ON NON-LINEAR HYDRODYNAMIC FLUCTUATIONS

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ON NON-LINEAR HYDRODYNAMIC FLUCTUATIONS

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CONTENTS

CHAPTER I : INTRODUCTION 7

- 1.1 Hydrodynamic fluctuations 7
- 1.2 Light scattering by a fluid: linear theory 8
- 1.3 Non-linear fluctuations: mode-mode coupling and long time tails 11
- 2.1 Outline of the analysis of chapters II to IV 12
- 2.2 Comments and remarks 17

CHAPTER II : HYDRODYNAMICS FOR AN IDEAL FLUID: HAMILTONIAN FORMALISM AND LIOUVILLE EQUATION 24

- 1 Introduction 24
- 2 The Clebsch representation for an ideal fluid 25
- 3 Variational principle for ideal fluids and Hamiltonian formulation 27
- 4 The Liouville equation 32
- 5 The equilibrium distribution 35
- 6 Discussion 37
- Appendix 37

CHAPTER III : NON-LINEAR HYDRODYNAMIC FLUCTUATIONS AROUND EQUILIBRIUM 41

- 1 Introduction 41
- 2 Hamiltonian form of ideal fluid hydrodynamics 43
- 3 Stochastic differential equations for hydrodynamic fluctuations 45
- 4 The Fokker-Planck equation 46
- 5 The form of the dissipative currents 50
- 6 The linear response to an external force 55
- Appendix A 59
- Appendix B 61

CHAPTER IV : A CANONICAL TRANSFORMATION RELATING THE LAGRANGIAN AND EULERIAN DESCRIPTION OF IDEAL HYDRODYNAMICS 65

1 Introduction 65

2 Variational principle for the Lagrangian description 66

3 Canonical transformation to Eulerian description 69

SAMENVATTING 75

CURRICULUM VITAE 78

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CHAPTER I

INTRODUCTION

It is the aim of this introductory chapter to serve as a motivation for the work described in chapters II to IV. Section 1 is intended to give some background on hydrodynamical fluctuations, while in section 2 the line of reasoning followed in chapters II and III is illustrated by an analogy.

1.1 *Hydrodynamic fluctuations*

In many physical relevant situations, the behaviour of fluids is very accurately described by the hydrodynamic equations. These are continuum equations, which means that the basic variables are field variables, i.e. are continuous (and in most cases differentiable) functions of coordinates and time. The smooth behaviour of physical quantities on all space and time scales is, of course, not in accordance with the molecular structure of nature. However, the use of a hydrodynamical description can often be justified by appealing to the difference in scales of time and length on which the molecular processes and the macroscopic rate of change of the fluid quantities take place in the following way: in most situations of practical interest, the properties of a fluid do not vary appreciably over distances of the order of the mean free path of the constituent particles of the fluid, say $10^{-9} - 10^{-10}$ m. Let us in that case divide the fluid into cells which are small on a macroscopic scale but large on this microscopic scale, and average the molecular motion of the constituent particles in each cell over some time interval which is much longer than the mean time between collisions of the particles but short on the time scale on which the macroscopic fluid properties change considerably. In that case, we may assume that each cell is in a state of local equilibrium, and that the state parameters in the cells are related in the same way as in equilibrium, i.e. via thermodynamic relations. In view of the fact that the macroscopic properties of the fluid are supposed to vary sufficiently slowly in space, the state parameters of neighbouring cells will be nearly the same. To calculate the evolution of the system for the analysis of phenomena taking place on these macroscopic length scales, it is then convenient to use instead of all state parameters of cells, the interpolated fields. These are the fields which are assumed to obey the hydrodynamic equations. In this way, the hydrodynamic description arises as a coarse-grained average in space and time of the molecular motion.

In deterministic continuum equations, like the hydrodynamical equations, no spontaneous fluctuations arise. As is well known, however, the molecular structure of nature also entails the occurrence of fluctuations. Correspondingly, the above picture of a fluid is therefore still too naive: the coarse-grained quantities which occur in the hydrodynamic description will be subject to small fluctuations. These fluctuations may still persist on the hydrodynamic level. It is the goal of hydrodynamic fluctuation theory to provide a framework to discuss fluctuation effects which are observed on time and length scales on which the hydrodynamic equations yield an accurate description of the fluid.

1.2 *Light scattering by a fluid: linear theory*

To illustrate some aspects of hydrodynamic fluctuation theory, we recall here briefly the important example of the scattering of light by a one-component fluid in equilibrium (see e.g. refs. 1-3). As sketched in fig. 1, the incoming light-waves with wave-vector $\vec{k}_0$ are scattered by such a fluid. If the intensity I of the waves which are scattered over an angle Θ are ana-

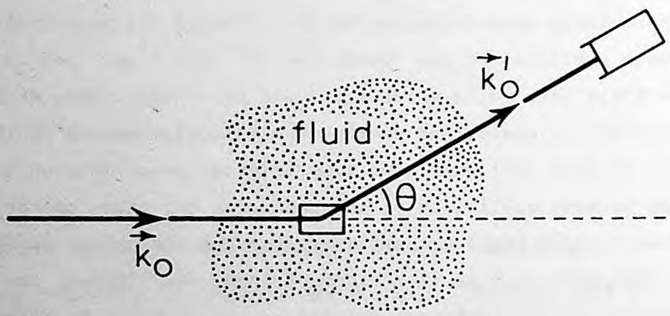

fig. 1. Light scattering by a fluid.

lysed as a function of ω, the frequency of these waves, qualitatively a spectrum of the form depicted in fig. 2 is found. There are three peaks of Lorentzian shape in the spectrum, one at ω_0, the frequency of the incoming waves, and two identical ones at $\omega = \omega_0 \pm ck$, where c is the velocity of sound in the fluid, $\vec{k}_0'$ the wave-vector of the scattered waves and

$k = |\vec{k}_o' - \vec{k}_o| \simeq 2k_o \sin\Theta/2$.

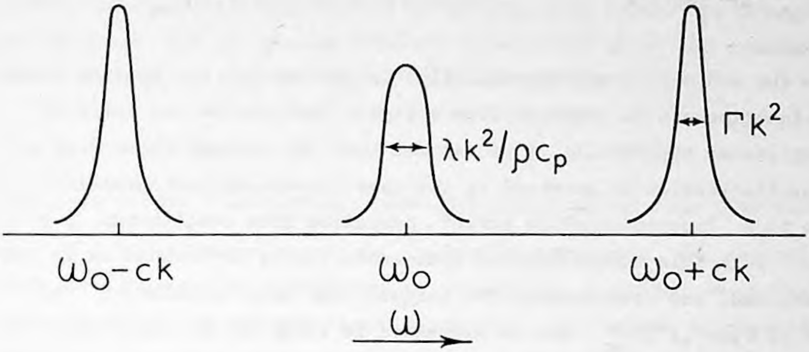

fig.2 Qualitative sketch of $I(\vec{k},\omega)$

Scattering of light can be caused by small variations in the dielectric constant ε . In a fluid such variations can occur due to variations $\Delta\rho$ of the fluid density. Since the wave-length of light is of the order of 10^{-6}m , the light scattering will be caused mainly by density fluctuations with a wave-length of this order of magnitude. On such length scales a hydrodynamic analysis should be applicable, and small variations $\Delta\varepsilon$ in the dielectric constant can therefore be expressed in terms of the density fluctuations by means of the thermodynamic relation

$$\Delta\varepsilon \simeq \left(\frac{\partial\varepsilon}{\partial\rho}\right)_T \Delta\rho \quad . \tag{1}$$

In this equation we have left out the temperature induced fluctuations of ε , since these are negligible in most fluids. From a more detailed analysis [2], one finds in fact that the spectrum $I(\vec{k},\omega)$, where $\vec{k} = \vec{k}_o' - \vec{k}_o$, is in good approximation related to the density-density correlation function of the fluid according to

$$I(\vec{k},\omega) = A\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T^2 \quad <\Delta\rho(\vec{k},\omega)\,\Delta\rho^*(\vec{k},\omega)>_{eq} \quad . \tag{2}$$

Here A is a factor depending on quantities such as the intensity and wavelength of the incoming light and the equilibrium value of the dielectric constant of the fluid, and $<\Delta\rho(\vec{k},\omega)\,\Delta\rho^*(\vec{k},\omega)>_{eq}$ is the Fourier transform of the equilibrium density auto-correlation function,

$$<\Delta\rho(\vec{k},\omega)\,\Delta\rho^{*}(\vec{k},\omega)>_{eq} =$$

$$\int d\vec{r}' \int dt'\; e^{-i\vec{k}\cdot\vec{r}' + i\omega t'} <\Delta\rho(\vec{r}+\vec{r}', t+t')\,\Delta\rho(\vec{r},t)>_{eq} \quad . \tag{3}$$

Within the context of hydrodynamic fluctuation theory, the density autocorrelation function can be computed from a linear analysis on the basis of Onsager's regression hypothesis: it is assumed that the average relaxation of a spontaneous fluctuation is governed by the same linearised hydrodynamical equations as is a "macroscopically large" deviation from equilibrium, e.g. a sound wave. With this assumption the time-evolution of a fluctuation is completely determined, and consequently the unequal time auto-correlation function $<\Delta\rho(\vec{r},t)\,\Delta\rho(\vec{r}',t')>_{eq}$ can be expressed in terms of the equal time correlation function $<\Delta\rho(\vec{r},t)\,\Delta\rho(\vec{r}',t)>_{eq}$. The latter can in turn be obtained from the thermodynamic equilibrium distribution P^{eq} for fluctuations [4,5]

$$P^{eq} \sim e^{\Delta S/k} \quad . \tag{4}$$

Here ΔS is the total change in the entropy of the system associated with a small deviation of the state parameters from the equilibrium state, and k is Boltzmann's constant. The equilibrium distribution (4) is sometimes called the Einstein distribution. In a linear fluctuation analysis, the entropy difference ΔS in eq.(4) is expanded up to second order in these deviations. In that case, P^{eq} becomes a Gaussian distribution, from which $<\Delta\rho(\vec{r},t)\,\Delta\rho(\vec{r}',t)>_{eq}$ can easily be computed. As discussed above the light scattering intensity then follows via the analysis of the unequal time correlation function on the basis of Onsager's regression hypothesis.

From a theoretical analysis as described above, one obtains indeed a spectrum of the form as depicted in fig.2 . It turns out that the central peak, the so-called Rayleigh line, represents the scattering of light on entropy fluctuations, whereas the two lines at $\omega_0 \pm ck$ are due to scattering on sound waves. Moreover, the width of the peaks (full width at half maximum height) is related to the transport coefficients: the width of the central peak is equal to $\lambda k^2/\rho c_p$, where λ is the thermal conductivity of the fluid, and where ρ and c_p are the equilibrium density and specific heat at constant pressure of the fluid, respectively. Analogously, the width of the sound peaks is equal to Γk^2 , where Γ is the sound attenuation coefficient, which depends among other things on λ and the shear and bulk viscosities η and η_v .

The above discussion clearly demonstrates that the hydrodynamic analysis of the intensity of light, scattered by density fluctuations, is not only of theoretical interest, but also of great practical value: the measurement of the spectrum $I(\vec{k},\omega)$ yields directly information about the transport coefficients as well as the velocity of sound of the fluid.

1.3 *Non-linear fluctuations: mode-mode coupling and long time tails*

The hydrodynamic equations are highly non-linear, and accordingly many phenomena observed in hydrodynamics can only be explained on the basis of a proper analysis of the non-linear equations. As regards fluctuations, studies have e.g. been made of fluctuations around non-equilibrium steady states. In these cases, one usually linearizes the equations around the steady state and then performs a linear fluctuation analysis, similar to the linear one around equilibrium. (Several approaches to the problem of light scattering by a fluid in a stationary temperature gradient 6) are of this kind). We will, however, concentrate on the non-linear relaxation of fluctuations. Studies of these aspects are commonly referred to as mode-mode coupling theories 7-15). This name stems from the fact that in a non-linear theory the Fourier modes of the fluctuating fields with a different wave-vector $\vec{k}$ are coupled to each other *). Mode-mode coupling goes back to the work of Fixman 7) on the shear viscosity of critical mixtures in 1962; the main impetus to this field was given in particular by Kawasaki 9,10).

Mode-mode coupling is of particular interest in the vicinity of critical points, where fluctuations are large and long-ranged. As an example, we again consider light scattering by a fluid. A measure for the distance from the critical point is the correlation length ξ. As we have seen in subsection 1.2 a linear fluctuation analysis showed that the linewidth of the Rayleigh line (the central peak) is $\lambda k^2/\rho c_p$. This will be a good approximation away from the critical point, if $k\xi \ll 1$. A linear theory does not suffice, however, if $k\xi \gg 1$. A detailed mode-mode coupling analysis has been given by Kawa-

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* The name mode-mode coupling is not very distinctive, as the coupling of modes is a general feature of non-linear theories. Indeed many aspects of mode-mode coupling calculations resemble quite closely those encountered in e.g. the theory of turbulence, where this coupling is also essential (see e.g. ref. 16-18).

saki10), who showed that for $k\xi \gg 1$, the linewidth should be proportional to k^3 . This and other predictions have turned out to be in good agreement with experiment.

We also recall that in 1967 Alder and Wainwright19) observed that in molecular dynamics simulations of a fluid, the velocity auto-correlation function of the particles did not decay exponentially for long times t , but as $t^{-d/2}$. Here d is the dimensionality of the system. This phenomenon can be explained within the framework of kinetic theory as well as from a hydrodynamic point of view. On the kinetic level, these so-called long time tails in dilute gases are caused by "ring-collisions".20) In the hydrodynamic picture, the long time tails result from the convective non-linearities in the conservation equations* (e.g. the term $\vec{v}\cdot\vec{\nabla}\vec{v}$ in the velocity equation), and they can therefore be calculated with the aid of a mode-mode coupling analysis.11,12,22)

In practice, non-linear generalized Langevin equations with random noise terms are often the starting point for mode-mode coupling calculations. In order to make the calculations tractable, it is usually assumed in addition that the random noise terms are Gaussian processes with white noise. It will be our aim to investigate the consistency of such a procedure, in particular for fluctuations around equilibrium. In the next section, we will illustrate by an analogy the main line of reasoning on which the analysis presented in the following chapters will be based.

2.1 *Outline of the analysis of chapters II to IV*

As stated in the last section, mode-mode coupling calculations are often based on the use of non-linear Langevin-type equations with "random forces" which are assumed to be Gaussian processes with white noise. In general, such a description is equivalent to one with a Fokker-Planck equation for a density distribution function of the state parameters. However, as stressed in particular by Van Kampen23,24), the Langevin approach has its limitations: use of Gaussian white noise amounts to the assumption that the fluctuations are due to infinitely many infinitely small jumps.25) Moreover, the whole concept of a nonlinear Langevin equation is ambiguous. Essentially, the problem is the follow-

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* In earlier work21) which was partly based on hydrodynamical considerations, non-linearities enter by an "apparently innocuous trick"13).

ing: often, we know that the evolution of the mean $\overline{\vec{x}}$ of a vector $\vec{x}$, of which the components specify the macroscopic state of some system, is governed by an equation of the form

$$\frac{d\overline{\vec{x}}}{dt} = \vec{g}(\overline{\vec{x}}) \quad , \tag{5}$$

where $\vec{g}(\overline{\vec{x}})$ is some non-linear function of $\overline{\vec{x}}$. In the Langevin approach, it is customary to assume that the fluctuating variable $\vec{x}$ obeys basically the same equation (5), but with a random force $\vec{f}$ with zero mean added to the right hand side,

$$\frac{d\vec{x}}{dt} = \vec{g}(\vec{x}) + \vec{f}(t) \quad , \tag{6}$$

where

$$\overline{\vec{f}(t)} = 0 \quad . \tag{7}$$

The term $\vec{f}$, so to speak, represents the change in $\vec{x}$ due to fluctuations. However, (6) and (7) imply that

$$\frac{d\overline{\vec{x}}}{dt} = \overline{\vec{g}(\vec{x})} \quad . \tag{8}$$

If $\vec{g}(\vec{x})$ is a linear function of $\vec{x}$, eqs.(5) and (8) coincide and are consistent with Onsager's regression hypothesis. If $\vec{g}(\vec{x})$ is non-linear, however, eqs.(5) and (8) are in conflict since $\overline{\vec{g}(\vec{x})} \neq \vec{g}(\overline{\vec{x}})$; we are therefore forced to conclude that the mere replacement of a non-linear equation for the mean by a non-linear Langevin equation for the fluctuating variable is not necessarily a clear-cut and consistent approach to fluctuation theory.*

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* In the linear case, the Langevin-approach is easily shown to be equivalent with the one described for linear fluctuations in section 1.2. In the non-linear case, the difference between eqs.(5) and (8) results in the so-called renormalization of transport coefficients. If $\vec{x}$ obeys eq.(6) with bare transport coefficients in $\vec{g}$, then $\overline{\vec{x}}$ obeys an equation involving renormalized transport coefficients. Nevertheless, the problem does remain that an equation for the fluctuating variable is not simply related to the equation for the mean.

A more proper starting point to discuss fluctuations, advocated in particular by Van Kampen[23,24)], is to assume (or show) that on a certain "mesoscopic" level (between microscopic and macroscopic), the stochastic evolution of the system can be appropriately described by a continuous Markov process (essentially, this means that the transition probabilities for the state parameters at time t depend only on the values of these parameters at that time t , but not on the history of the system at earlier times). If such a Markovian description is adopted, one can in principle perform a systematic expansion of the Master equation (the evolution equation for the probability distribution), which is based on the fact that the fluctuations are relatively small. From this expansion, one usually obtains a Fokker-Planck equation as a first approximation. However, higher order corrections are in principle present, and they can be dealt with in a systematic way. The limitations of the Fokker-Planck description can therefore be assessed.

The completion of the above program, to start with a Master-equation approach, is at present still too ambitious for hydrodynamics. In chapters II and III, we will therefore content ourselves with a more modest goal, that we will illustrate here with the example of a Brownian particle in a fluid.

Suppose we describe the behaviour of a one-dimensional Brownian particle in an external potential V with the equations

$$\frac{dq}{dt} = \frac{p}{m} = \left\{q,H\right\} \quad , \tag{9}$$

$$\frac{dp}{dt} = -\frac{dV(q)}{dq} + F^s + f = \left\{p,H\right\} + F^s + f \quad , \tag{10}$$

where the Hamiltonian H is given by

$$H = \frac{p^2}{2m} + V(q) \quad . \tag{11}$$

Here q is the coordinate of the particle, m its mass, p its momentum, V an external potential and $\left\{.,.\right\}$ the usual Poisson bracket. Eq.(9) and the first term on the right hand side of eq.(10) are just Hamilton's equations for a particle in an external potential. The terms F^s and f in eq.(10) are introduced to mimic the systematic frictional damping and the fluctuating force exerted by the fluid on the particle, respectively. Usually, one takes F^s linear in the velocity of the particle (for a spherical particle in three dimensions, it is customary to use Stokes' law for the friction term). Instead, we now do not specify the form of F^s , but investigate which form of F^s is compatible with the assumption that f is a Gaussian random force with zero mean and correlations

$$\overline{f(t)\,f(t')} = 2\,D\;\delta(t-t') \quad , \tag{12}$$

where D is a constant. To answer this question, we follow a line of argument which essentially goes back to Chandrasekhar.[26] As is well known[5,27,28] the above stochastic equations are equivalent with a Fokker-Planck equation for the probability distribution function P ,

$$\frac{\partial P}{\partial t} = \left\{H,P\right\} + \frac{\partial}{\partial p}\left(-F^{s}\,P + D\,\frac{\partial P}{\partial p}\right) \quad . \tag{13}$$

If there are no dissipation and fluctuations ($F^{s} = 0$ and $D = 0$) , this equation reduces to the Liouville equation for one particle.

We know furthermore from equilibrium statistical mechanics that the equilibrium distribution of the Brownian particle, which is in contact with a heat bath (the fluid), is given by the canonical distribution

$$P^{eq} = e^{(F-H)/kT} \quad , \tag{14}$$

where F is the free energy of the particle and T the temperature of the fluid. For consistency, P^{eq} should also be a stationary solution of eq.(13), and every distribution function P should approach P^{eq} for long times. Since P^{eq} satisfies $\left\{H,P^{eq}\right\} = 0$, the requirement that P^{eq} is a stationary solution of eq.(13) is equivalent to

$$\frac{\partial}{\partial p}\left[\left(-F^{s} + D\,\frac{\partial}{\partial p}\right)P^{eq}\right] = 0 \quad . \tag{15}$$

This equation implies that[*]

$$F^{s} = D\,\frac{\partial \ell n P^{eq}}{\partial p} = -\,\beta\,\frac{p}{m} \quad , \tag{16}$$

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[*] The general solution is $F^{s} = -\,Dp/mkT + h(q)/P^{eq}$ with h an arbitrary function. However, since P^{eq} is even in p , the second term is in conflict with the obvious symmetry requirement for F^{s} , $F^{s}(-p) = -F^{s}(p)$. Hence h must be identically zero (Similar arguments, i.e. detailed balance, are used in chapter III). In addition, it can be shown that every distribution function P approaches P^{eq} for long times, if eq.(16) holds.

where the friction coefficient β is given by

$$\beta = D/kT \quad . \tag{17}$$

Eq.(16) shows that F^s must be linear in the velocity of the particle, in order to make the Fokker-Planck equation (13) compatible with the equilibrium distribution (14). Eq.(17) represents the fluctuation dissipation theorem for the present case.

In essence, the above line of reasoning is also followed in chapters II and III for the case of hydrodynamic fluctuations. In chapter II, we discuss the Seliger and Whitham[29)] variational principle for ideal fluids, and show that a Hamiltonian formalism, leading to hydrodynamic Poisson-brackets, can be developed. This allows us to write the evolution equation for the probability distribution function for the hydrodynamic variables in the case of ideal flow (no dissipation) in the form of a Liouville equation. In chapter III, a Fokker-Planck equation is obtained from the stochastic differential equations for fluctuating hydrodynamics introduced by Landau and Lifshitz[4,30)]. With the aid of this Fokker-Planck equation we then investigate of which form the phenomenological laws must be for this approach to be consistent.

It is sometimes convenient to view the motion of a fluid as the motion of small fluid elements or cells. Accordingly, while one usually specifies the fluid fields as a function of time and of a fixed position in space, it is also possible to write the fluid variables as a function of time and of the "labels" of the material fluid elements. The former way to specify the fields is called the Eulerian description, the latter the Lagrangian. Since the Lagrangian description expresses the fact that physical quantities are more directly related to material elements - "the fluid particles" - than to fixed positions in space, the equations of motion in this description reflect closely those of particle mechanics. Indeed the variational principle for the Lagrangian description is essentially the same as the one for point particles. On the other hand, the Seliger and Whitham[29)] variational principle for the Eulerian description, discussed in chapter II, is more abstruse. We therefore come back to the relation between the two types of description in chapter IV, where we show that the two variational principles can be related by means of a canonical transformation. This transformation clarifies the variational principle for the Eulerian specification, in that it enables us to derive it directly from the more familiar but often less convenient one for the Lagrangian description.

2.2 *Comments and remarks*

We now indicate a number of complications encountered in carrying out for hydrodynamic fluctuations the program described above; we also make some remarks in connection with several points mentioned previously as well as on some of the results obtained in chapters II and III.

i) If $V(q) = a q^2$ the equations of motion (9)-(12) are fully linear and can be solved exactly. Alternatively, the Fokker-Planck equation (13) can be solved, and the approach of P to the Maxwell-Boltzmann distribution for long times can be verified. Similarly, the consistency of the linear analysis discussed in section 1.2 can be verified explicitly.

The term $V(q)$ in the above equations is of purely mechanical origin; its presence has no influence on eq.(12) and therefore not on the linear relation (16) and on the fluctuation-dissipation theorem (17) either. As we shall see, the reversible non-linearities in the hydrodynamical equations can be retained in the analysis of chapters II and III for similar reasons.

ii) Along the lines set forth above, one only checks the consistency of the description with results known from equilibrium statistical mechanics. If we aim at a justification of the equations (9)-(12) by starting with a Master-equation approach, a more detailed knowledge of the nature of the fluctuations is necessary. Consider e.g. within the context of Brownian motion the "Rayleigh piston", a heavy piston of mass M executing a one-dimensional Brownian motion due to the bombardment by light molecules with mass m .$^{31)}$ On the basis of the assumption that the velocity distribution of the light molecules is the Maxwell-Boltzmann distribution, it is possible to obtain expressions for the transition probabilities in a Master-equation approach* . Analogously, for a discussion of hydrodyna-

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* In this case the proper expansion parameter is the ratio m/M $^{23,25)}$; from the usual expansion of the master equation it is found that the equations (9)-(12) are a good approximation in the limit $m/M \to 0$ and on a time scale τ such that $\tau = m t/M$ is finite; in this limit the bombardment by the molecules gives rise to "infinitely many infinitely small" jumps in the velocity. In approaches based on the Boltzmann equation $^{32)}$ similar limits as above have to be taken in order to agree with eqs.(9)-(12).

mic fluctuations in a Master-equation approach one has to go back to the cell picture of the fluid described in section 1.1 . One then has to analyse the transition probabilities of the cell variables in terms of the exchange between cells of mass, energy, etc. due to molecular transfer 33),34) .

