

## Answers to the Quantum Information exercises

1. a) normalization  $|\alpha|^2 + |\beta|^2 = 1$  reduces the number of free parameters from 4 to 3, and then we can reduce it to 2 because the wave functions  $|\Psi\rangle$  and  $e^{i\phi}|\Psi\rangle$  describe the same state.  
 b) take  $\alpha = e^{i\phi_1} \cos(\theta/2)$  and  $\beta = e^{i\phi_2} \sin(\theta/2)$  to satisfy the normalization requirement, and then multiply the wave function by  $e^{-i(\phi_1+\phi_2)/2}$ , with  $\phi = (\phi_2 - \phi_1)/2$ .  
 c)  $|0\rangle$  has  $\theta = 0$ , so it lies on the north pole,  $|1\rangle$  has  $\theta = \pi$ , so it lies on the south pole.  
 d) a spin oriented along the  $x$  or  $y$ -axis lies on the equator ( $\theta = \pi/2$ ).  
 e)  $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ ,  $|\Psi\rangle = \{e^{-i\phi/2} \cos(\theta/2)|0\rangle + e^{i\phi/2} \sin(\theta/2)|1\rangle\}$ , then  $(\mathbf{n} \cdot \boldsymbol{\sigma})|\Psi\rangle = |\Psi\rangle$ .  
 f)  $|\psi\rangle = e^{-i\phi/2} \cos(\theta/2)|0\rangle + e^{i\phi/2} \sin(\theta/2)|1\rangle$ ,  $|\phi\rangle = e^{-i\phi'/2} \cos(\theta'/2)|0\rangle + e^{i\phi'/2} \sin(\theta'/2)|1\rangle$ ,  $F = \frac{1}{2} + \frac{1}{2} \sin \theta \sin \theta' \cos(\phi - \phi') + \frac{1}{2} \cos \theta \cos \theta'$ , average over the unit sphere gives  $\bar{F} = 1/2$ .  
 2. a)  $\langle \Psi_{\text{out}} | \Psi_{\text{out}} \rangle = \langle \Psi_{\text{in}} | U^\dagger U | \Psi_{\text{in}} \rangle = \langle \Psi_{\text{in}} | \Psi_{\text{in}} \rangle = 1$  for any  $|\Psi_{\text{in}}\rangle$ , hence  $U^\dagger = U^{-1}$ .  
 b) Pauli matrices are unitary,  $\sigma_\alpha^{-1} = \sigma_\alpha = \sigma_\alpha^\dagger$ . The matrix  $\sigma_\alpha$  rotates a point on the Bloch sphere around the  $\alpha$ -axis over an angle  $\pi$ .  
 c)  $\sigma_x \sigma_y = i\sigma_z$ , so only two of the three Pauli matrices are independent.  
 d) The Hadamard gate  $H = (\sigma_x + \sigma_z)/\sqrt{2}$  rotates over an angle  $\pi$  around the axis  $\mathbf{n} = (1, 0, 1)/\sqrt{2}$ .  
 e)  $H \cdot \sigma_x \cdot H = \sigma_z$ .  
 f) Since  $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = 1$ , the identity  $e^{-i(\theta/2)\mathbf{n} \cdot \boldsymbol{\sigma}} = \cos(\theta/2)\sigma_0 - i \sin(\theta/2)\mathbf{n} \cdot \boldsymbol{\sigma}$  follows upon Taylor expansion of the exponential function in powers of  $\theta$ . The operator  $R_{\mathbf{n}}(\theta)$  multiplies eigenfunctions of  $\mathbf{n} \cdot \boldsymbol{\sigma}$  by a phase factor, so the rotation is around the  $\mathbf{n}$  axis. Choose  $\mathbf{n} = (0, 0, 1)$  along the  $z$ -axis,  $|\Psi\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  along the  $x$ -axis,  $R_{\mathbf{n}}(\theta)|\Psi\rangle = \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}$ , to see that the rotation is over an angle  $\theta$ .  
 g)  $c \oplus ab$  switches  $0 \leftrightarrow 1$  for  $c$  if and only if  $ab = 1$ , so  $a$  and  $b$  both 1.  
 h) the product  $V^2 = \sigma_x$  flips qubit  $c$  if both  $a$  and  $b$  are 1; if both are 0 nothing happens to  $c$ , and if one of them is 1 and the other 0, then  $VV^+$  or  $V^+V$  acts on  $c$ , which is the identity operation.  
 3. a)  $\text{tr } \boldsymbol{\rho} = \langle \Psi | \Psi \rangle = 1$ ,  $\boldsymbol{\rho}^\dagger = |\Psi\rangle\langle\Psi| = \boldsymbol{\rho}$ ,  $\boldsymbol{\rho}^2 = |\Psi\rangle\langle\Psi||\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi| = \boldsymbol{\rho}$ . The eigenvalues of  $\boldsymbol{\rho}$  are real, they are equal to their square, and they sum to 1: only possibility is that all eigenvalues are 0 except a single eigenvalue which is 1.

b)  $\langle \mathbf{M} \rangle = \sum_n p_n \langle \Psi_n | \mathbf{M} | \Psi_n \rangle = \text{tr } \mathbf{M} \boldsymbol{\rho}$ .

c)  $\sum_n p_n = 1 \Rightarrow \text{tr } \boldsymbol{\rho} = 1$ ;  $p_n$  is real (and  $\geq 0$ ), so  $\rho$  is still Hermitian;  $\boldsymbol{\rho}^2 = \sum_n p_n^2 |\Psi_n\rangle \langle \Psi_n|$  (for orthogonal  $\Psi_n$ ); this is  $\neq \boldsymbol{\rho}$  unless all  $p_n$ 's are 0 except a single one.

d)  $\langle \Psi | \boldsymbol{\rho} | \Psi \rangle = \sum_n p_n |\langle \Psi | \Psi_n \rangle|^2 \geq 0$ . So the eigenvalues of  $\boldsymbol{\rho}$  are nonnegative and sum to 1.

e)  $(\sigma_0 + \mathbf{a} \cdot \boldsymbol{\sigma})/2$  is Hermitian, the trace is 1, and the eigenvalues are  $(1/2)(1 \pm |\mathbf{a}|^2)$ ; these must be non-negative, so  $|\mathbf{a}| \leq 1$ .

Complex conjugation:  $a_y \mapsto -a_y$ , so reflection in the  $x$ - $z$  plane.

Unitary transformation:  $U \rho U^\dagger$  with  $U = \exp(-\frac{1}{2} i \theta \mathbf{n} \cdot \boldsymbol{\sigma})$  rotates  $\mathbf{a}$  around  $\mathbf{n}$  by an angle  $\theta$ .

f) For a pure state  $\rho^2 = \rho$ , so  $|\mathbf{a}| = 1$ , the surface of the Bloch sphere

$$\rho = \{ \{ e^{-i\phi} \cos^2(\theta/2), \frac{1}{2} \sin \theta \}, \{ \frac{1}{2} \sin \theta, e^{i\phi/2} \sin^2(\theta/2) \} \}.$$

g)  $\frac{1}{2} \text{tr} (\sigma_0 + \mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{n} \cdot \boldsymbol{\sigma}) = \mathbf{a} \cdot \mathbf{n}$ . A collection of systems is needed to measure the expectation value, a single system is not sufficient.

h)  $F = \langle \psi | P_\uparrow | \psi \rangle^2 + \langle \psi | P_\downarrow | \psi \rangle^2 = \cos^4(\theta/2) + \sin^4(\theta/2)$ , averaged over the unit sphere this gives  $\bar{F} = 2/3$ .