ON THE THEORY OF ALPHA DECAY

P. J. BRUSSAARD

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN
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Promotor: PROF. DR S. R. DE GROOT

STELLINGEN

I

Tegen de formulering van alfa-emissie (ter berekening van de hoekverdeling) met behulp van storingstheorie kunnen bezwaren worden aangevoerd.

L. C. Biedenharn en M. E. Rose, Rev. mod. Phys. **25** (1953) 729, IIA en IIIB.
M. E. Rose, Elementary theory of angular momentum, Wiley (1957), § 33 en § 34.
Hoofdstuk II, § 2 van dit proefschrift.

II

Uit metingen van de hoekverdeling van alfadeeltjes van gerichte kernen kunnen gegevens verkregen worden over de interne structuur van de emitterende kernen.

Hoofdstuk II van dit proefschrift.

III

Het is van belang dat bij metingen van de hoekverdeling van de alfadeeltjes van gerichte kernen, gelijktijdig de hoekverdeling van de gammastraling gemeten wordt.

Hoofdstuk II, § 5 van dit proefschrift.

IV

De geometrische betekenis van quantummechanische berekeningen met Racah-algebra kan vaak verduidelijkt worden door beschouwingen van de klassieke limiet.

Hoofdstuk III van dit proefschrift.

V

Ter bepaling van de energieniveaux en de golffuncties van de nucleonen in een sferoidale kern heeft Nilsson berekeningen uitgevoerd, uitgaande van een potentiaalput van sferoidale vorm. Het verdient aanbeveling na te gaan in hoeverre de zo berekende dichtheidsverdeling en de potentiaalput consistent zijn.

S. G. Nilsson, Dan. mat. fys. Medd. **29** (1955) no. 16.

STELLINGEN

I

Tegen de formulering van [illegible] bij berekening van de hoekverdeling met behulp van [illegible] bezwaren worden aangevoerd.

[illegible] en M. E. Rose, Rev. mod. Phys. 29 (1957) 727, 728 en 1325.
M. E. Rose, Elementary theory of angular momentum, Wiley (1957) 123 en 128.
Hoofdstuk II, § 2 van dit proefschrift.

II

Uit metingen van de hoekverdeling van alfadeeltjes van gerichte kernen kunnen gegevens verkregen worden over de interne structuur van de [illegible] kernen.

Hoofdstuk II van dit proefschrift.

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Het is van belang dat bij metingen van de hoekverdeling van de alfadeeltjes van gerichte kernen, [illegible] de hoekverdeling van de gammastraling gemeten wordt.

Hoofdstuk II, § 4 van dit proefschrift.

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S. G. Nilsson, Dan. mat. fys. Medd. 29 (1955) no. 16.

VI

De bewering van Edmonds, dat de functie

$$\mathscr{Y}_{lm}(\vartheta, \varphi) = \Sigma_{m_1 m_2} C^{lm}_{l_1 m_1 l_2 m_2} \; Y_{l_1 m_1}(\vartheta, \varphi) \; Y_{l_2 m_2}(\vartheta, \varphi)$$

niet de symmetrie-eigenschap

$$\mathscr{Y}^*_{lm}(\vartheta, \varphi) = (-1)^m \, \mathscr{Y}_{l-m}(\vartheta, \varphi)$$

bezit, indien de bolfuncties $Y_{lm}(\vartheta, \varphi)$ deze eigenschap wel bezitten, is onjuist.

A. R. Edmonds, Angular momentum in quantum mechanics, CERN 55–26, Genève (1955), § 2.6 en § 5.4.

VII

De wijze waarop Lindhard een relaxatietijd τ invoert in de Boltzmann-vergelijking, is aan bedenkingen onderhevig.

J. Lindhard, Dan. mat. fys. Medd. **28** (1954) no. 8, p. 18.

VIII

Het is mogelijk dat de hoge waarde die Taylor en Dash vinden voor de viscositeit van ^{4}He boven het λ-punt, veroorzaakt wordt door een systematische fout in de meetmethode.

R. D. Taylor en J. G. Dash, Phys. Rev. **106** (1957) 398.

IX

De bewering van Pupke, dat iedere $m \times n$-matrix, waarvan alle elementen van nul verschillen, te schrijven is als het product van een $m \times 1$-kolommatrix en een $1 \times n$-rijmatrix, is onjuist.

H. Pupke, Einführung in die Matrizenrechnung, Deutsche Verlag der Wissenschaften, Berlin (1953), p. 9.

X

Het is niet juist dat, zoals Reulos opmerkt, de Maxwell-Lorentzvergelijkingen tot een contradictie leiden bij de verklaring van de unipolaire inductiemachine.