It should be mentioned that eqs. like (9)-(12) have also been derived in certain limits from microscopic models (see e.g. refs. 35,36) or by using Zwanzig-Mori projection operator techniques 37-39). However, to attain the same degree of rigour along these lines for the case of hydrodynamic fluctuations, seems at present an idle hope.

iii) Statistical continuum theories are often plagued by divergencies. Accordingly, in mode-mode coupling calculations one usually has to introduce a cut-off wavenumber k_c of the order of an inverse molecular length. In this way the short wavelength modes, of which the evolution is not governed by the hydrodynamical equations, are eliminated. Whether or not it is necessary to introduce a cut-off depends on the problem under consideration, and in particular on the relative importance of small distance scales. If e.g. the modes with wavevector $|\vec{k}| << k_c$ dominate the long time behaviour of the relevant quantities to be calculated, these will be insensitive to the precise value of the cut-off. Similar divergencies occur in chapters II and III where we want to write down a Fokker-Planck equation for fields. In this case, we remove the divergencies by a discretization procedure.

iv) If we consider two sets of thermodynamic variables $\alpha_1 \ldots \alpha_n$ and $\beta_1 \ldots \beta_n$, the probability distribution functions $P(\vec{\alpha})$ and $P(\vec{\beta})$ for these sets of variables are related by

$$P(\vec{\beta}) = P(\vec{\alpha}) \frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(\beta_1, \ldots, \beta_n)} \quad . \tag{18}$$

Here $\partial(\alpha_1, \ldots \alpha_n)/\partial(\beta_1, \ldots, \beta_n)$ denotes the Jacobian of the transformation $\vec{\alpha} \rightarrow \vec{\beta}$. In linear fluctuation theory, transformation of variables are linear and this Jacobian is a constant. In the non-linear case, however, the Jacobian is in general not a constant. Hence, if the probability distribution (4) is valid for the set $\vec{\alpha}$, then the probability distribution $P(\vec{\beta})$ is not of the same form. In non-linear problems the question therefore arises for which variables the Einstein distribution function (4) has to be used. It will be argued in chapter II that this distribution function should be used for the conserved variables (mass, momentum and energy). This choice would also most naturally arise from more microscopic considerations.

v) In chapter III, we split, following Landau and Lifshitz[4,30], the viscous pressure tensor $\overset{\rightrightarrows}{\Pi}$ and the heat current $\vec{J}$ appearing in the hydrodynamical equations, into systematic and random parts,

$$\overset{\rightrightarrows}{\Pi}(\vec{r},t) = \overset{\rightrightarrows}{\Pi}^{S}(\vec{r},t) + \overset{\rightrightarrows}{\Pi}^{R}(\vec{r},t) \quad , \quad \vec{J}(\vec{r},t) = \vec{J}^{S}(\vec{r},t) + \vec{J}^{R}(\vec{r},t) \quad , \tag{19}$$

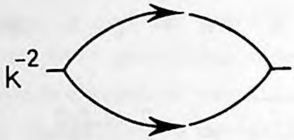
where $\overset{\rightrightarrows}{\Pi}^{R}$ and $\vec{J}^{R}$ are "random fields" , Gaussian processes with white noise. We will conclude that this can be done consistently provided that the systematic parts $\overset{\rightrightarrows}{\Pi}^{S}$ and $\vec{J}^{S}$ (to be compared with the term F^{S} in eq.(10)) are given by

$$\overset{\rightrightarrows}{\Pi}^{S}(\vec{r},t) = -2L\,\frac{\overline{\vec{\nabla}\vec{v}(\vec{r},t)}}{T(\vec{r})} - L_{v}\,\overset{\rightrightarrows}{U}\,\frac{\vec{\nabla}\cdot\vec{v}(\vec{r},t)}{T(\vec{r},t)} \quad , \tag{20}$$

$$\vec{J}^{S}(\vec{r},t) = L_{q}\,\vec{\nabla}\,\frac{1}{T(\vec{r},t)} \quad . \tag{21}$$

Here L , L_{v} and L_{q} are constants, $\vec{v}$ and T are the velocity and temperature fields of the fluid, $\overset{\rightrightarrows}{U}$ is the unit tensor and $\overline{\ldots}$ denotes the traceless symmetric part of a tensor. The fact that the Onsager coefficients L , L_{v} and L_{q} have to be constant is a severe limitation, for it implies e.g. that the bulk viscosity $\eta = L/T(\vec{r})$ is inversely proportional to the temperature. Such a dependence does not occur in real systems. It thus turns out that for the Landau-Lifshitz approach to non-linear hydrodynamic fluctuations to be consistent, severe restrictions on the form of the phenomenological laws have to be imposed. To rate this conclusion at its true value, it should be kept in mind that in thermodynamics of irreversible processes similar limitations are encountered[5]: stationary states are strictly speaking only states of minimum entropy production if the phenomenological coefficients have a specific dependence on the state variables. Thus, in the case of thermal conduction, it is precisely the coefficient L_{q} appearing in eq.(21) which has to be constant in order that the entropy production in the stationary state is minimal. If, however, the system is sufficiently close to global equilibrium, the theorem of minimum entropy production is still valid in good approximation since the phenomenological coefficients may be assumed to depend only on the overall equilibrium values of the state parameters. We shall comment further on the result expressed by eqs.(20) and (21) in chapter III (cf. also the next remark).

vi) That the conclusion of our analysis concerning the specific form of the phenomenological laws (cf. eqs.(20) and (21)) does not invalidate the analysis of the most dominant term in the long time tails (see section 1.3) can be seen as follows. Mode-mode coupling calculations are usually based on perturbation expansions. A typical diagram contributing to the renormalization of the transport coefficients is



Here —▸— stands for the propagator of the linear problem and the vertices —< are the interaction vertices arising from the non-linear coupling of the modes. The coupling strength related with a vertex is in general proportional to a certain power, μ say, of the wavenumber k. E.g. convective non-linearities like the term $\vec{v}\cdot\vec{\nabla}\vec{v}$ in the equation for the velocity field give rise to a coupling strength of the modes which is proportional to $k (\mu=1)$. The long time tails are associated with the small k behaviour of the diagrams. Since the propagators —▸— are exponentially damped as $\exp(-Dk^2t)$ where D is some transport coefficient, the asymptotic behaviour of the renormalized transport coefficient $D_R(\vec{k},t)$ for $Dk^2t \ll 1$ is essentially determined by integrals of the form

$$D_R(\vec{k},t) \sim \int dk'\, k'^{\,d-1+2(\mu-1)}\, e^{-2Dk'^2t} \sim t^{-d/2-(\mu-1)} \qquad (Dk^2t \ll 1) \quad . \qquad (22)$$

Here d is the dimensionality of the system. It follows from this result that the convective non-linearities $(\mu=1)$ give terms proportional to $t^{-d/2}$. On the other hand, non-linearities resulting from the necessity to take into account the proper form of the phenomenological laws correspond with vertices proportional to k^2, so that $\mu=2$ (remember that e.g. for an incompressible fluid one obtains from eq.(20) $\vec{\nabla}\cdot\vec{\Pi}^{s} = - L\, \vec{\nabla}\cdot(\overline{\vec{\nabla}\vec{v}}/T)$. Consequently, corrections from such terms decay faster than $t^{-d/2}$. We therefore conclude that our results regarding the specific form of the phenomenological laws do not affect the expressions obtained for the long time tails in the regime $D k^2 t \ll 1$.

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CHAPTER II

HYDRODYNAMICS FOR AN IDEAL FLUID: HAMILTONIAN FORMALISM AND LIOUVILLE-EQUATION

1. Introduction

Clebsch^{1,2}) was the first to derive, in 1859, the hydrodynamic equations for an ideal fluid from a variational principle for Euler coordinates. His derivation was, however, restricted to the case of an incompressible fluid. Later Bateman^{3}) showed that the analysis of Clebsch also applies to compressible fluids if the pressure is a function of density alone. Finally, in 1968, Seliger and Whitham^{4}) formulated a Lagrangian density for the most general case, i.e. taking also into account the dependence on entropy. From a Lagrangian formalism it is of course in general possible to go over to a Hamiltonian description. For hydrodynamics, this was done by Kronig and Thellung^{5,6}) in order to quantize the fluid equations. As they based their work on Bateman's analysis, their results only apply to the case of isentropic (or, alternatively, isothermal) flow.

Recently, there has been renewed interest in a Hamiltonian formulation of hydrodynamics. In an interesting paper Enz and Turski^{7}) considered hydrodynamic fluctuations on the basis, and with the limitations, of the formalism developed by Thellung^{6}).

In this paper we will develop a Hamiltonian formalism for the general case of Seliger and Whitham and discuss a number of statistical properties of an ideal fluid. This discussion will enable us to study in a subsequent paper nonlinear fluctuations in a real fluid.

In section 2 we discuss the Clebsch representation of the fluid velocity field. Seliger and Whitham's^{4}) variational principle, which is based on this representation, is reviewed in section 3. We then introduce a Hamiltonian description of hydrodynamics and define Poisson-brackets in terms of the canonical (nonphysical) fields in a standard way. From these one may derive Poisson-bracket expressions for the physical (hydrodynamic) fields. These turn out to be identical to those introduced on the basis of symmetry considerations and without the use of canonical fields by Dzyaloshinskii and Volovick[8]).

In section 4 we consider an ensemble of ideal fluids and study the evolution in time of the density distribution function in the "phase space" of physical fields. We conclude that if the physical fields are the mass-density, momentum-density and energy-density, the flow in the corresponding phase space is incompressible, so that the density distribution function obeys a Liouville-equation. The standard equilibrium distributions of fluctuation theory are of course stationary solutions of this equation. This is discussed in section 5.

2. The Clebsch representation for an ideal fluid

The behaviour of an ideal fluid is described by the five hydrodynamic equations (conservation laws)

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \rho \boldsymbol{v}, \tag{2.1}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} \cdot \rho \boldsymbol{v} \boldsymbol{v} - \boldsymbol{\nabla} p, \tag{2.2}$$

$$\frac{\partial \rho s}{\partial t} = -\boldsymbol{\nabla} \cdot \rho \boldsymbol{v} s, \tag{2.3}$$

where $\rho(\boldsymbol{r}, t)$ is the mass density, $\boldsymbol{v}(\boldsymbol{r}, t)$ the velocity, $p(\boldsymbol{r}, t)$ the hydrostatic pressure and $s(\boldsymbol{r}, t)$ the entropy per unit of mass. These equations can equivalently be written as

$$\rho \frac{\mathrm{d}v}{\mathrm{d}t} = \boldsymbol{\nabla} \cdot \boldsymbol{v}, \tag{2.4}$$

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} p, \tag{2.5}$$

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = 0, \tag{2.6}$$

where $v \equiv \rho^{-1}$ is the specific volume, and where

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \tag{2.7}$$

is the total time derivative.

In order to formulate a variational principle, as will be done in the next section, it is convenient to introduce the Clebsch representation[1,2])

$$\boldsymbol{v} = -\boldsymbol{\nabla}\phi_p - \lambda\boldsymbol{\nabla}\phi_\lambda - s\boldsymbol{\nabla}\phi_s, \tag{2.8}$$

where the three components of the velocity field are given in terms of four scalar fields as well as the entropy*. Clearly one of the fields is redundant. In fact, one could, in principle, always represent an arbitrary velocity field with the choice $\phi_s = 0$. As will become apparent below, however, it is convenient to choose ϕ_s unequal to zero if the flow is not isentropic (i.e. $\boldsymbol{\nabla}s \neq 0$). In this case it will be necessary that an additional equation specifies this field.

In the Clebsch representation, the vorticity of the fluid is given by

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \wedge \boldsymbol{v} = -\boldsymbol{\nabla}\lambda \wedge \boldsymbol{\nabla}\phi_\lambda - \boldsymbol{\nabla}s \wedge \boldsymbol{\nabla}\phi_s. \tag{2.9}$$

The total time derivative of the velocity field may then be written as

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}v^2 - \boldsymbol{v}\wedge\boldsymbol{\omega}\\ &= -\boldsymbol{\nabla}\left(\dot{\phi}_p + \lambda\dot{\phi}_\lambda + s\dot{\phi}_s - \frac{1}{2}v^2\right) + \left(\frac{\mathrm{d}\phi_\lambda}{\mathrm{d}t}\right)\boldsymbol{\nabla}\lambda - \left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)\boldsymbol{\nabla}\phi_\lambda + \left(\frac{\mathrm{d}\phi_s}{\mathrm{d}t}\right)\boldsymbol{\nabla}s,\end{aligned} \tag{2.10}$$

where partial time derivatives are denoted by a dot. In obtaining this equation, use has been made of eq. (2.6). On the other hand, eq. (2.5) may be written in the form

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\boldsymbol{\nabla}h + T\boldsymbol{\nabla}s, \tag{2.11}$$

where h is the enthalpy per unit of mass and T the temperature, and where we have used the thermodynamic relation

$$\frac{1}{\rho}\mathrm{d}p = \mathrm{d}h - T\,\mathrm{d}s. \tag{2.12}$$

The four scalar fields ϕ_p, λ, ϕ_λ and ϕ_s must therefore be chosen in such a way that the right-hand side of eq. (2.10) reduces to the right-hand side of eq. (2.11). This requirement is not sufficient to determine uniquely the equations satisfied by these fields. An appropriate and customary choice[4]) is

$$\dot{\phi}_p + \lambda\dot{\phi}_\lambda + s\dot{\phi}_s - \frac{1}{2}v^2 = h, \tag{2.13}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0, \tag{2.14}$$

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* For the special case of isentropic flow, the last term in eq. (2.8) may be absorbed into the first, so that $\boldsymbol{v} = -\boldsymbol{\nabla}\phi - \lambda\boldsymbol{\nabla}\phi_\lambda$ where $\phi = \phi_p + s\phi_s$.

$$\frac{\mathrm{d}\phi_\lambda}{\mathrm{d}t} = 0, \tag{2.15}$$

$$\frac{\mathrm{d}\phi_s}{\mathrm{d}t} = T. \tag{2.16}$$

The last equation may be viewed as the equation necessary to specify the redundant field.

The equations of motion for the six fields (ρ, s, λ, ϕ_ρ, ϕ_s and ϕ_λ) describing the system in the Clebsch representation, are then the eqs. (2.1), (2.2) and (2.13)–(2.16).

It should be noted that for stationary states (for which the physical fields ρ, v and s are independent of time) the fields ϕ_ρ, ϕ_λ and ϕ_s are not necessarily independent of time[4]). This is obvious for the case of a homogeneous fluid at rest for which it follows from eq. (2.16), that $\phi_s = Tt$. Alternatively, for steady flow Bernoulli's theorem states that $\frac{1}{2}v^2 + h$ is constant along a streamline, so that, according to eq. (2.13),

$$\dot{\phi}_\rho + \lambda\dot{\phi}_\lambda + s\dot{\phi}_s = \text{constant}$$

along a streamline. As this constant is in general different along different streamlines, it can not be set equal to zero.

In the next section we will review the derivation of the hydrodynamic equations from a variational principle using the Clebsch representation. We shall then also discuss the Hamiltonian formulation of the hydrodynamic equations.

3. Variational principle for ideal fluids and Hamiltonian formulation

Following Seliger and Whitham[4]), the six eqs. (2.1), (2.3) and (2.13)–(2.16) can be found as Euler–Lagrange equations from the variational principle

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\boldsymbol{r}\mathcal{L} = 0, \tag{3.1}$$

where the Lagrangian density $\mathcal{L}$ is defined as

$$\mathcal{L} = \rho\{\dot{\phi}_\rho + \lambda\dot{\phi}_\lambda + s\dot{\phi}_s - \tfrac{1}{2}(\nabla\phi_\rho + \lambda\nabla\phi_\lambda + s\nabla\phi_s)^2 - u(\rho, s)\}. \tag{3.2}$$

Here u is the internal energy per unit of mass, taken as a function of ρ and s, for which one has the thermodynamic relations

$$\left(\frac{\partial u}{\partial \rho}\right)_s = \frac{p}{\rho^2}, \quad \left(\frac{\partial u}{\partial s}\right)_\rho = T. \tag{3.3}$$

Eq. (3.1) must hold for arbitrary variations of the fields which vanish at t_1 and t_2 and at the boundary of the volume V occupied by the fluid. Thus one finds for arbitrary variations of the density ρ

$$\mathcal{L} = p. \tag{3.4}$$

This equation expresses the fact that the Lagrangian density reduces to the pressure on the extremal path and is equivalent to eq. (2.13) since $h = u + p/\rho$. For arbitrary variations with respect to ϕ_ρ, ϕ_s, ϕ_λ, λ and s one finds in the same way eqs. (2.1), (2.3) and (2.14)–(2.16) respectively as Euler–Lagrange equations. As discussed in the previous section, these equations are, together with eq. (3.4), the hydrodynamic equations in the Clebsch representation*.

Clebsch, in his masterful 1859 paper¹), showed already that a variational principle could be formulated for the case that the equation of motion for the velocity field of the fluid can be written in the form

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla\psi \tag{3.5}$$

if one uses the representation

$$\boldsymbol{v} = -\nabla\phi_\rho - \lambda\nabla\phi_\lambda. \tag{3.6}$$

Eq. (3.5) holds for a number of cases in which the hydrodynamic equations are constrained. Examples are: 1. the case of incompressible flow which was considered by Clebsch himself and for which $\psi = p/\rho$ (cf. eq. (2.5)); 2. the case of isentropic flow, for which $\psi = h$ (cf. eq. (2.12)); and 3. isothermal flow, for which $\psi = h - Ts$. The work of Bateman³) comprises both case 2 and 3. If the equations are not constrained however, the equation of motion is not of the form (3.5). It is then convenient to use the more general Clebsch representation, eq. (2.8).

From the Lagrangian formulation of the hydrodynamic equations one may go over to a Hamiltonian formalism. Contrary to the usual case however, we will obtain the same number of Hamiltonian equations, since the Lagrangian equations are already differential equations of first order in time. As the time derivatives of ρ, λ and s do not appear in the Lagrangian, one can only define the three momenta conjugate to the fields ϕ_ρ, ϕ_λ and ϕ_s,

$$\Pi_\rho \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_\rho} = \rho, \tag{3.7}$$

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* The derivation of the energy and momentum conservation laws on the basis of Noether's theorem is straightforward and in fact completely parallels the corresponding analysis by Thellung⁶) for isentropic flow, to which the reader is referred.

$$\Pi_\lambda \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\lambda} = \rho\lambda, \tag{3.8}$$

$$\Pi_s \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_s} = \rho s \equiv s_v. \tag{3.9}$$

It is seen that these momenta are the fields ρ, $\rho\lambda$ and s_v, the entropy per unit of volume. The Hamiltonian density, given by

$$\mathcal{H} \equiv \dot{\phi}_\rho \Pi_\rho + \dot{\phi}_\lambda \Pi_\lambda + \dot{\phi}_s \Pi_s - \mathcal{L}, \tag{3.10}$$

becomes

$$\mathcal{H} = \tfrac{1}{2}(\Pi_\rho \nabla\phi_\rho + \Pi_\lambda \nabla\phi_\lambda + \Pi_s \nabla\phi_s)^2/\Pi_\rho + \Pi_\rho u(\Pi_\rho, \Pi_s). \tag{3.11}$$

The hydrodynamic equations may now be found as Hamiltonian equations of motion:

$$\dot{\phi}_\rho = \frac{\partial \mathcal{H}}{\partial \Pi_\rho} = -\boldsymbol{v} \cdot \nabla\phi_\rho - \tfrac{1}{2}v^2 + \mu, \tag{3.12}$$

$$\dot{\phi}_\lambda = \frac{\partial \mathcal{H}}{\partial \Pi_\lambda} = -\boldsymbol{v} \cdot \nabla\phi_\lambda, \tag{3.13}$$

$$\dot{\phi}_s = \frac{\partial \mathcal{H}}{\partial \Pi_s} = -\boldsymbol{v} \cdot \nabla\phi_s + T, \tag{3.14}$$

$$\dot{\Pi}_\rho = \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla\phi_\rho} = -\nabla \cdot \Pi_\rho \boldsymbol{v}, \tag{3.15}$$

$$\dot{\Pi}_\lambda = \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla\phi_\lambda} = -\nabla \cdot \Pi_\lambda \boldsymbol{v}, \tag{3.16}$$

$$\dot{\Pi}_s = \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla\phi_s} = -\nabla \cdot \Pi_s \boldsymbol{v}, \tag{3.17}$$

where the chemical potential μ per unit of mass is given by

$$\mu \equiv u - Ts + p/\rho = \partial(\Pi_\rho u(\Pi_\rho, \Pi_s))/\partial \Pi_\rho. \tag{3.18}$$

Eqs. (3.13)–(3.17) are identical with eqs. (2.1), (2.3) and (2.14)–(2.16). Eq. (3.12) can be shown to be identical to eq. (2.13) by using once more eqs. (3.13) and (3.14).

As one would expect, the Hamiltonian density eq. (3.11) represents the total energy density e_v:

$$\mathcal{H} = e_v \equiv \tfrac{1}{2}j^2/\rho + \rho u, \tag{3.19}$$

with j the momentum density given by, cf. eqs. (2.8) and (3.7)–(3.9),

$$\boldsymbol{j} \equiv \rho \boldsymbol{v} = -\Pi_\rho \nabla\phi_\rho - \Pi_\lambda \nabla\phi_\lambda - \Pi_s \nabla\phi_s. \tag{3.20}$$

By introducing the total Hamiltonian

$$H \equiv \int \mathrm{d}r \mathcal{H}(r), \tag{3.21}$$

the Hamiltonian equations of motion (3.12)–(3.17) can also be written as

$$\dot{\phi}_k = \frac{\delta H}{\delta \Pi_k}, \quad \dot{\Pi}_k = -\frac{\delta H}{\delta \phi_k}, \quad \text{with } k = \rho, \lambda, s, \tag{3.22}$$

where the functional derivatives $\delta/\delta\phi_k$ and $\delta/\delta\Pi_k$ are defined in the standard way^{9}).

