, if it has, $\Lambda_{\lambda+-}$ is $||.||_2$-entire and satisfies (3.58) on C, so by Liouville's theorem,

$$\Lambda_{\lambda+-} = 0 \qquad \forall\lambda\in C, \tag{3.62}$$

from which (3.61) easily follows.)

We shall now consider the Fock space S-operator, defined by (2.75) (which should hold on $\tilde{D}$), which corresponds to (real-valued) scalar and electromagnetic fields in $S(R^4)$. It follows from (20,21,l.c.) that $\ell_s > 0$, where ℓ_s is defined as in subsection 2B. Thus, $\mathfrak{S}_\lambda$ exists for $\lambda\in(-\ell_s,\ell_s)$. If $\lambda\in(-\ell_s,\ell_s)$ is such that (2.76) holds then we require (2.77). Denoting the supremum of the numbers $\alpha > 0$ such that

$$\int dt||O(t,\lambda)|| < \tfrac{1}{2} \qquad \forall\lambda\in D_\alpha \tag{3.63}$$

(cf. I(4.36)) by ℓ' and setting

$$\ell_c = \min(\ell',\ell_s) \tag{3.64}$$

we have

<u>Theorem 3.4</u>. (i) For any $\lambda\in(-\ell_s,\ell_s)$ (2.76) holds true. For these λ one has for any $\phi\in D$:

$$\mathfrak{S}\phi = \det(1_{--}-\Lambda_{+-}{}^*\Lambda_{+-})^{\frac{1}{2}}:\exp(\Lambda_{+-}a^*b^*+\Lambda_{++}a^*a+\Lambda_{--}bb^*+\Lambda_{-+}ba):\phi. \tag{3.65}$$

(ii) $\mathfrak{S}_\lambda$ is strongly continuous on $(-\ell_s,\ell_s)$.

(iii) For any $\psi,\phi\in D$ the function $(\psi,\mathfrak{S}_\lambda\phi)$ on $(-\ell_c,\ell_c)$ has an analytic continuation to D_{ℓ_c}.

(iv) For any $\lambda\in(-\ell_s,\ell_s)$ $\mathfrak{S}_\lambda$ is causal, up to a phase factor, and Lorentz covariant.

Proof. It suffices to prove (iv). However, this statement easily follows from ITh.4.1. ∎

We remark that, by IThs.4.4,4.1, $\ell_s = \infty$ if A_μ is such that $\vec{A} = 0$ in some inertial frame. Hence, $\ell_c = \ell'$ for these fields.

We further mention that Bellissard (20) arrived at the expression (3.65) for $\mathfrak{S}_\lambda$ by using renormalization theory. He then showed that it can be defined on coherent states and proved several properties, like unitarity, causality up to a phase factor, Lorentz covariance and analyticity.

C. The connection with the Feynman-Dyson series

According to Th.3.4 $(\psi,\mathfrak{S}_\lambda\phi)$ can, for any $\psi,\phi\in D$, be expanded in a power series in λ, the convergence radius of which is greater than or equal to ℓ_c. We will now derive explicit expressions for the expansions of $v(\lambda)$ (defined by (2.80)) and of $(\psi,\mathcal{R}_\lambda\phi)$ (where $\mathcal{R}_\lambda$ is defined by (2.81)), and compare the result with the expressions obtained from the F.D. series (25).

We introduce a formal operator $M_\lambda(\lambda\in D_{\ell_c})$ by (2.82), in which

$$\Lambda_{\lambda\varepsilon\varepsilon'}(\vec{p},\vec{q}) = \sum_{n=1}^{\infty}\lambda^n\Lambda^{(n)}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) \tag{3.66}$$

where

$$\Lambda^{(n)}_{\varepsilon\varepsilon'}(\vec{p},\vec{q}) = 2\pi i\int dk_1..dk_{n-1}(2E_p)^{-\frac{1}{2}}\Big[\tilde{V}(\varepsilon p,k_1)\tilde{\Delta}_F(k_1)\tilde{V}(k_1,k_2)$$
$$..\tilde{\Delta}_F(k_{n-1})\tilde{V}(k_{n-1},\varepsilon' q)+\text{all }\tilde{A}_\mu\tilde{A}^\mu\text{-contractions}\Big](2E_q)^{-\frac{1}{2}} \tag{3.67}$$

$$\tilde{V}(k,k') \equiv \tilde{A}_\mu(k-k')(k^\mu+k'^\mu)+\tilde{A}_\mu(k-k') \tag{3.68}$$