R. Reulos, Archives des sciences physiques et naturelles (Genève) **10** (1957) 545.

XI

Uit metingen van de magnetische susceptibiliteit van Mn_3O_4 concluderen Sinha en Sinha dat het mangaan op de tetraederplaatsen covalent gebonden is. Deze gevolgtrekking is aanvechtbaar.

K. P. Sinha and A. P. B. Sinha, J. phys. Chem. **61** (1957) 758.
P. F. Bongers, Proefschrift, Leiden (1957).

XII

Een beperking van het aantal soorten belastingen in het Nederlandse belastingstelsel is gewenst.

VI

De bewering van Edmonds, dat de functie

$$\Psi_{lm}(\theta,\varphi) = \sum_{m_1 m_2} C^{l_1 l_2 l}_{m_1 m_2 m}\, Y_{l_1 m_1}(\theta,\varphi)\, Y_{l_2 m_2}(\theta,\varphi)$$

met de symmetrie-eigenschap

$$\Psi^*_{lm}(\theta,\varphi) = (-1)^m\, \Psi_{l\,-m}(\theta,\varphi)$$

bezit, indien de bolfuncties $Y_{lm}(\theta,\varphi)$ deze eigenschap ook bezitten, is onjuist.

A. R. Edmonds, Angular momentum in quantum mechanics, CERN 55-26, Genève (1955), § 2.5 en § 5.4.

VII

De wijze waarop Landshoff een permutatieindex ν invoert in de Boltzmann-vergelijking, is aan bedenkingen onderhevig.

J. Landshoff, Ten. ... Math. 23 (1954), p. 16.

VIII

Het is mogelijk dat de hoge waarde die Taylor en Dash vinden voor de viscositeit van ^4He boven het λ-punt, veroorzaakt wordt door een systematische fout in de meetmethode.

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De bewering van Papke, dat iedere $n \times n$-matrix, waarvan alle elementen van nul verschillen, te schrijven is als het product van een $n \times 1$-kolommatrix en een $1 \times n$-rijmatrix, is onjuist.

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X

Het is niet juist dat, zoals Rénier opmerkt, de Maxwell-Lorentz-vergelijkingen tot een contradictie leiden bij de verklaring van de unipolaire inductieverschijnselen.

R. Rénier, Archives des sciences physiques et naturelles (Genève) 10 (1957) 243.

XI

Uit metingen van de magnetische susceptibiliteit van Mn_3O_4 concluderen Sinha en Sinha dat het mangaan op de tetraederplaatsen covalent gebonden is. Deze gevolgtrekking is aanvechtbaar.

K. P. Sinha and A. P. B. Sinha, J. phys. Chem. 61 (1957) 758.
P. F. Bongers, Proefschrift, Leiden (1957).

XII

Een beperking van het aantal voorzitters ... in het Nederlandse telefoonnet is gewenst.

[illegible]

Op verzoek van de Faculteit der Wis- en Natuurkunde volgen hier enkele gegevens over mijn studie.

In 1948 legde ik het eindexamen gymnasium-β af aan het Stedelijk Gymnasium te Den Haag. In hetzelfde jaar begon ik mijn studie aan de Universiteit te Leiden en behaalde in februari 1952 het candidaatsexamen natuur- en wiskunde (*d*). Vervolgens studeerde ik voor het doctoraalexamen in het bijzonder bij Prof. Dr S. R. de Groot, Prof. Dr C. J. Gorter, Dr H. A. Tolhoek, Dr J. Korringa, Dr P. Mazur en Dr N. G. van Kampen. In maart 1955 legde ik het doctoraalexamen af met hoofdvak theoretische natuurkunde en bijvakken wiskunde en mechanica.

Inmiddels was ik sinds januari 1954 als medewerker van de "Stichting voor Fundamenteel Onderzoek der Materie" werkzaam op het Instituut-Lorentz voor theoretische natuurkunde, waar ik onder leiding van Dr H. A. Tolhoek onderzoek verrichtte op kernfysisch gebied. Dit onderzoek betrof voornamelijk verschillende aspecten van de theorie van de alfa-emissie (alfa-emissie en kernstraalbepaling, alfa-emissie van gerichte gedeformeerde atoomkernen, het verband tussen alfa-emissie en kernstructuur) en enkele mathematisch-fysische ontwikkelingen betreffende Clebsch-Gordancoefficienten, Racahcoefficienten en de coefficienten van de matrixrepresentatie van de rotatiegroep.

Van veel nut waren voor mij discussies met Prof. J. A. Wheeler, Prof. E. P. Wigner en de heer Chr. D. Hartogh op het Instituut-Lorentz en met Dr W. J. Huiskamp en de heer H. Postma op het Kamerlingh-Onnes Laboratorium.

De auteur is lid van de wetenschappelijke staf van de "Stichting voor Fundamenteel Onderzoek der Materie" (F.O.M.) die financieel wordt gesteund door de "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek" (Z.W.O.).