We now define Poisson-brackets

$$\{A, B\} \equiv \int \mathrm{d}r \sum_{k=\rho,\lambda,s} \left[\frac{\delta A}{\delta\phi_k(r)} \frac{\delta B}{\delta\Pi_k(r)} - \frac{\delta A}{\delta\Pi_k(r)} \frac{\delta B}{\delta\phi_k(r)}\right], \tag{3.23}$$

where A and B are arbitrary functionals of the canonical fields. The corresponding Poisson-bracket for either isothermal or isentropic flow was introduced recently by Enz and Turski^{7}). If we then consider the five physical fields ρ, j and e_v as components α_i of a five-dimensional vector field $\underset{\sim}{\alpha}$ such that

$$\begin{aligned} &\alpha_0 \equiv \rho, \\ &\alpha_i \equiv j_i, \quad \text{for } i = 1, 2, 3, \\ &\alpha_4 \equiv e_v, \end{aligned} \tag{3.24}$$

we can define the following basic Poisson-brackets:

$$L_{ij}(r, r') \equiv \{\alpha_i(r), \alpha_j(r')\} = -L_{ji}(r', r), \quad \text{for } i, j = 0, \dots, 4. \tag{3.25}$$

These matrix elements can be evaluated in a straightforward way (see appendix); one finds

$$\begin{aligned} &L_{00}(r, r') = 0, \\ &L_{0i}(r, r') = -L_{i0}(r', r) = -\frac{\partial}{\partial r_i}[\rho(r)\delta(r - r')], \quad \text{for } i = 1, 2, 3, \\ &L_{04}(r, r') = -L_{40}(r', r) = -\nabla \cdot [j(r)\delta(r - r')], \\ &L_{ij}(r, r') = -L_{ji}(r, r') = -\frac{\partial}{\partial r_j}[j_i(r)\delta(r - r')] \\ &\qquad\qquad + \frac{\partial}{\partial r'_i}[j_j(r')\delta(r - r')], \quad \text{for } i, j = 1, 2, 3, \\ &L_{i4}(r, r') = -L_{4i}(r', r) = -\nabla \cdot [v(r) j_i(r)\delta(r - r')] - \frac{\partial}{\partial r_i}[p(r)\delta(r - r')] \\ &\qquad\qquad + \frac{\partial}{\partial r'_i}[e_v(r')\delta(r - r')], \quad \text{for } i = 1, 2, 3, \\ &L_{44}(r, r') = -\nabla \cdot [v(r)(e_v(r) + p(r))\delta(r - r')] \\ &\qquad\qquad + \nabla' \cdot [v(r')(e_v(r') + p(r'))\delta(r - r')]. \end{aligned} \tag{3.26}$$

It is important to note that L depends on the six canonical fields only via the five physical fields. It is therefore possible, now that these basic Poisson-brackets have been evaluated within the canonical formalism, to define Poisson-brackets for functionals $A(\{\underline{\alpha}\})$ and $B(\{\underline{\alpha}\})$ of the physical fields entirely in terms of the matrix L without reference to the canonical fields:

$$\{A(\{\underline{\alpha}\}), B(\{\underline{\alpha}\})\} \equiv \sum_{i,j=0}^{4} \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}' \frac{\delta A}{\delta \alpha_i(\boldsymbol{r})} L_{ij}(\boldsymbol{r}, \boldsymbol{r}') \frac{\delta B}{\delta \alpha_j(\boldsymbol{r}')}. \tag{3.27}$$

With the help of these Poisson-brackets the hydrodynamic equations for the physical fields can be written in the compact form

$$\begin{aligned} \frac{\partial \alpha_i(\boldsymbol{r}, t)}{\partial t} &= \{\alpha_i(\boldsymbol{r}, t), H\} \\ &= \sum_j \int \mathrm{d}\boldsymbol{r}' L_{ij}(\boldsymbol{r}, \boldsymbol{r}') \frac{\delta H}{\delta \alpha_j(\boldsymbol{r}', t)} \\ &= \int \mathrm{d}\boldsymbol{r}' L_{i4}(\boldsymbol{r}, \boldsymbol{r}'), \quad \text{for } i = 0, \ldots, 4. \end{aligned} \tag{3.28}$$

Here we have used the fact that the Hamiltonian is a functional of α_4 only so that

$$\frac{\delta H}{\delta \alpha_i(\boldsymbol{r}')} = \delta_{i4}. \tag{3.29}$$

With relations (3.26) and the identification (3.24), eqs. (3.28) reduce to

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{j}, \tag{3.30}$$

$$\frac{\partial \boldsymbol{j}}{\partial t} = -\nabla \cdot [\boldsymbol{v}\boldsymbol{j} + p], \tag{3.31}$$

$$\frac{\partial e_v}{\partial t} = -\nabla \cdot [\boldsymbol{v}(e_v + p)]. \tag{3.32}$$

This set of equations is equivalent to the set (2.1)–(2.3). The use of the conserved quantities ρ, $\boldsymbol{j}$ and e_v will turn out to have certain advantages.

As a final remark, we note that for the total momentum

$$\boldsymbol{P} \equiv \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{j}(\boldsymbol{r}), \tag{3.33}$$

one finds with the aid of eqs. (3.26)

$$\{\boldsymbol{P}, \alpha_i(\boldsymbol{r})\} = \nabla \alpha_i(\boldsymbol{r}), \quad \text{for } i = 0, \ldots, 4. \tag{3.34}$$

Thus the total momentum is, as expected, the generator of spatial translations.

Recently Dzyaloshinskii and Volovick[8]) used this property as a starting point to arrive at expressions (3.28) for "hydrodynamic" Poisson-brackets without the use of the Hamiltonian formulation based on the Clebsch representation.

4. The Liouville equation

In a statistical description one considers an ensemble of fluid systems and introduces a density distribution $P(\{\underline{\alpha}(r)\}, t)$ in the "phase space" of physical fields. Since the hydrodynamic equations are first order in time, this density obeys a continuity or conservation equation which one would be tempted to write in the form

$$\frac{\partial P(\{\underline{\alpha}(r)\}, t)}{\partial t} = -\sum_i \int \mathrm{d}r \frac{\delta}{\delta \alpha_i(r)} \dot{\alpha}_i(r) P(\{\underline{\alpha}(r)\}, t). \tag{4.1}$$

However, strictly speaking this equation is a meaningless string of symbols, since it is not clear how the functional derivative $\delta\dot{\alpha}_i(r)/\delta\alpha_i(r)$ occurring in eq. (4.1) should be interpreted*.

A way to avoid this difficulty is to discretize the system in coordinate space, so that the phase space becomes of finite dimensionality**. To this end we divide the fluid into small cubic cells of size Δ^3. The position of a cell is denoted by $\boldsymbol{r} = \boldsymbol{n}\Delta$ where $\boldsymbol{n}$ is a vector of which the components are integer numbers. The state of the fluid in each cell is characterized by the five hydrodynamic variables

$$\underline{\alpha}_n = (\rho_n, \boldsymbol{j}_n, e_{v,n}), \tag{4.2}$$

the specific quantities in each cell. In the limit of vanishing cell-size these variables correspond to the previously defined hydrodynamic fields. The discrete hydrodynamic variables are postulated to obey equations of motion which are discrete analogues of the conservation laws (3.30)–(3.32), viz.

$$\frac{\partial \rho_n}{\partial t} = -\boldsymbol{\nabla}_n \cdot \boldsymbol{j}_n, \tag{4.3}$$

$$\frac{\partial \boldsymbol{j}_n}{\partial t} = -\boldsymbol{\nabla}_n \cdot [\boldsymbol{v}_n \boldsymbol{j}_n + p_n], \tag{4.4}$$

$$\frac{\partial e_{v,n}}{\partial t} = -\boldsymbol{\nabla}_n \cdot [(e_{v,n} + p_n)\boldsymbol{v}_n], \tag{4.5}$$

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* In the previous sections a similar difficulty did not occur since the functional derivatives occurring there had always a well-defined meaning.

** Discretization is in fact also necessary in order to normalize the distribution function.

where $v_n \equiv j_n/\rho_n$ is the discrete velocity and p_n the pressure in the cell labeled by n. The discrete operator ∇_n is defined as

$$\nabla_n A_n \equiv \sum_{i=1}^{3} \frac{\hat{e}_i}{2\Delta} [A_{n+\hat{e}_i} - A_{n-\hat{e}_i}], \tag{4.6}$$

with the $\hat{e}_i$'s unit vectors along the Cartesian axes. The continuity-equation for the distribution function $P(\{\underline{\alpha}_n\}, t)$ in the phase space spanned by the discrete set of variables (4.2) is

$$\frac{\partial P(\{\underline{\alpha}_n\}, t)}{\partial t} = -\sum_n \sum_{i=0}^{4} \frac{\partial}{\partial \alpha_{i,n}} \dot{\alpha}_{i,n} P(\{\underline{\alpha}_n\}, t). \tag{4.7}$$

It is now straightforward to show with the aid of eqs. (4.3)–(4.5) and the definition (4.6) that the following property holds:

$$\sum_i \frac{\partial \dot{\alpha}_{i,n}}{\partial \alpha_{i,n}} = 0. \tag{4.8}$$

This implies that the flow in *this particular phase space is incompressible.* Consequently eq. (4.7) reduces to

$$\frac{\partial P(\{\underline{\alpha}_n\}, t)}{\partial t} = -\sum_n \sum_{i=0}^{4} \dot{\alpha}_{i,n} \frac{\partial P(\{\underline{\alpha}_n\}, t)}{\partial \alpha_{i,n}}. \tag{4.9}$$

In the limit of continuous fields this equation becomes

$$\frac{\partial P(\{\underline{\alpha}(\mathbf{r})\}, t)}{\partial t} = -\sum_{i=0}^{4} \int \mathrm{d}\mathbf{r} \dot{\alpha}_i(\mathbf{r}) \frac{\delta P(\{\underline{\alpha}(\mathbf{r})\}, t)}{\delta \alpha_i(\mathbf{r})}, \tag{4.10}$$

provided that the limiting distribution function is a proper functional of $\underline{\alpha}(\mathbf{r})$[9]. Using also eqs. (3.27) and (3.28) this equation may also be written in the form

$$\frac{\partial P}{\partial t} = \{H, P\}. \tag{4.11}$$

This equation, in which functional derivatives have a well-defined meaning, is a Liouville-equation for the density distribution in the space of physical fields ρ, $\mathbf{j}$ and e_v. It can be used as the basis for the discussion of the statistical properties of an ideal fluid.

Clearly one could in principle also have derived a Liouville-equation for a distribution in the phase space of the six canonical fields. It is however the reduced density distribution in the space of five physical fields which is directly relevant for the evaluation of averages of physical quantities.

It should be stressed that while an equation of the form (4.7) is also valid for a transformed set of hydrodynamic variables $\underline{\boldsymbol{\beta}}_n(\underline{\boldsymbol{\alpha}}_n)$, the property (4.8) in this transformed set does not necessarily hold, nor is there then a Liouville-equation. Using eq. (4.8), one can indeed show in a straightforward way that

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = \sum_n \sum_{i,j,k} \dot{\alpha}_{k,n} \frac{\partial \alpha_{j,n}}{\partial \beta_{i,n}} \frac{\partial^2 \beta_{i,n}}{\partial \alpha_{j,n} \partial \alpha_{k,n}}. \tag{4.12}$$

Using also the identity[10])

$$\frac{\partial \ln \det \mathbf{G}}{\partial x} = \sum_{ij} G^{-1}_{ij} \frac{\partial G_{ji}}{\partial x}, \tag{4.13}$$

which holds for any matrix **G** which depends on a parameter x, eq. (4.12) can be re-written as

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = \sum_n \sum_i \dot{\alpha}_{i,n} \frac{\partial \ln J_n}{\partial \alpha_{i,n}} = \sum_n \sum_i \dot{\beta}_{i,n} \frac{\partial \ln J_n}{\partial \beta_{i,n}}, \tag{4.14}$$

where J_n is the Jacobian of the transformation $\underline{\beta}_n \rightarrow \underline{\alpha}_n$. Thus the flow in $\underline{\beta}$-space will only be incompressible if the Jacobian of the transformation to new hydrodynamic variables is a constant. Let us in particular consider three different sets of variables $\underline{\beta}$:

a. $\underline{\beta}_n = (\rho_n, j_n, u_{v,n})$.

Here $u_{v,n} = e_{v,n} - \frac{1}{2} j_n^2/\rho_n$ is the discrete internal energy per unit of volume. The Jacobian for this case is 1, so that the flow in the corresponding phase space is also incompressible.

b. $\underline{\beta}_n(\rho_n, v_n, e_{v,n})$.

Here $v_n \equiv j_n/\rho_n$ is the discrete velocity. The Jacobian of the transformation $\underline{\beta}_n \rightarrow \underline{\alpha}_n$ becomes in this case

$$J_n = \rho_n^{-3}, \tag{4.15}$$

and one has according to equation (4.14) the result

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = -3 \sum_n \dot{\rho}_n/\rho_n. \tag{4.16}$$

c. $\underline{\beta}_n = (\rho_n, j_n, s_{v,n})$.

Here $s_{v,n}$ is the discrete entropy per unit of volume, which is related to the energy density variable via the Gibbs-relation

$$T_n \, \mathrm{d}s_{v,n} = \mathrm{d}u_{v,n} - \mu_n \, \mathrm{d}\rho_n. \tag{4.17}$$

T_n and μ_n are the temperature and the chemical potential in cell n respectively. The Jacobian becomes in this case

$$J_n = T_n^{-1}. \tag{4.18}$$

Hence one obtains here

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = -\sum_n \left[\frac{\dot{s}_{v,n}}{T_n} \left(\frac{\partial T_n}{\partial s_{v,n}} \right)_\rho + \frac{\dot{\rho}_n}{T_n} \left(\frac{\partial T_n}{\partial \rho_n} \right)_{s_v} \right]. \tag{4.19}$$

Both in case 2 and 3 the flow in "phase space" is *not* incompressible and a Liouville-equation does not hold. Moreover, it is clear from the general form of eq. (4.14) and the results in the special cases considered above, that the "rate of compression" is either zero (e.g. case 1) or infinite (e.g. cases 2 and 3) in the continuum-limit. In the latter cases the continuum-limit of eq. (4.7) does not exist and we can not give a well-defined meaning to the functional derivative occurring in eq. (4.1)*.

A final remark must be made in connection with case c. The validity of eq. (4.8) for the original set of variables (4.2) is an immediate consequence of the fact that these variables obey the conservation laws (4.3)–(4.5). On the basis of these equations, using also eq. (4.17), the discrete entropy $s_{v,n}$ then satisfies an evolution-equation**

$$\begin{aligned} \frac{\partial s_{v,n}}{\partial t} = \frac{1}{T_n} [&- \boldsymbol{\nabla}_n \{ (\tfrac{1}{2}\rho_n v_n^2 + u_{v,n} + p_n) \boldsymbol{v}_n \} - \tfrac{1}{2} v_n^2 (\boldsymbol{\nabla}_n \cdot \boldsymbol{j}_n) \\ &+ \{ \boldsymbol{\nabla}_n \cdot (\boldsymbol{v}_n \boldsymbol{j}_n + \mathbf{p}_n) \} \cdot \boldsymbol{v}_n + \mu_n (\boldsymbol{\nabla}_n \cdot \boldsymbol{j}_n)]. \end{aligned} \tag{4.20}$$

Thus the discrete entropy is not conserved (even though eq. (4.20) reduces to eq. (2.3) in the continuum-limit). In this connection it should be remembered that discontinuities such as shock waves in an ideal fluid give rise to an increase of entropy^{11}). On the other hand, if one had started by postulating that ρ_n, $\boldsymbol{j}_n$ and $s_{v,n}$ obey the discrete analogues of eqs. (2.1)–(2.3), the discrete energy $e_{v,n}$ would not have been conserved. As only a description in which mass, momentum and energy are conserved seems satisfying, we have based our discussion on eqs. (4.3)–(4.5).

5. The equilibrium distribution

According to the Liouville-equation (4.11), the distribution function $P(\{\underline{\alpha}(r)\}, t)$ in the phase space of the basic physical fields (3.24) is stationary

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* An alternative way to give meaning to eq. (4.1) would have been to consider the phase space spanned by the Fourier-components of the particular set of hydrodynamic variables (3.24) and to introduce an ad hoc cut-off wave-vector for these variables. If such a procedure is carried out consistently, one obtains in a less transparent way results equivalent to those found above.

** One may easily verify that the result eq. (4.19) can also be obtained from eq. (4.20), together with eqs. (4.3) and (4.4).

if P is only a function of the constants of the motion of the system, such as its total energy E, its total entropy S and its total mass M

$$E = \int \mathrm{d}r e_v(r), \quad S = \int \mathrm{d}r s_v(r) \quad \text{and} \quad M = \int \mathrm{d}r \rho(r). \tag{5.1}$$

For a system which is energetically and materially insulated, the equilibrium distribution is given by the Einstein formula

$$P^{\mathrm{eq}}(\{\underline{\alpha}(r)\}) \sim \mathrm{e}^{S(\{\underline{\alpha}(r)\})/k}, \tag{5.2}$$

where k is Boltzmann's constant. This distribution can also be written in the form

$$P^{\mathrm{eq}}(\{\underline{\alpha}(r)\}) = P^{\mathrm{eq}}(\{\underline{\alpha}^0(r)\})\, \mathrm{e}^{\Delta S(\{\underline{\alpha}(r)\})}, \tag{5.3}$$

where

$$\Delta S(\{\underline{\alpha}(r)\}) \equiv S(\{\underline{\alpha}(r)\}) - S^0(\{\underline{\alpha}^0(r)\}) = \int \mathrm{d}r\, (s_v(\alpha(r)) - s_v(\underline{\alpha}^0(r))). \tag{5.4}$$

Here the superscript zero refers to the "equilibrium state" characterised by a uniform density and energy-field and a vanishing momentum density field.

If the system is in thermal contact with a heat-bath with temperature T_0 the equilibrium-distribution is[12])

$$P^{\mathrm{eq}}(\{\underline{\alpha}(r)\}) = P^{\mathrm{eq}}(\{\underline{\alpha}^0(r)\})\, \mathrm{e}^{-(\Delta E - T_0 \Delta S)/kT_0}, \tag{5.5}$$

while for a system which can in addition exchange mass with a reservoir, this distribution is

$$P^{\mathrm{eq}}(\{\underline{\alpha}(r)\}) = P^{\mathrm{eq}}(\{\underline{\alpha}^0(r)\})\, \mathrm{e}^{-(\Delta E - T_0 \Delta S - \mu_0 \Delta M)/kT_0}, \tag{5.6}$$

where μ_0 is the thermal chemical potential of the reservoir.

The distributions (5.5) and (5.6) follow immediately from eq. (5.3). One has indeed according to this equation for the system including the bath

$$P^{\mathrm{eq}} \sim \mathrm{e}^{(\Delta S + \Delta S_{\mathrm{bath}})/k}. \tag{5.7}$$

For the bath we may write in good approximation

$$\Delta S_{\mathrm{bath}} = \frac{1}{T_0} \Delta E_{\mathrm{bath}} - \frac{\mu_0}{T_0} \Delta M_{\mathrm{bath}}. \tag{5.8}$$

Since the total energy and mass of the system together with the bath are constant, $\Delta E_{\mathrm{bath}} = -\Delta E$ and $\Delta M_{\mathrm{bath}} = -\Delta M$, the distribution (5.7) reduces to the distribution (5.6). In the same way eq. (5.5) is found if the system is materially insulated, i.e. if $\Delta M = 0$.

It should be stressed that the distribution functions (5.3), (5.5) and (5.6) are equilibrium distribution functions in the phase space of the fields ρ, j and e_v. If one had considered a different phase space, e.g. the one spanned by the fields ρ, v and e_v, the above distributions must be divided by the Jacobian of the corresponding transformation*, which in general does not exist in the continuum-limit. For linearized hydrodynamics, this problem does not arise as the transformations considered are then always linear so that the Jacobian is a constant.

6. Discussion

In the preceding sections we have used the Hamiltonian formalism based on the canonical fields of the extended Clebsch representation introduced by Seliger and Whitham^{3}) to obtain expressions for hydrodynamic Poisson-brackets. With the help of these, we were able to formulate a Liouville-equation in the phase space of the particular set of (non-canonical) physical fields ρ, j and e_v. The well-known equilibrium distribution functions for the fluid fluctuations are stationary solutions of this Liouville-equation. This fact plays an important role in an analysis of nonlinear hydrodynamic fluctuations as we shall discuss in a subsequent paper.

Appendix

In this appendix we evaluate the Poisson-brackets defined by eq. (3.25) for the physical fields ρ, j and e_v, which are given in terms of the canonical fields by (cf. eqs. (3.7)–(3.9), (3.19) and (3.20))

$$\rho = \Pi_\rho, \tag{A.1}$$

$$j = -\Pi_\rho \nabla\phi_\rho - \Pi_\lambda \nabla\phi_\lambda - \Pi_s \nabla\phi_s, \tag{A.2}$$

$$e_v = \tfrac{1}{2}(\Pi_\rho \nabla\phi_\rho + \Pi_\lambda \nabla\phi_\lambda + \Pi_s \nabla\phi_s)^2/\Pi_\rho + u_v(\Pi_\rho, \Pi_s). \tag{A.3}$$

Here $u_v = \rho u$ is the internal energy per unit of volume. We will first evaluate the Poisson-brackets for ρ, j and u_v.

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* Einstein^{13}) in his original article on fluctuation theory includes in the distribution function in addition to the exponential factor $\exp(\Delta S/k)$ a function f which depends on the choice of the fluctuating variables and which plays the same role as the Jacobian in the above discussion. He then argues that if ΔS may be approximated by a quadratic function, the function f may be replaced by a constant f. This corresponds to the linear theory.

Since the Poisson-brackets of momenta vanish, we immediately obtain

$$L_{00} = \{\rho, \rho\} = 0, \tag{A.4}$$

$$\{\rho, u_v\} = 0, \tag{A.5}$$

$$\{u_v, u_v\} = 0. \tag{A.6}$$

From the definition (A.2) we get

$$\frac{\delta \mathbf{j}(\mathbf{r})}{\delta \phi_\alpha(\mathbf{r}')} = \nabla'(\Pi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')), \quad (\alpha = \rho, \lambda, s) \tag{A.7}$$

$$\frac{\delta \mathbf{j}(\mathbf{r})}{\delta \Pi_\alpha(\mathbf{r})} = -\nabla\phi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}'), \quad (\alpha = \rho, \lambda, s) \tag{A.8}$$

where $\nabla' = \partial/\partial \mathbf{r}'$. One therefore obtains

$$\begin{aligned} \{\Pi_\alpha(\mathbf{r}), j_i(\mathbf{r}')\} &= -\int \mathrm{d}\mathbf{r}'' \frac{\delta \Pi_\alpha(\mathbf{r})}{\delta \Pi_\alpha(\mathbf{r}'')} \frac{\delta j_i(\mathbf{r}')}{\delta \phi_\alpha(\mathbf{r}'')} \\ &= -\int \mathrm{d}\mathbf{r}'' \delta(\mathbf{r} - \mathbf{r}'')\nabla''_i(\Pi_\alpha(\mathbf{r}'')\delta(\mathbf{r} - \mathbf{r}'')) \\ &= -\nabla_i(\Pi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')), \quad (i = 1, 2, 3;\ \alpha = \rho, \lambda, s). \end{aligned} \tag{A.9}$$

Thus we have

$$L_{0i}(\mathbf{r}, \mathbf{r}') = \{\rho(\mathbf{r}), j_i(\mathbf{r}')\} = -\nabla_i(\rho(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')) \quad (i = 1, 2, 3). \tag{A.10}$$

Similarly we obtain

$$\begin{aligned} L_{ik}(\mathbf{r}, \mathbf{r}') = \{j_i(\mathbf{r}), j_k(\mathbf{r}')\} &= -\int \mathrm{d}\mathbf{r}'' \sum_{\alpha = \rho, \lambda, s} [\nabla''_i(\Pi_\alpha(\mathbf{r}'')\delta(\mathbf{r} - \mathbf{r}'')) \\ &\quad \times \nabla'_k\phi_\alpha(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r}'') \\ &\quad - \nabla_i\phi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'')\nabla''_k(\Pi_\alpha(\mathbf{r}'')\delta(\mathbf{r}' - \mathbf{r}''))] \\ &= -\sum_{\alpha = \rho, \lambda, s} [\nabla'_k\phi_\alpha(\mathbf{r}')\nabla'_i(\Pi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \\ &\quad - \nabla_i\phi_\alpha(\mathbf{r})\nabla_k(\Pi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'))] \\ &= -\sum_{\alpha = \rho, \lambda, s} [\nabla'_i(\Pi_\alpha(\mathbf{r}')\nabla'_k\phi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \\ &\quad - \nabla_k(\Pi_\alpha(\mathbf{r})\nabla_i\phi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'))] \\ &= \nabla'_i(j_k(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \\ &\quad - \nabla_k(j_i(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')) \quad (i, k = 1, 2, 3). \end{aligned} \tag{A.11}$$

From the thermodynamic relation

$$\mathrm{d}u_v = T\,\mathrm{d}s_v + \mu\,\mathrm{d}\rho, \tag{A.12}$$

one finds that the derivatives of u_v with respect to $s_v(= \rho s = \Pi_s)$ and $\rho(= \Pi_\rho)$

are T and μ respectively. With eq. (A.9) and the definition (3.13), one then gets

$$\begin{aligned}\{u_v(\mathbf{r}), j_i(\mathbf{r}')\} &= -\mu(\mathbf{r})\nabla_i(\rho(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) - T(\mathbf{r})\nabla_i(s_v(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) \\ &= -(u_v(\mathbf{r}) + p(\mathbf{r}))\nabla_i\delta(\mathbf{r}-\mathbf{r}') \\ &\quad -(\mu(\mathbf{r})\nabla_i\rho(\mathbf{r}) + T(\mathbf{r})\nabla_i s_v(\mathbf{r}))\delta(\mathbf{r}-\mathbf{r}') \quad (i = 1, 2, 3). \qquad \text{(A.13)}\end{aligned}$$

Using eq. (A.12) once more, we can rewrite eq. (A.13) as

$$\{u_v(\mathbf{r}), j_i(\mathbf{r}')\} = -\nabla_i(u_v(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) - p(\mathbf{r})\nabla_i\delta(\mathbf{r}-\mathbf{r}') \quad (i = 1, 2, 3). \qquad \text{(A.14)}$$

With the aid of the Poisson-brackets (A.4), (A.5), (A.11) and (A.14) we now find

$$\begin{aligned}L_{04}(\mathbf{r}, \mathbf{r}') &= \{\rho(\mathbf{r}), e_v(\mathbf{r}')\} = \{\rho(\mathbf{r}), j^2(\mathbf{r}')/2\rho(\mathbf{r}') + u_v(\mathbf{r}')\} \\ &= \{\rho(\mathbf{r}), \mathbf{j}(\mathbf{r}')\} \cdot \mathbf{v}(\mathbf{r}') = -\boldsymbol{\nabla} \cdot (\mathbf{j}(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')), \qquad \text{(A.15)}\end{aligned}$$

and

$$\begin{aligned}L_{i4}(\mathbf{r}, \mathbf{r}') &= \{j_i(\mathbf{r}), e_v(\mathbf{r}')\} = \{j_i(\mathbf{r}), j^2(\mathbf{r}')/2\rho(\mathbf{r}') + u_v(\mathbf{r}')\}, \\ &= \{j_i(\mathbf{r}), \mathbf{j}(\mathbf{r}')\} \cdot \mathbf{v}(\mathbf{r}') - \{j_i(\mathbf{r}), \rho(\mathbf{r}')\}\tfrac{1}{2}v^2(\mathbf{r}') + \{j_i(\mathbf{r}), u_v(\mathbf{r}')\}, \\ &= \nabla_i'(\mathbf{j}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) \cdot \mathbf{v}(\mathbf{r}') - \boldsymbol{\nabla}(j_i(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) \cdot \mathbf{v}(\mathbf{r}') \\ &\quad -\tfrac{1}{2}v^2(\mathbf{r}')\nabla_i'(\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) + \nabla_i'(u_v(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) + p(\mathbf{r}')\nabla_i'\delta(\mathbf{r}-\mathbf{r}'), \\ &= -\boldsymbol{\nabla} \cdot (\mathbf{j}(\mathbf{r})v_i(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) - \nabla_i(p(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) \\ &\quad + \nabla_i'(e_v(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) \quad (i = 1, 2, 3). \qquad \text{(A.16)}\end{aligned}$$

The Poisson-bracket $\{e_v, e_v\}$ can be written as

$$\begin{aligned}L_{44}(\mathbf{r}, \mathbf{r}') = \{e_v(\mathbf{r}), e_v(\mathbf{r}')\} &= -\tfrac{1}{2}v^2(\mathbf{r})\{\rho(\mathbf{r}), e_v(\mathbf{r}')\} + \mathbf{v}(\mathbf{r}) \cdot \{\mathbf{j}(\mathbf{r}), e_v(\mathbf{r}')\} \\ &\quad + \{u_v(\mathbf{r}), \mathbf{j}(\mathbf{r}')\} \cdot \mathbf{v}(\mathbf{r}'). \qquad \text{(A.17)}\end{aligned}$$

Upon substitution of eqs. (A.14), (A.15) and (A.16) into eq. (A.17) one finally arrives at

$$\begin{aligned}L_{44}(\mathbf{r}, \mathbf{r}') = \{e_v(\mathbf{r}), e_v(\mathbf{r}')\} &= -\boldsymbol{\nabla} \cdot [\mathbf{v}(\mathbf{r})(e_v(\mathbf{r}) + p(\mathbf{r}))\delta(\mathbf{r}-\mathbf{r}')] \\ &\quad + \boldsymbol{\nabla}' \cdot [\mathbf{v}(\mathbf{r}')(e_v(\mathbf{r}') + p(\mathbf{r}'))\delta(\mathbf{r}-\mathbf{r}')]. \qquad \text{(A.18)}\end{aligned}$$

This completes the derivation of the expressions (3.26).

