AN ELEMENTARY C^* ALGEBRA FIELD THEORY

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PROEFSCHRIFT

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The Electron-Positron Field, Coupled to External Electromagnetic Potentials, as an Elementary C^* Algebra Theory

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In recent years C^* algebra concepts have been suggested for dealing rigorously with the mathematical difficulties of Quantum Field Theory. To demonstrate some of the possibilities of these concepts we present an explicit and completely rigorous C^* algebra treatment of the simple model of the electron-positron field interacting with an external, classical electromagnetic field.

The usual results most of which cannot be derived in a rigorous way within the ordinary Fock–Hilbert space formalism, are obtained here in a straightforward and mathematically unobjectionable manner, in which divergent or ill-defined expressions are absent.

In particular no readjustments because of vacuum contributions have to be made in the resulting S operator and the terms of its perturbation series.

1. INTRODUCTION

In recent years a mathematically rigorous approach to general quantum field theory has been developed in terms of abstract C^* algebras. (*1*), (*2*), (*3*), (*4*). In this approach a quantum system is characterized by the structure of the set of its observable quantities, that form an abstract C^* algebra. States, as generalized expectation values, are positive linear functionals on that algebra, and physical transformations such as connected with symmetries and with evolution in time, are represented by structure preserving transformations, i.e., * automorphisms.

Some fundamental properties of relativistic local field theories have been established within an axiomatic framework along these lines, as proposed in (*4*), but one still has a long way to go if one wants to incorporate in it conventional Lagrangian field theory, with its specific dynamical prescriptions.

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In view of this, and in order to obtain more understanding of the concrete possibilities of C^* algebra ideas, it may be useful to try to test them on simpler, explicitly known models that exhibit some of the typical mathematical difficulties of general quantum field theory.

As an example of such a field theoretic model we will treat by C^* algebra methods the electron positron field, interacting with given external Maxwell potentials.

It is possible to give a completely rigorous and explicit treatment of this model, of its dynamics for finite times, and its assymptotic behavior leading to a well defined S operator. The usual physically meaningful results are obtained in a rigorous way, in which the divergences and ill-defined expressions inherent to the conventional formulation of the model, are absent.

2. THE MODEL AND SOME OF ITS PROPERTIES

The physical situation is that of a system of electrons and positrons, not interacting with each other but with a given classical electromagnetic field. In the usual nonrigorous language it is described by a spinor field $\psi(\mathbf{x}, t)$, satisfying the equal time anti-commutation relations

$$[\psi_\alpha(\mathbf{x}, t), \psi_\beta(\mathbf{x}', t)]_+ = [\psi_\alpha(\mathbf{x}, t), \psi_\beta(\mathbf{x}', t)]_+ = 0,$$
$$[\psi_\alpha(\mathbf{x}, t), \psi_\beta{}^*(\mathbf{x}', t)]_+ = \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{x}'), \tag{1}$$

and by a "second quantized" Hamiltonian

$$B = \int : \psi^*(\mathbf{x}, t) \left(-i \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x^k} + m\beta\right) \psi(\mathbf{x}, t) : d\mathbf{x}$$
$$+ e \int : \psi^*(\mathbf{x}, t)(A^0(\mathbf{x}) - \sum_{k=1}^{3} \alpha^k A^k(\mathbf{x}))\, \psi(\mathbf{x}, t) : d\mathbf{x}, \tag{2}$$

in which $A^0(\mathbf{x})$ and $A^k(\mathbf{x})$ are given nonquantized electromagnetic potentials and α^k, β the usual matrices from Dirac's theory.

The fields $\psi(\mathbf{x}, t)$ and $\psi^*(\mathbf{x}, t)$ are Heisenberg operators in the sense that they characterize the time development of the system according to

$$\psi(\mathbf{x}, t) = e^{iBt}\psi(\mathbf{x}, 0)\, e^{-iBt}. \tag{3}$$

This together with (1) and (2) gives the field equation

$$\frac{\partial}{\partial t} \psi = -i \left(-i \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x^k} + m\beta\right) \psi - ie \left(A^0(\mathbf{x}) - \sum_{k=1}^{3} \alpha^k A^k(\mathbf{x})\right) \psi. \tag{4}$$

This is of course the "second quantized" version of the wave equation introduced by Dirac in 1928, in order to improve the quantum mechanical description of an electron in a given electromagnetic field of force (*5, 6*). He conceived it as a direct relativistic generalization of the one particle Schrödinger equation, with the usual interpretation of $\psi(\mathbf{x}, t)$ as a quantum mechanical wave-function. The negative energy problem forced him later to modify the meaning of the equation. This resulted in his "hole theory" (*4, 8, 9*), very successful as a physical theory, but at the same time highly intuitive and difficult to formulate in a precise mathematical way. Later formulations by others stressed in a more straightforward way the many particle aspect inherent in the theory, by using the ideas of "second quantization", in which the wave-function $\psi(\mathbf{x}, t)$ becomes an "operator field" acting in a many particle space of particles and anti-particles. (*10, 11*). This has made it into a quantum field model; it is often presented as a first step in the study of general quantum electrodynamics from which it can be obtained by simplifying assumptions (*12, 13*). In this formulation, characterized by (1), (2), (4) it shares with quantum electrodynamics formal elegance but also some of its fundamental mathematical difficulties. Perturbative calculations lead, as usual to divergent expressions in the higher order S matrix elements. On the level of formal calculations these infinities are relatively harmless. There is essentially one type of divergent vacuum diagram, and its contributions to the S operator are taken care of by a simple although arbitrary and mathematically meaningless prescription. On a more fundamental level one find that it is very difficult to say what one of the basic elements of the theory, the total Hamiltonian B, (2) means in precise mathematical terms. Inspection of the interaction term B_1 of B, written in the usual way by means of the Fourier transforms of the fields as a creation-annihilation operator expression in momentum space shows that it contains an electron-positron pair creation-part, that would carry the vacuum state and any n-particle state out of the Fock–Hilbert space of square integrable many particle momentum wave-functions. Detailed investigations in this situation have indicated that in the case where there is only an electrostatic potential $A^0(\mathbf{x})$, it is still possible to define in the Fock–Hilbert space, in an indirect way a self adjoint operator that corresponds to the formal expression (2) and can be called the total Hamiltonian (*14–16*). In the general case where the potentials $A^k(\mathbf{x})$ are also present, this seems to be impossible however. In the case where B exists it has very awkward domain properties. The Dyson perturbation series in the interaction picture is purely formal therefore and has no meaning in a strict Hilbert space sense.

The C^* algebra formulation of this theory that will be presented in the following has none of these problems. It covers the general case of $A^0(\mathbf{x})$ and $A^k(\mathbf{x})$ and would equally well apply, as will be obvious, to a slightly more general situation in which for instance an extra term like

$$\frac{i}{2}\mu_0 \int \psi^*\beta \left(\sum_{\rho,\nu=1}^{4} \sigma^{\rho\nu} F_{\rho\nu} \right) \psi \, d\mathbf{x}$$

is present, representing a given anomalous magnetic moment of the particles. The same ideas could also be used to treat the case of time dependent potentials.

The C^* algebra formalism will give some insight into the nature of the problems of the conventional formulation; some of the above, rather loose remarks on these problems will be stated and proved in a precise way in this context. Except for the two theorems quoted in Section 5 only elementary material on C^* algebras and their representations will be needed.

3. THE ABSTRACT C^* ALGEBRA THEORY

The abstract algebraic field theory that will be constructed consists of two elements, a C^* algebra generated by the fields at a fixed time, and a one parameter group of *automorphisms, acting in this algebra and propagating the fields in time, in accordance with a rigorous form of (2), (3), and (4).

I. Starting point for the construction of a field algebra is (1). The first step in a rigorous definition of fields must be to smear them with test functions. It is sufficient for the purposes of this model to smear in space only; time will be retained as a separate variable. Fields will be therefore objects $\psi^*(f)$, $\psi(f)$, suggested by the symbolical relations

$$\psi^*(f) = \int \sum_{\alpha=1}^{4} \psi_\alpha^*(\mathbf{x})\, f_\alpha(\mathbf{x})\, d\mathbf{x}$$

$$\psi(f) = \int \sum_{\alpha=1}^{4} \psi_\alpha(\mathbf{x})\, \overline{f_\alpha(\mathbf{x})}\, d\mathbf{x}, \tag{5}$$

in which $f(\mathbf{x}) = f_\alpha(\mathbf{x})$, $\alpha = 1, 2, 3, 4$ are suitable complex testfunctions. They must satisfy the following relations, suggested by (1)

$$[\psi(f), \psi(g)]_+ = [\psi^*(f), \psi^*(g)]_+ = 0$$

$$[\psi(f), \psi^*(g)]_+ = \int \sum_{\alpha=1}^{4} \overline{f_\alpha(\mathbf{x})}\, g_\alpha(\mathbf{x})\, d\mathbf{x}. \tag{6}$$

These fields $\psi(f)$, $\psi^*(f)$ have to be defined as generating elements of an abstract C^* algebra, attached to a linear space of testfunctions $f(\mathbf{x})$. Properties of the fields can then be obtained from the study of this C^* algebra and its representations. Several authors have defined and investigated such an algebra of the canonical anti-commutation relations, using however slightly different approaches. See for instance (*17*, *2*, *18*, *19*). Fortunately these different approaches all lead, in the case of anti-commutation relations, to the same C^* algebra (for a concise review of some of the variations, see *(20)*). We will use the formulation of abstract fermion systems that can be found in *(19)*, to which we refer for details.

Abstract fermion-systems may be generated by a real Hilbert space $\mathscr{H}$, of even finite or infinite dimensions, with inner-product $(z_1, z_2)_R$ together with groups of orthogonal transformations T of $\mathscr{H}$. There exist a unique complex *algebra, $C_0(\mathscr{H})$, the Clifford algebra over $\mathscr{H}$, having the properties:

1^0: $C_0(\mathscr{H})$ contains a unit I $(\neq 0)$

2^0: There is a real-linear 1–1 map $R: \mathscr{H} \to C_0(\mathscr{H})$, the images $R(z)$ are self-adjoint and generate $C_0(H)$ algebraically

3^0: $[R(z_1), R(z_2)]_+ = 2(z_1, z_2)_R I$

$$\forall z_1, z_2 \in \mathscr{H}. \tag{7}$$

One may construct $C_0(\mathscr{H})$ as follows: Let $A(\mathscr{H})$ be the *complex* free tensor algebra over the real space $\mathscr{H}$. An involution operation is defined by

$$\left(\sum_{\text{finite sum}} \lambda_{i_1 \ldots i_n} z_{i_1} \otimes z_{i_2} \otimes \cdots \otimes z_{i_n}\right)^* = \sum_{\text{finite sum}} \overline{\lambda_{i_1 \ldots i_n}} z_{i_n} \otimes \cdots \otimes z_{i_2} \otimes z_{i_1}. \tag{8}$$

Let $\mathscr{I}$ be the two-sided *ideal generated by elements

$$(z_1 \otimes z_2 + z_2 \otimes z_1 - 2(z_1, z_2)_R I),$$

(i.e.: $\omega \in \mathscr{I}$ if:

$$\omega = \sum_{\text{finite sum}} u_j (z_{j_1} \otimes z_{j_2} + z_{j_2} \otimes z_{j_1} - 2(z_{j_1}, z_{j_2})_R I) u_j$$

for

$$u_j, v_j \in A(\mathscr{H}) \quad z_{j_i} \in \mathscr{H}).$$

Then $C_0(\mathscr{H}) = A(\mathscr{H})/\mathscr{I}$, with $R(z)$ the image of z under

$$A(\mathscr{H}) \to A(\mathscr{H})/\mathscr{I}. \tag{9}$$

To obtain a norm on $C_0(\mathscr{H})$, the algebra is represented as an operator algebra in Hilbert space, using in a standard fashion a special positive linear functional E_0 on $C_0(\mathscr{H})$, the central state, that may be given by:

$$E_0(I) = 1$$

$$E_0(R(z)) = 0, \quad \forall z \in \mathscr{H}.$$

$$E_0(R(z_1), \ldots, R(z_n)) = 0,$$

for all orthonormal sequences $z_1 \ldots, z_2$ from $\mathscr{H}$. (10)

The norm of $A \in C_0(\mathscr{H})$ is then the operator norm of the corresponding operator in this representation.