Dit proefschrift bevat de resultaten van onderzoek dat werd verricht in het kader van de werkgemeenschap "Kernfysica" van de "Stichting voor Fundamenteel Onderzoek der Materie".

CONTENTS

NOTE Formulae are numbered separately in each chapter. References to another chapter are given thus: III (2.1), III (3.1), III (A.2.4).

INTRODUCTION

The purpose of this thesis is the study of different aspects of the theory of alpha emission. In the first chapter the internal problem of the formation of the alpha particle from the nucleons in the nucleus is examined. The alpha particle is supposed to be formed out of four nucleons in definite shell model states. An approximate wave function for the system is then proposed, which leads to a boundary condition at the nuclear surface to be satisfied by the wave function for the alpha particle. From this boundary condition we are able to calculate absolute transition probabilities in alpha decay, thus relating the lifetime and the radius of the alpha emitting nucleus. In the second chapter of this thesis we consider the directional distribution of alpha radiation from oriented nuclei and give the general formulae for the directional distribution as a function of the degree of orientation of the ensemble of emitting nuclei. It is shown that one can deduce from the measurements on alpha directional distributions of oriented, spheroidally deformed nuclei whether the preferential emission of the alpha particles takes place from the poles or from the equator of the nucleus. This provides a possibility of obtaining independent experimental information concerning the preformation problem of alpha particles studied in the first chapter. In the classical limit, i.e. for "heavy" nuclei and large angular momenta, it is shown that the averaging procedures which are used for the derivation of the formulae for the directional distributions can be interpreted in a simple geometrical way. For this purpose some results of the next chapter are used. In the third chapter the classical limits (asymptotic expressions for large angular momenta) of Clebsch-Gordan coefficients, Racah coefficients and the coefficients of the matrix representation of the rotational group are investigated; this chapter may be read independently of the others.

The contents of this thesis are also published in Physica (Physica **23** (1957) 955; **24** (1958) 233; **24** (1958) 263). Some earlier work on alpha disintegration appeared in Physica **21** (1955) 449.

Chapter I

ON THE THEORY OF EMISSION OF ALPHA PARTICLES AS RELATED TO THE STRUCTURE OF THE NUCLEUS

Synopsis

The internal problem of the formation of the alpha particles emitted in alpha disintegration is considered. In this respect the mean free path of alpha particles in nuclear matter is important. A comparison is made with the optical model treatment of alpha particle scattering; the significance of the alpha particle potential well parameters is discussed. A short mean free path for the alpha particles in nuclear matter suggests that the alpha particle formation can be characterized approximately by a boundary condition on the nuclear surface, which further determines the alpha particle propagation in the external region. An approximate wave function representing a nucleus with shell structure emitting an alpha particle is proposed and provides an expression for this boundary condition. Such an expression can be used for the calculation of absolute as well as relative transition probabilities in alpha decay. The value for the radius of the alpha particle potential well, which is obtained in this way from the alpha decay probability is in good agreement with the radius of the potential well determined from alpha particle scattering. A summary is given of the experimental information which would be useful for a further test of the picture of alpha particle formation.

§ 1. *Introduction.* The theory of alpha disintegration may be divided into two parts:

(1) the *external problem* of propagation of the alpha particle once it has left the nuclear region,

(2) the *internal problem* concerning the formation of the alpha particle from the nucleons in the nucleus.

In this paper we shall be mainly concerned with the internal problem. The value of the original theory of alpha disintegration [1][2][3][4][5][6] given in 1928 consists mainly of a derivation of the lifetime-energy relation from the quantum mechanical barrier penetration formula. The situation concerning the internal problem has been unclear for a long time. Only recently a number of suggestions were made by Perlman, Ghiorso and Seaborg [7]) Rasmussen [8]), Thomas [9]), Griffing and Wheeler [10]) [11]), Bohr, Fröman and Mottelson [12]), Tolhoek and Brussaard [13]), Rasmussen and Segall [14]), Fröman [15]) and Mang [16]), which have a bearing on the topic of alpha particle formation and nuclear structure, to which this paper is also devoted. Furthermore it is useful to make some comparison of alpha disintegration and the scattering of energetic alpha particles by nuclei [17] [18].

In § 2 of this paper we give a discussion of some concepts used in the theory of alpha disintegration and also summarize some results of earlier papers. In § 3 we discuss the meaning of the data obtained by the analysis of alpha particle scattering. A proposal for an approximate solution of the internal problem (leading to a boundary condition for the alpha particle wave function at the nuclear surface) is given in § 4 and used for an estimate of the absolute transition probability for alpha disintegration. In § 5 a discussion is given of the experimental material, which would be useful for testing the ideas on the internal problem and making them more precise.