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CHAPTER III

NON-LINEAR HYDRODYNAMIC FLUCTUATIONS AROUND EQUILIBRIUM

1. Introduction

Linear fluctuation theory is well established and has been applied to hydrodynamics with considerable success[1]). Within the framework of such a linear theory, the hydrodynamic equations are *fully linearized.* Hydrodynamics, however, contains non-linear contributions from various sources. On the one hand, there are all those non-linear contributions which are due to the occurrence of reversible convective fluxes and also to the non-linear functional dependence of the thermodynamic functions on the thermodynamic state parameters. On the other hand, even if one wishes to remain within the domain of validity of non-equilibrium thermodynamics, in which case dissipative fluxes are linear functions of the gradients of the hydrodynamic fields, there occur additional non-linearities due to the dependence of the transport coefficients on the state variables, and to the presence of dissipative quadratic source terms such as the Rayleigh dissipation function.

Several methods have been developed to take into account the hydrodynamic non-linearities within the framework of a hydrodynamic fluctuation theory*. Thus the various approaches to what is generally known as

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* For fluctuations around stationary non-equilibrium states it is customary to first linearise the non-linear equations around the stationary state and then to perform a linear fluctuation analysis[2]). Such a procedure has, for instance, been used to discuss fluctuations near the Bénard instability[3]) and to discuss light-scattering in a stationary temperature gradient. A detailed discussion and an extensive list of references of the latter problem, which has also been studied by other methods[4]), can be found in ref. 5. It should be emphasized that in this paper we consider ***non-linear*** fluctuations around *equilibrium.*

mode–mode coupling[6]) have led to the introduction of the concept of renormalized transport coefficient, and to the calculation of the long-time behaviour, the long-time tails, exhibited by the hydrodynamic correlation functions[7]). One aspect, however, within the context of the different mode–mode coupling theories of the hydrodynamical equations, is that quite generally not all non-linearities occurring in these equations are retained in the analysis. For instance, the contribution of the Rayleigh dissipation term is usually neglected[8]). It is not clear whether such a procedure is consistent, and, in particular, whether the assumption which is usually made for the random fluxes occurring in the truncated equations, namely that they are Gaussian processes with white noise, is still compatible with the known equilibrium distribution of the hydrodynamic variables.

It is the aim of this paper to formulate a scheme for the description of hydrodynamic fluctuations around equilibrium, in which the non-linearities are retained in a consistent way and in which the assumption is made that the random fluxes are Gaussian processes with white noise. A preliminary version of this work was discussed in ref. 9. In spirit, our discussion is analogous to the one given by Enz and Turski[10]) who, however, restricted their treatment to the case in which temperature fluctuations do not occur, thereby omitting the energy-equation completely.

In a previous paper[11]), hereafter to be referred to as paper I, we developed a Hamiltonian formalism for the hydrodynamics of a one-component fluid. We also discussed the statistical description of such a system and derived the Liouville-equation for the density distribution in the "phase space" of the hydrodynamic fields. These statistical considerations shall now be extended to a real fluid in which one has the dissipative phenomena of heat conduction and viscous pressure.

In section 2 we summarize some of the results of paper I which we use in our subsequent discussion. In section 3 we introduce the stochastic differential equations for the hydrodynamic fields. The random fluxes are assumed to be Gaussian processes with white noise, while the form of the systematic parts of the dissipative currents remains unspecified. On the basis of this assumption we derive in section 4 the corresponding Fokker–Planck equation. This derivation is performed in a well-defined way by a discretization of the hydrodynamic fields. The Fokker–Planck equation which is obtained reduces to the Liouville-equation discussed in paper I in the continuum limit if the dissipative terms are neglected.

The form of the systematic part of the dissipative currents compatible with the known equilibrium distribution and with the assumption of Gaussian white noise for the random currents is derived in section 5. It is found that they have their usual form with the proviso that the Onsager coefficients (i.e. the viscosities divided by the temperature and the heat conductivity divided by the temperature squared) do not depend on the fluctuating variables.

Finally, in section 6, we study the linear response to a small external force density and establish the various fluctuation–dissipation theorems within the framework of non-linear fluctuating hydrodynamics. The renormalization of transport coefficients may be based on these equations.

2. Hamiltonian form of ideal fluid hydrodynamics

In this section we summarize some of the results of paper I, which will be of use in our subsequent discussion of hydrodynamic fluctuations.

We will describe the state of a one-component fluid by the five physical fields ρ, j and e_v, the mass-density, the momentum-density and the total energy-density respectively. The internal energy per unit mass u is related to e_v by the relation $e_v = \frac{1}{2}j^2/\rho + \rho u$. It is convenient to consider these five fields as components α_β of a five-dimensional vector $\boldsymbol{\alpha}$[*],

$$\alpha_0 \equiv \rho, \quad \alpha_i \equiv j_i \text{ for } i = 1, 2, 3, \quad \alpha_4 \equiv e_v. \tag{2.1}$$

As discussed in paper I, Poisson-brackets for these fields may be defined within the framework of a Hamiltonian formalism; these Poisson-brackets have the usual property

$$\{\alpha_i(\boldsymbol{r}), \alpha_j(\boldsymbol{r}')\} = -\{\alpha_j(\boldsymbol{r}'), \alpha_i(\boldsymbol{r})\}. \tag{2.2}$$

We now introduce a matrix L of which the elements coincide with the Poisson-brackets $\{\alpha_i, \alpha_j\}$ of the five fields (2.1). The elements of this matrix are explicitly

$$\begin{aligned}
&L_{00}(\boldsymbol{r}, \boldsymbol{r}') \equiv \{\alpha_0(\boldsymbol{r}), \alpha_0(\boldsymbol{r}')\} = 0,\\
&L_{0i}(\boldsymbol{r}, \boldsymbol{r}') \equiv \{\alpha_0(\boldsymbol{r}), \alpha_i(\boldsymbol{r}')\} = -\frac{\partial}{\partial r_i}[\rho(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')],\\
&L_{04}(\boldsymbol{r}, \boldsymbol{r}') \equiv \{\alpha_0(\boldsymbol{r}), \alpha_4(\boldsymbol{r}')\} = -\boldsymbol{\nabla}\cdot[\boldsymbol{j}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')],\\
&L_{ij}(\boldsymbol{r}, \boldsymbol{r}') \equiv \{\alpha_i(\boldsymbol{r}), \alpha_j(\boldsymbol{r}')\} = -\frac{\partial}{\partial r_j}[j_i(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')] + \frac{\partial}{\partial r'_i}[j_j(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')],
\end{aligned} \tag{2.3}$$

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[*] Throughout this paper, Greek indices (β, γ, etc.) run from 0 to 4, whereas Latin indices (i, j, etc.) can run from 1 to 3 and designate Cartesian components of three-dimensional vectors.

$$L_{i4}(\boldsymbol{r}, \boldsymbol{r}') \equiv \{\alpha_i(\boldsymbol{r}), \alpha_4(\boldsymbol{r}')\} = -\boldsymbol{\nabla} \cdot [\boldsymbol{v}(\boldsymbol{r}) j_i(\boldsymbol{r}) \delta(\boldsymbol{r} - \boldsymbol{r}')] - \frac{\partial}{\partial r_i} [p(\boldsymbol{r}) \delta(\boldsymbol{r} - \boldsymbol{r}')]$$
$$+ \frac{\partial}{\partial r'_i} [e_v(\boldsymbol{r}') \delta(\boldsymbol{r} - \boldsymbol{r}')],$$

$$L_{44}(\boldsymbol{r}, \boldsymbol{r}') \equiv \{\alpha_4(\boldsymbol{r}), \alpha_4(\boldsymbol{r}')\} = -\boldsymbol{\nabla} \cdot [\boldsymbol{v}(\boldsymbol{r})(e_v(\boldsymbol{r}) + p(\boldsymbol{r})) \delta(\boldsymbol{r} - \boldsymbol{r}')]$$
$$+ \boldsymbol{\nabla}' \cdot [\boldsymbol{v}(\boldsymbol{r}')(e_v(\boldsymbol{r}') + p(\boldsymbol{r}')) \delta(\boldsymbol{r} - \boldsymbol{r}')].$$

In these equations p denotes the hydrostatic pressure and $\boldsymbol{v}$ denotes the velocity, defined by $\boldsymbol{v} = \boldsymbol{j}/\rho$. The other elements of $\mathbf{L}$ can be obtained from the symmetry-relation (cf. eq. (2.2))

$$L_{ij}(\boldsymbol{r}, \boldsymbol{r}') = -L_{ji}(\boldsymbol{r}', \boldsymbol{r}). \tag{2.4}$$

Poisson-brackets for arbitrary functionals of the physical fields $A(\{\boldsymbol{\alpha}\})$ and $B(\{\boldsymbol{\alpha}\})$ are then given by

$$\{A(\{\boldsymbol{\alpha}\}), B(\{\boldsymbol{\alpha}\})\} = \sum_{\beta,\gamma=1}^{4} \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}' \frac{\delta A}{\delta \alpha_\beta(\boldsymbol{r})} L_{\beta\gamma}(\boldsymbol{r}, \boldsymbol{r}') \frac{\delta B}{\delta \alpha_\gamma(\boldsymbol{r}')}, \tag{2.5}$$

where the functional derivatives $\delta/\delta\alpha_\beta(\boldsymbol{r})$ are defined in the standard way. With these definitions, the hydrodynamic equations of motion for ideal flow may be written in the compact form

$$\frac{\partial \alpha_\beta}{\partial t} = \{\alpha_\beta, H\}. \tag{2.6}$$

The Hamiltonian H in this equation is equal to the total energy of the fluid,

$$H \equiv \int \mathrm{d}\boldsymbol{r} e_v(\boldsymbol{r}). \tag{2.7}$$

Since this equation implies that

$$\frac{\delta H}{\delta \alpha_\beta(\boldsymbol{r})} = \delta_{\beta 4}, \tag{2.8}$$

one finds from eqs. (2.6), (2.5) and (2.3) the equations of motion

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{j}, \tag{2.9}$$

$$\frac{\partial \boldsymbol{j}}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{v}\boldsymbol{j} - \boldsymbol{\nabla} p, \tag{2.10}$$

$$\frac{\partial e_v}{\partial t} = -\boldsymbol{\nabla} \cdot [\boldsymbol{v}(e_v + p)]. \tag{2.11}$$

These are the equations of conservation of mass, momentum and energy for an ideal fluid.

3. Stochastic differential equations for hydrodynamic fluctuations

In a non-ideal one component fluid one has the following conservation laws for the mass-density $\rho(r, t)$, the momentum-density $j(r, t)$ and the energy-density $e_v(r, t)$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j, \tag{3.1}$$

$$\frac{\partial j}{\partial t} = -\nabla \cdot [vj + p + \boldsymbol{\Pi}], \tag{3.2}$$

$$\frac{\partial e_v}{\partial t} = -\nabla \cdot [v(e_v + p) + v \cdot \boldsymbol{\Pi} + \boldsymbol{J}]. \tag{3.3}$$

Here $\boldsymbol{\Pi}(r, t)$ and $\boldsymbol{J}(r, t)$ are the symmetric viscous pressure tensor and heat current respectively. In macroscopic hydrodynamics the viscous pressure tensor and the heat current obey phenomenological laws which relate these quantities to the hydrodynamic fields and their gradients^{12}) so that eqs. (3.1)–(3.3) form a complete set of deterministic equations. In the context of fluctuation theory we shall consider these quantities to consist of systematic parts $\boldsymbol{\Pi}^{\mathrm{s}}$ and $\boldsymbol{J}^{\mathrm{s}}$, and random parts $\boldsymbol{\Pi}^{\mathrm{R}}$ and $\boldsymbol{J}^{\mathrm{R}}$,

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}^{\mathrm{s}} + \boldsymbol{\Pi}^{\mathrm{R}}, \tag{3.4}$$

$$\boldsymbol{J} = \boldsymbol{J}^{\mathrm{s}} + \boldsymbol{J}^{\mathrm{R}}. \tag{3.5}$$

The systematic viscous pressure tensor and heat current again obey phenomenological laws but now in terms of the fluctuating hydrodynamic fields. In order that the stochastic differential eqs. (3.1)–(3.3) with (3.4) and (3.5) may be solved, the stochastic properties of the random fluxes must be specified. We shall assume the processes $\boldsymbol{\Pi}^{\mathrm{R}}(r, t)$ and $\boldsymbol{J}^{\mathrm{R}}(r, t)$ to be Gaussian with zero mean,

$$\begin{aligned} &\overline{\boldsymbol{\Pi}^{\mathrm{R}}(r, t)} = 0, \\ &\overline{\boldsymbol{J}^{\mathrm{R}}(r, t)} = 0, \end{aligned} \tag{3.6}$$

and variances

$$\begin{aligned} \overline{\Pi^{\mathrm{R}}_{ij}(r, t)\Pi^{\mathrm{R}}_{kl}(r', t')} &= 2kL_{ijkl}\delta(r - r')\delta(t - t') \\ &\equiv 2k\left[L\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\,\delta_{ij}\delta_{kl}\right) + L_v\delta_{ij}\delta_{kl}\right]\delta(r - r')\delta(t - t'), \end{aligned} \tag{3.7}$$

$$\overline{\Pi^{\mathrm{R}}_{ij}(r, t)J^{\mathrm{R}}_{k}(r', t')} = 0, \tag{3.8}$$

$$\overline{J^{\mathrm{R}}_{i}(r, t)J^{\mathrm{R}}_{j}(r', t')} = 2kL_q\delta_{ij}\delta(r - r')\delta(t - t'). \tag{3.9}$$

In these equations averages denoted by a bar are taken over an ensemble of systems with the same initial conditions for the hydrodynamic fields. Eqs. (3.7) and (3.9) express the fact that the correlations of the random fluxes are local in time (white noise) and space. Due to isotropy, the strength of these correlations (which, for convenience, is expressed as a multiple of $2k$, where k is Boltzmann's constant) is completely determined by three scalar quantities L, L_v and L_q, which may still depend on the equilibrium temperature T_0 and the equilibrium density ρ_0 of the system.

Equations of the form (3.1)–(3.9) were first proposed by Landau and Lifshitz[13]), in combination with the phenomenological laws

$$\boldsymbol{\Pi}^{\mathrm{s}} = -2\eta\overline{\nabla \boldsymbol{v}} - \eta_v \nabla \cdot \boldsymbol{v}, \tag{3.10}$$

$$\boldsymbol{J}^{\mathrm{s}} = -\lambda \nabla T, \tag{3.11}$$

where η, η_v and λ are the viscosity, the volume viscosity and the heat conductivity respectively and where $\overline{\cdot\,\cdot}$ denotes the symmetric traceless part of a tensor. It is in their completely linearized version that eqs. (3.1)–(3.3) with eqs. (3.4)–(3.11) are usually referred to as the Landau–Lifshitz hydrodynamic equations. In this case it turns out that they form a consistent set of equations in the sense that all moments of the hydrodynamic fields, calculated in the fully linearized scheme, approach for long times their equilibrium values, provided that the following identification is made:

$$L = \eta T_0, \quad L_v = \eta_v T_0, \quad L_q = \lambda T_0^2. \tag{3.12}$$

The relations (3.12) represent the fluctuation–dissipation theorem in the fully linearized case.

It is our aim to investigate under what conditions, or, more precisely, for which form of the phenomenological laws for the systematic currents $\boldsymbol{\Pi}^{\mathrm{s}}$ and $\boldsymbol{J}^{\mathrm{s}}$, the *nonlinear* eqs. (3.1)–(3.3) are consistent with the assumptions (3.4)–(3.9), and the assumption of the Gaussian character of the random fluxes. To this end we shall in the next section derive the Fokker–Planck equation which is equivalent to our set of stochastic equations.

4. The Fokker–Planck equation

When considering the stochastic differential eqs. (3.1)–(3.3) together with eqs. (3.4)–(3.9), the problem arises whether these should be interpreted in the Stratonovich or Itô sense[14]), due to the presence of the term $\boldsymbol{v} \cdot \boldsymbol{\Pi}^{\mathrm{R}}$ in eq. (3.3). It is well known that it is possible, if these equations are interpreted in the Stratonovich sense, to derive the corresponding Itô equations and vice versa[14]). In appendix A we show, using the discretization rules introduced in paper I, that the Itô and Stratonovich equations are in this case identical, so that either of the interpretations may be used. We shall adopt here the Stratonovich interpretation, so that the usual rules of differential and integral calculus may be used.

We now turn to the derivation of the Fokker–Planck equation. In view of the difficulties encountered when writing down a conservation equation for the density distribution in the phase-space of physical fields (see paper I, where this is done in order to obtain the Liouville equation), it is necessary for such a derivation to discretize the system in coordinate space so that the phase-space becomes of finite dimensionality. As in paper I, we divide the fluid into small cubic cells of size Δ^3. We denote the position of a cell by $\boldsymbol{r} = \boldsymbol{n}\Delta$, where $\boldsymbol{n}$ is a vector of which the components are integer numbers. In each cell the state is characterized by the five hydrodynamic variables ρ_n, $\boldsymbol{j}_n$ and $e_{v,n}$, the specific quantities in each cell, which in the limit of vanishing cell-size correspond to the previously defined hydrodynamic fields. These discrete hydrodynamic variables, will obey the discrete analogues of the stochastic hydrodynamic eqs. (2.1)–(2.5), viz.

$$\frac{\partial \rho_n}{\partial t} = -\boldsymbol{\nabla}_n \cdot \boldsymbol{j}_n, \tag{4.1}$$

$$\frac{\partial \boldsymbol{j}_n}{\partial t} = -\boldsymbol{\nabla}_n \cdot (\boldsymbol{v}_n \boldsymbol{j}_n + p_n + \boldsymbol{\Pi}_n), \tag{4.2}$$

$$\frac{\partial e_{v,n}}{\partial t} = -\boldsymbol{\nabla}_n \cdot [\boldsymbol{v}_n(e_{v,n} + p_n) + \boldsymbol{v}_n \cdot \boldsymbol{\Pi}_n + \boldsymbol{J}_n], \tag{4.3}$$

$$\boldsymbol{\Pi}_n = \boldsymbol{\Pi}_n^{\mathrm{s}} + \boldsymbol{\Pi}_n^{\mathrm{R}}, \tag{4.4}$$

$$\boldsymbol{J}_n = \boldsymbol{J}_n^{\mathrm{s}} + \boldsymbol{J}_n^{\mathrm{R}}. \tag{4.5}$$

Here $\boldsymbol{\Pi}_n$ and $\boldsymbol{J}_n$ are the discrete viscous pressure tensor and heat current respectively, and the discrete gradient operator $\boldsymbol{\nabla}_n$ is defined as (cf. paper I, eq. (4.6))

$$\boldsymbol{\nabla}_n A_n \equiv \sum_{i=1}^{3} \frac{\hat{e}_i}{2\Delta} [A_{n+\hat{e}_i} - A_{n-\hat{e}_i}], \tag{4.6}$$

where the $\hat{e}_i$'s are unit vectors along the Cartesian axes.

