

Answers to the Quantum Information exercise 4

Exercise 4: Entanglement

a) $1 = \langle \Psi | \Psi \rangle = |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = \text{tr } cc^\dagger.$

b) singular value decomposition: there exist unitary 2×2 matrices U, V such that

$$c = U \begin{pmatrix} c'_{00} & 0 \\ 0 & c'_{11} \end{pmatrix} V.$$

The coefficients c'_{00} and c'_{11} are the so-called singular values of the matrix c (equal to the eigenvalues of c if c is Hermitian, but this does not need to be the case). We may choose c'_{00} and c'_{11} real ≥ 0 (by absorbing phase factors in U or V). Substitute into the expression for $|\Psi\rangle$,

$$|\Psi\rangle = \sum_{n,m=0}^1 c_{nm} |n\rangle |m\rangle = \sum_{n,m=0}^1 (U_{n0} c'_{00} V_{0m} + U_{n1} c'_{11} V_{1m}) |n\rangle |m\rangle$$

Perform a change of basis by a unitary transformation from $|\psi\rangle_A \mapsto U^\top |\psi\rangle_A$ and $|\psi\rangle_B \mapsto V |\psi\rangle_B$, and in this new basis one has $|\Psi\rangle = \sum_{n=0}^1 c'_{nn} |n\rangle_A |n\rangle_B$, which is the Schmidt decomposition.

c) $\text{tr}_B |\Psi\rangle \langle \Psi| = \sum_{k=0}^1 c_{ik} c_{jk}^* |i\rangle_A \langle j|_A \Rightarrow \rho_A = cc^\dagger$

d) A and B are entangled $\Leftrightarrow \rho_A^2 \neq \rho_A$; the eigenvalues of ρ_A are λ and $1 - \lambda$, with $0 \leq \lambda \leq 1/2$ (since ρ_A is Hermitian, non-negative definite, with unit trace). Take a basis where ρ_A is diagonal, to see that $\rho_A^2 \neq \rho_A \Leftrightarrow \lambda_1^2 \neq \lambda_1 \Leftrightarrow \lambda_1 \neq 0 \Leftrightarrow \det \rho_A \neq 0$.

e) $(\rho_B)_{ij} = \sum_k c_{ki} c_{kj}^* = (c^\dagger c)^\top$, $\det (c^\dagger c)^\top = \det c^\dagger c = \det cc^\dagger$.

f) $\det \rho_A = \lambda(1 - \lambda) \in [0, 1/4]$

g) $\rho_A = \frac{1}{4} \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} = \rho_B$; the singular values of the matrix $c = 2^{-3/2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ are $\lambda_\pm = (1 \pm \sqrt{3})2^{-3/2}$, so the Schmidt decomposition is $|\Psi\rangle = \lambda_+ |00\rangle + \lambda_- |11\rangle$.