§ 2. *Discussion of the basic concepts of the theory of alpha disintegration.* The potential which the alpha particle experiences is no problem if the alpha particle is outside the range of the nuclear forces of the nucleus; it is then simply the electric field of the nucleus. However, it is by no means simple to decide to what extent the influence of the nucleus on an alpha particle within the nuclear region can be described by a nuclear potential well. In this respect it would be of importance to know whether an alpha particle can move as a definite entity within nuclear matter. Here we could introduce a mean free path l_α of the alpha particle indicating the distance over which it travels before dissociating. l_α should be compared with the nuclear radius $R(\approx 8 \times 10^{-13}$ cm for heavy nuclei) and the nuclear surface thickness s ($\approx 1.3 \times 10^{-13}$ cm; R and s are considered as parameters characterizing the nuclear matter distribution, s being chosen as the distance over which the density falls from 80% to 20% of the central density). One may also compare l_α with the radius of the alpha particle which has a value of $\beta_\alpha \approx 1.6 \times 10^{-13}$ cm [19]).

The older versions of the theory of alpha disintegration assume that the alpha particle exists as such before being emitted and is contained in the nuclear potential well, which serves as a box. This picture would be plausible if $l_\alpha \gg R$. In this case one may expect alpha particles to move as definite sub-units in the nucleus (cf. the alpha particle model of the nucleus). Even for smaller values of l_α the notion of a nuclear potential well for alpha particles could be useful. However, one needs no longer to expect alpha particles to exist as sub-units of the nucleus with any appreciable probability, so that this picture could be compatible with a shell model wave function for the nucleus (this picture was used and elaborated in an earlier paper [13])). If finally l_α becomes very small ($l_\alpha \lesssim s$), this has as a consequence that alpha particles cannot move as well defined units within nuclear matter and the notion of a nuclear potential well may have only a meaning for the nuclear surface region, but will no longer be useful for the inner region of the nucleus.

It is difficult to determine l_α. A theoretical determination would present a very complex problem and we do not attempt here to solve it. The experimental data, which may be used for a determination of l_α, are the scattering cross sections of alpha particles (of kinetic energies of about 20 to 40 MeV) by nuclei. These data have been analyzed in terms of an optical model by Igo and Thaler [17]) and by Cheston and Glassgold [18]). They assume a complex potential well of which the imaginary part characterizes the absorption

$$V_c(r) = V(r) + iW(r) = (V_0 + iW_0)/\{1 + \exp[(r - R)/d]\}. \qquad (2.1)$$

The absorption due to a complex potential $V + iW$ can be alternatively expressed by the mean free path l_α (defined as the distance over which the intensity of a wave decreases by a factor e), related to it according to [17])

$$l_\alpha^{-2} = (4M/\hbar^2)T_{int}[\{(W/T_{int})^2 + 1\}^{\frac{1}{2}} - 1], \qquad (2.2)$$

where M is the mass of the alpha particle and T_{int} represents the local kinetic energy: $T_{int}(r) = E - \{V(r) + V_{coul}(r)\}$. Here E is the kinetic energy at large distance and V_{coul} is the potential energy of the alpha particle in the Coulomb field of the nucleus. In this way a mean free path $l_\alpha = (1.0 \text{ to } 1.5) \times 10^{-13}$ cm is found for the value of r at which $V(r) = -20$ MeV for a scattered alpha particle of 22 MeV. This means that l_α is short in the surface region and that we have the case mentioned at the end of the last paragraph, in which the central part of the potential is scarcely observable (cf. the discussion in § 3; we do not give any number for l_α for a value of r such that $r \ll R$, as such a number probably has no real significance). It should be realized that l_α (as well as $V(r)$) will in general depend on the energy of the alpha particle, so that l_α will probably have different values for alpha particle scattering and alpha disintegration. In this respect it would be interesting if an investigation were made to detect how l_α changes with the energy of the scattered alpha particles. However, we shall tentatively assume that l_α has a small value also for alpha disintegration.

We should like to point here to the dubious character of a way of reasoning which is sometimes presented in this context (assuming a value of l_α sufficiently large to speak of a nuclear potential well throughout the nucleus and thinking of one alpha particle in the well in the initial state): (a) the alpha particle is a boson being composed out of 4 nucleons, (b) the nuclear potential well is shallow (about 20 MeV; i.e. about the height of the Coulomb barrier at the nuclear surface minus the kinetic energy of the alpha particle at large distance from the nucleus) since the alpha particle should be in the lowest possible state in the potential well before emission. Otherwise a lower nuclear state could be reached by a transition of the alpha particle. This transition would not be forbidden because the Pauli principle does not act for a boson. This reasoning is subject to the following criticism: (a) the Ehrenfest-Oppenheimer theorem [20]) on the boson or fermion character of composite particles is subject to a limitation; it can no longer be applied if the composite particles penetrate each other or if they are penetrated by the kind of particles from which they are formed, (b) therefore an alpha particle within a nucleus cannot be considered as an independent boson; it should be analyzed in terms of nucleon states in order to see how the Pauli principle acts on it and there is no need for the alpha particle well to be as shallow as about 20 MeV.