In agreement with our previous assumption for the continuous stochastic fields $\boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t)$ and $\boldsymbol{J}^{\mathrm{R}}(\boldsymbol{r}, t)$, we assume that also the processes $\boldsymbol{\Pi}_n^{\mathrm{R}}(t)$ and $\boldsymbol{J}_n^{\mathrm{R}}(t)$ are Gaussian with zero mean,

$$\overline{\boldsymbol{\Pi}_n^{\mathrm{R}}(t)} = 0, \quad \overline{\boldsymbol{J}_n^{\mathrm{R}}(t)} = 0, \tag{4.7}$$

and have as variances the discrete analogues of eqs. (3.7)–(3.9),

$$\overline{\Pi^{\mathrm{R}}_{ij,n}(t)\Pi^{\mathrm{R}}_{kl,n'}(t')} = 2kL_{ijkl}\Delta^{-3}\delta_{nn'}\delta(t-t'), \tag{4.8}$$

$$\overline{\Pi^{\mathrm{R}}_{ij,n}(t)J^{\mathrm{R}}_{n'}(t')} = 0, \tag{4.9}$$

$$\overline{J^{\mathrm{R}}_{i,n}(t)J^{\mathrm{R}}_{j,n'}(t')} = 2kL_q\delta_{ij}\Delta^{-3}\delta_{nn'}\delta(t-t'). \tag{4.10}$$

With the aid of the notation (2.1) the five eqs. (4.1)–(4.3) together with eqs. (4.4) and (4.5) can be written in the compact form

$$\frac{\partial\alpha_{\beta,n}}{\partial t} = F^{\mathrm{irr}}_{\beta,n} + F^{\mathrm{rev}}_{\beta,n} + \sum_{n'}\sum_{\gamma,\delta=0}^{4} M_{\beta\gamma\delta,nn'}f_{\gamma\delta,n'}, \quad (\beta = 0, 1, 2, 3, 4). \tag{4.11}$$

In eq. (4.11) the quantities $F^{\mathrm{rev}}_{\beta,n}$ and $F^{\mathrm{irr}}_{\beta,n}$ denote the reversible and irreversible parts of the rates of change of the quantities $\alpha_{\beta,n}$ respectively:

$$F^{\mathrm{rev}}_{0,n} \equiv -\boldsymbol{\nabla}_n\cdot\boldsymbol{j}_n, \quad F^{\mathrm{rev}}_{i,n} \equiv -(\boldsymbol{\nabla}_n\cdot\boldsymbol{v}_n\boldsymbol{j}_n + \boldsymbol{\nabla}_n p_n)_i, \quad F^{\mathrm{rev}}_{4,n} \equiv -\boldsymbol{\nabla}_n\cdot\{(e_{v,n}+p_n)\boldsymbol{v}_n\}, \tag{4.12}$$

$$F^{\mathrm{irr}}_{0,n} \equiv 0, \quad F^{\mathrm{irr}}_{i,n} \equiv -(\boldsymbol{\nabla}_n\cdot\boldsymbol{\Pi}^{\mathrm{s}}_n)_i, \quad F^{\mathrm{irr}}_{4,n} \equiv -\boldsymbol{\nabla}_n\cdot\{\boldsymbol{\Pi}^{\mathrm{s}}_n\cdot\boldsymbol{v}_n + \boldsymbol{J}^{\mathrm{s}}_n\}, \tag{4.13}$$

while the quantities $M_{\beta\gamma\delta,nn'}$ and the random forces $f_{\beta\gamma,n}$ may be identified as

$$\begin{aligned}
&M_{0\beta\gamma,nn'} = M_{\beta0\gamma,nn'} = M_{\beta\gamma0,nn'} = M_{i4\beta,nn'} = M_{\beta\gamma4,nn'} = 0,\\
&M_{ijk,nn'} = -\delta_{ik}\nabla_{j,n}\delta_{nn'},\\
&M_{4jk,nn'} = -v_{k,n'}\nabla_{j,n}\delta_{nn'},\\
&M_{44i,nn'} = -\nabla_{i,n}\delta_{nn'}
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&f_{0\beta,n} = f_{\beta0,n} = f_{44,n} = f_{i4,n} = 0,\\
&f_{ij,n} = \Pi^{\mathrm{R}}_{ij,n},\\
&F_{4i,n} = J^{\mathrm{R}}_{i,n}.
\end{aligned} \tag{4.15}$$

The properties of the random forces $f_{\beta\gamma,n}$ follow from the identification (4.15). They are therefore Gaussian and have variances which follow from eqs. (4.7)–(4.9),

$$\overline{f_{\beta\gamma,n}(t)f_{\delta\epsilon,n'}(t')} = 2\Lambda_{\beta\gamma\delta\epsilon,nn'}\delta(t-t'), \tag{4.16}$$

where

$$\Lambda_{ijkl,nn'} = kL_{ijkl}\Delta^{-3}\delta_{nn'}, \tag{4.17}$$

$$\Lambda_{4i4j,nn'} = kL_q\delta_{ij}\Delta^{-3}\delta_{nn'}. \tag{4.18}$$

The other components of Λ are all zero.

For the distribution function $P(\{\boldsymbol{\alpha}_n\}, t)$ one may now derive the Fokker–Planck equation corresponding to the Stratonovich differential equation (4.11) in a standard way[14]). One obtains the equation

$$\frac{\partial P(\{\boldsymbol{\alpha}_n\}, t)}{\partial t} = \sum_n \sum_\beta \frac{\partial}{\partial \alpha_{\beta,n}} \Bigg[- F^{\text{rev}}_{\beta,n} - F^{\text{irr}}_{\beta,n} + \sum_{\substack{n',n''\\ n'''}} \sum_{\substack{\gamma,\gamma'\\ \delta,\delta'\\ \beta'}} M_{\beta\gamma\delta,nn'} \frac{\partial}{\partial \alpha_{\beta',n''}} M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n'n'''} \Bigg] P(\{\boldsymbol{\alpha}_n\}, t). \tag{4.19}$$

If one takes the dissipative terms, i.e. F^{irr} and Λ, equal to zero, this equation reduces to eq. (4.7) of paper I, which, as was discussed, yields the Liouville equation in the continuum limit. It may be verified (cf. also appendix A), using the explicit elements $M_{\beta\gamma\delta,nn'}$ and the definition (4.10), that

$$\sum_{\substack{n',n''\\ n'''}} \sum_{\substack{\gamma\delta\\ \gamma'\delta'\\ \beta'}} \left(\frac{\partial M_{\beta\gamma\delta,nn'}}{\partial \alpha_{\beta',n''}} \right) M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n'n'''} = 0. \tag{4.20}$$

The Fokker–Planck equation (4.19) may therefore also be written as

$$\frac{\partial P(\{\boldsymbol{\alpha}_n\}, t)}{\partial t} = \sum_n \sum_\beta \frac{\partial}{\partial \alpha_{\beta,n}} \left[- F^{\text{rev}}_{\beta,n} - F^{\text{irr}}_{\beta,n} + \sum_{n'} \sum_{\beta'} \frac{\partial}{\partial \alpha_{\beta',n'}} D_{\beta\beta',nn'} \right] P(\{\boldsymbol{\alpha}_n\}, t), \tag{4.21}$$

where the matrix of diffusion coefficients is given by

$$D_{\beta\beta',nn'} = \sum_{n''n'''} \sum_{\substack{\gamma\delta\\ \gamma'\delta'}} M_{\beta\gamma\delta,nn''} M_{\beta'\gamma'\delta',n'n'''} \Lambda_{\gamma\delta\gamma'\delta',n''n'''}. \tag{4.22}$$

With the aid of eqs. (4.14) and (4.18), one finds the explicit expressions

$$D_{0\beta,nn'} = D_{\beta0,nn'} = 0, \tag{4.23}$$

$$D_{\mathrm{ij},nn'} = D_{\mathrm{ji},n'n} = k\Delta^{-3} \left(L\boldsymbol{U} \boldsymbol{\nabla}_n \cdot \boldsymbol{\nabla}_{n'} \delta_{nn'} + \left(L_v + \frac{1}{3} L \right) \boldsymbol{\nabla}_n \boldsymbol{\nabla}_{n'} \delta_{nn'} \right)_{\mathrm{ij}}, \tag{4.24}$$

$$D_{\mathrm{i}4,nn'} = D_{4\mathrm{i},n'n} = k\Delta^{-3} (2L \boldsymbol{\nabla}_n \cdot \overline{\boldsymbol{\nabla}_{n'} \boldsymbol{v}_{n'}}\, \delta_{nn'} + L_v \boldsymbol{\nabla}_n \boldsymbol{\nabla}_{n'} \cdot \boldsymbol{V}_{n'} \delta_{nn'})_{\mathrm{i}}, \tag{4.25}$$

$$\begin{aligned} D_{44,nn'} = {} & k\Delta^{-3} L_q \boldsymbol{\nabla}_n \cdot \boldsymbol{\nabla}_{n'} \delta_{nn'} + 2k\Delta^{-3} L \overline{\boldsymbol{\nabla}_n \boldsymbol{v}_n} : \boldsymbol{\nabla}_{n'} \boldsymbol{v}_{n'} \delta_{nn} + k\Delta^{-3} L_v (\boldsymbol{\nabla}_n \cdot \boldsymbol{v}_n) \\ & \times (\boldsymbol{\nabla}_{n'} \cdot \boldsymbol{v}_{n'}) \delta_{nn'}. \end{aligned} \tag{4.26}$$

Here $\boldsymbol{U}$ denotes the unit tensor, while the gradient operators act on everything behind them.

The form (4.21) of the Fokker–Planck equation would also have been obtained if the equations (4.11) had been interpreted as stochastic differential equations in the Itô sense[14]). This in fact would suffice to conclude that, if (4.11) is interpreted as a Stratonovich equation, the corresponding Itô equation is unmodified and vice versa, as was already mentioned at the beginning of this section. The conclusion therefore is that eq. (4.11) leads to unambiguous results, independent of its interpretation either in the Itô or Stratonovich sense. We stress, however, that stochastic equations for functions of the variables ρ, j and e_v, as e.g. the entropy or the temperature*, do not necessarily get the same form on the basis of the two interpretations of eq. (4.11), although their physical content remains of course the same.

We finally note that the Fokker–Planck equation (4.19) may also be written in the form

$$\frac{\partial P(\{\boldsymbol{\alpha}_n\}, t)}{\partial t} = \sum_n \sum_\beta \left[-F^{\text{rev}}_{\beta,n} \frac{\partial}{\partial \alpha_{\beta,n}} + \frac{\partial}{\partial \alpha_{\beta,n}} \left(-F^{\text{irr}}_{\beta,n} + \sum_{n'} \sum_{\beta'} D_{\beta\beta',nn'} \frac{\partial}{\partial \alpha_{\beta',n'}} \right) \right] P(\{\boldsymbol{\alpha}_n\}, t). \qquad (4.27)$$

Here use has been made of the fact that (cf. also eq. (4.8) of paper I)

$$\sum_\beta \frac{\partial F^{\text{rev}}_{\beta,n}}{\partial \alpha_{\beta,n}} = 0, \qquad (4.28)$$

and that

$$\sum_{n'} \sum_{\beta'} \frac{\partial D_{\beta\beta',nn'}}{\partial \alpha_{\beta',n'}} = 0. \qquad (4.29)$$

These equations immediately follow from eqs. (4.12) and (4.23)–(4.26), together with the definition (4.6) of the discrete gradient operator.

5. The form of the dissipative currents

Until now we have left the form of the viscous pressure tensor and the heat current undetermined and have only assumed that they are functions of the hydrodynamic variables. We shall now determine the form of these currents compatible with our assumption that the random currents are Gaussian processes with white noise. It is on the basis of this assumption that we were able to derive in the last section a Fokker–Planck equation for the distribution function $P(\{\boldsymbol{\alpha}_n\}, t)$ of the hydrodynamic variables. For these variables the property of microscopic reversibility holds[12]). This implies that the fundamental solution of the Fokker–Planck equation $P(\{\boldsymbol{\alpha}_n\}, t \mid \{\boldsymbol{\alpha}'_n\})$, which gives the conditional probability density that the system is in the state $\{\boldsymbol{\alpha}_n\}$ at

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* In this connection, and also in relation to remarks made by Fox[15]), we discuss in appendix B the equation for the temperature.

time t if it is initially in the state $\{\alpha'_n\}$ at time zero, must satisfy the property

$$P(\{\alpha_n\}, t \mid \{\alpha'_n\}) P^{\mathrm{eq}}(\{\alpha'_n\}) = P(\{\tilde{\alpha}'_n\}, t \mid \{\tilde{\alpha}_n\}) P^{\mathrm{eq}}(\{\tilde{\alpha}_n\}). \tag{5.1}$$

Here $\tilde{\alpha}_{\beta,n} = \epsilon_\beta \alpha_{\beta,n}$, where $\epsilon_\beta = 1$ if $\alpha_{\beta,n}$ is even under time reversal and $\epsilon_\beta = -1$ if $\alpha_{\beta,n}$ is odd under time reversal, so that (cf. eq. (2.1)) $\tilde{\alpha}_{0,n} = \alpha_{0,n}$ and $\tilde{\alpha}_{i,n} = -\alpha_{i,n}$ and $\tilde{\alpha}_{4,n} = \alpha_{4,n}$.

From property (5.1) and the Fokker–Planck equation (4.27) the following set of conditions may be derived:

$$\sum_{n,\beta} F^{\mathrm{rev}}_{\beta,n} \frac{\partial P^{\mathrm{eq}}}{\partial \alpha_{\beta,n}} = 0 \tag{5.2}$$

for the equilibrium distribution, which is eq. (I.4.9) for the ideal fluid if P^{eq} is inserted, and

$$F^{\mathrm{irr}}_{\beta,n} = \sum_{n'} \sum_{\beta'} D_{\beta\beta',nn'} \frac{\partial \ln P^{\mathrm{eq}}}{\partial \alpha_{\beta',n'}}. \tag{5.3}$$

These are the so-called potential conditions[16]) as applied to the case considered here, in which eqs. (4.28) and (4.29) hold*.

We will analyse eqs. (5.2) and (5.3) in the continuum limit, in which the discretized variables become fields again. In this limit, we have e.g.

$$\lim_{\Delta\to 0} F^{\mathrm{rev}}_{0,n} = \lim_{\Delta\to 0} -\nabla_n \cdot j_n = -\nabla \cdot j(r) = \{\alpha_0(r), H\}. \tag{5.4}$$

Here, use has been made of eq. (4.12) and of the Poisson-bracket expressions defined in the first section (cf. eqs. (2.6) and (2.9)). Analogously, one finds for the other components of $F^{\mathrm{rev}}_{\beta,n}$

$$\lim_{\Delta\to 0} F^{\mathrm{rev}}_{\beta,n} = \{\alpha_\beta(r), H\}. \tag{5.5}$$

With the aid of this result, eq. (5.2) becomes in the continuum limit

$$\lim_{\Delta\to 0} \sum_n \sum_\beta F^{\mathrm{rev}}_{\beta,n} \frac{\partial P^{\mathrm{eq}}(\{\alpha_n\})}{\partial \alpha_{\beta,n}} = \sum_\beta \int \mathrm{d}r \{\alpha_\beta(r), H\} \frac{\delta P^{\mathrm{eq}}(\{\alpha(r)\})}{\delta \alpha_\beta(r)}$$
$$= \{P^{\mathrm{eq}}(\{\alpha(r)\}), H\} = 0, \tag{5.6}$$

provided that $P^{\mathrm{eq}}(\{\alpha_n\})$ converges to a proper functional $P^{\mathrm{eq}}(\{\alpha(r)\})$. This equation expresses the fact that P^{eq} should be a stationary solution of the Liouville-equation. As discussed in paper I, the Einstein distribution,

$$P^{\mathrm{eq}}(\{\alpha(r)\}) \sim \mathrm{e}^{S(\{\alpha(r)\})/k}, \tag{5.7}$$

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* In the derivation of eqs. (5.2) and (5.3) use is made of the fact that $F^{\mathrm{rev}}_\beta(\{\tilde{\alpha}\}) = -\epsilon_\beta F^{\mathrm{rev}}_\beta(\{\alpha\})$ (cf. eq. (4.12)) and of the property $F^{\mathrm{irr}}_\beta(\{\tilde{\alpha}\}) = \epsilon_\beta F^{\mathrm{irr}}_\beta(\{\alpha\})$.

where S is the total entropy, defined by

$$S(\{\boldsymbol{\alpha}(\boldsymbol{r})\}) \equiv \int \mathrm{d}\boldsymbol{r} s_v(\boldsymbol{r}), \tag{5.8}$$

is indeed a stationary solution. Similarly, the equilibrium distribution for a fluid in thermal contact with a heat bath at temperature T_0,

$$P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}) \sim \mathrm{e}^{-(H - T_0 S)/kT_0}, \tag{5.9}$$

and the equilibrium distribution for a fluid which can in addition exchange mass with a reservoir with chemical potential μ_0

$$P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}) \sim \mathrm{e}^{-(H - T_0 S - \mu_0 M)/kT_0}, \tag{5.10}$$

where M is the total mass of the fluid,

$$M \equiv \int \mathrm{d}\boldsymbol{r} \rho(\boldsymbol{r}), \tag{5.11}$$

satisfy eq. (5.6).

We now turn to the evaluation of eq. (5.3). In the continuum limit this equation becomes

$$F_\beta^{\mathrm{irr}}(\boldsymbol{r}) = \sum_{\beta'} \int \mathrm{d}\boldsymbol{r}' D_{\beta\beta'}(\boldsymbol{r}, \boldsymbol{r}') \frac{\delta \ln P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\})}{\delta \alpha_{\beta'}(\boldsymbol{r}')}, \tag{5.12}$$

where we have defined

$$F_\beta^{\mathrm{irr}}(\boldsymbol{r}) \equiv \lim_{\Delta \to 0} F_{\beta,n}^{\mathrm{irr}}, \tag{5.13}$$

$$D_{\beta\beta'}(\boldsymbol{r}, \boldsymbol{r}') = \lim_{\Delta \to 0} D_{\beta\beta',nn'}. \tag{5.14}$$

In view of the definitions (4.13), we obtain for $F_\beta^{\mathrm{irr}}(\boldsymbol{r})$

$$F_0^{\mathrm{irr}}(\boldsymbol{r}) = 0, \quad F_i^{\mathrm{irr}}(\boldsymbol{r}) = -(\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}^{\mathrm{s}})_i, \quad F_4^{\mathrm{irr}} = -\boldsymbol{\nabla} \cdot [\boldsymbol{\Pi}^{\mathrm{s}} \cdot \boldsymbol{v} + \boldsymbol{J}^{\mathrm{s}}], \tag{5.15}$$

and, from eqs. (4.23)–(4.26), for the elements of $D_{\beta\beta'}(\boldsymbol{r}, \boldsymbol{r}')$

$$D_{0\beta}(\boldsymbol{r}, \boldsymbol{r}') = D_{\beta 0}(\boldsymbol{r}, \boldsymbol{r}') = 0, \tag{5.16}$$

$$D_{ij}(\boldsymbol{r}, \boldsymbol{r}') = k\left(L\boldsymbol{U}\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}'\delta(\boldsymbol{r} - \boldsymbol{r}') + \left(L_v + \frac{1}{3}L\right)\boldsymbol{\nabla}\boldsymbol{\nabla}'\delta(\boldsymbol{r} - \boldsymbol{r}')\right)_{ij}, \tag{5.17}$$

$$D_{i4}(\boldsymbol{r}, \boldsymbol{r}') = D_{4i}(\boldsymbol{r}', \boldsymbol{r}) = k(2L\boldsymbol{\nabla} \cdot \overline{\boldsymbol{\nabla}'\boldsymbol{v}(\boldsymbol{r}')}\delta(\boldsymbol{r} - \boldsymbol{r}') + L_v \boldsymbol{\nabla}(\boldsymbol{\nabla}' \cdot \boldsymbol{v}(\boldsymbol{r}'))\delta(\boldsymbol{r} - \boldsymbol{r}')), \tag{5.18}$$

$$\begin{aligned} D_{44}(\boldsymbol{r}, \boldsymbol{r}') = {} & kL_q \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}'\delta(\boldsymbol{r} - \boldsymbol{r}') + 2kL\overline{\boldsymbol{\nabla}\boldsymbol{v}(\boldsymbol{r})} : \boldsymbol{\nabla}'\boldsymbol{v}(\boldsymbol{r}')\delta(\boldsymbol{r} - \boldsymbol{r}') \\ & + kL_v(\boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{r}))(\boldsymbol{\nabla}' \cdot \boldsymbol{v}(\boldsymbol{r}'))\delta(\boldsymbol{r} - \boldsymbol{r}'). \end{aligned} \tag{5.19}$$

One immediately sees that eq. (5.12) is trivially fulfilled for $\beta = 0$. In order to evaluate this equation for other values of β, we also need the functional derivatives of the logarithm of the equilibrium distribution. For the case of a fluid which is materially and energetically isolated, to which the equilibrium distribution (5.7) applies, these are

$$\frac{\delta \ln P^{\text{eq}}}{\delta \alpha_i(\boldsymbol{r})} = -k^{-1} \frac{v_i(\boldsymbol{r})}{T(\boldsymbol{r})}, \quad \frac{\delta \ln P^{\text{eq}}}{\delta \alpha_4(\boldsymbol{r})} = k^{-1} \frac{1}{T(\boldsymbol{r})}. \tag{5.20}$$

Substitution of eqs. (5.15)–(5.20) into eq. (5.12) yields

$$F_i^{\text{irr}}(\boldsymbol{r}) = -(\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}^{\text{s}}(\boldsymbol{r}))_i = \left(2L\boldsymbol{\nabla} \cdot \frac{\overline{\boldsymbol{\nabla} \boldsymbol{v}(\boldsymbol{r})}}{T(\boldsymbol{r})} + L_v \boldsymbol{\nabla} \frac{\boldsymbol{\nabla} \cdot \boldsymbol{v}}{T(\boldsymbol{r})}\right)_i, \tag{5.21}$$

$$\begin{aligned} F_4^{\text{irr}}(\boldsymbol{r}) &= -\boldsymbol{\nabla} \cdot (\boldsymbol{\Pi}^{\text{s}}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) + \boldsymbol{J}^{\text{s}}(\boldsymbol{r}) \\ &= -L_q \boldsymbol{\nabla}^2 \frac{1}{T(\boldsymbol{r})} + L\boldsymbol{\nabla} \cdot \left(\frac{\overline{\boldsymbol{\nabla} \boldsymbol{v}(\boldsymbol{r})}}{T(\boldsymbol{r})} \cdot \boldsymbol{v}(\boldsymbol{r})\right) + L_v \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{r})}{T(\boldsymbol{r})} \boldsymbol{v}(\boldsymbol{r})\right). \end{aligned} \tag{5.22}$$

Eqs. (5.21) and (5.22) lead to the identification

$$\boldsymbol{\Pi}^{\text{s}}(\boldsymbol{r}) = -2L \frac{\overline{\boldsymbol{\nabla} \boldsymbol{v}(\boldsymbol{r})}}{T(\boldsymbol{r})} - L_v \boldsymbol{U} \frac{\boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{r})}{T(\boldsymbol{r})}, \tag{5.23}$$

$$\boldsymbol{J}^{\text{s}}(\boldsymbol{r}) = L_q \boldsymbol{\nabla} \frac{1}{T(\boldsymbol{r})} = -\frac{L_q}{T^2(\boldsymbol{r})} \boldsymbol{\nabla} T(\boldsymbol{r}). \tag{5.24}$$

We have thus shown that the only expressions for the phenomenological laws compatible with the assumptions of Gaussian white noise for the random currents are those given by eqs. (5.23) and (5.24). These laws have the usual form of the phenomenological laws^{12}) in the sense that the dissipative currents are *linear in the gradients* of the state variables. It is, however, important to realize that in these expressions the Onsager coefficients L, L_v and L_q may only depend on the equilibrium quantities ρ_0 and T_0 and *not* on the fluctuating fields. As a consequence, the shear viscosity η and the bulk viscosity η_v, which are given by

$$\eta = \frac{L}{T(\boldsymbol{r})}, \quad \eta_v = \frac{L_v}{T(\boldsymbol{r})}, \tag{5.25}$$

are within the present scheme proportional to the inverse of the temperature, while the heat conductivity λ, given by

$$\lambda = \frac{L_q}{T^2(\boldsymbol{r})}, \tag{5.26}$$

is proportional to T^{-2}. In a real system, such temperature dependences of the transport coefficients do not occur. This implies that the assumption of

Gaussian white noise is, strictly speaking, not compatible with the physical phenomenological laws. The present scheme nevertheless is approximately valid if the temperature fluctuations may be considered to be sufficiently small so that the dissipative currents (5.23) and (5.24) can be linearized completely in the fluctuating fields*.

In the above analysis, use has been made of the Einstein formula (5.7) for the equilibrium distribution function. One easily checks however, that substitution of the equilibrium distributions (5.9) and (5.10) into eq. (5.12) leads to the same conclusions.

Even though our final results have been derived in the continuum limit, we emphasize the necessity of the discretization procedure for intermediate calculations. This is illustrated by the fact that, if one would immediately go over to the continuum limit in the Fokker–Planck equation (4.27), one of the terms arising would read

$$\sum_{\beta\gamma} \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}' \frac{\delta D_{\beta\gamma}(\boldsymbol{r}, \boldsymbol{r}')}{\delta\alpha_\beta(\boldsymbol{r})} \frac{\delta P}{\delta\alpha_\gamma(\boldsymbol{r}')}. \tag{5.27}$$

The interpretation of the functional derivative

$$\frac{\delta D_{\beta\gamma}(\boldsymbol{r}, \boldsymbol{r}')}{\delta\alpha_\beta(\boldsymbol{r})} \tag{5.28}$$

in this expression is not clear, however, due to the delta-function $\delta(\boldsymbol{r}-\boldsymbol{r}')$ in $D_{\beta\gamma}(\boldsymbol{r}, \boldsymbol{r}')$ (cf. eqs. (5.16)–(5.19)). This ambiguity has been avoided by the discretization procedure. For a more elaborate discussion of the discretization rules, especially in relation with the proper choice of variables, we refer to paper I. In this connection it may be noted that the difficulties encountered in a Fokker–Planck equation for fields are related to the well-known fact that equilibrium mean square fluctuations of state variables of a volume element are inversely proportional to the size of the volume element. In the continuum limit, such quantities diverge.

