ANSWERS TO THE EXAM QUANTUM INFORMATION, 18 JANUARY 2023

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* $\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$.
   The eigenvalues of ρ are $0, 0, 1/2, 1/2$.
   (b) The matrix of ρ is Hermitian, it has trace 1 and non-negative eigenvalues, so it satisfies the three requirements of a density matrix.
   (c) $\rho^2 \neq \rho$, so this represents a mixed state.

2. *(a)* $\rho \mapsto \frac{1}{2}\begin{pmatrix} 1 & 1-2p \\ 1-2p & 1 \end{pmatrix}$.
   (b) $P = \operatorname{Tr}\rho^2 = 1 - 2p(1-p)$.
   (c) A unitary transformation is of the form $\rho \mapsto U\rho U^\dagger$ with unitary U. It preserves the purity of ρ. This is not the case for the decoherence operation, so it is not a unitary transformation.

3. *(a)* $U^2 = UU^\dagger = UU^{-1} = I$
   (b) $U|\Psi_\pm\rangle = (U \pm U^2)|\Psi\rangle = (U \pm I)|\Psi\rangle = \pm|\Psi_\pm\rangle$, so $|\Psi_\pm\rangle$ is an eigenstate of U with eigenvalue ± 1.
   (c) The final two-qubit state before the measurement is $|0\rangle(I+U)|\Psi_{\text{in}}\rangle + |1\rangle(I-U)|\Psi_{\text{in}}\rangle$. If the first qubit is measured as 0, the final state is $|\Psi_+\rangle$, if it is measured as 1, the final state is $|\Psi_-\rangle$. Both are eigenstates of U. The corresponding eigenvalue is 1 if the first qubit is 0 and -1 if the first qubit is 1.

4. *(a)* If the basis choices of Alice and Bob coincide, their measurement results are the same, so they can use that series of shared bits to form the key.
   (b) Alice and Bob can correlate the outcomes of their measurements with different polarisation directions, to perform a Bell test for the entanglement of the photon pairs. The entanglement will be destroyed if a photon is intercepted and measured by Eve.
   (c) If Eve knows the basis in which the photon has been measured by Alice, she can measure it without disturbing the state and then retransmit it to Bob. Parts of the shared key will then be known to Eve.