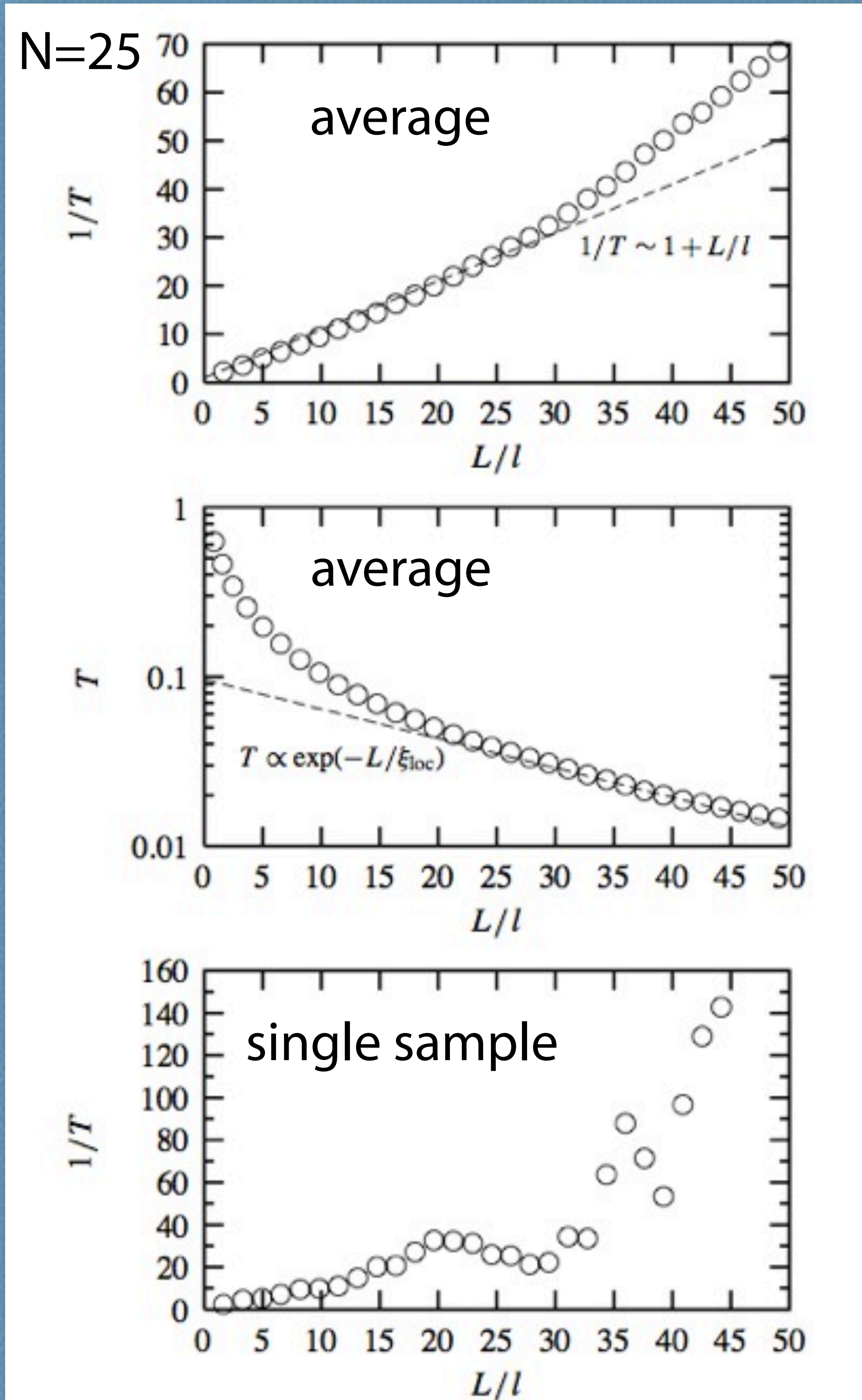
Random-matrix theory

III. localization & superconductivity

$$T = \mathrm{Tr}\, tt^\dagger = \sum_n T_n$$



disordered wire

metal: $T \sim 1/L$

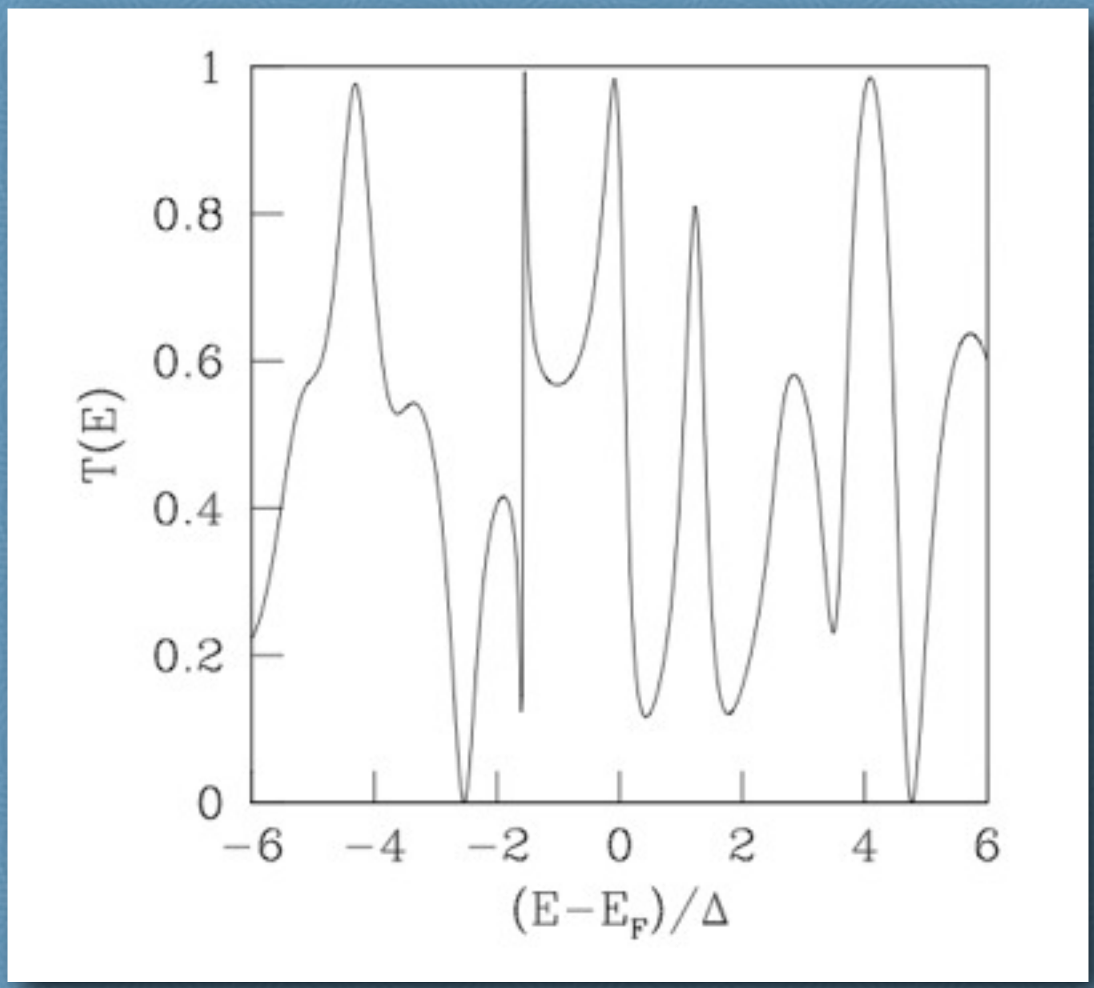
insulator: $T \sim e^{-L/\xi}$

