Answers to t,he Exam Quantum Theory, 13 January 2020

each item gives 2 points for a fully correct answer, grade = total $\times 9/24+1$

1. *a)* $\psi(p) = (2\pi\hbar)^{-1/2}\int_{-\infty} e^{-ipx/\hbar}\psi(x)dx$,

$$\int_{-\infty}^{\infty} |\psi(p)|^2 dp = (2\pi\hbar)^{-1}\int dp \int dx \int dx' e^{ip(x'-x)/\hbar}\psi(x)\psi^*(x')$$

$$= \int dx \int dx' \delta(x-x')\psi(x)\psi^*(x') = \int dx |\psi(x)|^2 = 1.$$

b)

$$\mathcal{T}\psi(p) = (2\pi\hbar)^{-1/2}\int_{-\infty} e^{-ipx/\hbar}\mathcal{T}\psi(x)dx = (2\pi\hbar)^{-1/2}\int_{-\infty} e^{-ipx/\hbar}\psi^*(x)dx$$

$$= (2\pi\hbar)^{-1/2}\left(\int_{-\infty} e^{ipx/\hbar}\psi(x)dx\right)^* = \psi^*(-p).$$

c) Kramers theorem requires that the time-reversal symmetry operator squares to -1, here $\mathcal{T}^2 = +1$ so it does not hold.

2. *(a)* $T_a\psi(x) = \psi(x) + \sum_{n=1}^{\infty}(a^n/n!)d^n\psi(x)/dx^n = \psi(x+a)$ (Taylor series).
(b) $H\psi(x) = \alpha\psi(x+a) + \alpha\psi(x-a)$, so hopping to the right and to the left with probability amplitude α. If α is complex we need $H = \alpha T_a + \alpha^* T_a^\dagger$ to ensure that H is Hermitian.
(c) $H = 2\alpha\cos(ap/\hbar)$, so $E(p) = 2\alpha\cos(ap/\hbar)$; the velocity has expectation value $v = dE/dp = -2(\alpha a/\hbar)\sin(ap/\hbar)$.

3. *(a)* since $aa^\dagger - a^\dagger a = 1$, we have $[a^\dagger a, H] = -\gamma|e\rangle\langle g|a + |g\rangle\langle e|a^\dagger$; moreover, since $\langle e|g\rangle = 0$, we have $[|e\rangle\langle e|, H] = \gamma(|e\rangle\langle g|a - |g\rangle\langle e|a^\dagger)$, $[|g\rangle\langle g|, H] = \gamma(-|e\rangle\langle g|a + |g\rangle\langle e|a^\dagger)$; combining this, gives
$[(a^\dagger a + \frac{1}{2}|e\rangle\langle e| - \frac{1}{2}|g\rangle\langle g|), H] = 0$.
The conserved quantity is the number of photons ($a^\dagger a$) plus the occupation number of the excited state of the atom, because $\frac{1}{2}|e\rangle\langle e| - \frac{1}{2}|g\rangle\langle g|$ increases by 1 when the atom makes the transition from ground state to excited state.
(b) $|\psi_1\rangle = |N_0\rangle|g\rangle$, $|\psi_2\rangle = |N_0-1\rangle|0\rangle|e\rangle$;
$\langle\psi_1|H|\psi_1\rangle = -\varepsilon/2 + (N_0+1/2)\hbar\omega$, $\langle\psi_2|H|\psi_2\rangle = +\varepsilon/2 + (N_0-1/2)\hbar\omega$, $\langle\psi_1|H|\psi_2\rangle = \gamma\langle N_0|a^\dagger|N_0-1\rangle = \gamma\sqrt{N_0}$, $\langle\psi_2|H|\psi_1\rangle = \gamma\langle N_0-1|a|N_0\rangle = \gamma\sqrt{N_0}$.
(c) At a given N_0 we may restrict H to the basis $|\psi_1\rangle$, $|\psi_2\rangle$, and the eigenstates are eigenvectors of M. The corresponding eigenvalues are $E_\pm$ = constant $\pm\sqrt{\gamma^2 N_0 + \delta^2/4}$, so $\delta E = \sqrt{4\gamma^2 N_0 + \delta^2}$. For $\gamma \to 0$ we have $\delta E = \delta = \varepsilon - \hbar\omega$, which is the energy difference between the states $|e, N_0-1\rangle$ and $|g, N_0\rangle$.

4. *a)* In the first equation we have summed the contributions
$\frac{1}{2}\hbar c|\boldsymbol{k}| = \frac{1}{2}\hbar c\sqrt{k_x^2 + k_y^2 + k_z^2}$ to the vacuum energy from the field amplitude $\phi(x,y,z) \propto \sin(n\pi x/L)e^{ik_y y + ik_z z}$. The field is a plane wave parallel to the plates with wave vector components k_y, k_z, and a sine perpendicular to the

plates with wave vector component $k_x = n\pi/L$, $n = 1, 2, 3, \ldots$, such that the amplitude vanishes on the metal plates (taken at $x = 0$ and $x = L$).
In the second equation we have carried out the integral over k_y, k_z in polar coordinates, $dk_y dk_z = 2\pi r dr = \pi dr^2$, and we have changed variables from $\pi^2 n^2/L^2 + r^2$ to u^2, with u ranging from $\pi|n|/L$ to ∞.
b) In the first step we have replaced $u^2 e^{-\epsilon u} = (d^2/d\epsilon^2)e^{-\epsilon u}$ and carried out the integral $\int e^{-\epsilon u} du = -\epsilon^{-1} e^{-\epsilon u}$; in the second step we summed the geometric series $\sum_{n=1}^{\infty} e^{-\epsilon\pi n/L} = e^{-\epsilon\pi n/L}(1 - e^{-\epsilon\pi/L})^{-1} = (e^{\epsilon\pi/L} - 1)^{-1}$.
c) $E_{\text{total}} = E(L_2 - L_1) + E(L_3 - L_2) = -(\hbar c\pi^2/1440)[(L_2 - L_1)^{-3} + (L_3 - L_2)^{-3}]$ plus terms of order ϵ^2 plus terms independent of L_2. Hence $F = -dE_{\text{total}}/dL_2 = -(\hbar c\pi^2/480)(L_2 - L_1)^{-4}$ in the limit $\epsilon \to 0$, $L_3 \to \infty$. (If we would include the polarization of the electromagnetic field we would get an answer that is twice as big.)