Completion under this norm makes $C_0(\mathscr{H})$ into a C^* algebra $C(\mathscr{H})$, the abstract fermion algebra over $\mathscr{H}$.

The algebra $C(\mathscr{H})$ is simple, all its (non-zero) *representations are faithful and norm-preserving. Its most useful property is: Each orthogonal transformation T of $\mathscr{H}$ induces a unique *automorphism ϕ_T in $C(\mathscr{H})$ such that $\phi_T(R(z)) = R(Tz)$, $\forall z \in \mathscr{H}$. A strongly continuous one parameter group of orthogonal transformations $T(t)$ induces a norm-continuous group of automorphisms ϕ_t in $C(\mathscr{H})$, (i.e., from $\lim_{t\to t_0} \| T_t z - T_{t_0} z \| = 0$, $\forall z \in \mathscr{H}$ follows $\lim_{t \to t_0} \| \phi_t(A) - \phi_{t_0}(A) \| = 0$, $\forall A \in C(\mathscr{H})$. The $\mathscr{H}$ that will be used is however more special, because it will be a complex Hilbert space, with complex inner product (z_1, z_2) such that $(\lambda z_1, z_2) = \bar{\lambda}(z_1, z_2)$). The elements of $\mathscr{H}$, with the same addition law and the same multiplication, but only by real numbers, form of course a real vector space, that is a real Hilbert space under the real inner product $(z_1, z_2)_R = \text{Re}(z_1, z_2)$. One constructs $C(\mathscr{H})$ over $\mathscr{H}$ considered as a real space in this way. The extra possibility of multiplying vectors with imaginary numbers, which has nothing to do with multiplication with imaginary numbers in the algebra $C(\mathscr{H})$ as a complex vector space is then used to define more convenient generators of $C(\mathscr{H})$:

$$\begin{aligned} \psi(z) &= \tfrac{1}{2}\{R(z) + iR(iz)\}, \\ \psi^*(z) &= \tfrac{1}{2}\{R(z) - iR(iz)\}. \end{aligned} \tag{11}$$

They have the following properties, that follow immediately from (7) 2, 3.

(1) $\psi^*(z)$ depends complex-linearly on $z \in \mathscr{H}$; $\psi(z)$ depends complex-antilinearly on $z \in \mathscr{H}$.

(2) $$\begin{aligned} &[\psi(z_1), \psi(z_2)]_+ = [\psi^*(z_1), \psi^*(z_2)]_+ = 0 \\ &[\psi(z_1), \psi^*(z_2)]_+ = (z_1, z_2)\, I \end{aligned} \tag{12}$$

Note that the automorphism ϕ_T, induced by a (real) orthogonal transformation T in $\mathscr{H}$ will in general not preserve the way $C(\mathscr{H})$ is generated by the $\psi(z)$; i.e., $\phi_T(\psi(z)) \neq \psi(Tz)$. However for the special case when T is also unitary with respect to (z_1, z_2), then $\phi_T(\psi(z)) = \psi(Tz)$.

We are now in a position to define for our model the fields $\psi(f)$, $\psi^*(f)$ and the algebra they generate:

Take $\mathscr{H}$ to be the complex L_2 space of functions $f_\alpha(\mathbf{x})$, $\alpha = 1, 2, 3, 4$ with the inner product

$$(f, g) = \int \sum_{\alpha=1}^{4} \overline{f_\alpha(\mathbf{x})}\, g_\alpha(\mathbf{x})\, d\mathbf{x}. \tag{13}$$

The field algebra is then $C(\mathscr{H})$, and the fields $\psi(f)$, $\psi^*(f)$ the generators defined by (11). Note that (12) corresponds with (5) and (6).

II. In order to obtain suitable automorphisms ϕ_t that represent the temporal evolution we employ Eq. (4) as a classical "c-number" equation. The expressions

$$H_0' = -i \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x^k} + m\beta$$

$$H' = H_0' + e \left\{A^0(\mathbf{x}) - \sum_{k=1}^{3} \alpha^k A^k(\mathbf{x})\right\}, \tag{14}$$

are formally self-adjoint with respect to the inner product (13) in $\mathscr{H}$. It is known, (*21*), that H_0' is essentially selfadjoint, for instance on the functions $f_\alpha(\mathbf{x})$ from the Schwartz space (S). This is also known for H', on the same domain provided the potentials $A^0(\mathbf{x})$, $A^k(\mathbf{x})$ are chosen in a physically reasonable way. For instance in the case of a purely electrostatic Coulomb potential

$$A^0(\mathbf{x}) = \frac{Ze}{|\mathbf{x}|} \qquad \text{for} \quad Z < 68,$$

(see (*21*)) and in the obvious case where $A^0(\mathbf{x})$, $A^k(\mathbf{x})$ are bounded, measurable functions. We assume that we are dealing with such a situation, so that we have H_0, H, the selfadjoint extensions of H_0', H'.

Equation (4), as a classical equation is then solved, in Hilbert space sense by:

$$f(t) = e^{-iHt}f(0). \tag{15}$$

The operators e^{-iHt} are unitary in $\mathscr{H}$, and therefore induce automorphisms ϕ_t in $C(\mathscr{H})$, such that

$$\phi_t(\psi(f)) = \psi(e^{iHt}f), \qquad \text{for all } f \in \mathscr{H}. \tag{16}$$

These ϕ_t are the automorphisms that describe the temporal evolution of the abstract quantum field, they correspond to relation (3).

Instead of letting ϕ_t act on the operators and having a generalized Heisenberg picture, one may define a generalized Schrödinger picture, where the states carry the temporal evolution by $E_t(A) = E(\phi_t(A))$.

It should be noted that the time dependent fields $\psi_t(f) = \phi_t(\psi(f))$ satisfy the field Eq. (4) in a precise generalized functions sense:

Using

$$\lim_{\Delta \to 0} \left\| e^{iHt}\left[\frac{(e^{iH\Delta} - 1)}{\Delta} - iH\right] f \right\| = 0,$$

$$\forall f \in \mathscr{D}(H),\ (\mathscr{D}(H) \text{ is the domain of } H)$$

and $\|\psi(f)\| = \|f\|$, one derives immediately:

$$\lim_{\Delta\to 0}\left\| \frac{\psi_{t+\Delta}(f) - \psi_t(f)}{\Delta} - \psi_t(iHf)\right\| = 0,$$

or

$$\frac{d}{dt}\,\psi_t(f) = -i\psi_t(Hf),$$

for all $f \in \mathscr{D}(H)$, and in the sense of norm convergence in $C(\mathscr{H})$.

It would be easy to derive from the foregoing a 4-dimensional Minkowski space formulation defining

$$\hat{\psi}(u) = \int_{-\infty}^{+\infty} \psi_t(f)\,\rho(t)\,dt,$$

for $u_\alpha(x) = f_\alpha(\mathrm{x})\rho(t)$

$$f_\alpha(x) = S(R^3), \qquad \rho(t) \in S(R^1) \quad \text{(Schwartz functions)},$$

and extending this to a suitable class of testfunctions on space-time. This would lead to formulation in terms of vacuum expectation values of products of fields. One can easily compute the anti-commutation relations for the fields at different times

$$[\hat{\psi}(u), \hat{\psi}(u')]_+ = [\hat{\psi}^*(u), \hat{\psi}^*(u')]_+ = 0,$$

and

$$[\hat{\psi}(u), \hat{\psi}^*(u')]_+ = \int_{-\infty}^{+\infty}\int dt\,dt'\,[\psi_t(f), \psi_{t'}^*(g)]_+\,\rho(t)\,\rho'(t')$$

$$= \int_{-\infty}^{+\infty}\int (f, e^{-iH(t-t')}g)\,\rho(t)\,\rho'(t')\,dt\,dt'.$$

This may be written symbolically as

$$\int_{-\infty}^{+\infty}\int -\,iu(x)\,S(x, x')\,\beta u(x')\,dx\,dx'$$

and leads to the usual singular anti-commutation function:

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')]_+ = -iS_{\alpha\beta}(x, x'),$$

for $\bar{\psi} = \psi^*\beta$. For the free equation (i.e., $A^0(\mathrm{x}) = 0$, $A^k(\mathrm{x}) = 0$), this singular function is well-known, for the perturbed (i.e., the full equation) it can be determined, but in both cases it is a generalized "*c*-number" function, completely determined by the classical equations.

It will be useful to distinguish explicitly in the following between the free and perturbed equation. We consider separately the automorphisms ϕ_t, as defined and ϕ_t^0 defined in the same way with H_0 instead of H.

4. HILBERT SPACE REPRESENTATIONS OF THE FIELDS

We will represent the abstract field theory, the system $\{C(\mathscr{H}), \phi_t\}$ or $\{C(\mathscr{H}), \phi_t^0\}$, as an operator theory in Hilbert space.

Definition. A physical representation of the abstract field system $\{C(\mathscr{H}), \phi_t\}$ is a *representation of $C(\mathscr{H})$ as an algebra of bounded operators in a Hilbert space $\mathscr{K}$, which may without essential loss of generality be assumed to be irreducible, and such that

1. There exist a one parameter, strongly continuous group of unitary operators e^{iBt} in $\mathscr{K}$ that implement ϕ_t

$$\phi_t(A) = e^{iBt}Ae^{-iBt} \qquad \text{for all } A \in C(\mathscr{H}),$$

(existence of Hamiltonian as a selfadjoint operator).

2. $B \geqslant 0$ (positivity of the energy).

3. There exist a unique vector Φ, (the vacuum) in $\mathscr{K}$ such that $e^{iBt}\Phi_0 = \Phi_0$, for all $t \in R^1$. (17)

(To avoid cumbersome notation, the same symbols denote objects from the abstract algebra $C(\mathscr{H})$ and their image under the representation, as long as this does not lead to confusion). The same definition applies of course to the free field $\{C(\mathscr{H}), \phi_t^0\}$.

It has been shown by M. Weinless (*22*), that an abstract quantum field system $\{C(\mathscr{H}), \phi_t\}$ in which the ϕ_t are connected with a continuous group of orthogonal transformations $0(t)$ in $\mathscr{H}$ always has a physical representation, and moreover that it is unique, up to unitary equivalence, provided $0(t)$ has no vectors $z \in \mathscr{H}$, $0(t)z = z$, for all t (This is true for e^{iH_0t} and may be assumed for e^{iHt}).

We construct this representation, explicitly, separately for the free and the perturbed field. For the free field this representation is of course well-known, although the representation is usually not very transparent. The construction that will be given is essentially the same for the two cases, no Fourier transformation of the fields is used because that would be helpful only in the free case.

It should be emphasized that the two representation spaces $\mathscr{K}_0$ and $\mathscr{K}$ are a priori completely unrelated. This is overlooked in the conventional formulation of the theory, where one tries (more or less implicitly) to force the free and the perturbed theory both together in one Hilbert space.

The construction will employ J.M. Cook's general, rigorous and "coordinate free" formulation of "second quantization" (*23*). This will be reviewed briefly first.

Let $\mathcal{M}$ be a given complex Hilbert space; physically the one-particle space, together with groups of unitary operators, representing one-particle transformations. An anti-symmetric many particle space is defined as

$$V_F(\mathcal{M}) = \mathcal{M}_0 \oplus \mathcal{M} \oplus (\mathcal{M} \otimes \mathcal{M})_A \oplus (\mathcal{M} \otimes \mathcal{M} \otimes \mathcal{M})_A \oplus \cdots, \tag{18}$$

in which $\mathcal{M}_0$ is a one dimensional space, with a unit vector Φ_0, (the no particle vector) and the subscript A means anti-symmetrization of the tensor products.