(cf. I§4B, I(2.49)). Arguing as in subsection 2C one infers that (2.85-91) hold true, with $-\Lambda^T_{\lambda--}\to\Lambda^T_{\lambda--}$ in (2.88). Hence,

Theorem 3.5. For any $\psi,\phi\in D$ and $\lambda\in D_{\ell_c}$:

$$(\psi,\mathcal{R}_\lambda\phi) = \lim_{N\to\infty}\sum_{n=0}^{N}\lambda^n(\psi,\mathcal{R}^{(n)}\phi) \tag{3.69}$$

where $\mathcal{R}^{(n)}$ is defined by (2.93). $\blacksquare$

The F.D. S-operator is given by (2.96), where (cf. (25))

$$\mathcal{L}_I(x) \equiv \lambda{:}\phi^{o*}(x)V(x)\phi^{o}(x){:}+\lambda^2{:}\phi^{o*}(x)(A_\mu A^\mu)(x)\phi^{o}(x){:} \tag{3.70}$$

$$V \equiv -i\overleftarrow{\partial}_\mu A^\mu + iA_\mu\overrightarrow{\partial}^\mu + A_\mu \tag{3.71}$$

and

$$(\Omega,T(\partial^k_{x_\mu}\phi^{o}(x)\partial^\ell_{y_\nu}\phi^{o*}(y))\Omega) \equiv \partial^k_{x_\mu}\partial^\ell_{y_\nu}(\Omega,T(\phi^{o}(x)\phi^{o*}(y))\Omega) \quad k,\ell = 0,1. \tag{3.72}$$

Proceeding as in subsection 2C, one obtains as the analogue of (2.99), using the relation

$$(\Omega,T(\phi^{o}(x)\phi^{o*}(y))\Omega) = -i\Delta_F(x-y) \tag{3.73}$$

and combinatorial arguments,

$$\begin{aligned} M^{(n,L)}_{F.D.} = i^L \sum_{\substack{j_1,\dots,j_L=1\\ j_1+\dots+j_L=n}}^{n-L+1} \int dx_1\dots dx_n\Big[{:}\Big(\phi^{o*}(x_1)V(x_1)\Delta_F(x_1-x_2)V(x_2)\\ \dots\Delta_F(x_{j_1-1}-x_{j_1})V(x_{j_1})\phi^{o}(x_{j_1})\Big)\dots\Big(\phi^{o*}(x_{j_1+\dots+j_{L-1}+1})V(x_{j_1+\dots+j_{L-1}+1})\\ \dots V(x_n)\phi^{o}(x_n)\Big){:}+\text{all } A_\mu A^\mu\text{-contractions}\Big]. \end{aligned} \tag{3.74}$$

An $A_\mu A^\mu$-contraction of the term in brackets is by definition the same term where one or several different triplets $V(x_i)\Delta_F(x_i-x_{i+1})V(x_{i+1})$ are replaced by $A_\mu(x_i)\delta(x_i-x_{i+1})A^\mu(x_{i+1})$ $(i = 1,\dots,n-1)$. Since

$$i\int dx_1\dots dx_\ell\Big[\phi^{o*}(x_1)V(x_1)\Delta_F(x_1-x_2)\dots V(x_\ell)\phi^{o}(x_\ell)+\text{all } A_\mu A^\mu\text{-contractions}\Big]=M^{(\ell)} \tag{3.75}$$

(2.101) follows. Thus, (2.102) holds. Regarding the meaning of this equality and regarding the relative S-matrix elements the same remarks can be made as in subsection 2C.