$$\xi = \beta N l$$

with large sample-to-sample fluctuations

Thouless number

$$g = \frac{\text{resonance width}}{\text{resonance spacing}}$$



$\sim$ conductance $[e^2/h]$

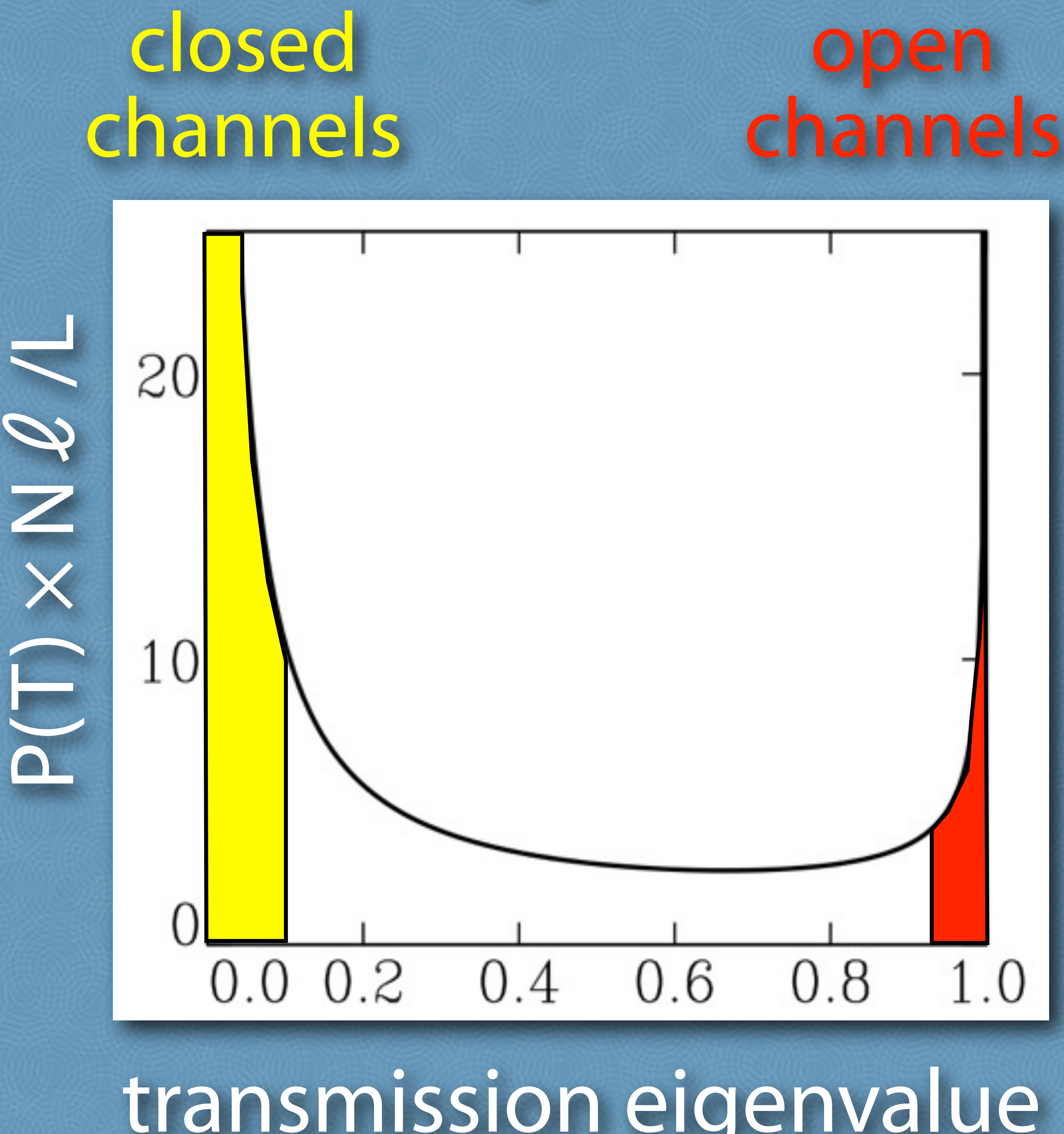
$$\frac{\hbar v \ell / L^2}{\hbar v / N L} \simeq N\ell/L$$