We finally remark that the quadratic term $\boldsymbol{\Pi} \cdot \boldsymbol{v}$, occurring in the energy-equation, has given rise to controversy. It has been argued^{15}) that this term, which plays an essential role in our analysis, is the origin of an inherent inconsistency of non-linear fluctuation theory. In appendix B we show, however, that this criticism is based on an erroneous argument.

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* In fact, according to eqs. (5.25) and (5.26), the transport coefficients must also be considered to be independent of the fluctuating density and to be only functions of ρ_0. Again this will in general only be legitimate in as far as density fluctuations are sufficiently small.

6. The linear response to an external force

In this section we will study the linear response to a (small) external force density. In the presence of an external potential $V(r, t)$, the hydrodynamic equations (3.1)–(3.3) become

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j, \tag{6.1}$$

$$\frac{\partial j}{\partial t} = -\nabla \cdot [vj + p + \boldsymbol{\Pi}] - \rho \nabla V, \tag{6.2}$$

$$\frac{\partial e_v}{\partial t} = -\nabla \cdot [v(e_v + p) + v \cdot \boldsymbol{\Pi} + \boldsymbol{J}] - j \cdot \nabla V, \tag{6.3}$$

where, as before, e_v is the total energy density in the absence of the force, i.e. $e_v = \frac{1}{2}j^2/\rho + u_v$.

We will analyse, along the lines of ref. 17, the linear response of the system to this potential using the Fokker–Planck equation equivalent with the above equations. As was discussed in the preceding sections, the Fokker–Planck equation should in principle be derived starting from a set of discretized equations. For conciseness, we will in this section not explicitly perform this program, but write the equations in a continuum-notation throughout. One may easily verify, however, that a more careful analysis, based on the discretization rules, leads to the same results.

As the terms in eqs. (6.2) and (6.3) arising from the external potential are of a purely mechanical nature, they already appear in the equations of motion of the ideal fluid. In the presence of the potential, the Hamiltonian H is given by

$$H(t) = H_0 + H_1(t), \tag{6.4}$$

with

$$H_0 \equiv \int \mathrm{d}r e_v(r), \quad H_1(t) \equiv \int \mathrm{d}r \rho(r) V(r, t). \tag{6.5}$$

The equations of motion (6.1)–(6.3) for the ideal fluid (i.e. $\boldsymbol{\Pi} = 0$, $\boldsymbol{J} = 0$) may with the aid of this Hamiltonian be written as

$$\frac{\partial \boldsymbol{\alpha}(r, t)}{\partial t} = \{\boldsymbol{\alpha}(r, t), H(t)\} = \{\boldsymbol{\alpha}(r, t), H_0\} + \{\boldsymbol{\alpha}(r, t), H_1(t)\}. \tag{6.6}$$

This may be verified, using the fact that

$$\frac{\delta H_0}{\delta \alpha_\beta(r)} = \delta_{\beta 4}, \quad \frac{\delta H_1}{\delta \alpha_\beta(r)} = V(r, t)\delta_{\beta 0}, \tag{6.7}$$

so that eq. (6.6) becomes

$$\frac{\partial \alpha_\beta(\boldsymbol{r}, t)}{\partial t} = \int \mathrm{d}\boldsymbol{r}' L_{\beta 4}(\boldsymbol{r}, \boldsymbol{r}') + \int \mathrm{d}\boldsymbol{r}' L_{\beta 0}(\boldsymbol{r}, \boldsymbol{r}') V(\boldsymbol{r}', t). \tag{6.8}$$

Upon substitution of the explicit expressions (2.3) for the elements $L_{\beta 4}$ and $L_{\beta 0}$ the ideal fluid equations immediately follow.

The Liouville equation now becomes*

$$\begin{aligned} \frac{\partial P(\{\alpha(\boldsymbol{r})\}, t)}{\partial t} &= \{H(t), P\} \equiv \mathsf{L}_0 P + \mathsf{L}_1(t) P, \\ &= \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}' \sum_{\beta=0}^{4} [L_{4\beta}(\boldsymbol{r}, \boldsymbol{r}') + V(\boldsymbol{r}, t) L_{0\beta}(\boldsymbol{r}, \boldsymbol{r}')] \frac{\delta P}{\delta \alpha_\beta(\boldsymbol{r}')}, \end{aligned} \tag{6.9}$$

where L_0 is the Liouville-operator in the absence of the external force and $\mathsf{L}_1(t)$ the contribution to the total Liouville operator due to the external force,

$$\mathsf{L}_0 P \equiv \{H_0, P\}, \quad \mathsf{L}_1(t) P \equiv \{H_1(t), P\}. \tag{6.10}$$

The definition of the Poisson brackets given in paper I, eq. (I. 3.23), immediately implies that $\{H(t), S\} = 0$. Consequently

$$(\mathsf{L}_0 + \mathsf{L}_1(t))\, \mathrm{e}^{-(H(t) - T_0 S)/kT_0} = \{H(t), \mathrm{e}^{-(H(t) - T_0 S)/kT_0}\} = 0. \tag{6.11}$$

For $V = 0$, this equation reduces to eq. (5.6) for the equilibrium distribution (5.9). By expanding eq. (6.11) up to linear order in V, one obtains

$$\mathsf{L}_1(t) P^{\mathrm{eq}} = \frac{1}{kT_0} \mathsf{L}_0 \Big(\int \mathrm{d}\boldsymbol{r} \rho(\boldsymbol{r}) V(\boldsymbol{r}, t) P^{\mathrm{eq}} \Big) = \frac{1}{kT_0} \int \mathrm{d}\boldsymbol{r} \nabla \cdot \boldsymbol{j}(\boldsymbol{r}) V(\boldsymbol{r}, t) P^{\mathrm{eq}}. \tag{6.12}$$

Here P^{eq} is the equilibrium distribution of the system in the absence of the external potential, which satisfies $\mathsf{L}_0 P^{\mathrm{eq}} = 0$.

We now return to the general case of a fluid with dissipation in the presence of an external potential. Instead of the Liouville equation (6.9), we then find for the evolution of the probability distribution a Fokker–Planck equation. Along similar lines as in the previous section one may show, on the basis of the fact that the correlation functions of the random currents are not modified by the external force, that the Fokker–Planck equation becomes

$$\frac{\partial P(\{\alpha(\boldsymbol{r})\}, t)}{\partial t} = (\mathsf{L}_0 + \mathsf{L}_1(t) + \mathsf{M}) P(\{\alpha(\boldsymbol{r})\}, t), \tag{6.13}$$

where M may formally be defined as

$$\mathsf{M} \equiv - \int \mathrm{d}\boldsymbol{r} \sum_\beta \frac{\delta}{\delta \alpha_\beta(\boldsymbol{r})} F_\beta^{\mathrm{irr}}(\boldsymbol{r}) + \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}' \sum_{\beta\beta'} \frac{\delta}{\delta \alpha_\beta(\boldsymbol{r})} D_{\beta\beta'}(\boldsymbol{r}, \boldsymbol{r}') \frac{\delta}{\delta \alpha_{\beta'}(\boldsymbol{r}')}. \tag{6.14}$$

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* Again the incompressibility of the flow in phase space is easily checked using the discretization rules.

A more precise definition of $\mathbb{M}$ should be given in the context of the discrete formulation, cf. section 4, in particular eq. (4.27).

If we write P as

$$P(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t) = P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}) + P^{1}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t), \tag{6.15}$$

where P^1 is the deviation from P^{eq}, one obtains from eq. (6.13) up to linear order in the external potential

$$\frac{\partial P^{1}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t)}{\partial t} = (\mathbb{L}_0 + \mathbb{M})P^{1}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t) + \mathbb{L}_1(t)P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}). \tag{6.16}$$

We will assume that $V(\boldsymbol{r}, t) \to 0$ for $t \to -\infty$ and that the system was initially in equilibrium. With the aid of eq. (6.12) the formal solution of eq. (6.16) can then be written as

$$P^{1}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t) = \frac{1}{kT_0}\int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}\, \exp\{(\mathbb{L}_0 + \mathbb{M})(t - t')\}\boldsymbol{\nabla}\cdot \boldsymbol{j}(\boldsymbol{r}) \\ \times V(\boldsymbol{r}, t)P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\}). \tag{6.17}$$

From this equation one easily calculates the average linear response to the perturbation. The average $\langle \boldsymbol{\alpha}(\boldsymbol{r}, t)\rangle$ is defined by

$$\langle \boldsymbol{\alpha}(\boldsymbol{r}, t)\rangle = \lim_{\Delta\to 0}\int \prod_{n} \mathrm{d}\boldsymbol{\alpha}_n \boldsymbol{\alpha}_n P(\{\boldsymbol{\alpha}_n\}, t) \\ \equiv \iint \mathrm{d}\{\boldsymbol{\alpha}(\boldsymbol{r})\}\boldsymbol{\alpha}(\boldsymbol{r})P(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t) \tag{6.18}$$

and the equilibrium correlation functions $\langle \boldsymbol{\alpha}(\boldsymbol{r}, t)\boldsymbol{\alpha}(\boldsymbol{r}', 0)\rangle_{\mathrm{eq}}$ by

$$\langle \boldsymbol{\alpha}(\boldsymbol{r}, t)\boldsymbol{\alpha}(\boldsymbol{r}', 0)\rangle_{\mathrm{eq}} \equiv \iint \mathrm{d}\{\boldsymbol{\alpha}(\boldsymbol{r})\} \iint \mathrm{d}\{\boldsymbol{\alpha}'(\boldsymbol{r})\}\boldsymbol{\alpha}(\boldsymbol{r})\boldsymbol{\alpha}'(\boldsymbol{r}')P(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t \mid \{\boldsymbol{\alpha}'(\boldsymbol{r})\}) \\ \times P^{\mathrm{eq}}(\{\boldsymbol{\alpha}'(\boldsymbol{r})\}), \tag{6.19}$$

where, as before, $P(\{\boldsymbol{\alpha}(\boldsymbol{r})\}, t \mid \{\boldsymbol{\alpha}'(\boldsymbol{r})\})$ is the fundamental solution of the Fokker–Planck equation in the absence of the potential.

From eq. (6.17), we obtain for

$$\Delta\langle \rho(\boldsymbol{r}, t)\rangle \equiv \langle \rho(\boldsymbol{r}, t)\rangle - \langle \rho(\boldsymbol{r}, t)\rangle_{\mathrm{eq}},$$

$$\Delta\langle \rho(\boldsymbol{r}, t)\rangle = \frac{1}{kT_0}\int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}'\Big[\iint \mathrm{d}\{\boldsymbol{\alpha}(\boldsymbol{r})\}\rho(\boldsymbol{r})\exp\{(\mathbb{L}_0 + \mathbb{M})(t - t')\} \\ \times \boldsymbol{\nabla}'\cdot \boldsymbol{j}(\boldsymbol{r}')P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\})\Big] V(\boldsymbol{r}', t'). \tag{6.20}$$

To proceed, we note that

$$P(\{\boldsymbol{\alpha}(\boldsymbol{r})\},0\,|\,\{\boldsymbol{\alpha}'(\boldsymbol{r})\}) = \lim_{\Delta\to 0}\prod_n \delta(\boldsymbol{\alpha}_n-\boldsymbol{\alpha}'_n) \equiv \delta(\{\boldsymbol{\alpha}(\boldsymbol{r})\}-\{\boldsymbol{\alpha}'(\boldsymbol{r})\}). \tag{6.21}$$

With the aid of this initial condition, we may rewrite the term between square brackets in eq. (6.20), viz.

$$\begin{aligned}
&\iint \mathrm{d}\{\boldsymbol{\alpha}(\boldsymbol{r})\}\rho(\boldsymbol{r})\exp\{(\mathbb{L}_0+\mathbb{M})(t-t')\}\nabla'\cdot\boldsymbol{j}(\boldsymbol{r}')P^{\mathrm{eq}}(\{\boldsymbol{\alpha}(\boldsymbol{r})\})\\
&\quad=\iint \mathrm{d}\{\boldsymbol{\alpha}(\boldsymbol{r})\}\iint \mathrm{d}\{\boldsymbol{\alpha}'(\boldsymbol{r})\}\rho(\boldsymbol{r})\exp\{(\mathbb{L}_0+\mathbb{M})(t-t')\}\nabla'\cdot\boldsymbol{j}'(\boldsymbol{r}')\\
&\qquad\times P(\{\boldsymbol{\alpha}(\boldsymbol{r})\},0\,|\,\{\boldsymbol{\alpha}'(\boldsymbol{r})\})P^{\mathrm{eq}}(\{\boldsymbol{\alpha}'(\boldsymbol{r})\})\\
&\quad=\iint \mathrm{d}\{\boldsymbol{\alpha}(\boldsymbol{r})\}\iint \mathrm{d}\{\boldsymbol{\alpha}'(\boldsymbol{r})\}\rho(\boldsymbol{r})\nabla'\cdot\boldsymbol{j}'(\boldsymbol{r}')P(\{\boldsymbol{\alpha}(\boldsymbol{r})\},t-t'\,|\,\{\boldsymbol{\alpha}'(\boldsymbol{r})\})\\
&\qquad\times P^{\mathrm{eq}}(\{\boldsymbol{\alpha}'(\boldsymbol{r})\})\\
&\quad=\langle\rho(\boldsymbol{r},t)\nabla'\cdot\boldsymbol{j}(\boldsymbol{r}',t')\rangle_{\mathrm{eq}}.
\end{aligned} \tag{6.22}$$

In the above, use was made of the definition (6.19) of equilibrium correlation functions and of the fact that the fundamental solution of the Fokker–Planck equation in the $V=0$ case is formally given by

$$P(\{\boldsymbol{\alpha}(\boldsymbol{r})\},t\,|\,\{\boldsymbol{\alpha}'(\boldsymbol{r})\}) = \exp\{(\mathbb{L}_0+\mathbb{M})t\}P(\{\boldsymbol{\alpha}(\boldsymbol{r})\},0\,|\,\{\boldsymbol{\alpha}'(\boldsymbol{r})\}). \tag{6.23}$$

Substitution of eq. (6.22) into eq. (6.20) yields

$$\begin{aligned}
\Delta\langle\rho(\boldsymbol{r},t)\rangle &= \frac{1}{kT_0}\int_{-\infty}^{t}\mathrm{d}t'\int \mathrm{d}\boldsymbol{r}'\langle\rho(\boldsymbol{r},t)\nabla'\cdot\boldsymbol{j}(\boldsymbol{r}',t')\rangle_{\mathrm{eq}}V(\boldsymbol{r}',t')\\
&=\int_{-\infty}^{\infty}\mathrm{d}t'\int \mathrm{d}\boldsymbol{r}'\boldsymbol{G}_{\rho j}(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\boldsymbol{K}(\boldsymbol{r}',t').
\end{aligned} \tag{6.24}$$

Here $\boldsymbol{K}(\boldsymbol{r},t)=-\nabla V(\boldsymbol{r},t)$ is the force per unit mass due to the external potential, and $\boldsymbol{G}_{\rho j}$ a Green's function defined by

$$\boldsymbol{G}_{\rho j}(\boldsymbol{r}-\boldsymbol{r}',t-t') \equiv \frac{1}{kT_0}\langle\rho(\boldsymbol{r},t)\boldsymbol{j}(\boldsymbol{r}',t')\rangle_{\mathrm{eq}}\Theta(t-t'). \tag{6.25}$$

In this definition, $\Theta(t-t')$ is the Heavyside step function.

One similarly obtains

$$\Delta\langle\boldsymbol{j}(\boldsymbol{r},t)\rangle = \int_{-\infty}^{+\infty}\mathrm{d}t'\int \mathrm{d}\boldsymbol{r}'\boldsymbol{G}_{jj}(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\boldsymbol{K}(\boldsymbol{r}',t'), \tag{6.26}$$

and

$$\Delta\langle e_v(\boldsymbol{r}, t)\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}' \boldsymbol{G}_{ej}(\boldsymbol{r}-\boldsymbol{r}', t-t') \cdot \boldsymbol{K}(\boldsymbol{r}', t'), \tag{6.27}$$

with

$$\boldsymbol{G}_{jj}(\boldsymbol{r}-\boldsymbol{r}', t-t') \equiv \frac{1}{kT_0} \langle \boldsymbol{j}(\boldsymbol{r}, t)\boldsymbol{j}(\boldsymbol{r}', t')\rangle_{\mathrm{eq}} \Theta(t-t') \tag{6.28}$$

and

$$\boldsymbol{G}_{ej}(\boldsymbol{r}-\boldsymbol{r}', t-t') \equiv \frac{1}{kT_0} \langle e_v(\boldsymbol{r}, t)\boldsymbol{j}(\boldsymbol{r}', t')\rangle_{\mathrm{eq}} \Theta(t-t'). \tag{6.29}$$

Expressions (6.24)–(6.29) represent the fluctuation dissipation theorem for the present case. The equilibrium correlation functions occurring in these theorems and giving the *linear* response to the external force, should be calculated using the *nonlinear* equations describing the equilibrium fluctuations of the fluid. These non-linear terms in the equations of motion lead to the so-called mode–mode coupling contributions to the above correlation functions. One obtains in that way within the framework of a hydrodynamic fluctuation theory on the one hand the long time tails of these functions and on the other hand the mode–mode coupling expressions for the renormalized transport coefficients which are of special interest in particular near the critical point.

We finally note that Ma[18]) derived a fluctuation-dissipation theorem similar to those given above for a somewhat more simple system obeying a non-linear stochastic differential equation using a graph-theoretical procedure.

Appendix A

In this appendix we show that the discretized hydrodynamic eqs. (4.1)–(4.5) may either be interpreted in the Itô or in the Stratonovich sense. As discussed e.g. by Arnold[14]), the Stratonovich equations

$$\text{(S)}\quad \frac{\mathrm{d}a_i}{\mathrm{d}t} = F_i + \sum_j M_{ij} f_j, \tag{A.1}$$

where F_i and M_{ij} depend on the a_i and where f_i is a random force with correlations

$$\overline{f_i(t) f_j(t')} = 2\Lambda_{ij}\delta(t-t'), \tag{A.2}$$

is equivalent to the Itô equation

$$\text{(I)}\quad \frac{\mathrm{d}a_i}{\mathrm{d}t} = F_i + \sum_j \left(\frac{\partial M_{ij}}{\partial a_k}\right) M_{kl}\Lambda_{jl} + \sum_j M_{ij} f_j. \tag{A.3}$$

Generalization to the case considered in this paper shows that the discretized Stratonovich stochastic hydrodynamic equations written in the form (4.11),

$$\text{(S)}\quad \frac{\partial \alpha_{\beta,n}}{\partial t} = F^{\text{rev}}_{\beta,n} + F^{\text{irr}}_{\beta,n} + \sum_{n'} \sum_{\gamma,\delta=0}^{4} M_{\beta\gamma\delta,nn'} f_{\gamma\delta,n'}, \tag{A.4}$$

are equivalent with the Itô equations

$$\begin{aligned}\text{(I)}\quad \frac{\partial \alpha_{\beta,n}}{\partial t} &= F^{\text{rev}}_{\beta,n} + F^{\text{irr}}_{\beta,n} + \sum_{n'} \sum_{\gamma\,\delta=0}^{4} M_{\beta\gamma\delta,nn'} f_{\gamma\delta,n'} \\ &\quad + \sum_{\substack{n'n''\\ n'''}} \sum_{\substack{\gamma\delta\\ \gamma'\delta'}} \left(\frac{\partial M_{\beta\gamma\delta,nn'}}{\partial \alpha_{\beta',n''}}\right) M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n'n'''}.\end{aligned} \tag{A.5}$$

Hence, if

$$\sum_{\substack{n'n''\\ n'''}} \sum_{\substack{\gamma\delta\\ \gamma'\delta'\\ \beta'}} \left(\frac{\partial M_{\beta\gamma\delta,nn'}}{\partial \alpha_{\beta',n''}}\right) M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n'n'''} = 0, \tag{A.6}$$

the Itô and Stratonovich equations are identical*.

To show that eq. (A.6) is indeed obeyed, we notice that the only elements of $M_{\beta\gamma\delta,nn'}$ which depend on the variables are the elements M_{4kl} (cf. eqs. (4.14)). Therefore, eq. (A.6) is trivially fulfilled for $\beta = 0, 1, 2, 3$. For $\beta = 4$ one gets, using eqs. (4.14) and (4.17),

$$\begin{aligned}&\sum_{\substack{n'n''\\ n'''}} \sum_{\substack{\gamma\delta\\ \gamma'\delta'}} \left(\frac{\partial M_{4\gamma\delta,nn'}}{\partial \alpha_{\beta',n''}}\right) M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n''n'''} \\ &\quad = \frac{k}{\Delta^3} \sum_{\substack{n'n''\\ n'''}} \sum_{\substack{ikl\\ mp}} \left(\frac{\partial M_{4kl,nn'}}{\partial j_{i,n'}}\right) M_{imp,n''n'''} L_{klmp}\, \delta_{n'n'''} \\ &\quad = \frac{k}{\Delta^3} \sum_{\substack{n'n''\\ n'''}} \sum_{\substack{ikl\\ mp}} \delta_{ip}\delta_{il} L_{klmp}\, \frac{1}{\rho_{n'}}\, (\nabla_{k,n}\delta_{nn'})(\nabla_{m,n''}\delta_{n''n'''})\delta_{n'n''}\delta_{n'n'''} \\ &\quad = 0.\end{aligned} \tag{A.7}$$

The last step follows immediately from the fact that

$$(\nabla_{m,n''}\delta_{n''n'''})\delta_{n'n''}\delta_{n'n'''} = 0, \tag{A.8}$$

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* The equivalence of the Itô and Stratonovich equations may of course also be established via the Fokker–Planck equation, cf. eq. (4.20) and the discussion after eq. (4.26).

since the term between brackets is only non-zero if $\boldsymbol{n}''' = \boldsymbol{n}'' \pm \hat{e}_m$, while the two other Kronecker delta's are only non-zero if $\boldsymbol{n}'' = \boldsymbol{n}'''$. Eq. (A.7) together with (A.6) establishes the equivalence of the Itô and Stratonovich equations in our case.

Appendix B

In this appendix, we analyse the equations which led Fox to conclude that an "approach [similar to the one presented in this paper] to non-linear hydrodynamic fluctuations should be doubted and perhaps discarded"[15]).

Consider the equation for the fluctuating temperature of the fluid

$$\rho c_v \frac{\mathrm{d}T}{\mathrm{d}t} = -T\left(\frac{\partial p}{\partial T}\right)_\rho \nabla \cdot \boldsymbol{v} - \boldsymbol{\Pi}^{\mathrm{s}} : \nabla \boldsymbol{v} - \boldsymbol{\Pi}^{\mathrm{R}} : \nabla \boldsymbol{v} - \nabla \cdot \boldsymbol{J}^{\mathrm{s}} - \nabla \cdot \boldsymbol{J}^{\mathrm{R}}. \tag{B.1}$$

Here c_v is the specific heat at constant volume of the fluid, and $\mathrm{d}/\mathrm{d}t$ is the total time derivative, defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \tag{B.2}$$

Upon linearization of eq. (B.1) one should, according to Fox, retain the term $\boldsymbol{\Pi}^{\mathrm{R}} : \nabla \boldsymbol{v}$ since, even though it is bilinear in $\boldsymbol{\Pi}^{\mathrm{R}}$ and $\boldsymbol{v}$, it is linear in $\boldsymbol{v}$, whereas the term $\boldsymbol{\Pi}^{\mathrm{s}} : \nabla \boldsymbol{v}$ should be neglected since $\boldsymbol{\Pi}^{\mathrm{s}}$ is of the order of $\boldsymbol{v}$, so that the whole term is of the order of $\boldsymbol{v}^2$. The author then proceeds to show that

$$\langle \boldsymbol{\Pi}^{\mathrm{R}} : \nabla \boldsymbol{v} \rangle_{\mathrm{eq}} = \infty, \tag{B.3}$$

on the basis of which result he concludes that the theory is inconsistent. In the above formula $\langle \ldots \rangle_{\mathrm{eq}}$ denotes an average over an equilibrium ensemble.