The analogue of Th.2.6 is:

Theorem 3.6. For any $\lambda\in D_{\ell_c}$ $v(\lambda)$ is given by (2.105), where d_n is defined by (2.106), and

$$a_k \equiv \sum_{n=1}^{[\frac{1}{2}k]-1}\frac{1}{2n}\sum_{\substack{i_1,\dots,j_n=1\\ i_1+\dots+j_n=k}}^{k-2n+1} \mathrm{Tr}\,\Lambda^{(i_1)}_{+-}{}^{*}\Lambda^{(j_1)}_{+-}\dots\Lambda^{(i_n)}_{+-}{}^{*}\Lambda^{(j_n)}_{+-} \qquad k \geq 2. \tag{3.76}$$

Proof. This follows as in subsection 2C from the relation (for $|\lambda|$ small enough)

$$v(\lambda) = \det(1_{--}-\Lambda^{*}_{\lambda+-}\Lambda_{\lambda+-})^{\frac{1}{2}} = \exp(-\tfrac{1}{2}\sum_{n=1}^{\infty} n^{-1}\sigma_n(\lambda)) \tag{3.77}$$

where σ_n is defined by (2.109). $\blacksquare$

As the analogue of (2.112) one obtains

$$a_2^{F.D.} = 2^{-1}\int dx_1 dx_2 V(x_1)\Delta_F(x_1-x_2)V(x_2)\Delta_F(x_2-x_1) \tag{3.78}$$

$$a_k^{F.D.} = k^{-1}\int dx_1..dx_k\Big[V(x_1)\Delta_F(x_1-x_2)V(x_2)..\Delta_F(x_k-x_1)+\text{all } A_\mu A^\mu\text{-contractions}\Big]$$
$$k \geq 3. \tag{3.79}$$

In time-momentum variables the integral is absolutely convergent if $k \geq 4$ and $A_\mu = 0$. However, we do not know whether in this case its real part equals a_k. But for this circumstance, similar remarks on $(\mathrm{Re})a_k^{F.D.}$ and $|(\Omega,S_\lambda^{F.D.}\Omega)|$ can be made as in subsection 2C. In order to see that (2.114) holds, transform (3.78-79) to energy-momentum variables and substitute

$$\tilde{\Delta}_F = \tilde{\Delta}_R+\tilde{\Delta}_-. \tag{3.80}$$

Using the relation

$$\tilde{\Delta}_-(p) = 2\pi i\theta(-p^0)\delta(p^2-m^2) \tag{3.81}$$

(cf. I(3.17)) and I(4.47) it then follows as in subsection 2C that

$$(\Omega,S_\lambda^{F.D.}\Omega) = \det(1_{--}+R_{\lambda--})^{-1}. \tag{3.82}$$

(Again, $R_{\lambda--}$ is actually not $||.||_1$-analytic in a neighbourhood of the origin.) Thus, in view of I(2.47) and (3.55), (2.114) holds true.

We close this section with the following Furry type theorem.

Theorem 3.7. Let $A_4 = 0$ and let A_μ be such that $\vec{A} = 0$ im some inertial frame. Then (2.124) holds true.

Proof. If $\vec{A} = 0$ then the charge conjugation operator satisfies (2.122-123) and (2.125-126) (cf. I§4B). Thus, using (2.127) and ITh. 4.1, (2.124) follows. $\blacksquare$

4. CONCLUDING REMARKS.

(1) It follows from I that time-independent electric and "pseudo-electric" fields (in the spin-$\frac{1}{2}$ case) resp. electric and scalar fields (in the spin-0 case), which are real-valued functions in $S(R^3)$, give rise to an evolution operator $U_\lambda(T_2,T_1)$ which is implementable in $\mathcal{F}_a$ resp. $\mathcal{F}_s$ for any $(\lambda,T_2,T_1)\in R^3$. It is easily seen that the resulting Fock space evolution operator (after normalization) has properties analogous to those mentioned in Ths. 2.3, 3.3 (mutatis mutandis: ℓ now depends on $|T_2-T_1|$). Similarly, the (pseudo-)unitary 1-parameter group $\exp(-iHt)(H\equiv H(1)$, cf. I(2.104)) leads to a family of unitary operators $\mathcal{U}(t)$, forming a projective representation of R; after normalization $\mathcal{U}(t)$ is strongly continuous for $t\in R$ ($|t|< ||V||^{-1}$) in the spin-0 (spin-$\frac{1}{2}$) case. Since such a representation is equivalent to a vector representation (30) there exists a phase function $c(t)$ such that

