EXAM QUANTUM INFORMATION, 2 DECEMBER 2025, 15.15–18.15 HOURS.

1. Consider the two-qubit operator

$$\rho = \tfrac{1}{4}(\sigma_z \otimes \sigma_z + I \otimes I).$$

The operator I is the single-qubit identity operator and σ_z is a Pauli matrix.

- *a)* Write ρ as a 4×4 matrix. What are its eigenvalues?
- *b)* Why is ρ a valid density matrix?
- *c)* Does ρ describe a pure state or a mixed state? Motivate your answer.

2. Consider the two-qubit state $|\psi\rangle = \sqrt{\frac{1}{3}}|0\rangle|0\rangle + \sqrt{\frac{2}{3}}|1\rangle|1\rangle$. Apply a CNOT operation to this state, with the first qubit as the control and the second as the target. Then apply a Hadamard operation on the first qubit only.

- *a)* Calculate the final state after these operations.
- *b)* Calculate the concurrence of the two qubits in the final state. Are they entangled or not?
- *c)* After these operations, the first qubit is given to Alice and the second qubit to Bob. Calculate the reduced density matrix of Alice's qubit. Is it a pure or a mixed state?

3. The "no-cloning theorem" says that it is impossible to copy an unknown quantum state.

- *a)* Formulate this theorem in mathematical terms.
- *b)* Give a proof of the theorem.
- *c)* Explain why there does not exist a unitary operator U such that for any state $|\psi\rangle$ it holds that

$$U|\psi\rangle|\psi\rangle = |\psi\rangle|0\rangle.$$

This means that it is impossible, given two identical copies of an unknown state, to delete one of them using quantum mechanical operations.

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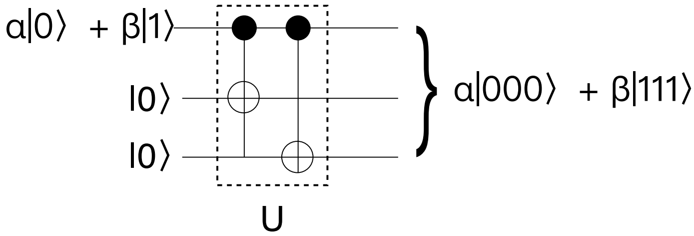
4. A qubit in the initial state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ interacts with the enviroment for certain time t; after that time it has suffered a bit-flip error with probability p.

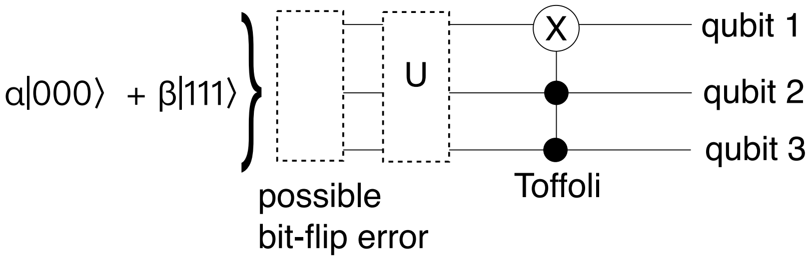
- *a)* What is the density matrix ρ_t of the qubit at time t?

  To protect the qubit from the error, the initial single-qubit state is encoded into three qubits by means of two CNOT gates, see the figure.



- *b)* The no-cloning theorem forbids you from making copies of an unknown state. Explain how the encoded state $\alpha|000\rangle + \beta|111\rangle$ differs from three copies of the initial state $\alpha|0\rangle + \beta|1\rangle$.

  For $p \ll 1$ you may neglect the possibility that more than a single qubit was flipped, so you may assume that at most one of the three qubits in the encoded state is flipped. The final state is first passed once more through the operator U, and then a Toffoli gate* is applied, with the second and third qubits as the controls (see figure).



- *c)* Explain why after this procedure the first qubit has returned to the initial state $\alpha|0\rangle + \beta|1\rangle$. Is the first qubit entangled with the other two qubits?

*The Toffoli gate is a controlled-controlled-not operation: the first qubit is flipped if and only if the second and third qubits are both equal to 1.