The above argument is however incorrect. Indeed, the total pressure tensor is a macroscopic variable which is even under time reversal (i.e. even in the velocities of the constitutive particles in a molecular description). This follows from the fact that this quantity is defined through the momentum conservation equation (3.2). As a consequence its equal time correlation function with the velocity (gradient), which is odd in the particle velocities, vanishes (see e.g. ref. 12). Therefore, using the decomposition of $\boldsymbol{\Pi}$ into $\boldsymbol{\Pi}^{\mathrm{s}}$ and $\boldsymbol{\Pi}^{\mathrm{R}}$ given by eq. (3.4), one has

$$\langle \boldsymbol{\Pi}(\boldsymbol{r}, t) : \nabla \boldsymbol{v}(\boldsymbol{r}, t) \rangle_{\mathrm{eq}} = \langle \boldsymbol{\Pi}^{\mathrm{s}}(\boldsymbol{r}, t) : \nabla \boldsymbol{v}(\boldsymbol{r}, t) \rangle_{\mathrm{eq}} + \langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t) : \nabla \boldsymbol{v}(\boldsymbol{r}, t) \rangle_{\mathrm{eq}} = 0. \tag{B.4}$$

Eq. (B.4) states that in equilibrium there is no heat production on the average. Since it follows from the phenomenological laws that

$$\boldsymbol{\Pi}^{\mathrm{s}}(\boldsymbol{r}, t) : \nabla \boldsymbol{v}(\boldsymbol{r}, t) < 0, \tag{B.5}$$

one finds by combining eqs. (B.4) and (B.5)

$$\langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t):\nabla \boldsymbol{v}(\boldsymbol{r}, t)\rangle_{\mathrm{eq}} > 0. \tag{B.6}$$

This inequality shows that the average temperature will decrease, if we neglect the term $\boldsymbol{\Pi}^{\mathrm{s}}:\nabla \boldsymbol{v}$ in eq. (B.1) while retaining the term $\boldsymbol{\Pi}^{\mathrm{R}}:\nabla \boldsymbol{v}$. The above equations also show that the so-called "bilinear term" $\boldsymbol{\Pi}^{\mathrm{R}}:\nabla \boldsymbol{v}$ is on the average of the same order of magnitude as the "quadratic term" $\boldsymbol{\Pi}^{\mathrm{s}}:\nabla \boldsymbol{v}$ and that is inconsistent to retain one and neglect the other.

In passing we remark that eq. (B.1) is only correct if interpreted as a Stratonovich equation, since in the derivation of eq. (B.1) from eqs. (3.1)–(3.3) the usual rules for the transformation of variables have been applied to the stochastic equations. These rules, however, do not apply to Itô equations. The inconsistency of eq. (B.1) as an Itô equation is already apparent from the fact that in Itô calculus

$$\text{Itô: } \langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t):\nabla \boldsymbol{v}(\boldsymbol{r}, t)\rangle_{\mathrm{eq}} = 0, \tag{B.7}$$

which is in contradiction with eq. (B.4).

We also note that expressions like $\langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t):\nabla \boldsymbol{v}(\boldsymbol{r}', t')\rangle$ are discontinuous at $t = t'$. In view of the Stratonovich interpretation needed here, the equal time correlation $\langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t):\nabla \boldsymbol{v}(\boldsymbol{r}, t)\rangle$ appearing in eq. (B.4) is *defined* by

$$\langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t):\nabla \boldsymbol{v}(\boldsymbol{r}, t)\rangle \equiv \lim_{\epsilon\to 0} \tfrac{1}{2}\langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t+\epsilon):\nabla \boldsymbol{v}(\boldsymbol{r}, t)\rangle + \tfrac{1}{2}\langle \boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r}, t-\epsilon):\nabla \boldsymbol{v}(\boldsymbol{r}, t)\rangle. \tag{B.8}$$

We shall now verify by a direct calculation that eq. (B.4) also follows without the explicit use of the more elegant argument of time-reversal symmetry. This is done for the following set of equations

$$\nabla\cdot \boldsymbol{v} = 0, \tag{B.9}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p - \nabla\cdot \boldsymbol{\Pi}^{\mathrm{s}} - \nabla\cdot \boldsymbol{\Pi}^{\mathrm{R}}, \tag{B.10}$$

$$\rho_0 c_{v_0} \frac{\partial T}{\partial t} = -\nabla\cdot \boldsymbol{J}^{\mathrm{s}} - \nabla\cdot \boldsymbol{J}^{\mathrm{R}} - \boldsymbol{\Pi}^{\mathrm{s}}:\nabla \boldsymbol{v} - \boldsymbol{\Pi}^{\mathrm{R}}:\nabla \boldsymbol{v}, \tag{B.11}$$

where $\boldsymbol{J}^{\mathrm{s}}$ and $\boldsymbol{\Pi}^{\mathrm{s}}$ now obey the linearized phenomenological laws (3.10) and (3.11), and where the correlations of $\boldsymbol{\Pi}^{\mathrm{R}}$ and $\boldsymbol{J}^{\mathrm{R}}$ are given by eqs. (3.6)–(3.9), together with eq. (3.12). From here on, equilibrium quantities are indicated by a subscript 0. Apart from the term $\boldsymbol{\Pi}^{\mathrm{s}}:\nabla \boldsymbol{v}$, eqs. (B.9)–(B.11) are the equations considered by Fox, specialized to the case of an incompressible fluid.

We define the Fourier transform of a function $f(\boldsymbol{r}, t)$ by

$$f(\boldsymbol{k}, \omega) = \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{r} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t} f(\boldsymbol{r}, t). \tag{B.12}$$

After substitution of eq. (3.11) into eq. (B.10) and Fourier-transformation, we obtain for the velocity fluctuations generated by the random stress tensor in an aged system

$$\boldsymbol{v}(\boldsymbol{k},\omega)=\frac{1}{\mathrm{i}\omega\rho_0+\eta\boldsymbol{k}^2}\,\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{k},\omega)\cdot(\boldsymbol{U}-\boldsymbol{k}\boldsymbol{k}/\boldsymbol{k}^2), \tag{B.13}$$

where use has been made of the fact that $\boldsymbol{k}\cdot\boldsymbol{v}(\boldsymbol{k},\omega)=0$. The equilibrium-correlations of $\boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{k},\omega)$ are given by

$$\langle\Pi^{\mathrm{R}}_{ij}(k,\omega)\Pi^{\mathrm{R}}_{kl}(k',\omega')\rangle_{\mathrm{eq}}=2kL_{ijkl}(2\pi)^4\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega'). \tag{B.14}$$

This equation follows from eq. (3.7) for an aged system in which the initial conditions for the hydrodynamic fields have been removed to time $t_0\to-\infty$. With the aid of these equations, one finds

$$\begin{aligned}\langle\boldsymbol{\Pi}^{\mathrm{s}}(\boldsymbol{r},t):\nabla\boldsymbol{v}(\boldsymbol{r},t)\rangle&=2\eta\int\frac{\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^8}\,\overline{\langle\boldsymbol{k}\boldsymbol{v}(\boldsymbol{k},\omega)}:\boldsymbol{k}'\boldsymbol{v}(\boldsymbol{k}',\omega')\rangle_{\mathrm{eq}}\\&\quad\times\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}+\mathrm{i}(\omega+\omega')t}\\&=-\int\frac{\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega}{(2\pi)^4}\frac{4kT_0\eta^2\boldsymbol{k}^4}{\omega^2\rho_0^2+\eta^2\boldsymbol{k}^4},\end{aligned} \tag{B.15}$$

and

$$\langle\boldsymbol{\Pi}^{\mathrm{R}}(\boldsymbol{r},t):\nabla\boldsymbol{v}(\boldsymbol{r},t)\rangle_{\mathrm{eq}}=\int\frac{\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega}{(2\pi)^4}\frac{4kT_0\eta\boldsymbol{k}^2}{-\mathrm{i}\omega\rho_0+\eta\boldsymbol{k}^2}=\int\frac{\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega}{(2\pi)^4}\frac{4kT_0\eta^2\boldsymbol{k}^4}{\omega^2\rho_0^2+\eta^2\boldsymbol{k}^4}. \tag{B.16}$$

Comparison of eqs. (B.15) and (B.16) shows indeed that eq. (B.4) is obeyed even though the integrals are divergent if no wave-vector cut-off, necessary to tame divergencies inherent to a continuum description, is introduced.

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CHAPTER IV

A CANONICAL TRANSFORMATION RELATING THE LAGRANGIAN AND EULERIAN DESCRIPTION OF IDEAL HYDRODYNAMICS

1. Introduction

In fluid mechanics, two ways exist to specify the fields. The one most often used is the Eulerian description, in which the fields are considered as functions of the position in space, and time. The Lagrangian description, on the other hand, is based on the observation that many quantities specifying the fluid refer more fundamentally to small identifiable pieces of matter, the "fluid particles". In the Lagrangian description one therefore considers all the fields as functions of the time and the label of the fluid particle, to which they pertain.

As the fluid equations for ideal flow in the Lagrangian specification reflect quite closely the equations of motion for ordinary point particles, it is not surprising that a variational principle for these equations is already known since and resembles the one for point-particles. In fact, the formulation of this variational principle is due to Lagrange himself. The formulation of a variational principle for the Eulerian way to specify the fields, however, has proceeded in steps and spanned a long period. Clebsch^{1}) was the first to introduce in 1859 a variational principle for the case of incompressible flow. His analysis however applies to other cases of constrained flow as well, such as the case of compressible isentropic flow, which was investigated in detail by Bateman^{2}) in 1929. It was only in 1968 that Seliger and Whitham extended Clebsch's variational principle to the most general case of compressible, nonisentropic flow. The Lagrangian they introduce is of a somewhat unexpected form; yet the corresponding Hamiltonian is just the energy and is therefore the same as the one found in the variational principle for the

Lagrangian description, as one might expect. Hence the question forces itself upon us whether the variational principles for either of the two ways of specifying the fields are not in fact one and the same, so that they may be transformed into each other by means of a canonical transformation. It is the purpose of this paper to show that such a canonical transformation actually exists, and that it enables us to derive Seliger and Whitham's[3]) variational principle from the more familiar variational principle of Lagrange. Our derivation will clarify the variational principle of Seliger and Whitham, in that the so called Clebsch representation and the new fields, introduced by Seliger and Whitham in order to arrive at a proper variational principle, will turn out to emerge in a natural way.

The plan of this paper is to treat the variational principle for the Lagrangian description first in section 2. We then discuss the canonical transformation that will yield the variational principle for the Eulerian description in section 3.

2. Variational principle for the Lagrangian description

In the Lagrangian description of ideal fluid mechanics, one brings out the fact that the fluid motion may be viewed as the motion of small identifiable fluid particles. Accordingly one considers the positions $\boldsymbol{q}$ of these fluid particles as a function of the time. If we label the fluid elements with their Cartesian coordinates $\boldsymbol{m} = (m_1, m_2, m_3)$ at some initial time t_0 (which we will take equal to zero), the positions $\boldsymbol{q}$ thus become functions of $\boldsymbol{m}$ and t. Other fields are given as functions of $\boldsymbol{m}$ and t as well.

Before we discuss the equations of motion of an ideal fluid in the Lagrange description, it is instructive to realize that the transition from the Lagrangian to the Eulerian description can be viewed as a time-dependent coordinate-transformation. For, in the Eulerian description, the fields are given as a function of a fixed position in space and of the time. A fixed position in space, however, can not only be given by the three Cartesian coordinates r_1, r_2 and r_3 of the position vector $\boldsymbol{r}$, but also by the label of the fluid particle which is at that moment at $\boldsymbol{r}$. This is expressed by the relation

$$\boldsymbol{r} = \boldsymbol{q}(\boldsymbol{m}, t), \tag{1}$$

or, equivalently,

$$\boldsymbol{m} = \tilde{\boldsymbol{q}}(\boldsymbol{r}, t), \tag{2}$$

where $\tilde{\boldsymbol{q}}$ is the inverse of the function $\boldsymbol{q}$. Eqs. (1) and (2) show that for the specification of fixed positions in space, $\boldsymbol{m}$ may be viewed as a set of non-Cartesian, time-dependent coordinates.

For the discussion of the fluid equations in the Lagrangian description, we will need the Jacobian

$$J(\boldsymbol{m}, t) = \frac{\partial(q_1, q_2, q_3)}{\partial(m_1, m_2, m_3)}, \tag{3}$$

and J_{ij}, the co-factor of $\partial q_i/\partial m_j$ in J. We also note the useful identity

$$\left(\frac{\partial \tilde{q}_i(\boldsymbol{r}, t)}{\partial r_j}\right)_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m},t)} = J_{ji}(\boldsymbol{m}, t)/J(\boldsymbol{m}, t) \quad \text{(Cramer's rule)} \tag{4}$$

and the fact the gradient operator may be written as

$$\nabla_i \equiv \frac{\partial}{\partial r_i} = J^{-1}(\boldsymbol{m}, t)\sum_{j=1}^{3} J_{ij}(\boldsymbol{m}, t)\frac{\partial}{\partial m_j}. \tag{5}$$

Let us consider a fluid element with mass $\rho_0(\boldsymbol{m})\,\mathrm{d}m_1\,\mathrm{d}m_2\,\mathrm{d}m_3$, where $\rho_0(\boldsymbol{m})$ is the mass density at $t = 0$. In the Lagrangian description, the equation of motion for such a fluid element is given by Newton's law as

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) = -\sum_j J_{ij}\frac{\partial p(\boldsymbol{m}, t)}{\partial m_j} = -J(\boldsymbol{m}, t)\nabla_i p(\boldsymbol{m}, t). \tag{6}$$

Here partial derivatives with respect to time are indicated by a dot. In order to get a closed description, the pressure p, which is a function of the mass-density ρ and the entropy per unit mass s, must be known. For ideal fluid motion, only the volume of a fluid element changes, whereas its mass and entropy remain the same. One thus has for the density

$$\rho(\boldsymbol{m}, t) = \rho_0(\boldsymbol{m})/J(\boldsymbol{m}, t). \tag{7}$$

and for the entropy per unit mass

$$s(\boldsymbol{m}, t) = s_0(\boldsymbol{m}). \tag{8}$$

Eqs. (6), (7) and (8), together with the function $p(\rho, s)$ form a closed set of equations.

The variational principle leading to eq. (6) is

$$\delta\int \mathrm{d}\boldsymbol{m}\int_{t_1}^{t_2}\mathrm{d}t\,\mathscr{L}(\boldsymbol{q}(\boldsymbol{m}, t)) \equiv \delta\int \mathrm{d}\boldsymbol{m}\int_{t_1}^{t_2}\mathrm{d}t[\rho_0(\boldsymbol{m})\dot{\boldsymbol{q}}^2(\boldsymbol{m}, t)/2 - \rho_0(\boldsymbol{m})u(\rho(\boldsymbol{m}, t), s_0(\boldsymbol{m}))] = 0. \tag{9}$$

This equation should hold for arbitrary variations $\delta\boldsymbol{q}$ that vanish at t_1 and t_2 and at the boundary of the region of integration. In eq. (9), the internal energy per unit mass u is a function of ρ and s, with derivatives

$$\left(\frac{\partial u}{\partial \rho}\right)_s = p/\rho^2, \tag{10}$$

$$\left(\frac{\partial u}{\partial s}\right)_\rho = T, \tag{11}$$

where T is the temperature.

As $\rho(\boldsymbol{m}, t)$ depends on the derivatives $\partial q_i/\partial m_j$ through eq. (7), the Euler–Lagrange equations corresponding to eq. (9) are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \sum_j \frac{\partial}{\partial m_j}\left(\frac{\partial \mathcal{L}}{\partial(\partial q_i/\partial m_j)}\right) = 0. \tag{12}$$

By using the explicit form of $\mathcal{L}$ and the thermodynamic relation (10) this equation becomes

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) - \sum_j \frac{\partial}{\partial m_j}\left[\frac{\rho_0(\boldsymbol{m})p(\boldsymbol{m}, t)}{\rho^2(\boldsymbol{m}, t)} \frac{\partial \rho(\boldsymbol{m}, t)}{\partial(\partial q_i/\partial m_j)}\right] = 0. \tag{13}$$

From eq. (7) we obtain

$$\frac{\partial \rho(\boldsymbol{m}, t)}{\partial(\partial q_i/\partial m_j)} = -\frac{\rho_0(\boldsymbol{m})}{J^2(\boldsymbol{m}, t)} \frac{\partial J(\boldsymbol{m}, t)}{\partial(\partial q_i/\partial m_j)} = -\frac{\rho^2(\boldsymbol{m}, t) J_{ij}(\boldsymbol{m}, t)}{\rho_0(\boldsymbol{m})}. \tag{14}$$

Upon substitution of eq. (14) into the Euler–Lagrange equation (13), one gets

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) = -\sum_j \frac{\partial}{\partial m_j}(p(\boldsymbol{m}, t) J_{ij}(\boldsymbol{m}, t)). \tag{15}$$

With the aid of the identity

$$\sum_j \frac{\partial J_{ij}}{\partial m_j} = 0, \tag{16}$$

which may easily be proved using eq. (4), eq. (15) finally reduces to

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) = -\sum_j J_{ij}(\boldsymbol{m}, t)\frac{\partial p(\boldsymbol{m}, t)}{\partial m_j}. \tag{17}$$

Comparison of eqs. (6) and (17) shows that the variational principle (9) is indeed consistent with the equations of motion for the ideal fluid in the Lagrangian description.

The transition to a Hamilton-formalism can be accomplished in the standard way. In terms of the generalized coördinate*) $\boldsymbol{q}$ and generalized momen-

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* In order to contrast the *field* $\boldsymbol{q}(\boldsymbol{m}, t)$ with the vectors $\boldsymbol{r}$, which denote points in a Cartesian coordinate system, we call $\boldsymbol{q}(\boldsymbol{m}, t)$ a generalized coordinate and $\boldsymbol{r}$ a position-vector with position-coordinates r_1, r_2 and r_3.

tum $\boldsymbol{p}$, defined by

$$\boldsymbol{p}(\boldsymbol{m}, t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}(\boldsymbol{m}, t)} = \rho_0(\boldsymbol{m})\dot{\boldsymbol{q}}(\boldsymbol{m}, t), \tag{18}$$

the Hamiltonian H reads

$$H \equiv \int \mathrm{d}\boldsymbol{m}\, \mathcal{H}(\boldsymbol{q}(\boldsymbol{m}, t), \boldsymbol{p}(\boldsymbol{m}, t)) \equiv \int \mathrm{d}\boldsymbol{m} \left[\tfrac{1}{2} \frac{\boldsymbol{p}^2(\boldsymbol{m}, t)}{\rho_0(\boldsymbol{m})} + \rho_0(\boldsymbol{m}) u(\rho(\boldsymbol{m}, t), s_0(\boldsymbol{m})) \right]. \tag{19}$$

The Hamilton equations read

$$\dot{\boldsymbol{q}}(\boldsymbol{m}, t) = \frac{\delta H}{\delta \boldsymbol{p}(\boldsymbol{m}, t)}, \quad \dot{\boldsymbol{p}}(\boldsymbol{m}, t) = -\frac{\delta H}{\delta \boldsymbol{q}(\boldsymbol{m}, t)}. \tag{20}$$

The functional derivatives used here are defined in the standard way (see e.g. Goldstein[4]). Using the explicit form of the Hamiltonian, these Hamilton equations become

$$\dot{q}_i(\boldsymbol{m}, t) = \frac{p_i(\boldsymbol{m}, t)}{\rho_0(\boldsymbol{m})}, \tag{21}$$

$$\dot{p}_i(\boldsymbol{m}, t) = \sum_j \frac{\partial}{\partial m_j} \left[\left(\frac{\partial u}{\partial \rho} \right)_s \frac{\partial \rho(\boldsymbol{m}, t)}{\partial(\partial q_i / \partial m_j)} \right]. \tag{22}$$

These equations are equivalent to the Euler–Lagrange equations (13); as was already discussed, eqs. (13) are in turn equivalent to the equations of motion, eq. (6).

We notice that the equation of motion for the fluid particles, eq. (6), resembles quite closely Newton's equation for point particles. Accordingly, the variational principle discussed above is completely analogous to the variational principle for point-particles. For we have seen that the Lagrangian just assumes its familiar form, namely "kinetic energy minus internal (potential) energy". Moreover, we remark that the Hamilton equations for $\dot{\boldsymbol{q}}$ (eq. (21)) is the same as the equation defining the generalized momentum $\boldsymbol{p}$; this one usually finds too. These analogies will be lost if we perform a transformation to the Eulerian description, as will be discussed in the next section.

3. Canonical transformation to Eulerian description

Instead of using the Lagrangian description, in which case one specifies fields with the label of the fluid particle to which they pertain, it is usually more convenient to employ the Eulerian specification of the fields. In the latter case, one considers the fields as functions of their position in space $\boldsymbol{r}$

and time. In the Eulerian description, the fluid flow is completely characterized by the five physical fields, the density $\rho(\boldsymbol{r}, t)$, the velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and the entropy $s(\boldsymbol{r}, t)$.

We have already discussed at the beginning of section 2 that the transition from the Lagrangian to the Eulerian description and vice versa may be viewed as a time-dependent coordinate-transformation. This transformation, which is expressed by eqs. (1) and (2), enables us to relate the density and entropy in the two types of specifications as

$$\rho(\boldsymbol{r}, t)\big|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} = \rho(\boldsymbol{m}, t), \quad \text{or} \quad \rho(\boldsymbol{r}, t) = \rho(\boldsymbol{m}, t)\big|_{\boldsymbol{m}=\bar{\boldsymbol{q}}(\boldsymbol{r}, t)}, \tag{23}$$

$$s(\boldsymbol{r}, t)\big|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} = s_0(\boldsymbol{m}), \quad \text{or} \quad s(\boldsymbol{r}, t) = s_0(\boldsymbol{m})\big|_{\boldsymbol{m}=\bar{\boldsymbol{q}}(\boldsymbol{r}, t)}. \tag{24}$$

As the velocity at the position $\boldsymbol{r}$ at time t is just equal to the velocity of the fluid element which is at $\boldsymbol{r}$ at that particular moment, we also have

$$\boldsymbol{v}(\boldsymbol{r}, t)\big|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} = \frac{\partial \boldsymbol{q}(\boldsymbol{m}, t)}{\partial t}, \quad \text{or} \quad \boldsymbol{v}(\boldsymbol{r}, t) = \frac{\partial \boldsymbol{q}(\boldsymbol{m}, t)}{\partial t}\bigg|_{\boldsymbol{m}=\bar{\boldsymbol{q}}(\boldsymbol{r}, t)}. \tag{25}$$

We will now show that the variational principle discussed in the previous section can be transformed by means of a canonical transformation into a variational principle for the fields $\rho(\boldsymbol{r}, t)$, $\boldsymbol{v}(\boldsymbol{r}, t)$ and $s(\boldsymbol{r}, t)$. In particular, we will look for a Hamiltonian formalism in which ρ, ρs ($\equiv s_v$, the entropy per unit volume) and the function $\rho\bar{q}_3$ play the role of the new generalized momenta $P_i(\boldsymbol{r}, t)$.

To be specific, we will consider a generating functional $F([\boldsymbol{q}(\boldsymbol{m}, t)], [\boldsymbol{Q}(\boldsymbol{r}, t)])$, where $\boldsymbol{Q}(\boldsymbol{r}, t)$ is the new generalized coordinate. The theory of canonical transformations for fields, which e.g. has been discussed in detail by Kobussen and Broer[5,6]), proceeds along the same lines as the familiar theory of canonical transformations for point-particles (see e.g. Goldstein[4])). For a generating functional of the above form, the generalized coordinates $\boldsymbol{p}(\boldsymbol{m}, t)$ and $\boldsymbol{P}(\boldsymbol{r}, t)$ are given in terms of $\boldsymbol{q}(\boldsymbol{m}, t)$ and $\boldsymbol{Q}(\boldsymbol{r}, t)$ by

$$p_i(\boldsymbol{m}, t) = \frac{\delta F}{\delta q_i(\boldsymbol{m}, t)}, \tag{26}$$

$$P_i(\boldsymbol{r}, t) = -\frac{\delta F}{\delta Q_i(\boldsymbol{r}, t)}, \tag{27}$$

whereas the Hamiltonian is invariant since the generating functional F does not depend explicitly on time.

