EXAM QUANTUM INFORMATION, 17 DECEMBER 2021, 14.15–17.15 HOURS.

1. Alice (A) and Bob (B) each have a qubit. The two-qubit state is

$$|\Psi\rangle = 2^{-1/2}|0\rangle_A|0\rangle_B + \tfrac{1}{2}|0\rangle_A|1\rangle_B + \tfrac{1}{2}|1\rangle_A|1\rangle_B.$$

- *a)* Are the two qubits entangled? Motivate your answer by calculating the concurrence of $|\Psi\rangle$.
- *b)* Alice measures her qubit. Suppose she finds $|0\rangle_A$. To which state does the qubit of Bob collapse? Call that state $|\psi_0\rangle_B$. Same question if Alice finds $|1\rangle_A$ as the outcome of her measurement. Call the resulting state of Bob $|\psi_1\rangle_B$.
- *c)* Bob does not know the outcome of the measurement of Alice's qubit. He describes the state of his qubit by the reduced density matrix ρ_B. Construct this object in terms of the states $|\psi_0\rangle_B$ and $|\psi_1\rangle_B$. Compare ρ_B with the reduced density matrix $\tilde{\rho}_B$ of Bob's qubit *before* Alice made her measurement.

2. The Hadamard gate is a single-qubit operation defined by

$$\mathrm{H}|0\rangle = 2^{-1/2}(|0\rangle + |1\rangle), \mathrm{H}|1\rangle = 2^{-1/2}(|0\rangle - |1\rangle).$$

- *a)* Is this operation unitary? Is it Hermitian? Explain your answer.
- *b)* The bit flip operation X is defined by $|0\rangle \mapsto |1\rangle$, $|1\rangle \mapsto |0\rangle$.
  A qubit is first passed through a Hadamard gate, followed by a bit flip and then passed again through a Hadamard gate. What is the effect on the state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$?
- *c)* The controlled-Z operation CZ is a two-qubit operation defined by

$$\mathrm{CZ}||00\rangle = |00\rangle,\ \ \mathrm{CZ}||01\rangle = |01\rangle,\ \ \mathrm{CZ}||10\rangle = |10\rangle,\ \ \mathrm{CZ}||11\rangle = -|11\rangle.$$

  Construct the circuit for the CZ operation by combining Hadamard and controlled-not (CNOT) operations.

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3. Alice and Charlie share an entangled qubit pair, in the *unknown* state $|\Psi\rangle_{AC} = \alpha|0\rangle_A|0\rangle_C + \beta|1\rangle_A|1\rangle_C$ (with $\alpha\beta \neq 0$). Alice also shares an entangled qubit pair with Bob, in the *known* state $|\Psi\rangle_{AB} = 2^{-1/2}|0\rangle_A|0\rangle_B + 2^{-1/2}|1\rangle_A|1\rangle_B$.

- *a)* Alice and Bob carry out the teleportation protocol. Draw the circuit for the gate operations that Alice will perform on her two qubits. Specify which qubit is entangled with Bob and which with Charlie.
- *b)* At the end of the gate operations Alice measures her two qubits and finds that they are both in the state $|0\rangle$. Write down the joint state of the four qubits (two with Alice, one with Bob, one with Charlie).
- *c)* Are the qubits of Bob and Charlie entangled? Motivate your answer.

4. We wish to protect the state of one qubit against the occurrence of an error of the phase-shift type (a σ_z error). For that purpose we encode our qubit into a three-qubit state, according to the rule

$$
\begin{aligned}
|0\rangle \to |\psi_0\rangle &\equiv 2^{-3/2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle), \\
|1\rangle \to |\psi_1\rangle &\equiv 2^{-3/2}(|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle).
\end{aligned} \tag{1}
$$

- *a)* Use CNOT and Hadamard gates to construct a circuit that encodes the state $\alpha|0\rangle + \beta|1\rangle$ as $\alpha|\psi_0\rangle + \beta|\psi_1\rangle$.
- *b)* Show that this encoded state is an eigenstate of each of the two operators $S_1 = \sigma_x \otimes \sigma_x \otimes 1$ and $S_2 = 1 \otimes \sigma_x \otimes \sigma_x$. What is the eigenvalue?
- *c)* Suppose we know that at most one of the three qubits in the encoded state has suffered a σ_z error. Show that the state remains an eigenstate of S_1 and S_2. Explain how a measurement of S_1 and S_2 allows you to determine on which of the qubits the error has occured — without disturbing the encoded state. A σ_z operation on the erroneous qubit then allows you to recover the original state.