# Answers to the Exam Quantum Information, 11 November 2019

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)* Orthogonal states are vectors $\hat{n}$ and $-\hat{n}$ on the Bloch sphere.
   *(b)* $\Omega = i\sigma_y$.
   *(c)* the universal NOT gate would have to perform a complex conjugation, which is not a unitary operation, so it is not a valid quantum gate.

2. *a)* $\langle\psi|\rho\psi\rangle = \sum_n p_n |\langle\psi|\psi_n\rangle|^2 \geq 0$.
   *b)* $d\rho/dt = \sum_n p_n(|d\Psi_n/dt\rangle\langle\Psi_n| + |\Psi_n\rangle\langle d\Psi_n/dt|) = (-i/\hbar)\sum_n p_n(H|\Psi_n\rangle\langle\Psi_n| - |\Psi_n\rangle\langle\Psi_n|H) = (-i/\hbar)[H,\rho]$.
   *c)* $\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$ so $\rho^2(t) - \rho(t) = e^{-iHt/\hbar}[\rho^2(0) - \rho(0)]e^{iHt/\hbar}$ and $\rho^2(0) - \rho(0) = e^{iHt/\hbar}[\rho^2(t) - \rho(t)]e^{-iHt/\hbar}$; hence $\rho^2(t) = \rho(t) \Leftrightarrow \rho^2(0) = \rho(0)$.

3. *a)* $|\Psi\rangle = \frac{1}{2}(|0\rangle_A + |1\rangle_A)(|0\rangle_B + |1\rangle_B)$ which factorizes into the product of a state of qubit $A$ and a state of qubit $B$, so they are not entangled.
   *b)* $\rho_A = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $\rho_A^2 = \rho_A$, so this is a pure state.
   *c)* first act on qubit $C$ with a Hadamard gate then perform a CNOT operation with qubit $C$ as the control and qubit $D$ as the target:
   $|\Phi\rangle \mapsto 2^{-1/2}(|0\rangle_C + |1\rangle_C)|0\rangle_D \mapsto 2^{-1/2}(|0\rangle_C|0\rangle_D + |1\rangle_C|1\rangle_D)$.

4. *a)* If $f(0) = f(1)$ the function $f(x)$ is not invertible, so the single-qubit gate $|x\rangle \mapsto |f(x)\rangle$ would not be a unitary operation.
   *b)* $|\Psi_1\rangle = |0\rangle|1\rangle$, $|\Psi_2\rangle = (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$, $|\Psi_3\rangle = |0\rangle(|f(0)\rangle - |1 \oplus f(0)\rangle) + |1\rangle(|f(1)\rangle - |1 \oplus f(1)\rangle)$, $|\Psi_4\rangle = (|0\rangle + |1\rangle)(|f(0)\rangle - |1 \oplus f(0)\rangle) + (|0\rangle - |1\rangle)(|f(1)\rangle - |1 \oplus f(1)\rangle)$.
   *c)* if $f(0) = f(1)$ the state $|\Psi_4\rangle \propto |0\rangle(|f(0)\rangle - |1 \oplus f(0)\rangle)$ so a measurement of the first qubit will always give the value 0; while if $f(0) \neq f(1)$ the state $|\Psi_4\rangle \propto |1\rangle(|f(0)\rangle - |1 \oplus f(0)\rangle)$, so a measurement of the first qubit will always give the value 1.