EXAM QUANTUM INFORMATION, 15 DECEMBER 2020, 13.30–17.00 HOURS.

1. Consider the two-qubit state $|\psi\rangle = \sqrt{\frac{1}{3}}|0\rangle|0\rangle + \sqrt{\frac{2}{3}}|1\rangle|1\rangle$. Apply a CNOT operation to this state, with the first qubit as the control and the second as the target. Then apply a Hadamard operation on the first qubit only.

- *a)* Calculate the final state after these operations.
- *b)* Calculate the concurrence of the two qubits in the final state. Are they entangled or not?
- *c)* After these operations, the first qubit is given to Alice and the second qubit to Bob. Calculate the reduced density matrix of Alice's qubit. Is it a pure or a mixed state?

2. The density matrix ρ of a system with Hamiltonian H evolves in time according to

$$i\hbar\frac{\partial}{\partial t}\rho(t) = [H, \rho(t)],$$

where $[A, B] \equiv AB - BA$ denotes the commutator.

- *a)* Given that $\mathrm{Tr}\,\rho(t) = 1$ at $t = 0$, prove that this normalization condition holds for all $t > 0$.
- *b)* Given that $\rho^2(t) = \rho(t)$ at $t = 0$, prove that this purity condition holds for all $t > 0$.
- *c)* A state with $\rho^2 = \rho$ can be described by a certain wave function $|\psi\rangle$. How are ρ and $|\psi\rangle$ related? Prove that $\rho|\psi\rangle = |\psi\rangle$.

3. Alice and Bob each have a single qubit. They know that one of the two qubits is in the state $|\psi_1\rangle$ and the other qubit is in the state $|\psi_2\rangle$. They don't know who has which qubit and they have no way to communicate.

- *a)* Suppose that Bob is told that $|\psi_1\rangle = \sqrt{\frac{1}{2}}|0\rangle + \sqrt{\frac{1}{2}}|1\rangle$ and $|\psi_2\rangle = \sqrt{\frac{1}{2}}|0\rangle - \sqrt{\frac{1}{2}}|1\rangle$. Explain how Bob can determine *with certainty* the state of his qubit, by performing any combination of unitary operations and measurements. If this is not possible, explain why not.
- *b)* Same question as in *a)*, but instead Bob is told that $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = \sqrt{\frac{1}{2}}|0\rangle + \sqrt{\frac{1}{2}}|1\rangle$.
- *c)* Explain why there does not exist a unitary operator U such that for any state $|\psi\rangle$ it holds that

$$U|\psi\rangle|\psi\rangle = |\psi\rangle|0\rangle.$$

This means that it is impossible, given two identical copies of an unknown state, to delete one of them using quantum mechanical operations.

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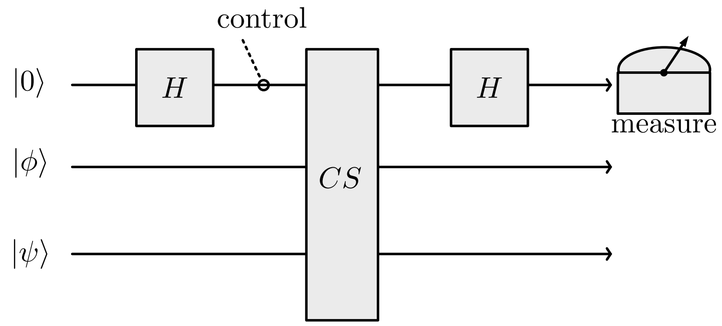
4. The controlled-SWAP gate CS is a three-qubit gate that exchanges ("swaps") the states of the second and third qubit if and only if the first qubit (the "control") is in the state 1:

$$CS|0\rangle|\phi\rangle|\psi\rangle \mapsto |0\rangle|\phi\rangle|\psi\rangle, \quad CS|1\rangle|\phi\rangle|\psi\rangle = |1\rangle|\psi\rangle|\phi\rangle.$$

The diagram shows a circuit where the control qubit starts out in the state $|0\rangle$ and is acted on with a Hadamard gate before and after the controlled-SWAP operation. Finally the control qubit is measured.



- *a)* Calculate the three-qubit state just before the final measurement.
- *b)* Calculate the probability that the final measurement of the control qubit gives the state $|0\rangle$.
- *c)* You have been told that the states $|\phi\rangle$ and $|\psi\rangle$ are either identical or orthogonal. The final measurement of the control qubit gives the state $|1\rangle$. What can you conclude, are the states $|\phi\rangle$ and $|\psi\rangle$ identical or orthogonal?