For each $z \in \mathcal{M}$ one defines an annihilation operator $C(z)$ by extension of

$$\begin{aligned} &\Phi_0 \to 0 \\ &z' \to (z, z')\,\Phi_0 \\ &\sum_{\text{perm }\sigma} \operatorname{sign} \sigma(z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)}) \to \sqrt{n} \sum_{\sigma} \operatorname{sign} \sigma(z, z_{\sigma(1)})(z_{\sigma(2)} \otimes \cdots \otimes z_{\sigma(n)}) \end{aligned} \tag{19}$$

for all finite sets $z_1, \ldots, z_n \in \mathcal{M}$.

This defines a bounded operator $C(z)$ in $V_F(\mathcal{M})$ with $\| C(z)\| = \| z \|$; a creation operator for each z is defined as $C^*(z)$. Note that the norm closed operator algebra generated by all the $C(z)$, $C^*(z)$ is a special representation, the Fock–Cook representation of the algebra $C(\mathcal{M})$, as defined earlier, however that is not the point of view taken in this section. Operators in $\mathcal{M}$ induce operators in $V_F(\mathcal{M})$ in two ways:

1. A unitary U gives a unitary $\Gamma(U)$ in $V_F(\mathcal{M})$ by extension of

$$\begin{aligned} &\Phi_0 \to \Phi_0 \\ &\sum_{\sigma} \operatorname{sign} \sigma(z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)}) \to \sum_{\sigma} \operatorname{sign} \sigma(Uz_{\sigma(1)} \otimes \cdots \otimes Uz_{\sigma(n)}). \end{aligned} \tag{20}$$

2. A self-adjoint A in $\mathcal{M}$ gives a self-adjoint $\Omega(A)$ in $V_F(\mathcal{H})$ by extension of

$$\begin{aligned} &\Phi_0 \to 0 \\ &\sum_{\sigma} \operatorname{sign} \sigma(z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)}) \to \sum_{\sigma} \operatorname{sign} \sigma(Az_{\sigma(1)} \otimes z_{\sigma(2)} \otimes \cdots \otimes z_{\sigma(n)}) \\ &\qquad + \sum_{\sigma} \operatorname{sign} \sigma(z_{\sigma(1)} \otimes Az_{\sigma(2)} \otimes \cdots \otimes z_{\sigma(n)}) + \cdots \\ &\qquad + \sum_{\sigma} \operatorname{sign} \sigma(z_{\sigma(1)} \otimes \cdots \otimes Az_{\sigma(n)}). \end{aligned} \tag{21}$$

One has

$$A \geqslant 0 \Leftrightarrow \Omega(A) \geqslant 0, \tag{22}$$

and the relations

$$\Gamma(e^{iAt}) = e^{i\Omega(A)t} \tag{23}$$

$$e^{i\Omega(A)t}\Omega(B)\, e^{-i\Omega(A)t} = \Omega(e^{iAt}Be^{-iAt}), \tag{24}$$

for A, B self-adjoint in $\mathscr{M}$; $t \in R^1$.

$$e^{i\Omega(A)t}C(z)\, e^{-i\Omega(A)t} = C(e^{iAt}z), \tag{25}$$

for all $z \in \mathscr{M}$, $t \in R^1$.

So far this would be sufficient for a formulation of "second quantization" as it is used for instance in non-relativistic many body theory, where the one-particle space $\mathscr{M}$ is the obvious space of one-particle wave functions. This is not the case for relativistic equations. Dirac's equation cannot be interpreted as a single particle equation in the sense of the non-relativistic Schrödinger equation. This fact led to hole theory and later to a quantum field formulation. For this reason we introduced in Section 3 the more general concept of Clifford algebras, which allows the use of transformations that are only real orthogonal instead of unitary in the physical one-particle space. $\mathscr{H}$ is not the physical one-particle space, but a convenient mathematical space to define the fields and their transformations. We now construct the physical representation spaces:

I. The Free Field $\{C(\mathscr{H}), \phi_t^0\}$

Let $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, where $\mathscr{H}_+$ and $\mathscr{H}_-$ are the subspaces connected with the positive and negative part of the spectrum of H_0. (Corresponding projections P_+, P_-.) We use a conjugation operator U in $\mathscr{H}$ (i.e., U is anti-unitary and $U^2 = 1$) satisfying:

$$H_0U = -UH_0, \tag{26}$$

or equivalently

$$UP_+ = P_-U. \tag{27}$$

Any U having these properties could be used, different ones leading to equivalent representations; we take U to be the charge conjugation, uniquely defined, up to a phase factor by

$$(Uf)_\alpha(\mathbf{x}) = \sum_\beta C_{\alpha\beta} \overline{f_\beta(\mathbf{x})}, \tag{28}$$

$C_{\alpha\beta}$ a unitary matrix with

$$C^{-1}\gamma^\mu C = -\bar{\gamma}^\mu, \tag{29}$$

or

$$C^{-1}\beta C = -\bar{\beta}, \qquad C^{-1}\alpha^{\mu}C = \bar{\alpha}^{\mu}, \tag{30}$$

where

$$(\bar{\beta})_{\rho\nu} = \bar{\beta}_{\rho\nu}\,.$$

We simplify this by assuming the γ^{μ}, β to be in a Majorana representation, in that case $C = 1$, and charge conjugation is just complex conjugation of $f(x)$.

Define a physical one-particle space $\mathscr{H}' = \mathscr{H}_p \oplus \mathscr{H}_a$, where $\mathscr{H}_p$, $\mathscr{H}_a$ are copies of $\mathscr{H}_+$, I_p, I_a the identification maps $\mathscr{H}_+ \to \mathscr{H}_p$, $\mathscr{H}_+ \to \mathscr{H}_a$ and P_p, P_a projections on $\mathscr{H}_p$, $\mathscr{H}_a$.

Define a (real) orthogonal map γ:

$$\gamma : \mathscr{H} \to \mathscr{H}',$$

as

$$\gamma = I_p P_+ + I_a U P_-, \tag{31}$$

and then

$$\gamma^{-1} = I_p^{-1} P_p + U I_p^{-1} P_a\,. \tag{32}$$

The representation space $\mathscr{K}_0$ is now the many particle space over $\mathscr{H}'$:

$$\mathscr{K}_0 = V_F(\mathscr{H}') \tag{33}$$

We define operators in $\mathscr{K}_0$;

$$\psi^{(+)}(f) = C(\gamma P_+ f),$$

annihilation of a particle;

$$\psi^{(-)}(f) = C^*(\gamma P_- f),$$

creation of an antiparticle;

$$\psi^{(+)*}(f) = C^*(\gamma P_+ f),$$

creation of a particle;

$$\psi^{(-)*}(f) = C(\gamma P_- f),$$

annihilation of an antiparticle;

$$(\text{for all } f \in \mathscr{H}), \tag{34}$$

and then field operators.

$$\psi(f) = \psi^{(+)}(f) + \psi^{(-)}(f) \tag{35}$$

$$\psi^*(f) = \psi^{(+)*}(f) + \psi^{(-)*}(f).$$

It is easy to check that these field operators satisfy (12) and represent the abstract elements $\psi(f)$, $\psi^*(f)$ from $C(\mathscr{H})$; the norm closed operator algebra they generate represents $C(\mathscr{H})$. We verify that this is the physical representation: note that

$$\begin{aligned}\gamma e^{iH_0t}\gamma^{-1} &= \gamma e^{iH_0t}P_+\gamma^{-1} + \gamma e^{iH_0t}P_-\gamma^{-1} \\ &= I_pP_+e^{iH_0t}I_pP_p + I_aUP_-e^{iH_0t}UI_a^{-1}P_a \\ &= I_pP_+e^{iH_0t}I_p^{-1}P_p + I_aP_+e^{iH_0t}I_a^{-1}P_a \qquad (35) \\ &= \exp[i(I_pP_+H_0I_p^{-1})\,P_pt]\,P_p + \exp[i(I_aP_+H_0I_a^{-1})\,P_at]\,P_a \\ &= \exp[i\{(I_pP_+H_0I_p^{-1})\,P_p + (I_aP_+H_0I_a^{-1})\,P_a\}\,t],\end{aligned}$$

then for $f \in \mathscr{H}$

$$\begin{aligned}\phi_t^0[\psi(f)] &= \psi_t^0(f) = \psi(e^{iH_0t}f) = \psi^{(+)}(e^{iH_0t}f) + \psi^{(-)}(e^{iH_0t}f) \\ &= C(\gamma P_+e^{iH_0t}f) + C^*(\gamma P_-e^{iH_0t}f) \\ &= C(\gamma e^{iH_0t}\gamma^{-1}\gamma P_+f) + C^*(\gamma e^{iH_0t}\gamma^{-1}\gamma P_-f)\end{aligned}$$

using (35) and (25)

$$= e^{iB_0t}C(\gamma P_+f)\,e^{-iB_0t} + e^{iB_0t}C^*(\gamma P_-f)\,e^{-iB_0t} = e^{iB_0t}\psi(f)\,e^{-iB_0t} \qquad (36)$$

with $B_0 = \Omega(I_pH_0P_+I_p^{-1}P_p + I_aP_+H_0I_a^{-1}P_a)$, because

$$I_pH_0P_+I_p^{-1}P_p + I_aP_+H_0I_a^{-1}P_a \geqslant 0,$$

we have $B_0 \geqslant 0$. (using (22)), and of course $B_0\Phi_0 = 0$.

Note that the state vectors in this "physical" many-particle space $\mathscr{K}_0$ are nothing else than sequences of antisymmetric wave functions (separately for particles and antiparticles). In the Schrödinger picture they are essentially linear combinations of products of positive energy solutions of the free Dirac equation.

Note also that the many-particle description in this case is really superfluous, because all n-particle and m-anti-particle subspaces are invariant under time development. If one introduces the interaction in this same space $\mathscr{K}_0$ (in the cases where this is rigorously possible), this is no longer true, the different subspaces mix, there is a continuous creation and annihilation of particle-anti-particle pairs.

It is in this sense that the positivity requirement for the energy makes a one-particle formulation of the full Dirac equation impossible and forces it to be a true many-particle or quantum field equation.

II. The Perturbed Field $\{C(\mathscr{H}), \phi_t\}$

One expresses the dependence of H on the charge e by writing $H(e)$, one then may consider also $H(-e)$. Analogous to the free field case we define:

$$\mathscr{H} = \mathscr{H}_+^{(e)} \oplus \mathscr{H}_-^{(e)},\ P_+^{(e)},\ P_-^{(e)}$$

$$\mathscr{H} = \mathscr{H}_+^{(-e)} \oplus \mathscr{H}_-^{(-e)},\ P_+^{(-e)},\ P_-^{(-e)}.$$

We have

$$UH(e) = -H(-e)U, \tag{37}$$

or

$$\begin{aligned} UP_+^{(e)} &= P_-^{(-e)}U \\ UP_-^{(e)} &= P_+^{(-e)}U, \end{aligned} \tag{38}$$

define the physical one-particle space:

$$\mathscr{H}'' = \mathscr{H}_p' \oplus \mathscr{H}_a', \tag{39}$$

with projections P_p', P_a' on $\mathscr{H}_p'$, $\mathscr{H}_a'$. The $\mathscr{H}_p'$ and $\mathscr{H}_a'$ are copies of $\mathscr{H}_+^{(e)}$ and $\mathscr{H}_+^{(-e)}$ with the identification mappings

$$I_p' : \mathscr{H}_+^{(e)} \to \mathscr{H}_p'$$

$$I_a' : \mathscr{H}_+^{(-e)} \to \mathscr{H}_a',$$

analogous to γ there is an orthogonal ν

$$\nu : \mathscr{H} \to \mathscr{H}''$$

$$\nu = I_p'P_+^{(e)} + I_a'UP_-^{(e)} \tag{40}$$

$$\nu^{-1} = I_p'^{-1}P_p' + UI_a'^{-1}P_a', \tag{41}$$

the representation space is defined as

$$\mathscr{K} = V_F(\mathscr{H}'') \tag{42}$$

The fields are defined similarly as in (34), (35)

$$\psi(f) = C(\nu P_+^{(e)} f) + C^*(\nu P_-^{(e)} f) \tag{43}$$

$\psi(f)$, $\psi^*(f)$ have again all the properties required by the definitions.