If it is assumed that (a) the nucleons of the nucleus are (at least predominantly) in individual particle shell model states and (b) that no states, representing alpha particles moving within the nucleus, are admixed to the nuclear wave function with an appreciable probability, the problem arises whether the emitted alpha particle can be said to be formed from nucleons in definite orbits. If during alpha emission the nucleons disappear from four shell model states, one can say that the alpha particle was formed from the four nucleons in these states. We might then try to obtain a description of the alpha emission by taking only the wave functions of the nucleons from which the alpha particle is formed and by assuming further that the influence of the other nucleons is simply described by potential wells for the transforming nucleons and the alpha particle respectively. As far as experimental evidence on the shell model configurations for the initial and final states of an alpha transition is available, it does not disagree with this picture. The fact that alpha transitions often lead to the ground state or a low lying excited state of the daughter nucleus shows that the shell model states involved in the alpha emission are the states with the highest energies of the nucleons.

The preceding picture can be elaborated as well for spherical nuclei as for nuclei with an intrinsic spheroidal deformation. In order to have a definite picture we shall go into some more detail for the case of the spheroidally deformed nuclei. The projection Ω of the total angular momentum of a nucleon on the nuclear symmetry axis is a good quantum number, characerizing the independent particle motion. The eigenstates of the nucleons cf. Nilsson [21]), Moszkowski [22]), Gottfried [23])) are doubly degenerate, having the same energy for $\pm\,\Omega$. By K, we denote, as is customary, the component of the total angular momentum of the nucleus along the nuclear symmetry axis. A schematic representation of the nucleon states is given in fig. 1. On the basis of the preceding picture one may conjecture (cf. [12])) that alpha particle formation will be favored, if the two neutrons involved in the alpha emission have quantum numbers $\pm\,\Omega_n$ and the two protons involved have quantum numbers $\pm\,\Omega_p$: the wave functions of such nucleon pairs have the largest possible overlap, which seems favorable for alpha particle formation (this is formulated with more precision in § 4). Such favored alpha emission would be represented in the schematic fig. 1 by alpha particle formation from the nucleon states 1, 2, 3, 4; the alpha particle which is formed would be characterized by $m' = 0$ (m' being the component of the angular momentum of the alpha particle along the nuclear symmetry axis) on or just outside the nuclear surface and the alpha transition by the selection rules (1) $\Delta K = 0$ and (2) no change of parity. If the propagation of the alpha particle in the external region can be determined from a boundary condition on the nuclear surface, the boundary condition can be chosen to be real in this case of favored alpha emission (cf. (4.17)). Furthermore one would also

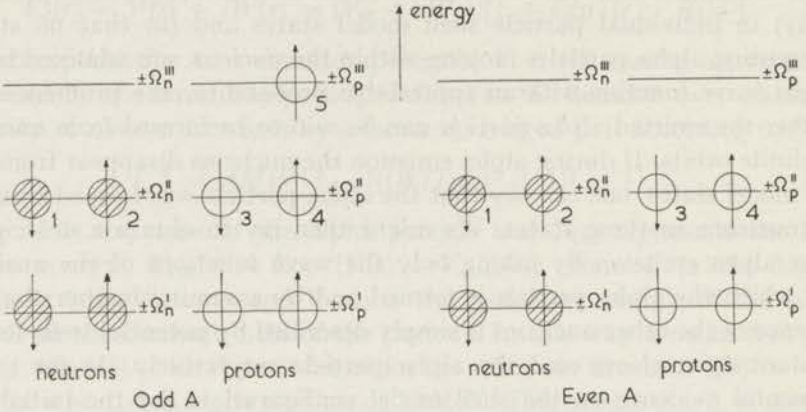

Fig. 1. *Schematic representation of alpha particle formation* from shell model states of nucleons (for spheroidal nuclei). These states are characterized by a quantum number Ω. It is supposed that favored alpha emission occurs if the alpha particle is formed from two sets of paired nucleons, such as 1, 2, 3, 4. For odd-A nuclei an unfavored transition occurs if the alpha particle is formed from the nucleons 1, 2, 3, 5; such a transition may lead to the ground state of the daughter nucleus.

expect that emission of an alpha particle from an odd-A nucleus formed from the nucleons 1, 2, 3, 5 would be possible (and it may very well present the most energetic alpha transition to the ground state of the nucleus); however, a smaller intrinsic probability should be expected for such alpha emission since the states 3 and 5 will not overlap too well. Such alpha emission may be called unfavored (in such transitions K, as well as the parity may change). Finally it can be remarked that the favored alpha emission from an even-even and an odd-A nucleus should be very analogous, if in both cases the alpha particle is formed from the same nucleon states (say, states 1, 2, 3, 4 in fig. 1 as well for odd A as for even A).