$$c(t)\mathcal{U}(t) = \exp(-iBt) \qquad \forall t\in R \tag{4.1}$$

with B self-adjoint. B can be regarded as the perturbed Hamiltonian in Fock space.

Provided that the classical S-operator

$$S = \operatorname*{s.lim}_{t\to\infty} U(t,0)\ \operatorname*{s.lim}_{t'\to-\infty} U(0,t') \tag{4.2}$$

exists and is unitary (and, in the spin-0 case, pseudo-unitary as well), one has

$$S_{\substack{+-\\-+}} = 0 \tag{4.3}$$

so the Fock space S-operator $\mathfrak{S}$ then exists and

$$\mathfrak{S} = \Gamma(\tilde{S}) \tag{4.4}$$

(cf. B(4.23-26)). Thus, for time-independent external fields, perturbation theory for $\mathfrak{S}$ amounts to investigating the Born series connected with S. It can be seen that the Feynman-Dyson series formally leads to the same result if vacuum diagrams are omitted.

(2) There exists a remarkable symmetry between the operators R and F (cf. I§2): If for some $(\lambda,T_2,T_1)\in(-\ell,\ell)\times\tilde{R}^2$ $U_\lambda(T_2,T_1)$ is implementable in Fock space the operator $U'_\lambda(T_2,T_1)$ defined by I(2.50) resp. I(2.58) is implementable in the "wrong statistics Fock space" in virtue of IThs. 2.10-11

and B (and vice versa). The resulting unitary operator $\mathcal{U}'$ is given by the r.h.s. of (2.63) (spin-0) resp. (3.54) (spin-$\frac{1}{2}$) with $\Lambda \to \Lambda'$, where Λ' is defined by I(2.51) resp. I(2.59). Clearly, one could prove analogues of Ths. 2.3-4 resp. Ths. 3.3-4 for $\mathcal{U}'$. Observe that $\mathcal{U}'$ does not satisfy (2.64) and that the wrong statistics "S-operator" $\mathfrak{S}'$ is Lorentz covariant, but non-causal; the perturbation expansion of its matrix elements is determined by the functions at the r.h.s. of (3.67) resp. (2.84) with $\tilde{\Delta}_F$ resp. $\tilde{S}_F$ replaced by $\tilde{\Delta}_R$ resp. $\tilde{S}_R$.

(3) It would be worthwhile to use second-quantized operators like the momentum cutoff interaction Hamiltonian as starting point for an investigation of the problems considered in this paper. (In the time-independent case an interesting result in this context has been obtained by Palmer (16).) The methods and ideas from constructive quantum field theory which could then be used might in particular lead to a deeper understanding of the Feynman-Dyson series (especially of the divergent vacuum diagrams).

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SAMENVATTING

In dit proefschrift worden op wiskundig precieze wijze theorieën onderzocht waarin een gequantiseerd veld wisselwerkt met niet-gequantiseerde uitwendige velden. Er wordt aangenomen dat het gequantiseerde veld correspondeert met geladen elementaire deeltjes.

In hoofdstuk 1 worden in een algemeen kader de veldoperatortransformaties bestudeerd die in deze theorieën optreden. Het belangrijkste resultaat is een eenvoudige uitdrukking voor de normale vorm van de unitaire Fock ruimte operator die een dergelijke transformatie implementeert. Verder worden reeds bekende noodzakelijke en voldoende voorwaarden voor implementeerbaarheid op een nieuwe en aanzienlijk eenvoudiger manier afgeleid.