Straightforward calculations show:

$$\phi_t[\psi(f)] = \psi_t(f) = \psi(e^{iHt}f) = e^{iBt}\psi(f)\, e^{-iBt} \quad \forall f \in \mathscr{H}, \tag{44}$$

with

$$B = \Omega(I_p' H(e)\, P_+^{(e)} I_p'^{-1} P_p' + I_a' H(-e)\, P_+^{(-e)} I_a'^{-1} P_a'), \tag{45}$$

so $B \geqslant 0$ and of course $B\Phi_0 = 0$ (Φ_0 the no-particle vector in $\mathscr{K}$).

Note that $\mathscr{K}$ is again a space of antisymmetric wave functions, however in the Schrödinger picture, the particles are represented by positive energy solutions of Dirac's Eq. (4), while the anti-particles are represented by positive energy solutions of the equation obtained from (4) by reversing the sign of the charge e. This agrees with the physical idea of an anti-particle as behaving in the same way as a particle but with opposite charge.

5. THE TOTAL HAMILTONIAN IN THE FOCK SPACE OF THE FREE FIELDS

As we have seen, the natural physical, i.e., positive energy representations of the free and perturbed time dependent fields are in two different Hilbert spaces, $\mathscr{K}_0$ and $\mathscr{K}$. It is in these, *a priori* unrelated spaces that the formal expressions for the free and total Hamiltonians are defined as positive self-adjoint operators, B_0 in $\mathscr{K}_0$, B in $\mathscr{K}$.

In the conventional formulation one would assume (implicitly) that there is only one space, $\mathscr{K}_0$, in which both B_0 and B act. Because the interaction term as an expression in momentum space creation and annihilation operators does not represent an operator in $\mathscr{K}_0$, the total Hamiltonian B is not defined as such an expression. There may however still exist a self-adjoint operator, defined in a different way, that can be considered to be the total Hamiltonian. Before trying to decide whether this is the case, one has of course to agree on a general definition of what makes a Hamiltonian, choosing a precise interpretation of the heuristic properties of the formal "second quantized" expression (2).

We base our definition on the observation that formally B is determined, uniquely up to an additive real constant, by its commutation relations with the (irreducible) fields $\psi(\mathbf{x})$, $\psi^*(\mathbf{x})$. These follow from (1) and (2), and are e.g., for $\psi(\mathbf{x})$:

$$[B, \psi(\mathbf{x})] = \left\{\left(\sum_{k=1}^{3} i\alpha^k \frac{\partial}{\partial x^k} - m\beta\right) - e(A^0(\mathbf{x}) - \alpha^k A^k(\mathbf{x}))\right\} \psi(\mathbf{x}), \tag{46}$$

for the smeared fields (5), this becomes

$$[B, \psi(f)] = \left[B, \int \sum_{\mu=1}^{4} \psi_\mu(\mathbf{x})\, \overline{f_\mu(\mathbf{x})}\, d\mathbf{x}\right] = -\int \sum_{\nu=1}^{4} \psi_\nu(\mathbf{x}) \overline{(Hf)_\nu\, (\mathbf{x})}\, d\mathbf{x} = -\psi(Hf) \tag{47}$$

In order to avoid domain problems, we bring this in an exponentiated form and arrive at the following rigorous relation that a self-adjoint operator B should satisfy if it is to be called the total Hamiltonian:

$$e^{iBt}\psi(f)\, e^{-iBt} = \psi(e^{iHt}f), \qquad \text{for all } f \in \mathscr{H},\ t \in R^1. \tag{48}$$

If such a B exists, it is determined uniquely, up to a additive real constant by (48).

We will derive a necessary and sufficient condition on the potential $A^0(\mathbf{x})$, $A^k(\mathbf{x})$ for the existence of such a B. We need two theorems for this:

THEOREM (1). (D. Shale, W. F. Stinespring (*24*)). *Let $\mathscr{M}$ be a complex Hilbert space, $C(\mathscr{M})$ the C^* algebra over $\mathscr{M}$ as defined in Section 3; T a real orthogonal operator in $\mathscr{M}$ (i.e., with respect to $\mathscr{M}$ as a real Hilbert space), ϕ_T the *automorphism of $C(\mathscr{M})$ such that $\phi_T(R(z)) = R(Tz)$, for all $z \in \mathscr{M}$. Then ϕ_T is unitary implementable in the Fock–Cook representation of $C(\mathscr{M})$, with respect to $\mathscr{M}$ as a complex space, if and only if $Ti - iT$ is a real Hilbert–Schmidt operator in $\mathscr{M}$.*

THEOREM (2). (R. Kallman (*25*)): *Let $\mathscr{A}$ be a separable C^* algebra in a separable Hilbert space $\mathscr{L}$; let the weak closure of $\mathscr{A}$ be all bounded operators in $\mathscr{L}$. Let ϕ_t be a one parameter group of *automorphisms, norm continuous in t, i.e.: $\lim_{t\to t_0} \| \phi_t(A) - \phi_{t_0}(A)\| = 0$, t, $t_0 \in R_1$, $A \in \mathscr{A}$. Let there exist unitary operators $W(t)$ in $\mathscr{L}$, $t \in R'$, such that $\phi_t(A) = W(t)\, AW(t)^*$; $\forall A \in \mathscr{A}$. Then there exists a strongly continuous one parameter group e^{iBt} that implements the automorphism*

$$\phi_t(A) = e^{iBt}Ae^{-iBt}, \qquad A \in \mathscr{A}.$$

THEOREM (3). *There exists a self-adjoint operator B in $\mathscr{K}_0$, satisfying* (48) *if and only if the potentials $A^0(\mathbf{x})$, $A^k(\mathbf{x})$ are such that $P_+e^{-iHt}P_-$ is a (complex linear) Hilbert-Schmidt operator in $\mathscr{H}$.*

Proof. In $\mathscr{K}_0$ we have a representation of the algebra $C(\mathscr{H})$ and automorphisms ϕ_t such that $\phi_t(\psi(f)) = \psi(e^{iHt}f)$. The question of existence of B is the question whether these ϕ_t are unitary implementable (u.i.) for all t; If this is so, they can be implemented by a strongly continuous one-parameter group of unitary operators e^{iBt} because the ϕ_t are such that Theorem (2) can be applied.

The field operator algebra is a representation of the abstract algebra $C(\mathscr{H})$, but can also be considered as a representation of the algebra $C(\mathscr{H}')$, by identifying the self-adjoint generators

$$R(f) = R_F(\gamma f) \tag{49}$$

$$R(f) = \psi(f) + \psi^*(f) \tag{50}$$

$$R_F(z) = C(z) + C^*(z), \tag{51}$$

for all $f \in \mathscr{H}$, $z \in \mathscr{H}'$; and γ, $C(z)$ etc. as defined in Section 4. From (50), (49):

$$\begin{aligned}\psi(f) &= \frac{1}{2} R(f) + \frac{i}{2} R(if) \\ &= \frac{1}{2} R_F(\gamma f) + \frac{i}{2} R_F(\gamma i f)\end{aligned} \tag{52}$$

$$\psi(e^{iHt} f) = \frac{1}{2} R_F(\gamma e^{iHt} f) + \frac{i}{2} R_F(\gamma i e^{iHt} f), \tag{53}$$

Call $\gamma e^{iHt} \gamma^{-1} = 0(t)$, a group of operators in $\mathscr{H}'$, that are only real linear, and therefore not unitary but only real orthogonal. Call $\gamma i \gamma^{-1} = \mathscr{J}$; then $\mathscr{J} = i(P_p - P_a)$, unitary in $\mathscr{H}'$. Now $\psi(f) \to \psi(e^{iHt} f)$ is (u.i.), $\forall f \in \mathscr{H}$ whenever

$$\frac{1}{2} R_F(z) + \frac{i}{2} R_F(\mathscr{J} z) \to \frac{1}{2} R_F(0(t)\, z) + \frac{i}{2} R_F(\mathscr{J} 0(t)\, z), \tag{54}$$

is (u.i.) for all $z \in \mathscr{H}'$. Because $\mathscr{J}$ is unitary in $\mathscr{H}'$. $R_F(z) \to R_F(0(t)z)$ is (u.i.) is equivalent to $R_F(\mathscr{J} z) \to R_F(\mathscr{J} 0(t) z)$ is (u.i.) $\forall z \in \mathscr{H}'$, and therefore (5.4) is equivalent to

$$R_F(z) \to R_F(0(t)z) \qquad \text{is (u.i.)}, \tag{55}$$

for all $z \in \mathscr{H}'$ (and all t). Using Theorem (1), this is equivalent to $0(t)i - i0(t)$ a real Hilbert Schmidt operator, (H.S.), in $\mathscr{H}'$, or because $\gamma : \mathscr{H} \to \mathscr{H}'$ is orthogonal, $\gamma^{-1}(0(t)i - i0(t))\gamma =$ a real (H.S.) operator in $\mathscr{H}$. Now

$$\begin{aligned}\gamma^{-1}(0(t)i - i0(t))\gamma &= e^{iHt}\gamma^{-1}i\gamma - \gamma^{-1}i\gamma e^{iHt} = e^{iHt}(iP_+ - iP_-) - (iP_+ - iP_-)e^{iHt} \\ &= -2i(P_+ e^{iHt} P_- - P_- e^{iHt} P_+).\end{aligned}$$

This is a complex linear operator, so we have $P_+ e^{iHt} P_- - P_- e^{iHt} P_+$ must be (complex) (H.S.) in $\mathscr{H}$, for all t. This is equivalent to $P_+ e^{-iHt} P_-$ is (H.S.) for all t. Q.E.D. As an application of this criterion we prove:

THEOREM (4). *The Hamiltonian B exist in $\mathscr{K}_0$, in the sense of* (48), *if*

1°: $A^k(\mathbf{x}) = 0$

2°: $A^0(\mathbf{x})$ *is a bounded L_2 function whose fourier transform*

$$\hat{A}^0(\mathbf{s}) = \frac{1}{2\pi} \int e^{-i\mathbf{s}\cdot\mathbf{x}} A^0(\mathbf{x})\, d\mathbf{x},$$

is continuous except possibly at $\mathbf{s} = 0$, *and has the following behaviour at* $0, \infty$: $\exists$ *positive numbers* $C_1, C_2, \epsilon_1, \epsilon_2, \delta, N$, *such that*

$$|\hat{A}^0(\mathbf{s})| \leqslant C_1 |\mathbf{s}|^{-5/2+\epsilon_1} \quad \text{for} \quad 0 < |\mathbf{s}| < \delta$$

$$|\hat{A}^0(\mathbf{s})| \leqslant C_2 |\mathbf{s}|^{-2+\epsilon_2} \quad \text{for} \quad N < |\mathbf{s}|.$$

Proof. Introduce $M(s) = \max_{|\mathbf{s}|=s} e\hat{A}^0(\mathbf{s})$; this is a continuous function for $s > 0$, and has the same properties for $s \to 0, \infty$ as $\hat{A}^0(\mathbf{s})$.

We prove first several lemmas:

LEMMA (1). H_0 *self adjoint,* H_1 *bounded (so* $H = H_0 + H_1$ *self-adjoint and* $\mathscr{D}(H) = \mathscr{D}(H_0)$) *then*

$$e^{iH_0t}e^{-iHt} = 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, H_1(t_1) \cdots H_1(t_n), \tag{56}$$

(*Dyson series for finite times*),

with $H_1(s) = e^{iH_0s}H_1e^{-iHs}$. *The convergence is with respect to operator norms, and is uniform in t for* $|t| < C$, *for all* $C > 0$.