In order to test this picture experimentally one needs well analyzed alpha disintegration schemes with spin and parity assignments. Furthermore one can obtain a sort of intrinsic alpha particle formation probability [12] [15] [24] in the following way: define as $P_0(Z, E)$ the smoothed out Geiger-Nuttal law for even-even nuclei for ground state to ground state transitions (supposedly favored); this function is of the form

$$^{10}\log P_0(Z, E) = C(Z) - D(Z)E^{-\frac{1}{2}}. \tag{2.3}$$

According to Bohr, Fröman and Mottelson [12] [15] one may define an F-value for any alpha emission of odd- or even-A nuclei, as

$$F = P_\alpha / P_0(Z, E), \tag{2.4}$$

where P_α is the measured transition probability and $P_0(Z, E)$ has the value according to (2.3). One should expect that the so defined F-value is a sort of intrinsic formation probability, which should be largest for the favored alpha transitions. Thus one should be able to test the above picture by checking in alpha disintegration schemes (with spin and parity assignments) whether high F-values correspond to the selection rules: $\Delta K = 0$, no change of parity. Although there are some cases in which this is confirmed [15] (p. 51), one should like to have more extensive experimental information.

§ 3. *The optical model interpretation of alpha particle scattering.* Although various models have been proposed [17] [18] [25] [26] [27] [28] [29] [30] for the description of alpha particle scattering, the optical model (description by a complex potential well) seems most adequate. Alpha particle scattering data were analyzed with the aid of the optical model by Igo and Thaler [17] and by Cheston and Glassgold [18]. These authors noticed that the elastic scattering cross sections are insensitive to large changes in the inner part of the potential well. The same data for the scattering of 22 MeV alpha particles by Ag could be fitted about equally well [18] with the parameters (cf. (2.1)): $V_0 = -50$ MeV; $W_0 = -20$ MeV; $R = 7.5 \times 10^{-13}$ cm; $d = 0.60 \times 10^{-13}$ cm and $V_0 = -150$ MeV; $W_0 = -20$ MeV; $R = 7.09 \times 10^{-13}$ cm; $d = 0.60 \times \times 10^{-13}$ cm. The inner parts of the potential wells are very different for both sets of parameters, although the outward tails do not differ very much. This can be understood as a consequence of the strong absorption by the imaginary part of the potential, which prevents the alpha particle waves from reaching the central part of the nucleus with any appreciable amplitude.

This argument for the insensitivity of the scattering cross sections for the inner part of the potential well can easily be brought into a more quantitative form, if the shape of the potential allows the application of the WKB-method to the Schrödinger equation. We show this in the following way. In the optical model the wave function in the nucleus must be a solution of the Schrödinger equation with a complex potential energy

$$-[\hbar^2/(2M)]\,\Delta\psi + \{V_c(r) - E\}\,\psi = 0, \tag{3.1}$$

where

$$V_c(r) = V(r) + iW(r), \tag{3.2}$$

$V(r)$ and $W(r)$ being real functions. If we substitute

$$\psi = \textstyle\sum_{lm} (1/r)\, u_{lm}(r)\, Y_{lm}(\vartheta, \varphi), \tag{3.3}$$

we find for the radial part $u_{lm}(r)$

$$- [\hbar^2/(2M)](\mathrm{d}^2/\mathrm{d}r^2)\, u_{lm}(r) + \\ + \{V(r) + iW(r) - E + \hbar^2 l(l+1)/(2Mr^2)\}\, u_{lm}(r) = 0. \tag{3.4}$$

Also in this case of a complex potential energy we may use the WKB-solution just as in the real case (cf., e.g., Furry [31]). However, the WKB-connection formulae (at the turning points of the wave function) may be different [31] [32], but these formulae are not needed here since we are interested only in the insensitivity of the solution for a change in the real part of the inner core of the potential well. The fact that the potential is complex has the consequence that no rigorous distinction between exponential and oscillatory region exists (the wave number generally being complex, neither real nor pure imaginary). However, we shall suppose that W is not too large in comparison with V and define the exponential and oscillatory regions according to the sign of the real part of the expression in parentheses $\{\}$ in (3.5).