insulator: $g<1 \rightarrow L>N\ell$

$$\xi \simeq N\ell$$

absence of diffusion for N=1

bimodal density of transmission eigenvalues



$N\ell/L$ ~ # of open channels (N_{eff})

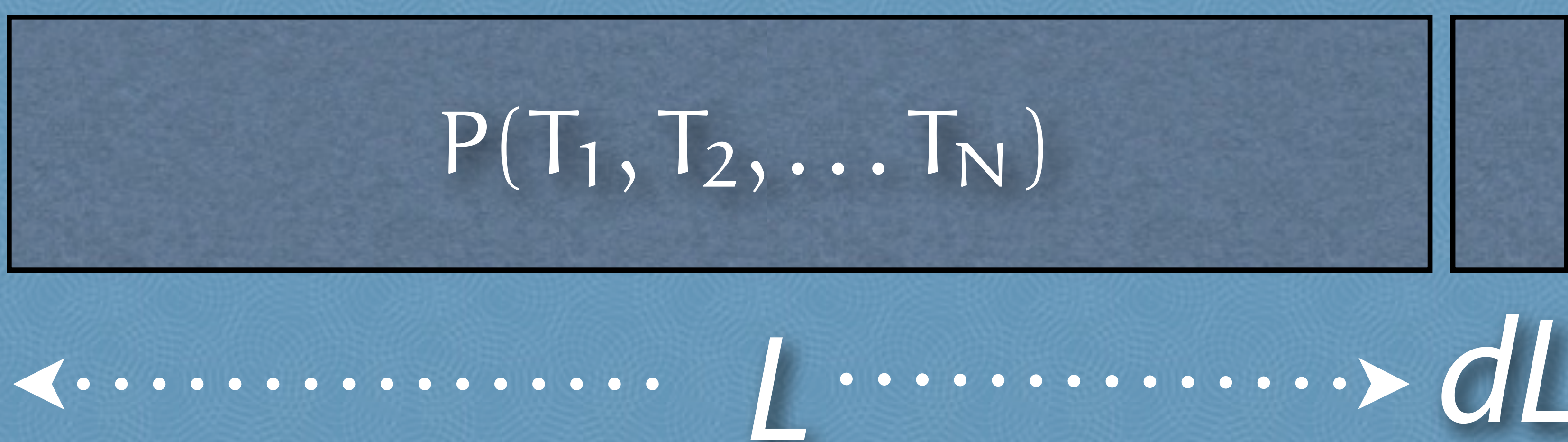
$g \sim N_{eff}$ Thouless #

$$\rho(T) = \frac{Nl}{2}\frac{1}{T\sqrt{1-T}}$$

$$e^{-2L/l} \lesssim T \leqslant 1$$

DMPK scaling equation

Dorokhov-Mello-Pereyra-Kumar



$$T_n \to T_n + \delta T_n$$

$$\langle \delta T_n \rangle = A_n \times dL$$

$$\langle \delta T_n \delta T_m \rangle = B_{nm} \times dL$$

drift-diffusion (Fokker-Planck) eq.

$$\frac{\partial P}{\partial L} = \sum_n \frac{\partial}{\partial T_n} \left(-A_n P + \frac{1}{2} \sum_m \frac{\partial}{\partial T_m} B_{nm} P \right)$$

Intermezzo: solution of the DMPK equation

$$\frac{\partial P}{\partial s} = \frac{1}{2\gamma}\sum_{n=1}^{N}\frac{\partial}{\partial x_n}\left(\frac{\partial P}{\partial x_n} + \beta P\frac{\partial \Omega}{\partial x_n}\right)$$

$$\Omega = -\sum_{i=1}^{N}\sum_{j=i+1}^{N}\ln|\sinh^2 x_j - \sinh^2 x_i| - \frac{1}{\beta}\sum_{i=1}^{N}\ln|\sinh 2x_i|$$

$$s = L/l,\quad T_n = 1/\cosh^2 x_n,\quad \gamma = \beta N + 2 - \beta$$

uniform diffusion constant in the *x*-variables

"Lyapunov exponents"

exact solution for β=2

Rejaei & CB 1993

mapping to free-fermion problem

$$P = \Psi e^{-\Omega/2} \Rightarrow -\frac{\partial \Psi}{\partial s} = \mathcal{H}\Psi,$$