Proof. Define sequence

$$\begin{aligned} R_0(t) &= 1 \\ R_1(t) &= -i \int_0^t H_1(t')\, dt' \\ &\;\;\cdots \\ R_n(t) &= -i \int_0^t H_1(t')\, R_{n-1}(t')\, dt', \\ &\qquad\qquad n = 2, 3, \ldots . \end{aligned} \tag{57}$$

Each term is a well defined bounded operator, for all $t \in R^1$, and strongly continuous in t. (The integrals are in the strong sense, for properties of such integrals, see (*26*).)

Consider $|t| \leqslant C$, $C > 0$, fixed

$$\|R_0(t)\| = 1,$$

$$\|R_1(t)\| \leqslant \|H_1\|\,|t| \leqslant \|H_1\|\, C$$

$$\|R_2(t)\| \leqslant \|H_1\| \int_0^t \|R_1(t')\|\, dt' \leqslant \frac{\|H_1\|^2}{2} C^2$$

general

$$\|R_n(t)\| \leqslant \frac{\|H_1\|^n}{n!} C^n.$$

This proves the convergence in operator norm of the right hand side of (56); since

$$F(t) = \sum_{n=0}^{\infty} R_n(t),$$

the convergence is uniform on all finite t-intervals, the $R_n(t)$ are strongly continuous; therefore $F(t)$ is strongly continuous in t. Call $U_n(t) = \sum_{j=0}^{n} R_j(t)$, then from (57),

$$U_n(t) = 1 - i \int_0^t H_1(t')\, U_{n-1}(t')\, dt'$$

$$n = 1, 2, \ldots .$$

Taking $\lim_{n\to\infty}$ on both sides, again using the uniform convergence of $U_n(t) \to F(t)$ we have for $F(t)$:

$$F(t) = 1 - i \int_0^t H_1(t')\, F(t')\, dt'. \tag{58}$$

As an equation, this has a unique strongly continuous solution. (Let $\Delta(t)$ be the difference of two such solutions, then

$$\Delta(t) = -i \int_0^t H_1(t')\, \Delta(t')\, dt'; \qquad \text{for } |t| \leqslant C, \quad C > 0;$$

iteration gives:

$$\| \Delta(t)\, \Phi \| \leqslant \frac{\| H_1 \|^n}{n!} |t|^n \max_{|t'| \leqslant C} \| F(t')\, \Phi \|$$

for all $\Phi \in \mathscr{H}$, $n = 1, 2, \ldots$ so $\Delta(t) = 0, \forall t$.) (58) is satisfied by $e^{iH_0 t} e^{-iHt}$: For $\Phi \in \mathscr{D}(H) = \mathscr{D}(H_0) = \mathscr{D}$:

$$\begin{aligned}
&\left\| \left\{ \frac{e^{-H_0 h} e^{-iHh} - 1}{h} + iH_1 \right\} \Phi \right\| \\
&= \left\| \left\{ \frac{e^{iH_0 h}(e^{-iHh} - 1) + e^{-iH_0 h} - 1}{h} + iH - iH_0 \right\} \Phi \right\| \\
&\leqslant \left\| \left\{ e^{iH_0 h} \left(\frac{e^{-iHh} - 1}{h} \right) + iH \right\} \Phi \right\| + \left\| \left\{ \frac{e^{iH_0 h} - 1}{h} - iH_0 \right\} \Phi \right\|.
\end{aligned} \tag{59}$$

$$\frac{e^{-iHh} - 1}{h} \Phi = -iH\Phi + \Phi_h, \quad \text{with} \quad s = \lim_{h\to 0} \Phi_h = 0$$

so

$$\leqslant \| (e^{-iH_0 h} - 1)\, H\Phi \| + \| \Phi h \| + \left\| \left\{ \frac{e^{-iH_0 h} - 1}{h} - iH_0 \right\} \Phi \right\|.$$

All three terms go to 0 for $h \to 0$, so

$$s\text{-}\lim_{h\to 0} \frac{e^{iH_0h}e^{-iHh} - 1}{h} \Phi = -iH_1\Phi \qquad \forall\Phi \in \mathscr{D}.$$

From this

$$s\text{-}\lim_{h\to 0} \frac{e^{iH_0(t+h)}e^{-iH(t+h)} - e^{iH_0t}e^{-iHt}}{h} = -ie^{iH_0t}H_1e^{-iHt}\Phi \qquad \forall\Phi \in \mathscr{D}.$$

Or

$$\frac{d}{dt}\{e^{iH_0t}e^{-iHt}\Phi\} = -ie^{iH_0t}H_1e^{-iHt}\Phi \qquad \forall\Phi \in \mathscr{D}.$$

Both sides can be integrated on $[0, t]$ and extended to all $\Phi \in \mathscr{D}$

$$e^{iH_0t}e^{-iHt} = 1 - i\int_0^t H_1(t')\, e^{iH_0t'}e^{-iHt'}\, dt',$$

so

$$F(t) = e^{iH_0t}e^{-iHt}.$$

LEMMA (2). *Let $B(t)$ be a Hilbert–Schmidt operator for all $t \in [a, b]$, $B(t)$ is strongly continuous in $t \in [a, b]$ and $\| B(t)\|_2 \leqslant f(t)$, $\forall t \in [a, b]$ for some $f(t) \geqslant 0$, integrable on $[a, b]$. Then $A = \int_a^b B(t)\, dt$ is* (H.S.) *and*

$$\| A \|_2 \leqslant \int_a^b \| B(t)\|_2 \, dt.$$

(We use (H.S.) for Hilbert–Schmidt and denote the Hilbert–Schmidt norm of a (H.S.) operator B by $\| B \|_2$.)

Proof. Choose orthonormal base Φ_n in $\mathscr{H}$.

$$\| B(t)\|_2 = \left(\sum_{p=1}^{\infty} \| B(t)\, \Phi_p \|^2\right)^{1/2}$$

$$= \lim_{n\to\infty} \left(\sum_{p=1}^{n} \| B(t)\, \Phi_p \|^2\right)^{1/2}.$$

The term $\| B(t)\|_2$ is a limit of continuous functions, all bounded by an integrable function $f(t)$; therefore $\| B(t)\|_2$ is integrable on $[a, b]$.

$$F_{nm}(t', t'') = \sum_{p,q=1}^{n,m} |(\Phi_p, B(t')\Phi_q)(B(t'')\Phi_q, \Phi_p)|$$
$$\leqslant \sum_{p,q=1}^{\infty} |(\Phi_p, B(t')\Phi_q)(B(t'')\Phi_q, \Phi_p)|$$
$$\leqslant \| B(t')\|_2 \| B(t'')\|_2 .$$
$$\sum_{p,q}^{n,m} |(\Phi_p, A\Phi_q)|^2 \leqslant \int_a^b dt' \int_a^b dt'' F_{nm}(t', t'')$$
$$\leqslant \int_a^b dt' \int_a^b dt'' \| B(t')\|_2 \| B(t'')\|_2$$
$$= \left(\int_a^b \| B(t)\|_2 \, dt\right)^2 .$$

Therefore
$$\sum_{p,q=1}^{\infty} |(\Phi_p, A\Phi_q)|^2 = \| A \|_2^2 \leqslant \left(\int_a^b \| B(t)\|_2 \, dt\right)^2 < \infty.$$

LEMMA (3). *The H and H_0 as in Lemma* (1), *P_- a spectral projection of H_0, $P_+ = 1 - P_-$; If $P_+R_1(t)P_- = -i\int_0^t P_+H_1(t')P_- \, dt'$ is* (H.S.), *and $\| P_+R_1(t)P_- \|$ is bounded, for $|t| \leqslant C$, $C > 0$, then $P_+e^{-iHt}P_-$ is* (H.S.) *for $|t| \leqslant C$.*

Proof. $\| P_+R_1(t)P_- \|_2 \leqslant A$, $|t| \leqslant C$ for $n \geqslant 1$,
$$P_+R_{n+1}(t)P_- = -i\int_0^t P_+H_1(t')R_n(t')P_- \, dt'$$
$$= -i\int_0^t P_+H_1(t')P_+R_n(t')P_- \, dt'$$
$$-i\int_0^t P_+H_1(t')P_-R_n(t')P_- \, dt'.$$

Suppose that $P_+R_n(t')P_-$ is (H.S.) for $|t'| \leqslant C$, and that $\| P_+R_n(t')P_- \|_2$ is bounded by an integrable function for $|t'| \leqslant C$. Then the first integral is (H.S.) by Lemma (2) and also
$$\left\| -i\int_0^t P_+H_1(t')P_+R_n(t')P_- \, dt' \right\|_2 \leqslant \| H_1 \| \int_0^t \| P_+R_n(t')P_- \|_2 \, dt'.$$

The second term can be written as
$$\int_0^t \frac{d}{dt'}\{P_+R_1(t')P_-\} R_n(t')P_- \, dt'$$
$$= P_+R_1(t)P_-R_n(t)P_- + i\int_0^t P_+R_1(t')P_-H_1(t')R_{n-1}(t') \, dt'.$$

This shows that it is also (H.S.), it's (H.S.) norm

$$\leqslant A \parallel R_n(t) \parallel + A \int_0^t \parallel H_1 \parallel \parallel R_{n-1}(t') \parallel dt'$$
$$\leqslant 2A \frac{\parallel H_1 \parallel^n}{n!} C^n.$$

Therefore, by induction all terms $P_+ R_n(t) P_-$ are (H.S.) for $|t| \leqslant C$ and

$$\parallel P_+ R_{n+1}(t) P_- \parallel_2 \leqslant \parallel H_1 \parallel \int_0^t \parallel P_+ R_n(t') P_- \parallel_2 dt' + 2A \frac{\parallel H_1 \parallel^n}{n!} C^n.$$

Repeated application of this gives

$$\parallel P_+ R_n(t) P_- \parallel_2 \leqslant (2n-1) A \frac{\parallel H_1 \parallel^{n-1}}{(n-1)!} C^{n-1}$$

$$\sum_{n=1}^{k} \parallel P_+ R_n(t) P_- \parallel_2 \leqslant \sum_{n=1}^{k} A(2n-1) \frac{\parallel H_1 \parallel^{n-1}}{(n-1)!} C^{n-1}$$

$$= \sum_{n=0}^{k-1} A \frac{\parallel H_1 \parallel^n}{n!} C^n + 2A \sum_{n=0}^{k-1} n \frac{\parallel H_1 \parallel^n}{n!} C^n.$$

The first term is

$$\leqslant A \sum_{n=0}^{\infty} \frac{\parallel H_1 \parallel^n}{n!} C^n = A e^{\parallel H_1 \parallel C}.$$

The second term

$$2A \parallel H_1 \parallel C \sum_{n=1}^{k-1} \frac{\parallel H_1 \parallel^{n-1}}{(n-1)!} C^{n-1} \leqslant 2A \parallel H_1 \parallel C e^{\parallel H_1 \parallel C},$$

so

$$\sum_{n=1}^{k} \parallel P_+ R_1(t) P_- \parallel_2 \leqslant A(1 + 2 \parallel H_1 \parallel C)\, e^{\parallel H_1 \parallel C}.$$

The series converges in the (H.S.) norm, the sum is (H.S.); it converges also in the operator norm to $P_+ e^{iH_0 t} e^{-iHt} P_-$; the limits are the same, therefore $P_+ e^{iH_0 t} e^{-iHt}$ and finally $P_+ e^{-iHt} P_-$ is (H.S.) for $|t| \leqslant C$. Q.E.D.

