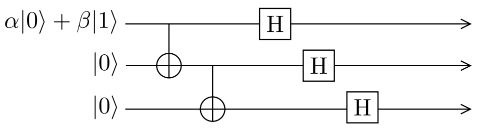
**Answers to the Exam Quantum Information, 19 December 2019**
each item gives 2 points for a fully correct answer, grade = total × 9/24 + 1

1. *(a)* the first qubit is 0 with probability $1/3 + 1/6 = 1/2$.
   *(b)* $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle + i\sqrt{\frac{1}{3}}|1\rangle$, or $\rho = \frac{1}{3}\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 1 \end{pmatrix}$;
   this is the pure state $|\psi\rangle$, and indeed $\rho^2 = \rho$.
   *(c)* the concurrence equals $C = 2|-i\sqrt{1/12} + i\sqrt{1/24}| \neq 0$, so the state is entangled.

2. *(a)* CNOT operation with the first qubit as the control and the second qubit as the target.
   *(b)* the no-cloning theorem forbids copying an unknown state, but the copy of $|\psi_1\rangle$ would have been $(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$, which is different from $\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$, so the no-cloning theorem is not violated.
   *(c)* suppose there is a unitary $U$ such that $|\psi\rangle|0\rangle \mapsto |\psi\rangle|\psi\rangle$; then $|0\rangle|0\rangle \mapsto |0\rangle|0\rangle$ and $|1\rangle|0\rangle \mapsto |1\rangle|1\rangle$, hence $(|0\rangle + |1\rangle)|0\rangle \mapsto |0\rangle|0\rangle + |1\rangle|1\rangle$, which is different from the copied state $(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$.

3. *(a)* To complete the teleportation protocal Alice must measure her two qubits and communicate the outcome to Bob. The outcome consists of the four possibilities $00, 01, 10, 11$, so this is two bits of classical information that Alice has to communicate to Bob in order to transmit the unknown state of her qubit.
   *(b)* If $b_2 = 0$, the two-qubit state just before Bob carries out the operations is $|0\rangle|0\rangle + (-1)^{b_1}|1\rangle|1\rangle$; after the CNOT operation it is $|0\rangle|0\rangle + (-1)^{b_1}|1\rangle|0\rangle$, and then after the Hadamard it finally is $|0\rangle|0\rangle + |1\rangle|0\rangle + (-1)^{b_1}|0\rangle|0\rangle - (-1)^{b_1}|1\rangle|0\rangle$.
   If $b_2 = 1$, the two-qubit state just before Bob carries out the operations is $(-1)^{b_1}|1\rangle|0\rangle + |0\rangle|1\rangle$; after the CNOT operation it is $(-1)^{b_1}|1\rangle|1\rangle + |0\rangle|1\rangle$, and then after the Hadamard it finally is $(-1)^{b_1}|0\rangle|1\rangle - (-1)^{b_1}|1\rangle|1\rangle + |0\rangle|1\rangle + |1\rangle|1\rangle$.
   *(c)* if $b_1 = 0, b_2 = 0$ the final state is $|0\rangle|0\rangle$ and Bob measures $0, 0$, while if $b_1 = 1, b_2 = 0$ the final state is $|1\rangle|0\rangle$ and Bob measures $1, 0$;
   if $b_1 = 0, b_2 = 1$ the final state is $|0\rangle|1\rangle$ and Bob measures $0, 1$, while if $b_1 = 1, b_2 = 1$ the final state is $|1\rangle|1\rangle$ and Bob measures $1, 1$;
   so indeed, in all cases he measures $b_1, b_2$.

4. *(a)*
   
   
   (combination of CNOT and Hadamard operations)
   *(b)* $S_1|\psi_1\rangle = |\psi_1\rangle$, $S_2|\psi_2\rangle = (-1)^2|\psi_2\rangle = |\psi_2\rangle$, because $\sigma_x(|0\rangle + |1\rangle) = |0\rangle + |1\rangle$ and $\sigma_x(|0\rangle - |1\rangle) = -(|0\rangle + |1\rangle)$. Hence $\alpha|\psi_0\rangle + \beta|\psi_1\rangle$ is an eigenstate of both $S_1$ and $S_2$ with eigenvalue 1.

*(c)* For a $\sigma_z$ error on the first qubit a measurement of $S_1$ gives outcome $-1$ and a measurement of $S_2$ gives outcome $+1$; for an error on the second qubit the outcomes are $-1, -1$ and for an error on the third qubit the outcomes are $+1, -1$. The erroneous states remain eigenstates of $S_1$ and $S_2$, so the measurement does not perturb them.