$$\mathcal{H} = -\frac{1}{2\gamma}\sum_i \left(\frac{\partial^2}{\partial x_i^2} + \frac{1}{\sinh^2 2x_i}\right) + \frac{\beta(\beta-2)}{2\gamma}\sum_{i<j}\frac{\sinh^2 2x_j + \sinh^2 2x_i}{(\cosh 2x_j - \cosh 2x_i)^2}$$

variation on Calogero-Sutherland Hamiltonian
(without translational invariance)

interaction vanishes for β=2

β=2 solution is Slater determinant:

$$P = C(s)\prod_{i<j}(\sinh^2 x_j - \sinh^2 x_i)\prod_i \sinh 2x_i$$

$$\times\ \mathrm{Det}\left[\int_0^\infty dk\, e^{-k^2 s/4N}\tanh(\tfrac{1}{2}\pi k)k^{2m-1}P_{\frac{1}{2}(ik-1)}(\cosh 2x_n)\right]$$

toroidal function

simplifies to a Bessel function for k≫1

metallic regime $s/N = L/Nl \ll 1$

$$P = C(s)\prod_{i<j}\left[(\sinh^2 x_j - \sinh^2 x_i)(x_j^2 - x_i^2)\right]$$

$$\times\prod_i\left[\exp(-x_i^2 N/s)(x_i \sinh 2x_i)^{1/2}\right]$$

nonlogarithmic eigenvalue repulsion

generalized to β=1,4 by Caselle

$$P = C(s)\,\mathrm{exp}\Big[-\beta\Big(\sum_{i<j} u(x_i, x_j) + \sum_i V(x_i)\Big)\Big],$$

$$u(x_i, x_j) = -\tfrac{1}{2}\ln|\sinh^2 x_j - \sinh^2 x_i| - \tfrac{1}{2}\ln|x_j^2 - x_i^2|,$$

$$V(x) = \tfrac{1}{2}(\gamma/s)\beta^{-1}x^2 - \tfrac{1}{2}\beta^{-1}\ln|x\sinh 2x|.$$

interaction with "image charge"

$$u(x, x') = \mathcal{U}(x - x') + \mathcal{U}(x + x')$$

$$\mathcal{U}(x) = -\tfrac{1}{2}\ln|2x\sinh x|$$

localization

$$L \gg Nl \Rightarrow 1 \ll x_1 \ll x_2 \ll \cdots x_N$$

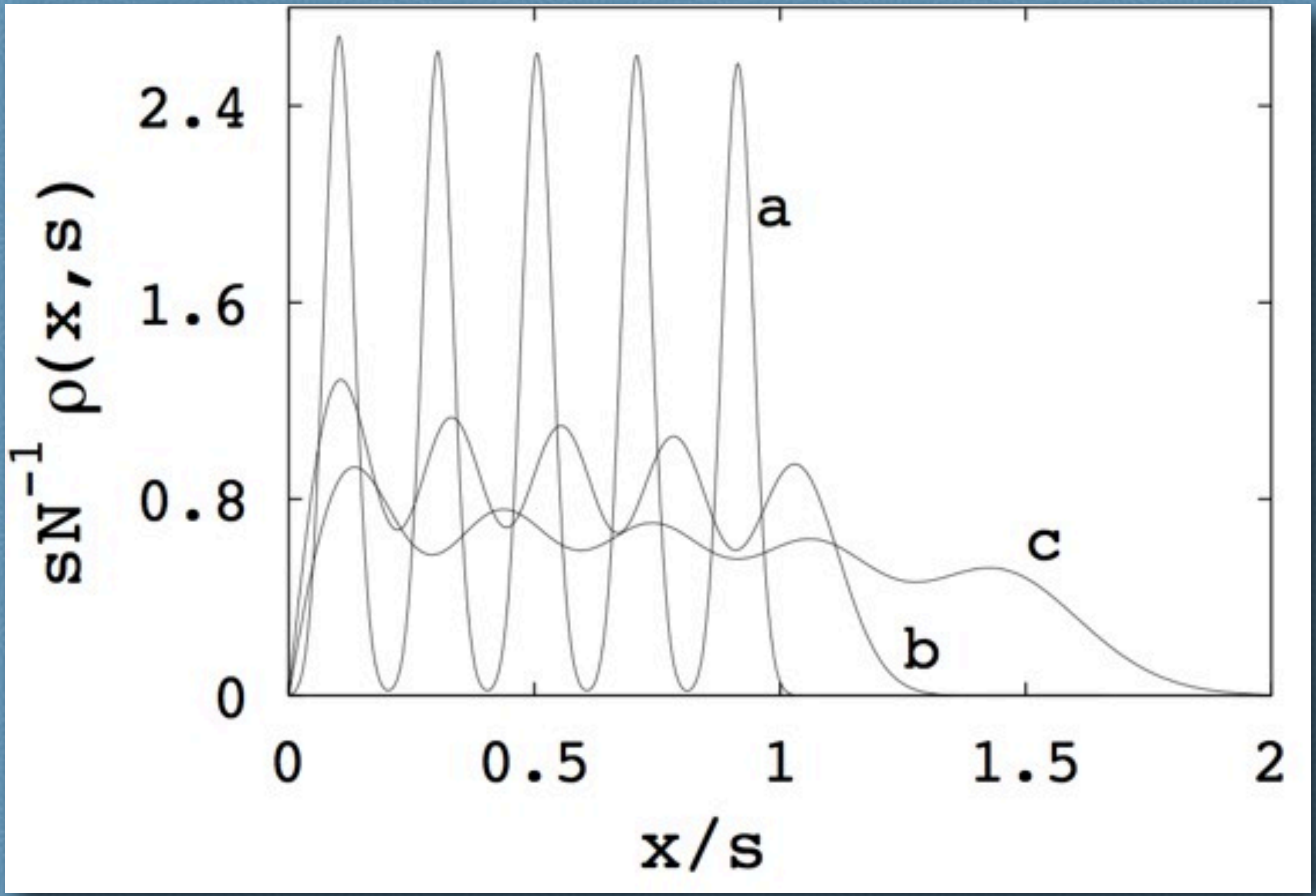
$$\Omega \approx -2\beta^{-1} \sum_{n=1}^{N} (1 + \beta n - \beta) x_n$$