To prove the theorem we only have to show that $P_+ R_1(t) P_-$ is (H.S.) for all t, and with the properties of $A^0(\mathbf{x})$, $A^k(\mathbf{x})$ as stated.
We use the "momentum representation" for $\mathscr{H}$, by means of the Fourier transform

$$f_\mu(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{k}\cdot\mathbf{x}} f_\mu(\mathbf{x})\, d\mathbf{x}.$$

$\mathscr{H}$ is now the L_2 space of functions $f_\mu(\mathbf{k})$, $\mu = 1, 2, 3, 4.$, $H_0 = \boldsymbol{\alpha} \cdot \mathbf{k} + m\beta$ ($\boldsymbol{\alpha} = \alpha^1, \alpha^2, \alpha^3$), $P_\pm$ are represented by

$$P_\pm(\mathbf{k}) = \frac{1}{2}\left(1 \pm \frac{\boldsymbol{\alpha} \cdot \mathbf{k} + m\beta}{\omega(\mathbf{k})}\right),$$

for $\omega(\mathbf{k}) = (\mathbf{k}^2 + m^2)^{1/2}$, and H_1 is given by the integral kernel $e\hat{A}^0(\mathbf{k}_1 - \mathbf{k}_2)$. The operator $R_+R_1(t)\,P_- = -i\int_0^t P_+H_1(t')\,P_-\,dt'$ has the kernel

$$-ie\int_0^t P_+(\mathbf{k}_1)\,e^{i\omega(\mathbf{k}_1)t'}\,\hat{A}^0(\mathbf{k}_1 - \mathbf{k}_2)\,P_-(\mathbf{k}_2)\,e^{i\omega(\mathbf{k}_2)t'}\,dt'$$

$$= eP_+(\mathbf{k}_1)\,P_-(\mathbf{k}_2)\,\hat{A}^0(\mathbf{k}_1 - \mathbf{k}_2)\,\frac{1 - \exp[i\{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)\}\,t]}{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)}.$$

$P_+R_1(t)\,P_-$ is (H.S.) if

$$\iint_{-\infty}^{+\infty} d\mathbf{k}_1\,d\mathbf{k}_2\,|\,e\hat{A}^0(\mathbf{k}_1 - \mathbf{k}_2)|^2 \left|\frac{1 - \exp[i\{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)\}\,t]}{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)}\right|^2$$

$$\cdot\,\mathrm{Tr}\{P_+(\mathbf{k}_1)\,P_-(\mathbf{k}_2)\} < \infty.$$

Using well known trace properties of α^k, β

$$\iint_{-\infty}^{+\infty} d\mathbf{k}_1\,d\mathbf{k}_2\left(1 - \frac{\mathbf{k}_1 \cdot \mathbf{k}_2 + m^2}{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)}\right)|\,e\hat{A}^0(\mathbf{k}_1 - \mathbf{k}_2)|^2$$

$$\cdot\left|\frac{1 - \exp[i(\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2))\,t]}{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)}\right|^2 < \infty.$$

It is sufficient to prove

$$\iint_{-\infty}^{+\infty}\left(1 - \frac{\mathbf{k}_1 \cdot \mathbf{k}_2 + m^2}{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)}\right) M(|\,\mathbf{k}_1 - \mathbf{k}_2\,|)^2$$

$$\cdot\left|\frac{1 - \exp[i(\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2))\,t]}{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)}\right|^2 d\mathbf{k}_1\,d\mathbf{k}_2 < \infty.$$

Introduce new variables: $\mathbf{k}_s = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k}_d = \mathbf{k}_1 - \mathbf{k}_2$.

The integrand is a function of $k_s = |\,\mathbf{k}_s\,|$, $k_d = |\,\mathbf{k}_d\,|$ and $u = \cos(\mathbf{k}_s, \mathbf{k}_d)$; after introducing polar variables and integrating over some of these, one gets the condition

$$\int_0^\infty\int_0^\infty\int_{-1}^{+1} f(k_s, k_d, u)\,g(k_s, k_d, u)^2\,M(k_d)^2\,k_s^2k_d^2\,dk_s\,dk_d\,du < \infty$$

with

$$f(k_s, k_d, u) = 1 - \frac{k_s^2 - k_d^2 + 4m^2}{(a^2 - b^2u^2)^{1/2}},$$

$$g(k_s, k_d, u) = \left| \frac{1 - \exp[\frac{1}{2}i\{(a + bu)^{1/2} + (a - bu)^{1/2}\} t]}{\frac{1}{2}\{(a + bu)^{1/2} + (a - bu)^{1/2}\}} \right|,$$

$$a = k_s^2 + k_d^2 + 4m^2, \qquad b = 2k_s k_d$$

$$k_s \geqslant 0, \qquad k_d \geqslant 0, \qquad m > 0, \qquad -1 \leqslant u \leqslant +1.$$

We investigate first the integral

$$\int_0^\infty \int_{-1}^{+1} fg^2 k_s^2 \, dk_s \, du. \tag{60}$$

We have the following useful inequalities

1: $a > b \geqslant 0$, so $a - bu$, $a + bu$, $a^2 - b^2u^2 > 0$ for all admitted values of k_s, k_d, u. (61)

2: $0 < (a^2 - b^2)^{1/2} \leqslant (a^2 - b^2u^2)^{1/2} \leqslant a$. (62)

3: $0 < (a + b)^{1/2} + (a - b)^{1/2} \leqslant (a + bu)^{1/2} + (a - bu)^{1/2} \leqslant 2a^{1/2}$. (63)

4: $\left| \dfrac{k_s^2 - k_d^2 + 4m^2}{(a^2 - b^2)^{1/2}} \right| \leqslant 1.$

From these there follow inequalities for f, g:

(a) for $f(k_s, k_d, u)$:

1: if $k_s^2 + 4m^2 > k_d^2$:

$$0 \leqslant 1 - \frac{k_s^2 - k_d^2 + 4m^2}{(a^2 - b^2)^{1/2}} \leqslant f(k_s, k_d, u) \leqslant \frac{2k_d^2}{k_s^2 + k_d^2 + 4m^2} < 1. \tag{64}$$

2: if $k_s^2 + 4m^2 = k_d^2$: $f(k_s, k_d, u) = 1$. (65)

3: if $k_s^2 + 4m^2 < k_d^2$:

$$1 < \frac{2k_d^2}{k_s^2 + k_d^2 + 4m^2} \leqslant f(k_s, k_d, u) \leqslant 1 - \frac{k_s^2 - k_d^2 + 4m^2}{(a^2 - b^2)^{1/2}} < 2. \tag{66}$$

(b) for $g(k_s, k_d, u)$:

$$0 \leqslant g(k_s, k_d, u) \leqslant \left| \frac{\exp[(i/2)\{(a + bu)^{1/2} + (a - bu)^{1/2}\} t] - 1}{\frac{1}{2}\{(a + b)^{1/2} + (a - b)^{1/2}\}} \right| \tag{67}$$

$$\leqslant \frac{4}{(a + b)^{1/2} + (a - b)^{1/2}}.$$

Using (64) (67) we have for large k_s, i.e., for $k_s^2 > k_d^2 - 4m^2$

$$0 \leqslant fg^2 k_s^2 \leqslant \frac{2k_s^2 k_d^2}{k_s^2 + k_d^2 + 4m^2} \frac{16}{\{((k_s + k_d)^2 + 4m^2)^{1/2} + ((k_s - k_d)^2 + 4m^2))^{1/2}\}^2}$$

for $k_s \to \infty$, this behaves like $8k_d^2/k_s^2$, so (60) converges, for all $k_d \geqslant 0$. We now investigate

$$\int_0^\infty M(k_d)^2 A(k_d) k_d^2 \, dk_d \tag{68}$$

with

$$A(k_d) = \int_0^\infty \int_{-1}^{+1} fg^2 k_s^2 \, dk_s \, du.$$

We prove a few properties of $A(k_d)$

1: $A(k_d)$ is continuous for all $k_d > 0$; for $0 \leqslant k_s \leqslant k_d$ we have (64) (65) (66): $0 \leqslant f(k_s, k_d, u) \leqslant 2$, for $k_s \geqslant k_d$, because of (64)

$$0 \leqslant f(k_s, k_d, u) \leqslant \frac{2k_d^2}{k_s^2 + k_d^2 + 4m^2} \leqslant \frac{2k_d^2}{k_s^2}.$$

From (67):

$$0 \leqslant g(k_s, k_d, u)^2 \leqslant \frac{8}{k_s^2} \quad \text{so} \quad 0 \leqslant fg^2 k_s^2 \leqslant h(k_s, k_d),$$

with

$$\begin{aligned} h(k_s, k_d) &= 16 \qquad \text{for} \quad 0 \leqslant k_s \leqslant k_d \\ &= \frac{16k_d^2}{k_s^2} \qquad \text{for} \quad k_d \leqslant k_s. \end{aligned} \tag{69}$$

The term $fg^2 k_s^2$ is a C^∞ function in k_s, k_d, u, so $\int_{-1}^{+1} fg^2 \, du$ is a C^∞ function in k_s, k_d, and $A_N(k_d) = \int_0^N dk_s \, k_s^2 \int_{-1}^{+1} fg^2 \, du$ is C^∞ in k_d for every $0 < N < \infty$. For $N > k_d > 0$:

$$| A(k_d) - A_N(k_d)| \leqslant 2 \int_N^\infty h(k_s, k_d) \, dk_s = \frac{32}{N} k_d^2. \tag{70}$$

For $0 < k_d, k_d' < N$:

$$\begin{aligned} &| A(k_d') - A(k_d)| \\ &\leqslant | A(k_d') - A_N(k_d')| + | A_N(k_d') - A_N(k_d)| + | A(k_d) - A_N(k_d)| \\ &\leqslant 32 \frac{(k_d^2 + k_d'^2)}{N} + | A_N(k_d') - A_N(k_d)|, \end{aligned}$$

and for $k_d, k_d' < M < N$ this is

$$\leqslant 64\,\frac{M^2}{N} + |\,A_N(k_d') - A_N(k_d)|. \tag{71}$$

Using continuity of all $A_N(k_d)$, (71) gives in the usual way continuity of $A(k_d)$ for all $k_d > 0$

2: $\lim_{k_d \to 0} A(k_d)/k_d^2$ exist. Write $f(k_s, k_d, u)$ as

$$\frac{4k_d^2\{k_s^2(1-u^2) + 4m^2\}}{(a^2 - b^2u^2)^{1/2}\,\{(a^2 - b^2u^2)^{1/2} + k_s^2 - k_d^2 + 4m^2\}},$$

one obtains

$$\lim_{k_d \to 0} \frac{fg^2k_s^2}{k_d^2} = 2\left(1 - \frac{k_s^2u^2}{k_s^2 + 4m^2}\right)\frac{|\,1 - \exp[i(k_s^2 + 4m^2)^{1/2}\,t]|^2}{(k_s^2 + 4m^2)}, \tag{72}$$

for $k_d < 2m$, from (64) (67)

$$\frac{fg^2k_s^2}{k_d^2} \leqslant \frac{32}{k_s^2 + k_d^2 + 4m^2}\,\frac{1}{\{(a+b)^{1/2} + (a-b)^{1/2}\}^2}$$

$$\leqslant \frac{32}{(k_s^2 + 4m^2)}\,\frac{1}{2(k_s^2 + 4m^2)}.$$

This is integrable on $-1 \leqslant u \leqslant 1$, $0 \leqslant k_s < \infty$, so the dominated convergence theorem of Lebesgue gives

$$\lim_{k_d \to 0} \frac{A(k_d)}{k_d^2} = \int_{-1}^{+1} du \int_0^\infty dk_s \lim_{k_d \to 0} \frac{fg^2k_s^2}{k_d^2}.$$

3: $(A(k_d))/k_d$ remains bounded for $k_d \to \infty$ because

$$A(k_d) \leqslant 2\int_0^\infty h(k_s, k_d)\, dk_s = 64k_d.$$

This proves the convergence of (68) and therefore $P_+R_1(t)\,P_-$ is (H.S.), its (H.S.) norm bounded by a constant, independent of t; this proves the theorem.

