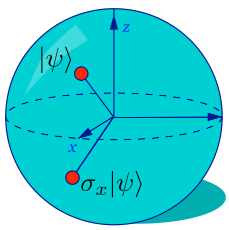
ANSWERS TO THE EXAM QUANTUM INFORMATION, 16 NOVEMBER 2020

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* Ω is a CNOT operation with the first qubit as the control and the second qubit as the target. If Ω acts on $(\alpha|0\rangle + \beta|1\rangle)|0\rangle$ the resulting state is $\alpha|0\rangle|0\rangle+\beta|1\rangle|1\rangle$, which is different from the copied state $(\alpha|0\rangle+\beta|1\rangle)|0\rangle)(\alpha|0\rangle+\beta|1\rangle)|0\rangle)$ (the cross terms are missing).
   b) U is a Hadamard operation on the first and the second qubit. When U acts on $|1\rangle|1\rangle$ the outcome is $\frac{1}{2}|0\rangle|0\rangle - \frac{1}{2}|0\rangle|1\rangle - \frac{1}{2}|1\rangle|0\rangle + \frac{1}{2}|1\rangle|1\rangle$.
   c) The concurrence is the twice the absolute value of the determinant of the coefficient matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is zero. If the first qubit is measured the second qubit remains in the state $2^{-1/2}(|0\rangle+|1\rangle)$ — irrespective of the result of the measurement outcome.

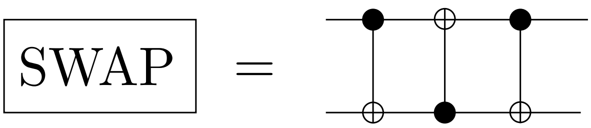
2. *a)* a density matrix should be Hermitian, it should have trace 1, and all its eigenvalues should be ≥ 0.
   b) ρ_1 is Hermitian, has trace 1, and eigenvalues 0,1/2,1/2, so it is a valid density matrix; ρ_2 has a negative eigenvalue $-1/2$ and the trace of ρ_3 is 2, so neither is a valid density matrix.
   c) a pure state has $\hat{\rho}^2 = \hat{\rho}$, only $\hat{\rho}_5$ satisfies: it is the density matrix $|\psi\rangle\langle\psi|$ of the pure state $|\psi\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$.

3. *a)* The σ_x operation rotates a point on the Bloch sphere by π around the x-axis. The rotated state is not orthogonal to the original state (an orthogonal state would be located at the inverted point $\vec{r} \mapsto -\vec{r}$).

   b) $\sigma_x = e^{-i\pi/2}e^{i(\pi/2)\sigma_x} \Rightarrow \sqrt{\sigma_x} = e^{-i\pi/4}e^{i(\pi/4)\sigma_x} = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$

   c)

4. *a)* $|0\rangle_A|0\rangle_{B_1}|0\rangle_{B_2}|0\rangle_C \mapsto \frac{1}{2}(|0\rangle_A|0\rangle_{B_1} + |1\rangle_A|1\rangle_{B_1})(|0\rangle_C|0\rangle_{B_2} + |1\rangle_C|1\rangle_{B_2})$; the qubit A is entangled with qubit B_1, the qubit C is entangled with qubit B_2.
   b) before the measurement, the four-qubit state (ignoring the normalization constant) is $|0\rangle_A|0\rangle_C(|0\rangle_{B_1} + |1\rangle_{B_1})|0\rangle_{B_2} + |1\rangle_A|0\rangle_C(|0\rangle_{B_1} - |1\rangle_{B_1})|1\rangle_{B_2} + |0\rangle_A|1\rangle_C(|0\rangle_{B_1} + |1\rangle_{B_1})|1\rangle_{B_2} + |1\rangle_A|1\rangle_C(|0\rangle_{B_1} - |1\rangle_{B_1})|0\rangle_{B_2}$. After the measurement, dependent on the measurement outcome p_1 of qubit B_1 and $p_2$of qubit B_2 the state of qubits A and C is:

$$|\psi_{00}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_C + |1\rangle_A|1\rangle_C) \ \text{if} \ (p_1,p_2) = (0,0)$$

$$|\psi_{01}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_C + |1\rangle_A|0\rangle_C) \text{ if } (p_1,p_2) = (0,1)$$

$$|\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_C - |1\rangle_A|1\rangle_C) \text{ if } (p_1,p_2) = (1,0)$$

$$|\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_C - |1\rangle_A|0\rangle_C) \text{ if } (p_1,p_2) = (1,1)$$

all for states are entangled
c) The density matrix of the qubits A and C without knowledge of p_1,p_2 is the sum of the density matrices of these four entangled states, each with weight 1/4:

$$\rho = \frac{1}{4}|\psi_{00}\rangle\langle\psi_{00}| + \frac{1}{4}|\psi_{01}\rangle\langle\psi_{01}| + \frac{1}{4}|\psi_{10}\rangle\langle\psi_{10}| + \frac{1}{4}|\psi_{11}\rangle\langle\psi_{11}|.$$

The cross terms cancel, what remains is

$$\rho = \frac{1}{4}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|),$$

which is the 4×4 unit matrix.