P becomes a product of Gaussians

$$P \approx \left(\frac{\gamma l}{2\pi L} \right)^{N/2} \prod_{n=1}^{N} \exp \left[-\frac{\gamma l}{2L} (x_n - L/\xi_n)^2 \right],$$

$$\xi_n = \gamma l (1 + \beta n - \beta)^{-1}$$

"crystallization" of the transmission eigenvalues



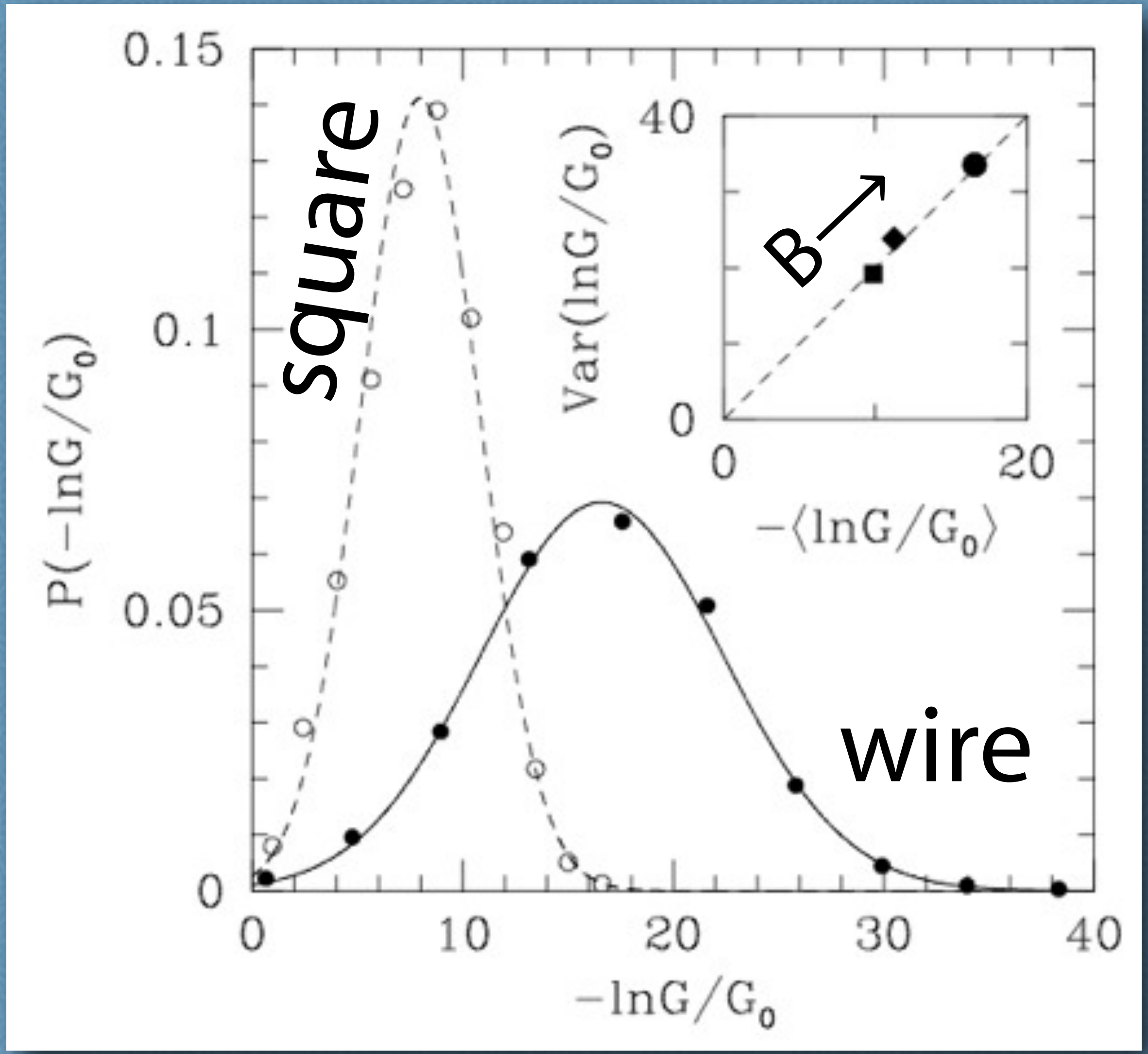
N=5
L/l=100,10,2

note: equal spacing of the x_n's

lognormal distribution of the conductance in the insulating regime

$$-\langle \ln G/G_0 \rangle = \frac{1}{2}\,\mathrm{Var}\,(\ln G/G_0) = 2L/\gamma l$$