The WKB-solution for $u_{lm}(r)$ deduced from (3.4) has the following form in the oscillatory region (and a similar expression in the exponential region)

$$v_{lm}(r) = \frac{\exp\,[i \int^r \{E - V(r) - \hbar^2 l(l+1)/(2Mr^2) - iW(r)\}^{\frac{1}{2}}\, \mathrm{d}r]}{\{E - V(r) - \hbar^2 l(l+1)/(2Mr^2) - iW(r)\}^{\frac{1}{4}}}. \tag{3.5}$$

(3.5) implies that we have two (oscillatory) solutions, one $v^{I}_{lm}(r)$ with decreasing amplitude for decreasing values of r and the other one $v^{II}_{lm}(r)$ with increasing amplitude for decreasing values of r. (cf. fig. 2 for a qualitative comparison of the behavior of the wave functions in some cases). The linear superposition of these two solutions, which represents the wave function in the case of the scattering of alpha particles, is determined by the boundary condition at the origin, namely

$$u_{lm}(r) = 0 \text{ for } r = 0. \tag{3.6}$$

If the absorption is sufficiently strong, the condition (3.6) implies that the solution $v^{II}_{lm}(r)$ will be practically absent. Therefore the behavior of the solution $u_{lm}(r) \approx v^{I}_{lm}(r)$ in the neighborhood of the nuclear surface does not change if the inner part of the potential well is altered. The behavior of $u_{lm}(r)$ for $r > R$ is fixed if the boundary conditions on the nuclear surface are given. The asymptotic behavior of $u_{lm}(r)$ for $r \to \infty$ determines the scattering phases and hence the scattering cross sections, which turn out to be insensitive to changes of the inner part of the potential if there is strong absorption. This confirms the conjecture formulated in the beginning of this section on a somewhat more intuitive basis.

In this way it is understandable that different potential wells (with strong absorption) can all give reasonable fits to the scattering cross sections, if they are about the same in the outer region, even if they differ much in the inner part of the potential well. This inner part of the well is a quantity which seems to be nearly "unobservable".

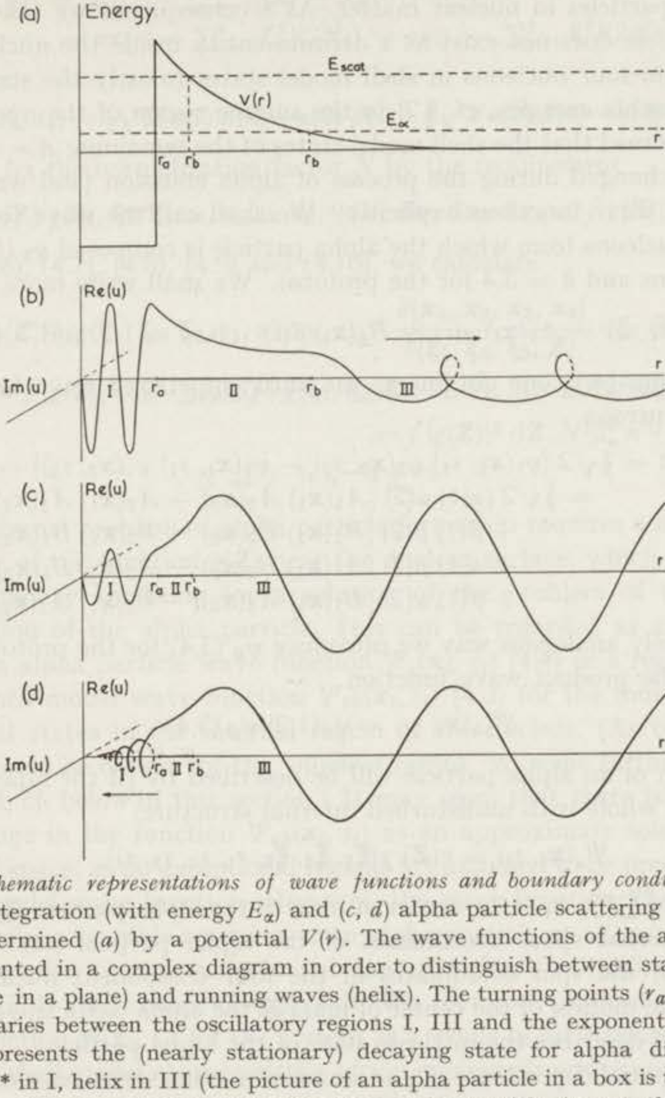

Fig. 2. *Schematic representations of wave functions and boundary conditions:* (*b*) for alpha disintegration (with energy E_α) and (*c, d*) alpha particle scattering (with energy E_{scat}), determined (*a*) by a potential $V(r)$. The wave functions of the alpha particle are represented in a complex diagram in order to distinguish between standing waves (sine curve in a plane) and running waves (helix). The turning points (r_a, r_b, r_b') form the boundaries between the oscillatory regions I, III and the exponential region II. Fig. 2*b* represents the (nearly stationary) decaying state for alpha disintegration: sine curve * in I, helix in III (the picture of an alpha particle in a box is taken for this figure). Fig. 2*c* represents the scattering of an alpha particle, in case there is no absorption within the nucleus: sine curves in I and III; boundary condition $u = 0$ for $r = 0$. Fig. 2*d* represents such a scattering for strong absorption within the nuclear potential well: sine curve * in III, helix for ingoing wave in I; the boundary condition $u = 0$ for $r = 0$ can be replaced for strong absorption by the condition: ingoing wave only inside the nuclear surface. (* sine curve, situated nearly but not entirely in one plane).

