

Answers to Quantum Information exercise 5 and 6

Exercise 5: Teleportation

a) the matrix c in these four states is given, respectively, by $2^{-1/2}\mathbf{1}$, $2^{-1/2}\sigma_x$, $2^{-1/2}\sigma_z$, $2^{-1/2}i\sigma_y$, so in each case $cc^\dagger = \frac{1}{2}\mathbf{1} \Rightarrow \det cc^\dagger = 1/4$, hence the concurrence is 1.

b) Hadamard gate: $|nm\rangle \mapsto 2^{-1/2}(|0m\rangle + (-1)^n|1m\rangle)$; CNOT gate: $|0m\rangle \mapsto |0m\rangle$, $|1m\rangle \mapsto |1\bar{m}\rangle$, with $\bar{m} = m + 1$ modulo 2.

c) a single measurement would give $|0\rangle$ or $|1\rangle$, without any information on α, β .

d) after CNOT (with the unknown qubit as the control):

$$|\Psi\rangle = \alpha(|000\rangle + |110\rangle) + \beta(|011\rangle + |101\rangle),$$

ignoring normalization factors; after Hadamard gate

$$|\Psi\rangle = \alpha(|000\rangle + |001\rangle + |110\rangle + |111\rangle) + \beta(|010\rangle - |011\rangle + |100\rangle - |101\rangle)$$

after measurement, with outcome

$$\begin{aligned} 00 : |\Psi\rangle &= \alpha|000\rangle + \beta|100\rangle, \\ 01 : |\Psi\rangle &= \alpha|001\rangle - \beta|101\rangle, \\ 10 : |\Psi\rangle &= \alpha|110\rangle + \beta|010\rangle, \\ 11 : |\Psi\rangle &= \alpha|111\rangle - \beta|011\rangle. \end{aligned} \tag{1}$$

Each outcome appears with probability $1/4$.

e) Bob does nothing if Alice tells him 00, he performs a σ_z operation if he hears 01, a σ_x operation for 10, and a σ_y operation for 11.

f) $\text{tr}_A|\Psi\rangle\langle\Psi| = \frac{1}{2}(|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|) + \frac{1}{2}(|\alpha|^2|1\rangle\langle 1| + |\beta|^2|0\rangle\langle 0|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.
no information on α, β is contained in ρ_B .

g) suppose the cloning gate exists; take two orthonormal states $|\Psi\rangle$ and $|\Psi'\rangle$ and perform the cloning operation on the sum

$$2^{-1/2}U|\Psi + \Psi'\rangle|0\rangle = 2^{-1}|\Psi + \Psi'\rangle|\Psi + \Psi'\rangle = 2^{-1}|\Psi\rangle|\Psi\rangle + 2^{-1}|\Psi'\rangle|\Psi'\rangle + 2^{-1}|\Psi\rangle|\Psi'\rangle + 2^{-1}|\Psi'\rangle|\Psi\rangle.$$

Because a unitary operator is linear, we should be able to obtain the same state by splitting up the cloning operation,

$$2^{-1/2}U|\Psi + \Psi'\rangle|0\rangle = 2^{-1/2}U|\Psi\rangle|0\rangle + 2^{-1/2}U|\Psi'\rangle|0\rangle = 2^{-1/2}|\Psi\rangle|\Psi\rangle + 2^{-1/2}|\Psi'\rangle|\Psi'\rangle.$$

These two outcomes are distinct, as one can see, for example by taking the inner product with $|\Psi\rangle|\Psi'\rangle$, you get $1/2$ in the first case and 0 in the second case.

Exercise 6: Bell inequality

a) The Schmidt decomposition allows to choose real α, β .

b) $\mathbf{c}$ has length one because $\mathbf{a} + \mathbf{a}'$ has length $2 \cos \theta$; to show that $\mathbf{c}'$ has length one we need to show that $\mathbf{a} - \mathbf{a}'$ has length $2 \sin \theta$: this is true because $|\mathbf{a} \pm \mathbf{a}'|^2 = 2 \pm 2\mathbf{a} \cdot \mathbf{a}'$, so $|\mathbf{a} + \mathbf{a}'|^2 + |\mathbf{a} - \mathbf{a}'|^2 = 4 = (2 \cos \theta)^2 + (2 \sin \theta)^2$.

To prove that $\mathbf{c}$ and $\mathbf{c}'$ are orthogonal: $\mathbf{c} \cdot \mathbf{c}' \propto (\mathbf{a} + \mathbf{a}') \cdot (\mathbf{a} - \mathbf{a}') = 0$

c) $\max_{\theta} (x \cos \theta + y \sin \theta) = \sqrt{x^2 + y^2}$, because

$$x \cos \theta + y \sin \theta = \sqrt{x^2 + y^2} (\cos \phi \cos \theta + \sin \phi \sin \theta) = \sqrt{x^2 + y^2} \cos(\theta - \phi).$$

d)

$$\begin{aligned} \frac{1}{4} (\max_{\theta} \mathcal{B})^2 &= \langle \Psi | (\sigma_x \sin \gamma + \sigma_z \cos \gamma)_A (\sigma_x \sin \phi + \sigma_z \cos \phi)_B | \Psi \rangle^2 \\ &\quad + \langle \Psi | (-\sigma_x \cos \gamma + \sigma_z \cos \gamma)_A (\sigma_x \sin \phi' + \sigma_z \cos \phi')_B | \Psi \rangle^2 \\ &= (\cos \gamma \cos \phi + 2\alpha\beta \sin \gamma \sin \phi)^2 + (\sin \gamma \cos \phi' - 2\alpha\beta \cos \gamma \sin \phi')^2. \end{aligned} \tag{2}$$

Maximize over ϕ and ϕ' ,

$$\frac{1}{4} \mathcal{B}_{\max}^2 = \cos^2 \gamma + 4\alpha^2 \beta^2 \sin^2 \gamma + \sin^2 \gamma + 4\alpha^2 \beta^2 \cos^2 \gamma = 1 + 4\alpha^2 \beta^2.$$

e) We have $\{P_a + P_{a'}, P_a - P_{a'}\} = \{2, 0\}$ or $\{0, 2\}$ or $\{-2, 0\}$ or $\{0, -2\}$, so $(P_a + P_{a'})P_b + (P_a - P_{a'})P_{b'}$ is $\pm 2P_b$ or $\pm 2P_{b'}$, hence this combination can only take the values ± 2 .

f) The average of a quantity that can only take the values $+2$ or -2 is at most $+2$.