In hoofdstuk 2 wordt eerst een algemene storingstheorie gepresenteerd die gebruikt kan worden om de wisselwerking van relativistische geladen deeltjes met uitwendige velden te bestuderen. Deze theorie wordt dan toegepast op de klassieke Dirac en Klein-Gordon theorieën. Voor uitwendige velden die worden beschreven door testfuncties op het ruimte-tijd continuüm wordt bewezen dat geretardeerde en geavanceerde elementaire oplossingen bestaan. Verder wordt het verband tussen deze oplossingen en de tijdsevolutie vastgesteld. Na aangetoond te hebben dat de klassieke verstrooiingsoperator Lorentz covariant en causaal is onderzoeken we tenslotte de implementeerbaarheid in de Fock ruimte van de tijdsevolutie. Elektrische velden blijken aanleiding te geven tot een implementeerbare tijdsevolutie, terwijl dit voor andere uitwendige velden in het algemeen niet het geval is.

In hoofdstuk 3 worden de resultaten van de eerste twee hoofdstukken toegepast op de gequantiseerde Dirac en Klein-Gordon vergelijkingen met uitwendige velden. Deze vergelijkingen worden opgelost en er wordt aangetoond dat het interpolerend veld lokaal is en aan de Yang-Feldman vergelijkingen voldoet. We laten vervolgens zien dat twee uit de literatuur bekende formalismen voor de beschouwde theorieën tot dezelfde unitaire verstrooiingsoperator in de Fock ruimte leiden. Na bestudering van de eigenschappen van deze verstrooiingsoperator en van de tijdsevolutie wordt een divergentievrije storingsreeks voor de verstrooiingsoperator afgeleid. Deze storingsreeks wordt dan vergeleken met de Feynman-Dyson reeks voor de formele verstrooiingsoperator. De relatieve verstrooiingsamplitudes van de formele theorie blijken gelijk te zijn aan die van de wiskundig precieze theorie. De modulus van de niet-gedefinieerde ("divergente") vacuümverwachtingswaarde van de formele verstrooiingsoperator moet dan worden gedefinieerd als de modulus van het pendant uit de precieze theorie om aan de eis te voldoen dat de formele verstrooiingsoperator correspondeert met een unitaire operator in de Fock ruimte. Tenslotte wordt aangetoond dat in de beschouwde theorieën een precies analogon van de formele stelling van Furry bestaat.

STELLINGEN

1. In de klassieke Dirac en Klein-Gordon theorie is de S-operator die de verstrooiing aan een extern veld A_μ beschrijft gelijk aan de S-operator voor het externe veld $A_\mu + \partial_\mu \Lambda$, waarbij $A_0, .., A_3$ en Λ testfuncties op het ruimte-tijd continuüm zijn. Dientengevolge is ook de S-operator in de gequantiseerde Dirac en Klein-Gordon theorie ijkinvariant.

2. Er zijn goede redenen om aan te nemen dat de faseverschuivingsdubbelzinnigheden, die optreden bij de beschrijving van elastische spin-0-spin-0 verstrooiing m.b.v. een eindig aantal partiële golven, corresponderen met curves in de vectorruimte van de faseverschuivingen.

   F.A.Berends, S.N.M.Ruijsenaars: Nuclear Physics B56(1973)507-524.

3. Er bestaan differentiële werkzame doorsneden en polarizaties voor elastische spin-0-spin-$\frac{1}{2}$ verstrooiing die beschreven kunnen worden door 4 verschillende drietallen faseverschuivingen δ_{0+}, δ_{1+} en δ_{1-}.

   F.A. Berends, S.N.M.Ruijsenaars: Nuclear Physics B56(1973)525-535.

4. Het is onjuist dat de oplossing van de differentiaalvergelijking $\dot{f}(t) = g(t,\lambda)$ $(t,\lambda \in R)$ met randvoorwaarde $f(0) = 0$ continu is in λ als de functie g continu is in t en λ afzonderlijk.

