Exam Quantum Information, 3 November 2025, 9–12 hours.

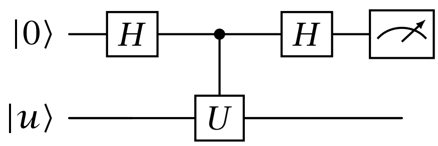
1. A qubit is prepared in a statistical mixture of the eigenstates of the Pauli-X operator, described by the density matrix:

$$\rho_0 = \tfrac{1}{2}|+\rangle\langle+| + \tfrac{1}{2}|-\rangle\langle-|,$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

- *a)* Write ρ_0 as a 2×2 matrix in the computational basis $\{|0\rangle, |1\rangle\}$. Is this a pure or a mixed state? Calculate its purity $P = \mathrm{Tr}(\rho_0^2)$.
- *b)* The qubit undergoes a *phase-damping* decoherence process, described by the mapping
$$\rho \mapsto (1-p)\rho + p\sigma_z\rho\sigma_z$$
for some probability $p \in [0,1]$. Calculate the new density matrix after this process has been applied to ρ_0. Has the purity changed?
- *c)* Same question, as in *b*), but now starting from the initial density matrix $\rho_+ = |+\rangle\langle+|$. What is the effect of the decoherence process, has the purity changed?

2. Consider a unitary operator U with an eigenvector $|u\rangle$ such that $U|u\rangle = e^{i\phi}|u\rangle$. We want to estimate the phase ϕ. We use the following circuit, where the top qubit is the control qubit and the bottom qubit is the target, initialized in the state $|u\rangle$.



The circuit consists of a Hadamard gate on the control qubit, a controlled-U operation (meaning, apply U on the target if and only if the control is 1), then another Hadamard gate on the control qubit, followed by a measurement of the control.

- *a)* Calculate the two-qubit state of the system just before the final Hadamard gate.
- *b)* After the final Hadamard gate is applied, what are the probabilities of measuring the control qubit in the state $|0\rangle$ and in the state $|1\rangle$?
- *c)* Suppose you are able to run this circuit many times, using many copies of the state $|u\rangle$. How would you estimate the unknown phase ϕ? Are there different phases in the interval $[-\pi, \pi)$ which you will not be able to distinguish?

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3. Alice and Charlie share an entangled qubit pair, in the *unknown* state $|\Psi\rangle_{AC} = \alpha|0\rangle_A|0\rangle_C + \beta|1\rangle_A|1\rangle_C$ (with $\alpha\beta \neq 0$). Alice also shares an entangled qubit pair with Bob, in the *known* state $|\Psi\rangle_{AB} = 2^{-1/2}|0\rangle_A|0\rangle_B + 2^{-1/2}|1\rangle_A|1\rangle_B$.

- *a)* Alice and Bob carry out the teleportation protocol. Draw the circuit for the gate operations that Alice will perform on her two qubits. Specify which qubit is entangled with Bob and which with Charlie.
- *b)* At the end of the gate operations Alice measures her two qubits and finds that they are both in the state $|0\rangle$. Write down the joint state of the four qubits (two with Alice, one with Bob, one with Charlie).
- *c)* Are the qubits of Bob and Charlie entangled? Motivate your answer.

4. We wish to detect (but not necessarily correct) a bit-flip error on a single qubit. For this purpose, we encode the qubit into a two-qubit state according to:

$$|0\rangle \mapsto |\phi_0\rangle = |0\rangle\,|0\rangle\,, \quad |1\rangle \mapsto |\phi_1\rangle = |1\rangle\,|1\rangle\,.$$

- *a)* Construct a circuit using CNOT gates that encodes the state $\alpha\,|0\rangle + \beta\,|1\rangle$ as $\alpha\,|\phi_0\rangle + \beta\,|\phi_1\rangle$.
- *b)* Show that the encoded state is an eigenstate of the parity operator $P = \sigma_z \otimes \sigma_z$. What is the eigenvalue?
- *c)* Suppose that at most one bit-flip error (a σ_x operation) has occurred on one of the two qubits. Show that a measurement of the parity operator P reveals whether an error has occurred, without changing the encoded state. Why can't we use that information to correct the error?