localization length $\xi = \gamma l = (\beta N + 2 - \beta) l$



N=10

$\xi(\beta = 2)/\xi(\beta = 1) = 20/11$

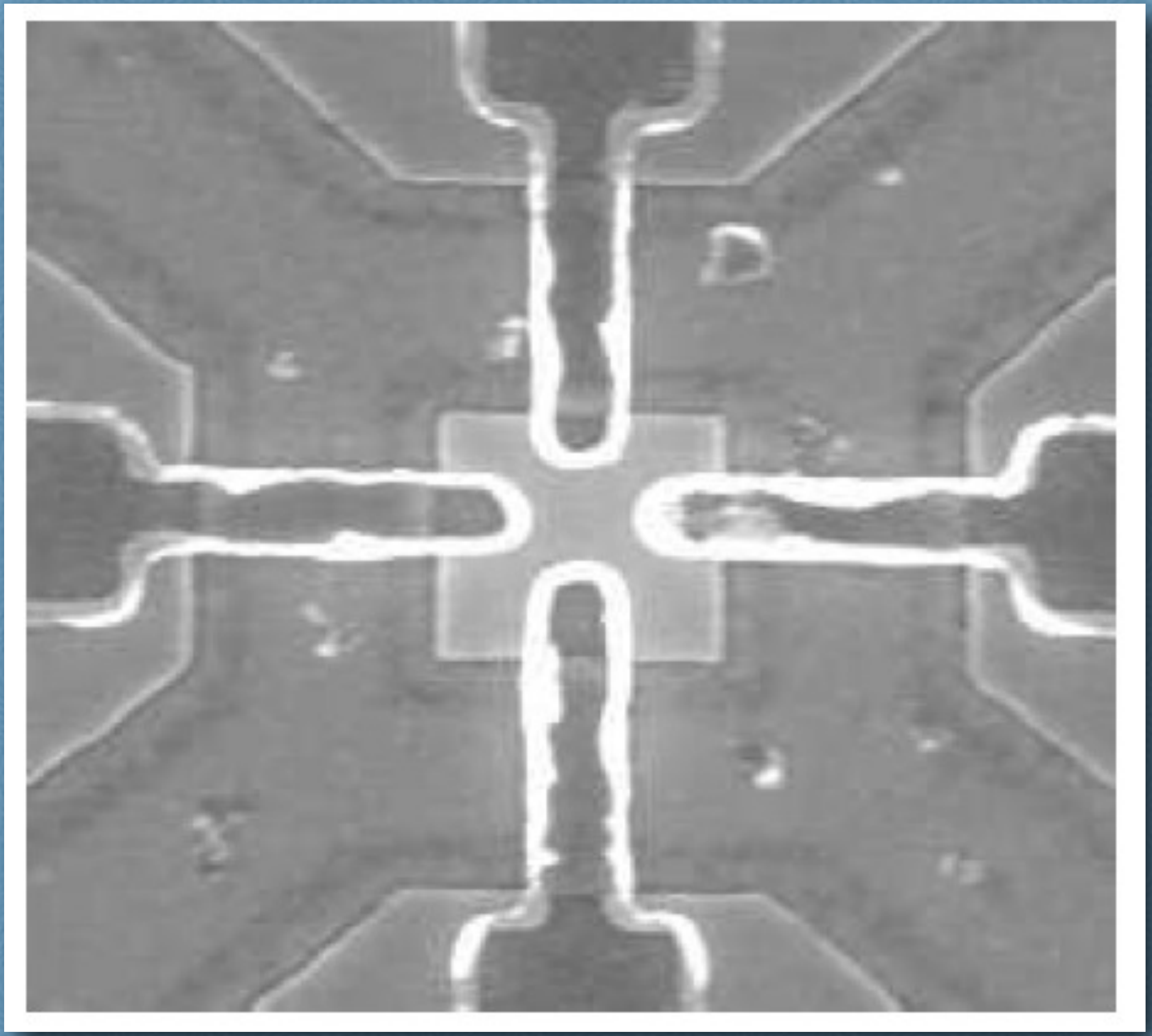
DMPK equation only valid in a wire geometry (quasi-1D); no theory for the 2D square geometry

lognormal distribution of the conductance in the insulating regime

$$-\langle \ln G/G_0 \rangle = \tfrac{1}{2}\,\mathrm{Var}\,(\ln G/G_0) = 2L/\gamma l$$