The same sort of question as was answered in Theorem (3) and (4) about B can be asked about the interaction term B_1 separately, using its commutation relations with $\psi(\mathbf{x})$, $\psi^*(\mathbf{x})$ in the same way as for B, one defines B_1 as the self-adjoint operator such that

$$e^{iB_1\tau}\psi(f)\,e^{-iB_1\tau} = \psi(e^{iH_1\tau}f), \qquad \text{for all } f \in \mathscr{H}, \quad \tau \in R_1 \tag{72}$$

It seems very unlikely that the potentials can be chosen such that this B_1 exists. This is clear from the following theorem.

THEOREM (5). *Let H_1 be such that*

1: $A^k(\mathbf{x}) = 0, k = 1, 2, 3.$

2: $A^0(\mathbf{x})$ *is a C^∞ function with compact support (non-empty). Then there exists no self-adjoint operator B_1, satisfying (72) in $\mathscr{K}_0$. (Note that for this case B_0 and B both exist in $\mathscr{K}_0$, according to the preceding theorem.)*

Proof. A trivial generalization of Theorem (3) shows that B_1 exists in $\mathscr{K}_0$, if and only if

$$P_+ e^{iA^0(\mathbf{x})\tau} P_- \text{ is (H.S.)}, \tag{73}$$

or

$$P_+ G(\mathbf{x}) P_- \quad \text{is} \quad \text{(H.S.)}, \qquad \text{with} \quad G(\mathbf{x}) = 1 - e^{iA^0(\mathbf{x})\tau}. \tag{74}$$

The function $G(\mathbf{x})$ is also a C^∞ function with compact support, and its fourier transform $\hat{G}(\mathbf{s})$ is in the space (S) of Schwartz. We prove that the kernel

$$P_+(\mathbf{k}_1) P_-(\mathbf{k}_2)\, \hat{G}(\mathbf{k}_1 - \mathbf{k}_2) \quad \text{is not} \quad \text{(H.S.)}$$

or that the integral

$$\iint \left(1 - \frac{\mathbf{k}_1\mathbf{k}_2 + m^2}{\omega(\mathbf{k}_1)\, \omega(\mathbf{k}_2)}\right) |\, \hat{G}(\mathbf{k}_1 - \mathbf{k}_2)|^2 \, d\mathbf{k}_1 \, d\mathbf{k}_2 \,, \tag{75}$$

diverges. Introduce $\mathbf{p}_1 = \mathbf{k}_1$, $\mathbf{p}_2 = \mathbf{k}_1 - \mathbf{k}_2$, we have the repeated integral

$$\int_{-\infty}^{+\infty} d\mathbf{k}_2 \,|\, \hat{G}(\mathbf{p}_2)|^2 \int_{-\infty}^{+\infty} \left(1 - \frac{\mathbf{p}_1(\mathbf{p}_1 - \mathbf{p}_2) + m^2}{\omega(\mathbf{p}_1)\, \omega(\mathbf{p}_1 - \mathbf{p}_2)}\right) d\mathbf{p}_1 \,.$$

If (75) converges, then

$$\int_{-\infty}^{+\infty} \left(1 - \frac{\mathbf{p}_1(\mathbf{p}_1 - \mathbf{p}_2) + m^2}{\omega(\mathbf{p}_1)\, \omega(\mathbf{p}_1 - \mathbf{p}_2)}\right) d\mathbf{p}_1 < \infty \tag{76}$$

(Fubini, for almost all $\mathbf{p}_2$).
Using $p_1 = |\,\mathbf{p}_1\,|$, $p_2 = |\,\mathbf{p}_2\,|$, and $u = \cos(\mathbf{p}_1, \mathbf{p}_2)$, then this can be written as a repeated integral over $p_1^2 \, dp_1 \, du$ and (76) implies

$$\int_0^\infty \left(1 - \frac{p_1^2 - p_1 p_2 u + m^2}{(p_1^2 + m^2)^{1/2}\, (p_1^2 + p_2^2 - 2p_1 p_2 u + m^2)^{1/2}}\right) p_1^2 \, dp_1 < \infty \tag{78}$$

(for almost all p_2, u.)

It is easy to verify that the integrand has a limit for $p_1 \to \infty$, it is $\frac{1}{2}p_2{}^2u^2$ which contradicts (78). Q.E.D.

6. SCATTERING THEORY, S-OPERATOR

In the formal derivation of the S-operator and its perturbation series, one uses an interaction picture,

$$\Phi^{\text{int}}(t) = e^{iB_0t}e^{-iBt}\Phi(0), \tag{79}$$

the evolution operator in this picture

$$U(t, t_0) = e^{iB_0t}e^{-iB(t-t_0)}e^{-iB_0t_0}, \tag{80}$$

is supposed to give the S-operator by

$$S = \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} U(t, t_0).$$

Because as we have seen, B in general does not make sense in the Hilbert space of the free field, where B_0 is defined, $U(t, t_0)$ is meaningless, and the question whether $U(t, t_0)$ converges to an S-operator cannot even be asked. The essential ideas of this procedure can however easily be formulated rigorously in the algebraic description. We need some assumptions on the scattering behavior of the classical equation. We are not interested here in this classical behavior as such, but want to show how it determines the scattering theory of the corresponding quantum field completely. The following assumptions serve this prupose, but are rather restrictive on the potentials $A^0(\mathrm{x})$, $A^k(\mathrm{p})$ see (*27*). Weaker assumptions would involve us in inessential complication that we want to avoid. Suppose $A^0(\mathrm{x})$, $A^k(\mathrm{p})$ are such that

$$\lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t} = W_\pm \tag{81}$$

exist as strong limits in $\mathscr{H}$, and are unitary. The classical S-operator is then

$$S_{\text{cl}} = W_+{}^*W_- . \tag{82}$$

A definition of a generalized interaction picture for the field is suggested by the idea of $E(A)$ as a generalized expectation value $(\Phi, A\Phi)$. For each state $E(A)$ of $C(\mathscr{H})$ (considered as a state at $t = 0$)

$$E_t^{\text{int}}(A) = E(\phi_t\phi^0_{-t}(A)). \tag{83}$$

The functional $E_t^{\text{int}}(A)$ is determined by its values on all products of fields.

$$E_t^{\text{int}}(\psi(f_1) \cdots \psi(f_n)) = E(\phi_t\phi_{-t}^0[\psi(f_1)] \cdots \phi_t\phi_{-t}^0[\psi(f_n)])$$

$$= E(\psi(e^{iHt}e^{-iH_0t}f_1) \cdots \psi(e^{iHt}e^{-iH_0t}f_n)),$$

because of (81), of the norm-continuity of $\psi(f)$ as a function of $f \in \mathscr{H}$, and of the continuity of $E(A)$ we have

$$\lim_{t\to\pm\infty} E_t^{\text{int}}(\psi(f_1) \cdots \psi(f_n)) = E(\psi(W_\pm f_1) \cdots \psi(W_\pm f_n)).$$

For each $E(A)$ there exist therefore states $E_\pm(A) = \lim_{t\to\pm\infty} E_t^{\text{int}}(A)$. The outcoming state is then given by

$$E_+(\psi(f_1) \cdots \psi(f_n)) = (\Phi_-, \psi(S_{\text{cl}}^* f_1) \cdots \psi(S_{\text{cl}}^* f_n)\, \Phi_-)$$

Because S_{cl} is a unitary operator in $\mathscr{H}$, commuting with H_0 the transformation $\psi(f) \to \psi(S_{\text{cl}}^* f)$ can be unitarily implemented in $\mathscr{K}_0$. ($P_\pm S_{\text{cl}}^* P_\mp = 0$). This means that there exist a unitary operator S in $\mathscr{K}_0$, determined uniquely, up to a phase factor such that

$$\psi(S_{\text{cl}}^* f) = S^* \psi(f)\, S \qquad \text{for all} \quad f \in \mathscr{H}. \tag{84}$$

Use the arguments leading to (36). In $\mathscr{K}_0$:

$$\psi(S_{\text{cl}}^* f) = C(\gamma P_+ S_{\text{cl}}^* f) + C^*(\gamma P_- S_{\text{cl}}^* f)$$

(with (23), (24))

$$= \Gamma^{-1}(\gamma S_{\text{cl}}\gamma^{-1})\, C(\gamma P_+ f)\, \Gamma(\gamma S_{\text{cl}}\gamma^{-1})$$

$$+ \Gamma^{-1}(\gamma S_{\text{cl}}\gamma^{-1})\, C^*(\gamma P_- f)\, \Gamma(\gamma S_{\text{cl}}\gamma^{-1})$$

$$= \Gamma^{-1}(\gamma S_{\text{cl}}\gamma^{-1})\, \psi(f)\, \Gamma(\gamma S_{\text{cl}}\gamma^{-1}) \tag{85}$$

So

$$S = \Gamma(\gamma S_{\text{cl}}\gamma^{-1}) = \Gamma(S'), \tag{86}$$

and

$$S' = \gamma S_{\text{cl}}\gamma^{-1} = I_p(P_+ S_{\text{cl}})\, I_p^{-1} P_p + I_a(UP_- S_{\text{cl}} U)\, I_a^{-1} P_a\,. \tag{87}$$

Because of (26) (37), $S_{\text{cl}}(e)U = US_{\text{cl}}(-e)$, so (86) can also be written as

$$S' = I_p(P_+ S_{\text{cl}}(e))\, I_p^{-1} P_p + I_a(P_+ S_{\text{cl}}(-e))\, I_a^{-1} P_a\,, \tag{88}$$

which makes its physical meaning clear. Consider now an incoming state $E_-(A)$ that is given by a vector Φ_- in the free Fock space $\mathscr{K}_0$

$$E_-(A) = (\Phi_-, A\Phi_-)$$

The corresponding outgoing state is then given by

$$E_+(\psi(f_1) \cdots \psi(f_n)) = E_-(\psi(S_{\text{cl}}^* f_1) \cdots \psi(S_{\text{cl}}^* f_n))$$

$$= (\Phi_-, \psi(S_{\text{cl}}^* f_1) \cdots \psi(S_{\text{cl}}^* f_n)\, \Phi_-) = (S\Phi_-, \psi(f_1) \cdots \psi(f_n)\, S\Phi_-),$$

so $E_+(A)$ is also given as a vector state in $\mathscr{K}_0$:

$$E_+(A) = (\Phi_+, A\Phi_+) \quad \forall A \in C(\mathscr{H}); \qquad \text{with} \quad \Phi_+ = S\Phi_-$$

(Note that the arbitrariness of a phase factor in S in (83) has been used to obtain $S\Phi_0 = \Phi_0$, Φ_0 vacuum of free Fock space.)

The following observation on the physical interpretation of this result should be made: As we noted earlier the Dirac Equation (4) describes essentially a many-particle situation; this is due to the requirement of positive energy. The operator e^{-iBt}, when it exists in $\mathscr{K}_0$, mixes subspaces of different particle and anti-particle numbers N_p, N_a. Pairs of particles and antiparticles are created and annihilated continuously; only the difference $N_p - N_a$ is constant (charge conservation).

However, for the limit $t \to \pm\infty$, the theory reduces to a one-particle situation. The scattering operator S that we have derived is a product operator $\Gamma(S'_{\text{cl}})$ as defined in (20), generated by a one-particle operator $S'_{\text{cl}} = \gamma S_{\text{cl}} \gamma^{-1}$ in the physical one-particle space $\mathscr{H}'$. Such a product operator does not create or annihilate particles or anti-particles; N_p and N_a are conserved separately. Each particle or anti-particle is individually scattered by the electromagnetic field, completely independent of the others. This is an assymptotic result, for $t \to \pm\infty$ and comes from a dynamical situation for finite times that is more complicated. Nevertheless once this result has been obtained the many particle space $\mathscr{K}_0$ becomes superfluous; all physical information on S in $\mathscr{K}_0$ is contained in S'_{cl}, acting in $\mathscr{H}'$. Moreover because "S'_{cl} has no matrix elements between $\mathscr{H}_p$ and $\mathscr{H}_a$" one can consider separately the parts of S'_{cl} that scatter a particle and an antiparticle.

