Answers to the Exam Quantum Information, 2 December 2025

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* $\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$.
   The eigenvalues of ρ are $0, 0, 1/2, 1/2$.
   (b) The matrix of ρ is Hermitian, it has trace 1 and non-negative eigenvalues, so it satisfies the three requirements of a density matrix.
   (c) $\rho^2 \neq \rho$, so this represents a mixed state.

2. *a)* $\text{CNOT}|\Psi\rangle = \sqrt{\frac{1}{3}}|0\rangle|0\rangle + \sqrt{\frac{2}{3}}|1\rangle|0\rangle$, then applying a Hadamard on the first qubit gives $\sqrt{\frac{1}{6}}(|0\rangle + |1\rangle)|0\rangle + \sqrt{\frac{1}{3}}(|0\rangle - |1\rangle)|0\rangle = c_+|0\rangle|0\rangle + c_-|1\rangle|0\rangle$, with $c_\pm = \sqrt{\frac{1}{6}} \pm \sqrt{\frac{1}{3}}$.
   b) concurrence is 0, the qubits are not entangled.
   c) the final state is a product state of $|0\rangle$ for the second qubit and $|\psi_A\rangle = c_+|0\rangle + c_-|1\rangle$ for the first qubit, so the reduced density matrix of the first qubit (with Alice) is $\rho = |\psi_A\rangle\langle\psi_A|$. This is a pure state.

3. *a)* there exists no unitary operator U such that for any pure state $|\phi\rangle$

$$U|\phi\rangle_A|0\rangle_B = e^{i\alpha(\phi)}|\phi\rangle_A|\phi\rangle_B$$

   b) Proof:

$$\begin{aligned}
\langle\phi|\psi\rangle &= \langle 0|_B\langle\phi|_A|\psi\rangle_A|0\rangle_B = \langle 0|_B\langle\phi|_A U^\dagger U|\psi\rangle_A|0\rangle_B \\
&= e^{i(\alpha(\psi)-\alpha(\phi))}\langle\phi|_B\langle\phi|_A|\psi\rangle_A|\psi\rangle_B \\
&= e^{i(\alpha(\psi)-\alpha(\phi))}\langle\phi|\psi\rangle^2 \\
&\Rightarrow |\langle\phi|\psi\rangle| = |\langle\phi|\psi\rangle|^2 \Rightarrow |\langle\phi|\psi\rangle| = 0 \text{ or } 1.
\end{aligned}$$

   This can not be the case for two arbitrary states.
   (c) A unitary operator is invertible, the inverse of the no-deleting statement is the no-cloning theorem:

$$U^\dagger|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle.$$

   So if such a U would exist, we would also be able to clone an arbitrary state using the operator $U^\dagger$, which is forbidden.

4. *a)* $\rho_t = (1-p)|\psi\rangle\langle\psi| + p|\psi'\rangle\langle\psi'|$, with $|\psi'\rangle = \sigma_x|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$.
   b) the copied state would be $(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$, it has cross-terms that do not appear in the encoded state $\alpha|000\rangle + \beta|111\rangle$.
   c) the final state when no error has occurred is $(\alpha|0\rangle+\beta|1\rangle)|0\rangle|0\rangle$; if an error occurred on the first qubit it is $(\alpha|0\rangle + \beta|1\rangle)|1\rangle|1\rangle$; if an error occurred on the second qubit it is $(\alpha|0\rangle + \beta|1\rangle)|1\rangle|0\rangle$; if an error occurred on the third qubit it is $(\alpha|0\rangle + \beta|1\rangle)|0\rangle|1\rangle$; so the state of the first qubit factors out from the state of the second and third qubits, it is not entangled with them.