localization length $\xi = \gamma l = (\beta N + 2 - \beta) l$

Proximity effect



Bogoliubov-De Gennes Hamiltonian

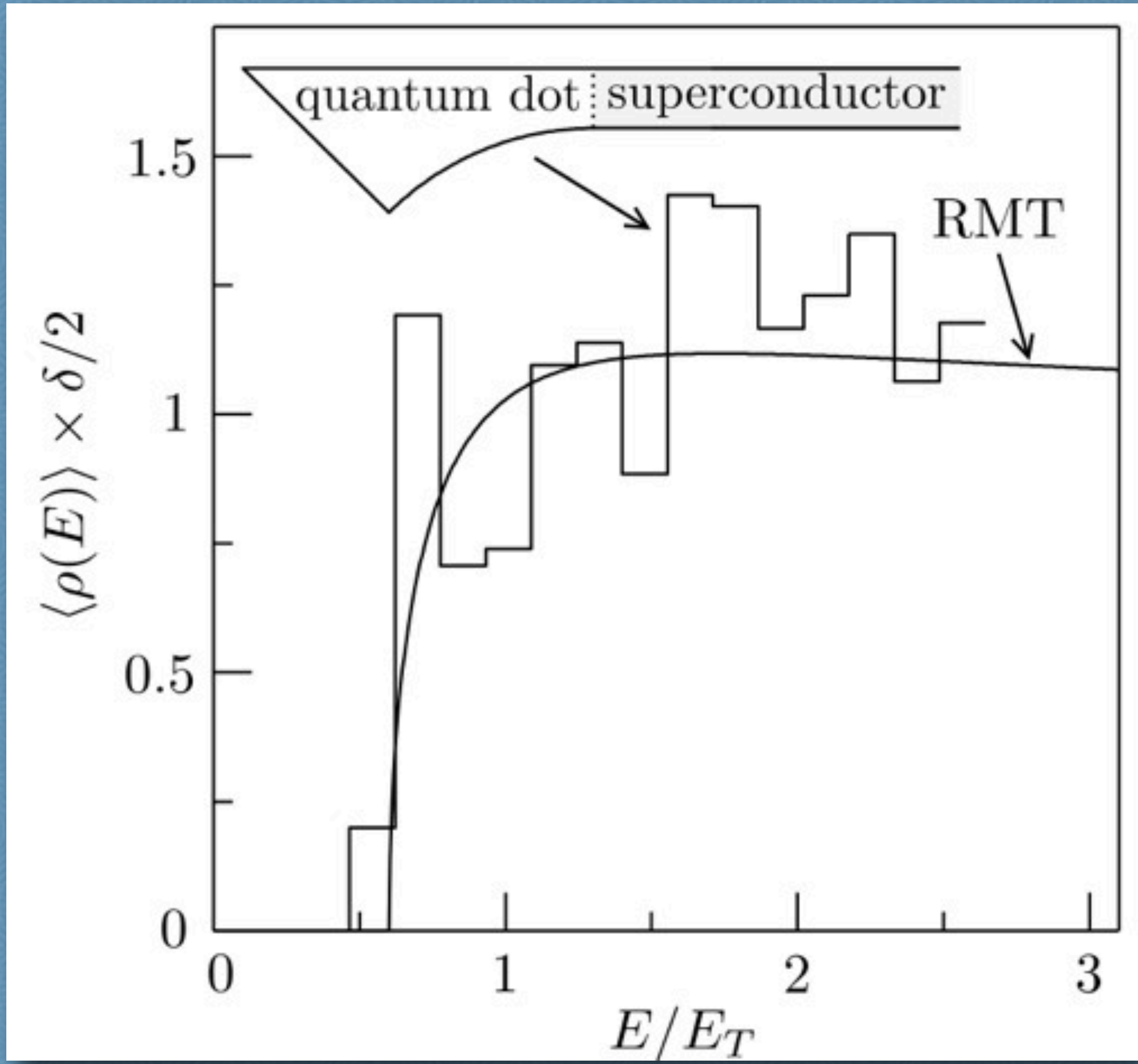
$$H = \begin{pmatrix} H_0 & \Delta \\ \Delta & -H_0^* \end{pmatrix}$$

RMT: H_0 random $M \times M$ matrix from GOE

Δ fixed $M \times M$ matrix of rank $N \ll M$

particle-hole symmetry: $\rho(E) = \rho(-E)$

level repulsion opens up a gap of order $N\delta$ (Thouless energy)