One has $S'_{\text{cl}} P_p = I_p(P_+ S_{\text{cl}})\, I_p^{-1} P_p$, acting in $\mathscr{H}_p$, which is identical to $P_+ S_{\text{cl}}$ in $\mathscr{H}_+$; so the scattering of a particle is described by the positive energy part of the classical S-operator.

One has $S'_{\text{cl}} P_a = I_a = I_a(UP_- S_{\text{cl}} U) I_a^{-1} P_a$ acting in $\mathscr{H}_a$, which is identical to $UP_- S_{\text{cl}} U$ in $\mathscr{H}_+$ and is obtained from $P_- S_{\text{cl}}^-$ in $\mathscr{H}_-$, so the scattering of an anti-particle is described by the negative energy part of the classical S-operator, provided a transformation using the charge conjugation U is performed.

If one writes the dependence on e again explicitly one has

$$Ue^{-iH(e)t} = e^{-iH(-e)t}U,$$

so

$$UP_{-}S_{\text{cl}}(e)U = P_{+}US_{\text{cl}}(e)U = P_{+}S_{\text{cl}}(-e).$$

This means that the scattering of an anti-particle can also be given by the positive energy part of the classical S-operator for opposite charge $-e$.

(We are aware that we are using a somewhat more abstract language than is standard in most of the physical literature on this and related problems. For rigorous investigations in the essentially mathematical pecularities of field theory this seems to be unavoidable. Restatement of the results in the more usual language of integral-kernels, "S-matrix elements", in momentum and spin variables is not difficult and is left to the reader.)

Finally something must be said about the status of the perturbation series for the S-operator in this model. As we have noted in the beginning of this paragraph, this series and the procedure leading to it is purely formal, there is no way of giving it a precise meaning in terms of operators in the Hilbert space $\mathscr{K}_0$. Still one can use this formal collection of integral expressions, some of which are infinite for approximative calculation of S. The reason for this can be understood from the following: Calculation of the scattering operator S in $\mathscr{K}_0$, can be reduced to calculation of S'_{cl} in $\mathscr{H}'$, and therefore to S_{cl} in $\mathscr{H}$. The S_{cl} has a perturbation series

$$S_{\text{cl}} = 1 - i\int_{-\infty}^{+\infty} H_1(t)\,dt + \frac{(-i)^2}{2!}\iint_{-\infty}^{+\infty} T\{H_1(t_1)\,H_1(t_2)\}\,dt_1\,dt_2 + \cdots.$$

This will exist and converge to S_{cl} under certain physically reasonable assumptions on the potentials $A^0(\mathbf{x})$, $A^k(\mathbf{x})$; it has none of the problems of the corresponding formal series of S in $\mathscr{K}_0$.

"Matrix elements" of S_{cl} (and therefore of S) can be approximated by terms of this series; it is not difficult to verify that one will get the same expressions as from the formal series in $\mathscr{K}_0$, except that the expressions corresponding to vacuum diagrams are automatically absent.

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SAMENVATTING

In dit proefschrift wordt een eenvoudig model uit de quantumveldentheorie op mathematisch strenge wijze beschreven met behulp van C^*-algebrabegrippen.

De situatie die door dit model beschreven wordt, is die van een systeem van electronen en positronen die niet met elkaar in wisselwerking zijn maar wel met een gegeven uitwendig, niet gequantiseerd electromagnetisch veld.

De beschrijving van dit model op de gebruikelijke wijze met operatoren en vectoren in de Fock-Hilbertruimte vertoont enkele van de voor de quantumveldentheorie karakteristieke divergenties. De totale Hamiltoniaan is een formele uitdrukking in de veldoperatoren en is niet gedefinieerd als zelfgeadjungeerde operator. Dientengevolge heeft de daaruit afgeleide storingstheorie geen mathematische betekenis. De reeks die, althans formeel, voor de S-operator kan worden opgeschreven, bevat divergente vacuumtermen. Deze termen kunnen worden geëlimineerd en een bruikbaar resultaat kan worden verkregen. De daarbij gebruikte procedure kan echter niet worden gemotiveerd en is ook mathematisch hoogst onbevredigend. Voor dit model is het mogelijk met behulp van C^*-algebrabegrippen hetzelfde resultaat te verkrijgen, maar dan op een eenduidig bepaalde en wiskundig strenge manier.

In § 3 worden de twee elementen waaruit een C^*-algebrabeschrijving van een quantumveld bestaat, geconstrueerd, n.l. de abstracte algebra van veldoperatoren en de groep van automorphismen die deze algebra in de tijd transformeert.

De veldoperatorenalgebra is een bijzonder geval van een algebra voortgebracht door anticommutatierelaties. De tijdsevolutie-automorphismen zijn afkomstig van de tijdsevolutie van oplossingen van de veldvergelijking, beschouwd als klassieke partiële differentiaalvergelijking.

In § 4 worden representaties van dit abstracte quantumveld in Hilbertruimtes onderzocht. Er bestaat een eenduidig bepaalde representatie met positieve energie; voor elk gegeven stel electromagnetische potentialen dat aan redelijke eisen voldoet. Deze representatie wordt geconstrueerd, evenals die van het vrije veld.

In § 5 wordt aangetoond dat in sommige gevallen er toch een zelfgeadjungeerde operator bestaat, in de Fockruimte, die opgevat kan worden als totale Hamiltoniaan. Het definitiegebied van deze operator is echter zodanig dat hij niet gebruikt kan worden voor storingstheorie.

In § 6 wordt de verstrooiingstheorie van het model behandeld. Een interactiebeeld in de gewone zin kan in het algemeen niet worden gedefinieerd. Het is echter wel mogelijk een gegeneraliseerd interactiebeeld in het C^*-algebra-systeem in te voeren. Als de oplossingen van de klassieke vergelijking een bevredigend asymptotisch gedrag vertonen, leidt dit gegeneraliseerde interactiebeeld tot een verstrooiingsautomorphisme, waarvan blijkt dat het gerepresenteerd kan worden door een unitaire S-operator in de Fockruimte.

Curriculum vitae

Op verzoek van de Faculteit der Wiskunde en Natuurwetenschappen volgen hier enige gegevens over mijn studie.

In 1957 legde ik het eindexamen H.B.S.-B af aan het St. Montfort College te Rotterdam. In hetzelfde jaar begon ik mijn studie aan de Rijksuniversiteit te Leiden, alwaar ik in 1960 het candidaatsexamen a′ (natuurkunde, wiskunde en sterrekunde) aflegde. In 1964 legde ik het doctoraal examen af met als hoofdvak theoretische natuurkunde en als bijvakken wiskunde en klassieke mechanica. Gedurende de studie hiervoor volgde ik, wat betreft de natuurkunde, colleges o.a. van Prof. Dr J. A. M. Cox, Prof. Dr S. R. de Groot en Prof. Dr P. Mazur, en voor de wiskunde colleges van o.a. Prof. Dr W. T. van Est, Prof. Dr C. Visser en Prof. Dr A. C. Zaanen.

Gedurende zes maanden van het studiejaar 1961–1962 was ik werkzaam aan het Instituut voor Atoomenergie te Kjeller, Noorwegen, en verrichtte in de groep van Dr K. Abrahams experimenteel onderzoek aan verstrooiing van neutronen. Hiertoe was ik in staat gesteld door een studietoelage van het Reactor Centrum Nederland, mij verleend op voordracht van Prof. Dr J. A. Goedkoop.

In 1963 trad ik in dienst bij de Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.) en werd opgenomen in de werkgroep hoge-energiefysica (H III-L) onder leiding van Prof. Dr J. A. M. Cox, op het Instituut-Lorentz voor theoretische natuurkunde te Leiden. Na een algemene oriëntatie op het gebied van de elementairedeeltjesfysica richtte ik mijn belangstelling in het bijzonder op de quantumveldentheorie en de daaraan verbonden wiskundige problemen, met name op mogelijkheden om veldentheoretische modellen te beschrijven met C^* algebras,

Van 1967 tot 1969 stelde de Stichting F.O.M. mij door een studiebeurs in staat om mijn werkzaamheden op dit gebied voort te zetten aan het Massachusetts Institute of Technology, waar ik werkte bij Prof. I. E. Segal.

STELLINGEN

1

Er bestaat in de Fockruimte van het vrije Diracveld geen zelfgeadjungeerde operator die de totale lading in een eindig volume beschrijft en die correspondeert met de gebruikelijke formele operatoruitdrukking voor die grootheid.

Paragraaf 5 van dit proefschrift.

2

Een essentiële wiskundige moeilijkheid die optreedt bij het gebruik van een indefiniet 'inwendig product', zoals dat van Gupta en Bleuler voor het vrije fotonveld, wordt niet opgelost door zo'n product te definiëren met behulp van een metrische operator in een ruimte met een echt, definiet inwendig product.

S. N. Gupta. Proc. Phys. Soc. (London) A 63, 681 (1950)
K. Bleuler. Helv. Phys. Acta 23, 564 (1950)
K. L. Pandit. Nuovo Cimento, Ser. 10, Suppl. vol. 11 (1959)

3

In verband met recente ontwikkelingen in de quantumveldentheorie zou het wenselijk zijn een mathematisch-strenge spectraaltheorie te ontwikkelen voor operatoren in een 'Hilbertruimte met indefiniete metriek'. Deze ruimte zou dan kunnen worden opgevat als een locaal-convexe topologische ruimte, voorzien van een continue, hermitische, niet-ontaarde vorm.

4

De wijze waarop Wigner bij zijn beschouwingen over de Lorentz-groep het begrip 'representation up to a factor' hanteert, is verwarrend en maakt niet duidelijk of hiermee een 'projectieve representatie' dan wel een 'meerwaardige representatie' wordt bedoeld.

E. P. Wigner, Ann. of Math. 40, 149 (1937).

5

De bewering van Domb en Wyles dat de soortelijke-warmtecurve van Gd VO_4, zoals deze is gemeten door Cashion en medewerkers, dicht bij de Néel temperatuur een abnormale vorm heeft, die zou wijzen op een sterke koppeling tussen het spinsysteem en het rooster, berust op een aanvechtbare interpretatie van de meetresultaten.

C. Domb, J. A. Wyles. J. Phys. C (Solid State Physics) 2, 2435 (1969)
J. D. Cashion at al. Proc. Colloque Int. du C.N.R.S.: Sur les Elements des Terres Rares (1969). To be published.

6

Wegens het feit dat de huidige experimentele kennis van het proces $K^+ \rightarrow \pi^+ \pi^- e^+ \nu$ diverse interpretaties openlaat, is het niet verwonderlijk dat het model van Roberts en Wagner dit proces kan beschrijven.

R. G. Roberts, F. Wagner. In: Proc. Topical Conference on Weak Interactions. CERN 69–7, p. 281.

7

Men kan een voorbeeld geven van een oneindig-dimensionale Banach-Liealgebra die een abstracte Liegroep karakteriseert, maar die toch niet continu geïnjecteerd kan worden in de Liealgebra van begrensde lineaire operatoren van een Banachruimte.

8

De invloed van crossrelaxatie-verschijnselen op kernspin-roosterrelaxatie en dynamische kernpolarisatie kan uitstekend worden onderzocht in met zink verdund $NiSiF_6 \cdot 6H_2O$.

9

Men zegt vaak dat een getal 'verdwijnt' als men bedoelt dat het gelijk is aan nul. Dit is een betreurenswaardig anglicisme.

10

Het verdient aanbeveling een voorwaardelijke kiesdrempel in te stellen die het optreden van een nieuwe politieke partij niet belemmert en pas in werking treedt als zo'n partij voor de tweede maal aan dezelfde soort verkiezingen deelneemt.

P. J. M. Bongaarts