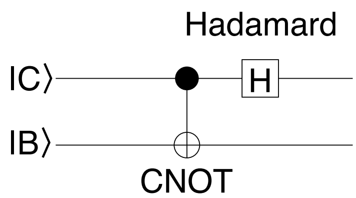
Answers to the Exam Quantum Information, 3 November 2025

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* $\rho_0 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$; this is a mixed state, $\rho_0^2 \neq \rho_0$; the purity is $P = 1/2$.
   (b) since $\rho_0 = \frac{1}{2}I$ and $\sigma_z^2 = I$, we have $\rho_p = (1-p)\frac{1}{2}I + p\frac{1}{2}I = \frac{1}{2}I = \rho_0$. No change in the density matrix, purity remains unchanged at 1/2.
   (c) Since $\sigma_z|+\rangle = |-\rangle$, the decoherence process maps $|+\rangle\langle+|$ onto
   $$\rho_p = (1-p)|+\rangle\langle+| + p|-\rangle\langle-|.$$
   We have
   $$\rho_p^2 = (1-p)^2|+\rangle\langle+| + p^2|-\rangle\langle-| \Rightarrow P = \mathrm{Tr}\,\rho_p^2 = (1-p)^2 + p^2.$$
   The purity has been reduced below the initial purity of 1, down to 1/2 for $p = 1/2$.

2. *a)* $|0\rangle|u\rangle \mapsto 2^{-1/2}(|0\rangle + |1\rangle)|u\rangle \mapsto 2^{-1/2}(|0\rangle|u\rangle + |1\rangle U|u\rangle) = 2^{-1/2}(|0\rangle|u\rangle + e^{i\phi}|1\rangle|u\rangle)$
   b) the final state before measurement is $\frac{1}{2}(|0\rangle + |1\rangle)|u\rangle + \frac{1}{2}e^{i\phi}(|0\rangle - |1\rangle)|u\rangle$, so the measurement of the first qubit gives 0 with probability $\frac{1}{4}|1+e^{i\phi}|^2 = \cos^2(\phi/2)$ and 1 with probability $\frac{1}{4}|1-e^{i\phi}|^2 = \sin^2(\phi/2)$.
   c) Repeated measurements give you the average of the number of times you measure 1, equating this average to $\sin^2(\phi/2)$ gives you an estimate for $|\phi|$. You cannot distinguish $\pm\phi$.

3. *a)* The circuit contains a CNOT operation (with the qubit shared with Charlie as the control and the qubit shared with Bob as the target), followed by a Hadamard operation (on the qubit shared with Charlie). See the diagram, where $|B\rangle$ indicates the qubit shared with Bob and $|C\rangle$ the qubit shared with Charlie.

   b) Alice's qubit has been teleported to Bob, so the final state is $|0\rangle_A|0\rangle_A(\alpha|0\rangle_B|0\rangle_C + \beta|1\rangle_B|1\rangle_C)$.
   c) Yes, the qubits of Bob and Charlie are entangled, if $\alpha\beta \neq 0$.

4. *a)* The circuit is a CNOT gate with $\alpha|0\rangle + \beta|1\rangle$ as the control and $|0\rangle$ as the target, the output is $|\Psi\rangle = \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$.
   b) since $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$, one has $\sigma_z \otimes \sigma_z|\Psi\rangle = |\Psi\rangle$, the state is an eigenstate of the parity operator with eigenvalue $+1$.
   c) since $\sigma_z\sigma_x = -\sigma_x\sigma_z$, a bit flip error on a single qubit changes the parity,

the state remains an eigenstate of P but now with eigenvalue -1. So a measurement of P does not change the encoded state, it tells us if a bit-flip error has occured, without disclosing which qubit has flipped ($P = -1$ irrespective of whether the first or the second qubit has flipped).