$E_T = N\delta/4\pi$

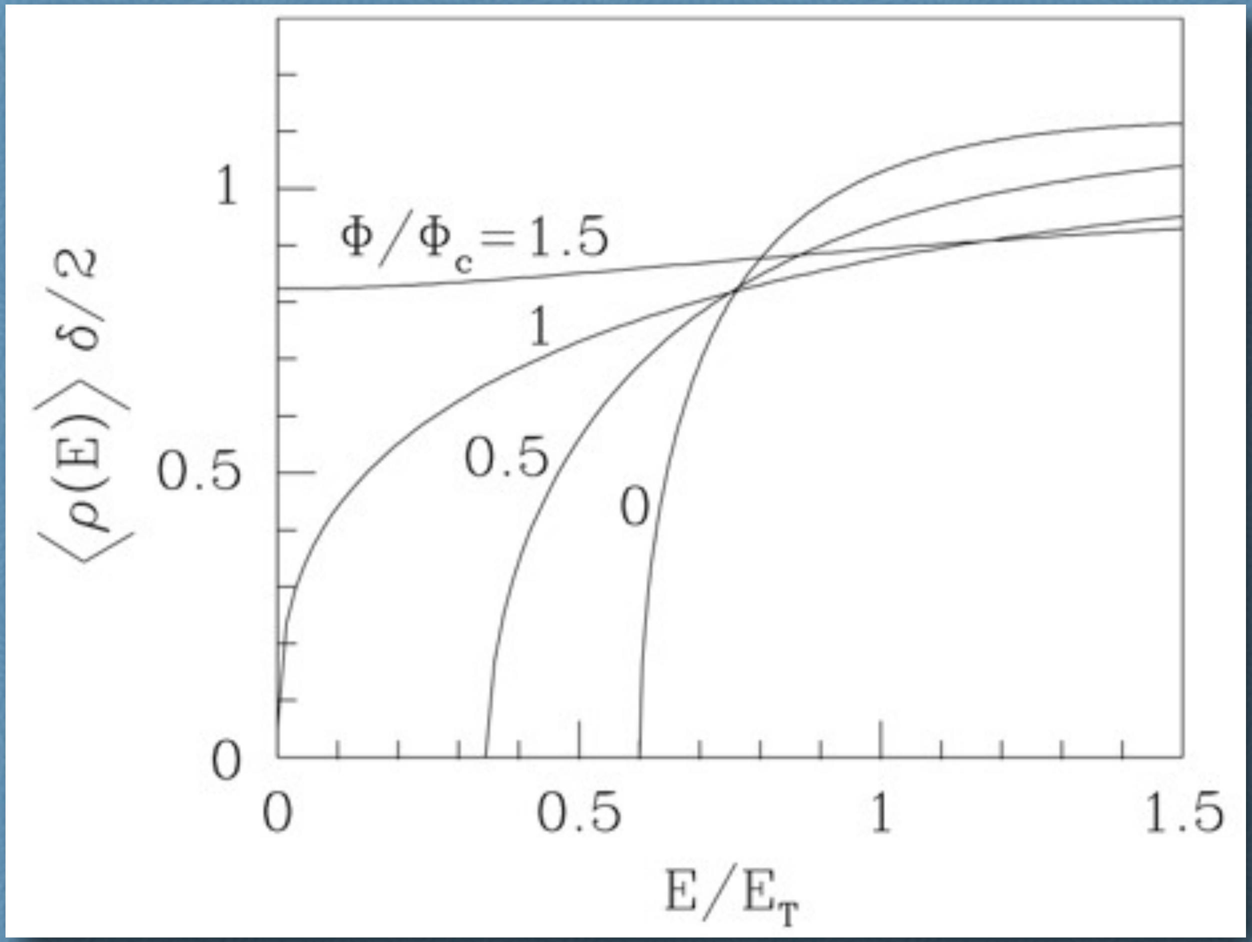
$E_{\text{gap}} = \gamma^{5/2} N\delta/2\pi$

mesoscopic effect of level repulsion (on a scale $N\delta \gg \delta$)

Melsen, Brouwer, Frahm & CB (1996)

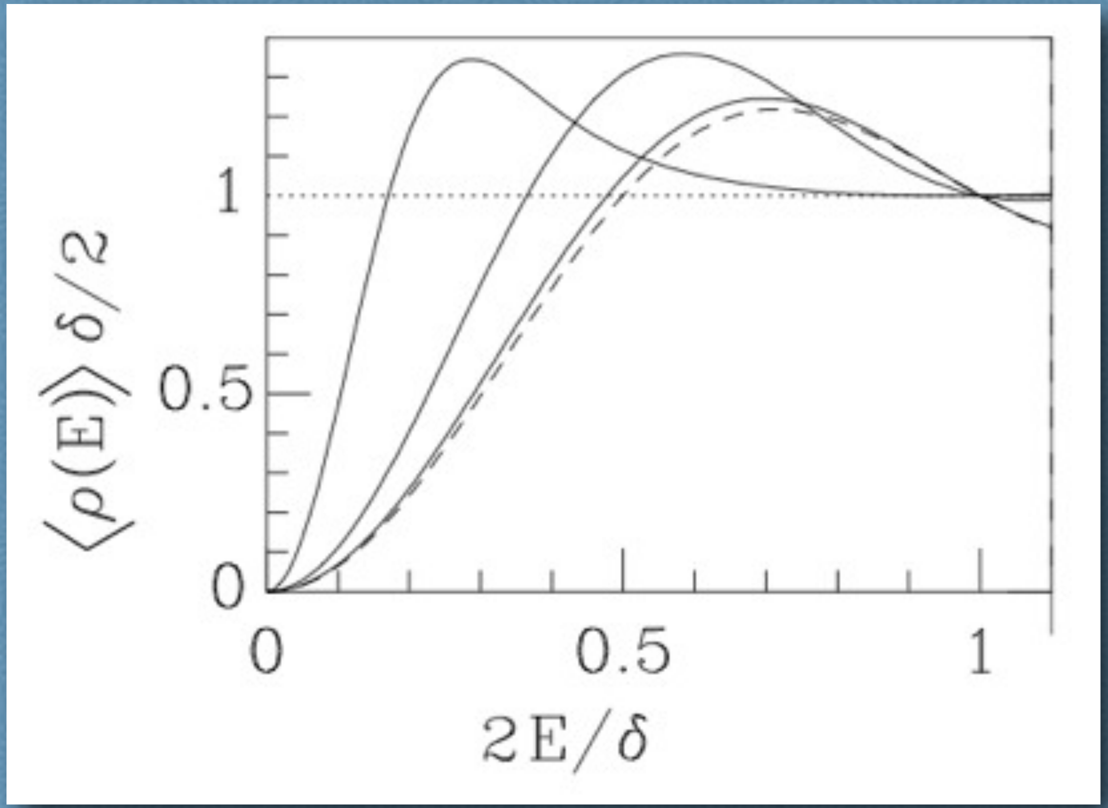
effect of a magnetic field

H_0 from GUE (instead of GOE)



level repulsion becomes microscopic

(gap reduced from Nδ to δ)



remaining soft gap from $\pm E$ symmetry

Altland & Zirnbauer (1996)