   N.Dunford, J.T.Schwartz: Linear Operators, Part II, p. 1284.

5. M.b.v. de kernstelling is in te zien dat er een natuurlijke injectie bestaat van de begrensde operatoren op $L^2(R)$ in de getemperde distributies over R^2. Het verdient aanbeveling de eigenschappen van deze afbeelding te onderzoeken.

6. De een-dimensionale klassieke Dirac vergelijking met elektrische potentialen, beschreven door begrensde meetbare reële functies met constante waarden links en rechts van een begrensd interval, leidt tot hetzelfde paradoxale verstrooiingsgedrag als het door Klein beschouwde geval.

S.N.M.Ruijsenaars, P.J.M.Bongaarts: Scattering theory for one-dimensional step potentials. Leiden University preprint 19'
O.Klein: Zeitschrift der Physik 53(1929)157-165.

7. De verstrooiing aan de door Dosch, Jensen en Müller onderzochte potentialen verschilt in kwalitatief opzicht essentieel met de verstrooiing aan de door Klein beschouwde potentiaal. Het is daarom wenselijk de benaming Klein paradox te reserveren voor het laatstgenoemde verstrooiingsgedrag.

H.G.Dosch, J.H.D. Jensen, V.F.Müller: Einige Bemerkungen zum Klein'schen Paradoxon. Physica Norvegica 5(1971)151-162.

8. De Klein paradox wordt niet opgelost door van een een-deeltjesformulering van de verstrooiing over te gaan op een veel-deeltjesformulering.

P.J.M.Bongaarts, S.N.M.Ruijsenaars: The Klein paradox as a many particle problem. Leiden Universtiy preprint 1975.

9. De eis, dat de formele Feynman-Dyson S-operator voor de in dit proefschrift beschouwde theorieën correspondeert met een unitaire operator, legt de absolute waarde van zijn vacuümverwachtingswaarde geheel vast. Het verdient aanbeveling om na te gaan of de gebruikelijke Feynman technieken om het reële gedeelte van de met de vacuümdiagrammen corresponderende integralen te berekenen een waarde opleveren die hiermee in overeenstemming is.

Dit proefschrift, hoofdstuk 3, §§ 2C, 3C.

10. In de Fock ruimte voor vrije geladen spin-0 deeltjes bestaat geen tijdsomkeeroperator met de door Källen en Gasiorowicz geëiste eigenschappen.

G.Källen: Elementarteilchenphysik, eqs. (12.92).

S.Gasiorowicz: Elementary particle physics, eqs. (1.124).

11. In de Fock ruimte voor vrije spin-$\frac{1}{2}$ Majorana deeltjes bestaat geen zelfgeadjungeerde partiteitsoperator die de veldoperator op lokale wijze transformeert.

12. Hegerfeldt heeft recentelijk aangetoond dat een relativistisch vrij deeltje slechts op één tijdstip in zijn evolutie gelokaliseerd kan zijn. Hierdoor vervalt het voornaamste bezwaar tegen de Newton-Wigner plaatsoperator. Mede omdat deze operator tot een causale snelheidsoperator leidt verdient hij de voorkeur boven andere in de literatuur voorgestelde plaatsoperatoren.

G.Hegerfeldt: Remark on Causality and Particle Localization. Heidelberg University preprint 1974.

A.Kalnay: The Localization Problem. In: Studies in the Foundations, Methodology and Philosophy of Science. Vol.4, Problems in the Foundations of Physics.

13. Het is wenselijk tot een schatting te komen van de kans op met de Newton-Wigner plaatsoperator corresponderende acausale effecten voor deeltjes met de massa van het elektron.

14. De voornaamste structuur in de thermomodulatiespectra van zilver rond 4 eV dient niet te worden toegeschreven aan overgangen van de d band naar de Fermi energie maar veeleer aan overgangen tussen geleidingsbanden.

R.Rosei, C.H.Culp, J.H.Weaver: Physical Review B10(1974)484-489.

S.N.M.Ruijsenaars

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