The following form turns out to be appropriate

$$\begin{aligned} F = -\int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{m}[\{&\rho_0(\boldsymbol{m})Q_1(\boldsymbol{r}, t) + \rho_0(\boldsymbol{m})s_0(\boldsymbol{m})Q_2(\boldsymbol{r}, t) \\ &+ \rho_0(\boldsymbol{m})m_3Q_3(\boldsymbol{r}, t)\}\delta(\boldsymbol{r} - \boldsymbol{q}(\boldsymbol{m}, t))]. \end{aligned} \tag{28}$$

We will first evaluate eq. (27). By making use of the identity

$$\delta(\boldsymbol{r} - \boldsymbol{q}(\boldsymbol{m}, t)) = J^{-1}(\boldsymbol{m}, t)\delta(\boldsymbol{m} - \bar{\boldsymbol{q}}(\boldsymbol{r}, t)), \tag{29}$$

we obtain with the aid of eqs. (7) and (23)

$$P_1(\boldsymbol{r}, t) = -\frac{\delta F}{\delta Q_1(\boldsymbol{r}, t)} = \left.\frac{\rho_0(\boldsymbol{m})}{J(\boldsymbol{m}, t)}\right|_{\boldsymbol{m}=\bar{\boldsymbol{q}}(\boldsymbol{r}, t)} = \rho(\boldsymbol{m}, t)|_{\boldsymbol{m}=\bar{\boldsymbol{q}}(\boldsymbol{r}, t)} = \rho(\boldsymbol{r}, t). \tag{30}$$

Similarly we get

$$P_2(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)s(\boldsymbol{r}, t) = s_v(\boldsymbol{r}, t), \tag{31}$$

$$P_3(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)\bar{q}_3(\boldsymbol{r}, t). \tag{32}$$

Indeed the mass density and the entropy density are two of the new generalized momenta. We will comment on eq. (32) later.

Next we evaluate eq. (26). By first performing the $\boldsymbol{r}$-integration in the expression (28), we obtain

$$\begin{aligned} p(\boldsymbol{m}, t) &= -\frac{\delta}{\delta q_i(\boldsymbol{m}, t)} \int \mathrm{d}\boldsymbol{m} [\rho_0(\boldsymbol{m}) Q_1(\boldsymbol{r}, t)|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} + \rho_0(\boldsymbol{m}) s_0(\boldsymbol{m}) Q_2(\boldsymbol{r}, t)|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} \\ &\quad + \rho_0(\boldsymbol{m}) m_3 Q_3(\boldsymbol{r}, t)|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)}] \\ &= -\rho_0(\boldsymbol{m}) \left.\frac{\partial Q_1(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} - \rho_0(\boldsymbol{m}) s_0(\boldsymbol{m}) \left.\frac{\partial Q_2(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} \\ &\quad - \rho_0(\boldsymbol{m}) m_3 \left.\frac{\partial Q_3(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)}. \end{aligned} \tag{33}$$

By using eq. (21) and the identification (25), eq. (33) can be rewritten in a more transparent way, viz.

$$\boldsymbol{v}(\boldsymbol{r}, t) = -\boldsymbol{\nabla} Q_1(\boldsymbol{r}, t) - s(\boldsymbol{r}, t)\boldsymbol{\nabla} Q_2(\boldsymbol{r}, t) - \bar{q}_3(\boldsymbol{r}, t)\boldsymbol{\nabla} Q_3(\boldsymbol{r}, t). \tag{34}$$

This is the form of the so-called Clebsch representation* for the velocity field which was first used by Seliger and Whitham[3]).

If we write the Hamiltonian as a functional of the new fields with the aid of eqs. (30)–(33), the new Hamilton density turns out to be equal to the energy

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* Clebsch[1]) introduced his way of representing the velocity field for the case that the term $s\boldsymbol{\nabla} Q_2$ may be omitted.

density in the Eulerian specification,

$$\begin{aligned} H &= \int \mathrm{d}\boldsymbol{r}\, \mathscr{H}(\boldsymbol{Q}(\boldsymbol{r},t), \boldsymbol{P}(\boldsymbol{r},t)), \\ &= \int \mathrm{d}\boldsymbol{r} \left[\Big(\sum_i P_i(\boldsymbol{r},t) \nabla Q_i(\boldsymbol{r},t) \Big)^2 \Big/ 2P_1(\boldsymbol{r},t) + P_1(\boldsymbol{r},t)\, u\Big(P_1(\boldsymbol{r},t), \frac{P_2(\boldsymbol{r},t)}{P_1(\boldsymbol{r},t)} \Big) \right], \\ &= \int \mathrm{d}\boldsymbol{r} [\tfrac{1}{2}\rho(\boldsymbol{r},t) \boldsymbol{v}^2(\boldsymbol{r},t) + \rho(\boldsymbol{r},t)\, u(\rho(\boldsymbol{r},t), s(\boldsymbol{r},t))]. \end{aligned} \tag{35}$$

The variational principle for the Eulerian description of ideal fluids that we derived thus consists of the Hamiltonian (35), together with the Clebsch representation (34) and the identifications (30)–(32). The corresponding Hamilton equations are (since from now on all functions depend on $\boldsymbol{r}$ and t, we will not write these arguments explicitly)

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} \rightarrow \begin{cases} \dot{Q}_1 = -\boldsymbol{v} \cdot \nabla Q_1 - \tfrac{1}{2}\boldsymbol{v}^2 + \mu, & (36) \\ \dot{Q}_2 = -\boldsymbol{v} \cdot \nabla Q_2 + T, & (37) \\ \dot{Q}_3 = -\boldsymbol{v} \cdot \nabla Q_3, & (38) \end{cases}$$

$$\dot{P}_i = -\frac{\delta H}{\delta Q_i} \rightarrow \begin{cases} \dot{P}_1 = -\nabla \cdot P_1 \boldsymbol{v}, \quad \text{or } \dot{\rho} = -\nabla \cdot \rho \boldsymbol{v}, & (39) \\ \dot{P}_2 = -\nabla \cdot P_2 \boldsymbol{v}, \quad \text{or } \dot{s}_v = -\nabla \cdot s_v \boldsymbol{v}, & (40) \\ \dot{P}_3 = -\nabla \cdot P_3 \boldsymbol{v}. & (41) \end{cases}$$

In eq. (36) we introduced the chemical potential per unit mass, defined by

$$\mu = u - Ts + p/\rho = \left(\frac{\partial \rho u}{\partial \rho} \right)_{s_v}. \tag{42}$$

Eqs. (39) and (40) are just the well-known mass and entropy conservation equations. With the aid of eq. (39) and the identification (32), eqs. (40) and (41) may also be written in the form

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \tag{43}$$

$$\frac{\mathrm{d}\tilde{q}_3}{\mathrm{d}t} = 0, \tag{44}$$

where the total time derivative $\mathrm{d}/\mathrm{d}t$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \tag{45}$$

Eq. (43) expresses the fact that if we follow a fluid element, its entropy per unit mass remains constant. In view of the relation $m_3 = \tilde{q}_3(r, t)$, we can similarly conclude that eq. (44) gives expression to the fact that the third label also remains constant in following a fluid particle*.

For the equation of motion for the velocity-field we obtain with the aid of eqs. (34), (43) and (44)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\nabla \frac{\mathrm{d}Q_1}{\mathrm{d}t} - s\nabla \frac{\mathrm{d}Q_2}{\mathrm{d}t} - \tilde{q}_3\nabla \frac{\mathrm{d}Q_3}{\mathrm{d}t} + (\nabla v)\cdot(\nabla Q_1 + s\nabla Q_2 + \tilde{q}_3\nabla Q_3). \tag{46}$$

The fourth term in the right-hand side of eq. (46), which is due to the fact that the gradient operator and the operator d/dt do not commute, is equal to $-\frac{1}{2}\nabla v^2$ (cf. eq. (34)). By substituting eqs. (36)–(38) into eq. (46), and using also the Gibbs–Duhem relation, we finally get

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\nabla\mu - s\nabla T = -\frac{1}{\rho}\nabla p. \tag{47}$$

Hence the Hamilton equations (36)–(41) indeed yield the Euler equation for the velocity-field. We have thus shown that the canonical transformation we introduced yields a variational principle, from which the well-known equations for $\rho(r, t)$, $v(r, t)$ and $s(r, t)$ follow. In addition, a sixth equation is found, which expresses the fact that the third label is conserved.

Let us retrace the origin of the fact that Q_3 is also conserved (cf. eq. (38)), i.e. remains constant if we follow a fluid particle. In the Lagrangian description the internal energy depends on m_1, m_2 and m_3 only via the *two* known functions $\rho_0(\boldsymbol{m})$ and $s_0(\boldsymbol{m})$ (cf. eq. (9)). Consequently, there is then a gauge-freedom, namely a freedom to choose, say, the third label m_3. This gauge-freedom is still present in the Eulerian description, in that u does not depend on $\tilde{q}_3$. This in turn implies that Q_3 is conserved, and we therefore conclude that the fact that Q_3 is conserved is related to the gauge-freedom.

From the Hamiltonian density (35), the Lagrangian of Seliger and Whitham[3]) may be obtained. The Lagrangian density and Hamiltonian density are related by

$$\mathcal{L} = \sum_i P_i\dot{Q}_i - \mathcal{H}. \tag{48}$$

If we use the identifications (30)–(32) to write the Lagrangian density in this

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* We remark that a canonical transformation, for which the three functions $\tilde{q}_i$ become the new generalized coordinates, was discussed by Kobussen and Broer[5,7]). Although in their case the three equations expressing conservation of label are found, no equations for ρ and s are then obtained.

equation as a function of ρ, s, $\tilde{q}_3$, Q_1, Q_2 and Q_3, we obtain

$$\mathscr{L} = \rho[\dot{Q}_1 + s\dot{Q}_2 + \tilde{q}_3\dot{Q}_3 - \tfrac{1}{2}v^2 - u(\rho, s)]. \tag{49}$$

This Lagrangian density, together with the Clebsch representation (34), was introduced some years ago by Seliger and Whitham[3]). One easily checks that the Euler–Lagrange equations corresponding to it are equivalent to the Hamilton equations (36)–(41).

It was originally suggested by Lin[8]), that equations, expressing label conservation, should be added to the hydrodynamic equations for ρ, v and s, in order to be able to arrive at a proper variational principle. Later, Seliger and Whitham[3]) argued that a variational principle can be formulated for the hydrodynamic equations in the Eulerian description if they are supplemented with an equation that expresses the conservation of only one of the labels. Accordingly, they introduced the field $\tilde{q}_3$ and arrived at the Lagrangian density (49). In view of the fact that our derivation fully supports the interpretation of $\tilde{q}_3$ as suggested by Lin, we may conclude that Lin has essentially made the right point.

We finally remark that Poisson-brackets for the hydrodynamic fields may be introduced with the aid of the generalized coordinates and momenta given in this section. This is discussed in a recent paper by the author and Bedeaux and Mazur[9]).

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SAMENVATTING

Gedurende de laatste tien à vijftien jaar is de belangstelling voor de bestudering van niet-lineaire bewegingsvergelijkingen sterk toegenomen. Ook in de statistische fysica is men sterker gaan beseffen dat veel transporteigenschappen essentieel beïnvloed worden door niet-lineariteiten. Deze laatste effecten zijn vooral bestudeerd binnen het kader van de zogenaamde 'mode-mode coupling' theorie, waaraan met name door Kawasaki veel is bijgedragen. In de praktijk vormen macroscopische transportvergelijkingen het uitgangspunt van deze theorie. Zelfs indien men binnen het geldigheidsgebied van de niet-evenwichts thermodynamica blijft, zodat de dissipatieve stromen lineair in de thermodynamische krachten zijn, bevatten de macroscopische wetten niet-lineariteiten van verschillende oorsprong: enerzijds zijn er de niet-lineaire bijdragen ten gevolge van convectieve stromen en de niet-lineaire relaties tussen de thermodynamische grootheden, anderzijds komen er ook kwadratische dissipatieve brontermen voor, zoals de Rayleigh dissipatie functie, en kunnen de transportcoëfficiënten een belangrijke afhankelijkheid van de toestandsparameters vertonen. Al dergelijke niet-lineariteiten hebben tot gevolg dat de evolutie van een 'Fourier-mode' met golf-vector $\vec{k}$ gekoppeld is met die van 'Fourier-modes' met andere golf-vectoren. De invloed van deze koppeling in fluctuerende systemen wordt onderzocht in de 'mode-mode coupling' theorie.

Verschillende macroscopische formuleringen van 'mode-mode coupling' theorie kunnen het uitgangspunt vormen voor berekeningen gebaseerd op deze theorie. In veel gevallen gaat men uit van niet-lineaire gegeneraliseerde Langevin vergelijkingen waarin fluctuerende 'krachten' voorkomen waarvan verondersteld wordt dat ze Gaussische processen zijn met een wit ruis-spectrum. Het is het doel van dit proefschrift de consistentie van een dergelijke aanpak nader te onderzoeken voor het speciale geval van hydrodynamische fluctuaties.

In hoofdstuk I van dit proefschrift wordt de hierboven beschreven aanleiding voor de analyse van niet-lineaire Langevin vergelijkingen nader toegelicht. Tevens wordt er gewezen op de analogie tussen de beschrijving van hydrodynamische fluctuaties met bepaalde niet-lineaire stochastische differentiaal vergelijkingen en de analyse van de Brownse beweging van een deeltje in een uitwendige potentiaal op basis van een Langevin vergelijking. Zoals in hoofdstuk I nader uiteengezet wordt, loopt de analyse van de Langevin vergelijkingen voor hydrodynamische fluctuaties in hoofdstukken II en III in grote lijnen parallel aan de volgende beschouwing:

De standaard beschrijving van de beweging van een Browns deeltje met behulp van een Langevin vergelijking is equivalent aan die met een Fokker-Planck vergelijking voor de snelheid en de positie van dit deeltje. Zonder dissipatie en fluctuaties reduceert deze Fokker-Planck vergelijking tot de Liouville vergelijking. De evenwichtsverdeling voor het Brownse deeltje is de canonische verdeling. Deze laatste is reeds een stationaire oplossing van de Liouville vergelijking; de eis dat zij ook de evenwichtsoplossing van de Fokker-Planck vergelijking is, resulteert in een restrictie op de uitdrukking die gebruikt kan worden voor de systematische frictie kracht. De verkregen relatie schrijft voor dat de frictie kracht om consistentie redenen lineair moet zijn in de snelheid van het Brownse deeltje. Op analoge wijze wordt in dit proefschrift onderzocht welke restricties gelden voor de fenomenologische wetten in de stochastische differentiaal vergelijkingen voor hydrodynamische fluctuaties.

In hoofdstuk II wordt het variatie principe van Seliger en Whitham voor ideale vloeistofstroming bediscussieerd. Op basis hiervan wordt een Hamilton formalisme ontwikkeld en worden hydrodynamische Poisson-haken geïntroduceerd. Met behulp van deze Poisson-haken wordt de evolutie vergelijking voor de waarschijnlijkheidsverdeling van de hydrodynamische variabelen in het geval van ideale stroming geschreven in de vorm van een Liouville vergelijking. In hoofdstuk III wordt vervolgens de Fokker-Planck vergelijking afgeleid die equivalent is met de door Landau en Lifshitz geïntroduceerde stochastische differentiaal vergelijkingen voor de beschrijving van hydrodynamische fluctuaties. Aan de hand van deze Fokker-Planck vergelijking wordt dan onderzocht van welke vorm de fenomenologische wetten moeten zijn wil deze aanpak consistent zijn.

Het is soms zinvol de beweging van een vloeistof op te vatten als de collectieve beweging van kleine vloeistof elementjes. Dienovereenkomstig is het ook mogelijk de vloeistof variabelen te schrijven als functie van de tijd en de 'labels' van de vloeistof elementjes, in plaats van als functie van de tijd en een vaste positie in de ruimte. De laatstgenoemde manier om de velden te specificeren wordt de Eulerse beschrijvingswijze genoemd, de eerstgenoemde de Lagrange-beschrijving. Omdat de Lagrange beschrijving uitdrukking geeft aan het feit dat fysische grootheden eigenlijk beter betrokken kunnen worden op de materiële vloeistof elementjes — de 'vloeistof-deeltjes' — dan op vaste posities in de ruimte, lijken de bewegingsvergelijkingen in deze beschrijvingswijze sterk op die, welke in de deeltjes mechanica voorkomen. Het variatie principe voor de Lagrange-beschrijving is dan ook in essentie hetzelfde als dat voor puntdeeltjes.

Het in hoofdstuk II besproken variatie principe van Seliger en Whitham voor de Eulerse beschrijving is daarentegen veel ondoorzichtiger. In hoofdstuk IV wordt daarom teruggekomen op de relatie tussen de twee beschrijvingswijzen: er wordt aangetoond dat de twee variatie principes gerelateerd kunnen worden via een canonische transformatie. Deze transformatie verduidelijkt de betekenis van het variatie principe voor de Eulerse beschrijving door het rechtstreeks af te leiden uit het meer vertrouwde maar niet altijd bruikbare variatieprincipe voor de Lagrange-beschrijving.

CURRICULUM VITAE

van

Wim van Saarloos

geboren te Franeker op 28 maart 1955

Ingevolge de wens van de Faculteit der Wiskunde en Natuurwetenschappen volgt hier een overzicht van mijn studie.

Na het behalen van het einddiploma Gymnasium β aan het Stedelijk Gymnasium te Leeuwarden ben ik in 1973 Technische Natuurkunde gaan studeren aan de Technische Hogeschool te Delft. In oktober 1977 legde ik het kandidaatsexamen af, en in oktober 1978 behaalde ik het ingenieursdiploma. Tijdens het vierde en vijfde jaar van mijn studie werkte ik in de vakgroep theoretische natuurkunde onder leiding van Prof. Dr. J.M.J. van Leeuwen, en nam daarbij deel aan onderzoek naar de renormalisatie-groep methode voor kritische verschijnselen. Op 1 november 1978 trad ik als wetenschappelijk assistent in dienst bij de Rijksuniversiteit te Leiden om aan het Instituut-Lorentz voor Theoretische Natuurkunde onderzoek te gaan doen op het gebied van de fluctuerende hydrodynamica. Dit onderzoek, waarvan de resultaten beschreven zijn in dit proefschrift, verrichtte ik onder leiding van Prof. Dr. P. Mazur en Dr. D. Bedeaux. Daarnaast werkte ik met Prof. Dr. P. Mazur aan een analyse van de hydrodynamische interacties die in een suspensie optreden tussen bolvormige deeltjes.

Aan het onderwijs droeg ik bij door in het academisch jaar 1980/1981 Dr. D. Bedeaux te assisteren bij de organisatie van een studentenseminarium over hydrodynamica. Tevens hielp ik bij het afnemen van tentamens en gaf ik enkele werkcolleges.

Tijdens mijn studie-periode te Delft werkte ik als CERN Summer Student drie maanden in de experimentele groep van Dr. P. Darriulat te Genève. In 1979 werd ik in staat gesteld deel te nemen aan de zomerschool 'Recent Advances in Statistical Mechanics' in Poiana Braşov, Roemenië. In 1980 nam ik deel aan de NUFFIC Zomerschool 'Fundamental Problems in Statistical Mechanics V' in Enschede, en bezocht ik de IUPAP Statphys XIV conferentie in Edmonton, Canada. De toekenning van een studiereis 1982 door de Koninklijke/Shell maakt het mij mogelijk in april 1982 een reis langs wetenschappelijke instituten in de Verenigde Staten te ondernemen.

Leiden, maart 1982

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9. P. Mazur and W. van Saarloos
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   Leiden, preprint (1982)

Voor hulp bij de typografische voorbereiding van dit proefschrift wil ik mijn erkentelijkheid betuigen aan Mevr. M. de C. Bridgen, die het typewerk verzorgde, en aan dhr. W.F. Tegelaar, die de tekeningen maakte.

STELLINGEN

1. Uit de Oseen vergelijking volgt dat een bol met straal a die in een incompressibele vloeistof met dichtheid ρ beweegt met snelheid $\vec{u}$ en hoeksnelheid $\vec{\omega}$, een Magnus-Robins kracht ondervindt ter grootte van $\frac{2}{3}\pi a^3 \rho\, \vec{\omega} \wedge \vec{u}$.

2. Bij de berekening van hydrodynamische interacties tussen bollen in de quasistatische Stokes benadering, levert de methode van de geïnduceerde krachten een compacte uitdrukking voor de ontwikkeling van de mobiliteitstensor in machten van R^{-1}, de inverse afstand tussen de bollen. Uit de verkregen formule is direct af te lezen welke machten van R^{-1} in deze ontwikkeling in geen geval voor kunnen komen.

   P. Mazur en W. van Saarloos, in druk.

3. Felderhof berekent de diffusiecoëfficiënt van bollen in een verdunde suspensie door de hydrodynamische paarinteracties mee te nemen tot op orde R^{-7} (met R de afstand tussen de bollen). Het is echter twijfelachtig dat hogere orde bijdragen zonder meer verwaarloosd kunnen worden.

   B.U. Felderhof, J. Phys. <u>A11</u> (1978) 929

4. Een Isingmodel op een hexagonaal rooster met zuiver imaginaire naastebuur interacties iJ, met J in het interval $[0, \pi k_B T/6]$ heeft een reële toestandssom en een kritisch punt voor $J = \pi k_B T/6$ met kritische exponent $\eta = \frac{1}{2}$ voor de correlatiefunctie.

5. Wil men in de differentiële renormalisatiemethode van Hilhorst, Schick en Van Leeuwen de gelineariseerde renormalisatiestroming in verband brengen met de kritische eigenschappen van d-dimensionale systemen met homogene (niet-plaats-afhankelijke) interacties, dan dient men in deze methode tenminste $d+1$ parameters te introduceren.

   H.J. Hilhorst, M. Schick en J.M.J. van Leeuwen, Phys. Rev. <u>B19</u>(1979)2749
   W. van Saarloos, in druk.

6. Differentiële renormalisatieschema's waarin de twee te vergelijken roostersystemen een gemeenschappelijk deelrooster hebben, zijn niet in overeenstemming te brengen met de schalingseis voor de correlatielengte van de renormalisatiegroeptheorie.

7. De constatering van Siegert, Vezzetti en Leff dat de berekening van de vrije energie van één-dimensionale spinmodellen met behulp van blok-spin renormalisatiemethoden niet essentieel verschilt van die met behulp van de overdrachtsmatrix, geldt niet voor de differentiële renormalisatiemethode.

A.J.F. Siegert, D.J. Vezzetti en H.S. Leff, Physica 89A (1977) 427

8. Indien voor de functie $F(\vec{r})$ geldt $F(\vec{r}) = f(\hat{r})\delta(r-1)$ met $\hat{r}$ een eenheidsvector in de richting van $\vec{r}$, dan kan het zinvol zijn voor de Fouriergetransformeerde $F(\vec{k}) \equiv \int d\vec{r}\, e^{-i\,\vec{k}\cdot\hat{r}}\, F(\vec{r})$ de ontwikkeling

$$F(\vec{k}) = \sum_{\ell=o}^{\infty} i^{\ell}\, \frac{(2\ell+1)!!}{\ell!} \left(\overline{\frac{\partial^{\ell}}{\partial \vec{k}^{\ell}}}\, \frac{\sin k}{k} \right) \odot F^{(\ell)}$$

te gebruiken in plaats van de vaak toegepaste ontwikkeling naar machten van $\vec{k}$. Hierin staat $\overline{\;..\;}$ voor het irreducibele deel van een tensor en $\odot$ voor een ℓ-voudige contractie, terwijl $F^{(\ell)}$ gedefinieerd is als

$$F^{(\ell)} \equiv \int d\hat{r}\; \overline{\hat{r}^{\ell}}\; f(\hat{r}) \,.$$

P. Mazur en W. van Saarloos, in druk.

9. Voor een beter begrip van de spinglas fase verdient het aanbeveling om het frequentieafhankelijke deel van de wisselstroomsusceptibiliteit van AuFe als functie van de temperatuur te meten.

10. "Making measurements of pressure, temperature and volume of gases [...] makes an esoteric theory such as the kinetic theory of gases more plausible, and illustrates its explanatory potential".
Door het doen van bovenstaande bewering gaan Biggins en Henderson te licht voorbij aan het feit dat ondanks dergelijke metingen Boltzmann's kinetische gastheorie aan het eind van de 19$^{\text{de}}$ eeuw door velen niet geaccepteerd werd.

D.R. Biggins en I. Henderson, Phys. Educ. 13 (1978) 438

11. De gerichtheid van de litteratuur over de Magnus-Robins kracht (de kracht die bijvoorbeeld een roterende bal, die door de lucht beweegt, ondervindt in de richting loodrecht op de bewegingsrichting en de rotatieas) is in de loop der tijd verschoven. Deze verandering kan niet worden toegeschreven aan een afnemende belangstelling voor militair gericht fysisch onderzoek, maar dient juist begrepen te worden in het licht van de toenemende perfectionering van militair wapentuig.

L. Euler, The True Principles of Gunnery, 1777

B. Robins, New Principles of Gunnery, ed. Hutton. 1805

G. Magnus, Uber die Abweichung der Geschosse, und eine auffallende Erscheinung bei rotierenden Körpern, Poggendorff's Annalen 88 (1853) 1

L. Prandtl, Magnuseffekt und Windkraftschiff, Naturwissenschaften 13 (1925) 93

C.B. Daish, The Physics of Ball Games, The English University Press, London, 1972

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