Exam Quantum Theory, 13 January 2020, 10.15–13.15 hours.

1. The wave function of a particle in position representation is $\psi(x)$, normalized to unity: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. (The particle is spinless and confined to the x-axis.)

- *a)* What is the corresponding wave function $\psi(p)$ in momentum representation? Verify that it is also normalized to unity.
- *b)* The time-reversal operation $\mathcal{T}$ in position representation is $\mathcal{T}\psi(x) = \psi^*(x)$. What is the corresponding time-reversal operation in momentum representation?
- *c)* The Kramers theorem says that, under certain conditions, the eigenstates in the presence of time reversal symmetry are twofold degenerate. Does Kramers theorem apply in this case? Motivate your answer.

2. Consider the operator $T_a = e^{iap/\hbar}$, where a is a real constant and $p = -i\hbar d/dx$ is the momentum operator along the x-axis.

- *a)* Explain why T_a is the operator for translation over a distance a.

  An electron in a chain of atoms at positions $x_n = na$, $n = 0, \pm 1, \pm 2, \ldots$, has wave function amplitude ψ_n on atom n. The electron can move along the chain by hopping from one atom to the next.

- *b)* Explain why the Hamiltonian

$$H = \alpha(T_a + T_a^\dagger)$$

  describes the electron motion for a real constant α. How should H look like if α was complex?

- *c)* Calculate the energy spectrum of H. (You may still assume a real α.) How does the expectation value v of the velocity of the electron depend on its momentum p?

3. In this problem we will investigate a two-level system (ground state $|g\rangle$ at energy $-\varepsilon/2$, excited state $|e\rangle$ at energy $\varepsilon/2$) interacting with a harmonic oscillator (frequency ω, bosonic creation operator $a^\dagger$). The Hamiltonian is

$$H = -\frac{\varepsilon}{2}|g\rangle\langle g| + \frac{\varepsilon}{2}|e\rangle\langle e| + \hbar\omega(a^\dagger a + \tfrac{1}{2}) + \gamma\big(|e\rangle\langle g|a + |g\rangle\langle e|a^\dagger\big),$$

where γ is the interaction energy. This socalled Jaynes-Cummings model describes how an atom (modelled by the two-level system) can be excited by absorbing a photon in an optical cavity (modelled by the harmonic oscillator).

- *a)* Show that the operator

$$N = a^\dagger a + \tfrac{1}{2}|e\rangle\langle e| - \tfrac{1}{2}|g\rangle\langle g|$$

  commutes with H. What is the corresponding conserved quantity?

  Because N is conserved, we can assume it has a definite value N_0 and restrict the Hilbert space to just two states: a state $|\psi_1\rangle = |g, N_0\rangle$ where the atom is in the ground state and the cavity contains N_0 photons, and a state $|\psi_2\rangle = |e, N_0 - 1\rangle$ where the atom is in the excited state and the cavity contains $N_0 - 1$ photons.

- *b)* Calculate the four matrix elements $\langle\psi_1|H|\psi_1\rangle$, $\langle\psi_1|H|\psi_2\rangle$, $\langle\psi_2|H|\psi_1\rangle$, $\langle\psi_2|H|\psi_2\rangle$.
  Help: Recall that the harmonic oscillator at energy $(n+\frac{1}{2})\hbar\omega$ is in the state $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$.

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The four matrix elements can be combined into the 2×2 matrix

$$M = \text{constant} \times I + \begin{pmatrix} -\delta/2 & \gamma\sqrt{N_0} \\ \gamma\sqrt{N_0} & \delta/2 \end{pmatrix},$$

where I is the unit matrix and $\delta = \varepsilon - \hbar\omega$.

- *c)* The two eigenstates of H at a given N_0 have an energy difference δE. Calculate δE and check that $\delta E \to \varepsilon - \hbar\omega$ in the limit $\gamma \to 0$.

4. In class we studied the Casimir effect with two metal plates, here we consider instead three parallel metal plates, at $x = L_1$, at $x = L_2$, and at $x = L_3$, with $0 < L_1 < L_2 < L_3$. The plates have infinite extent in the y-z plane. We treat the electromagnetic field as a scalar field $\phi(x, y, z)$ (ignoring polarization). In vacuum the quantum fluctuations result in an energy per unit area given by $E_{\text{total}} = E(L_2 - L_1) + E(L_3 - L_2)$, where the function $E(L)$ is given by

$$E(L) = \tfrac{1}{2}\hbar c \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sqrt{\pi^2 n^2/L^2 + k_y^2 + k_z^2} \tag{1a}$$

$$= \frac{\hbar c}{4\pi} \sum_{n=1}^{\infty} \int_{\pi n/L}^{\infty} u^2\, du. \tag{1b}$$

- *a)* Explain how to arrive at each of these two equations (1a) and (1b) for $E(L)$.

  Because the integral over u diverges, we insert a factor $e^{-\epsilon u}$ with $\epsilon > 0$ and calculate as follows:

$$E(L, \epsilon) = \frac{\hbar c}{4\pi} \sum_{n=1}^{\infty} \int_{\pi n/L}^{\infty} e^{-\epsilon u} u^2\, du \tag{2a}$$

$$= \frac{\hbar c}{4\pi} \frac{d^2}{d\epsilon^2} \frac{1}{\epsilon} \sum_{n=1}^{\infty} e^{-\epsilon\pi n/L} \tag{2b}$$

$$= \frac{\hbar c}{4\pi} \frac{d^2}{d\epsilon^2} \frac{1}{\epsilon} \frac{1}{e^{\epsilon\pi/L} - 1}. \tag{2c}$$

- *b)* Explain the two steps in this calculation, from Eq. (2a) to (2b) and from Eq. (2b) to (2c).

  We now take an infinitesimally small ϵ and expand Eq. (2c) to arrive at

$$E(L, \epsilon) = \frac{\hbar c}{4\pi} \left(\frac{6L}{\pi\epsilon^4} - \frac{1}{\epsilon^3} - \frac{\pi^3}{360L^3} + \text{order}\,(\epsilon^2) \right). \tag{3}$$

- *c)* Calculate the derivative $F = -dE_{\text{total}}/dL_2$ the limit $\epsilon \to 0$, $L_3 \to \infty$. This is the force per unit area which attracts the second plate to the first plate, separated by a distance $L_2 - L_1$.