§ 4. *Theory of alpha emission based on boundary conditions at the nuclear surface.* In this section we formulate a proposal for fitting the alpha particle wave function in the exterior region to a shell model wave function in the nucleus. This proposal is based on the assumption of a short mean free path for alpha particles in nuclear matter. As a consequence we take that the alpha particle does not exist as a definite entity inside the nucleus but is formed from four nucleons in shell model states (mostly the states of the highest possible energies, cf. § 2) in the surface region of the nucleus. It is further assumed that the shell model states of the remaining $A-4$ nucleons remain unchanged during the process of alpha emission (and we shall not write their wave functions explicitly). We shall call the wave functions of the four nucleons from which the alpha particle is composed ψ_k ($k = 1,2$ for the neutrons and $k = 3,4$ for the protons). We shall write more explicitly

$$\psi_k(\boldsymbol{x}_i, s_i) = A_k(\boldsymbol{x}_i)\,\alpha(i) + B_k(\boldsymbol{x}_i)\,\beta(i) \quad (i, k = 1,2 \text{ and } 3,4). \tag{4.1}$$

Starting from (4.1) one obtains as the antisymmetrized wave function for the two neutrons

$$\begin{aligned}\psi_n(1,2) &= \tfrac{1}{2}\sqrt{2}\,\{\psi_1(\boldsymbol{x}_1, s_1)\,\psi_2(\boldsymbol{x}_2, s_2) - \psi_2(\boldsymbol{x}_1, s_1)\,\psi_1(\boldsymbol{x}_2, s_2)\} = \\ &= \tfrac{1}{2}\sqrt{2}\,\{\alpha(1)\,\alpha(2)\,[A_1(\boldsymbol{x}_1)\,A_2(\boldsymbol{x}_2) - A_2(\boldsymbol{x}_1)\,A_1(\boldsymbol{x}_2)] + \\ &\quad + \beta(1)\,\beta(2)\,[B_1(\boldsymbol{x}_1)\,B_2(\boldsymbol{x}_2) - B_2(\boldsymbol{x}_1)\,B_1(\boldsymbol{x}_2)] + \\ &\quad + \alpha(1)\,\beta(2)\,[A_1(\boldsymbol{x}_1)\,B_2(\boldsymbol{x}_2) - A_2(\boldsymbol{x}_1)\,B_1(\boldsymbol{x}_2)] + \\ &\quad + \beta(1)\,\alpha(2)\,[B_1(\boldsymbol{x}_1)\,A_2(\boldsymbol{x}_2) - B_2(\boldsymbol{x}_1)\,A_1(\boldsymbol{x}_2)]\}.\end{aligned} \tag{4.2}$$

In an entirely analogous way we introduce ψ_p (3,4) for the protons and we write for the product wave function

$$\Psi_{sh}(\boldsymbol{x}_i, s_i) = \psi_n(1,2)\,\psi_p\,(3,4). \tag{4.3}$$

The motion of an alpha particle will be described by (if the alpha particle moves as a whole with undisturbed internal structure)

$$\Psi_\alpha(\boldsymbol{x}_i, s_i) = g(\boldsymbol{Z})\,\chi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, s_1, s_2, s_3, s_4), \tag{4.4}$$

where $\boldsymbol{Z} = \frac{1}{4}\sum_{i=1}^{4} \boldsymbol{x}_i$ is the center of mass of the alpha particle, $\boldsymbol{\xi}_j\,(j=1,2,3)$ are the internal space coordinates of the alpha particle (see below), s_i ($i = 1,2,3,4$) the spin coordinates of the four component nucleons, $g(\boldsymbol{Z})$ describes the motion of the center of mass of the alpha particle, $\chi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, s_1, s_2, s_3, s_4)$ describes the intrinsic state of the alpha particle.

For the spatial dependence of χ a Gaussian function seems reasonable [16][19] (for other choices cf. [33]). We introduce the relative coordinates

$$\boldsymbol{\zeta}_i = \boldsymbol{x}_i - \boldsymbol{Z} \quad (i = 1,2,3,4) \tag{4.5}$$

and the internal coordinates

$$\begin{aligned}&\boldsymbol{\xi}_1 = \tfrac{1}{2}\sqrt{2}(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) = \tfrac{1}{2}\sqrt{2}(\boldsymbol{x}_1 - \boldsymbol{x}_2),\ \boldsymbol{\xi}_2 = \tfrac{1}{2}\sqrt{2}(\boldsymbol{\zeta}_3 - \boldsymbol{\zeta}_4) = \tfrac{1}{2}\sqrt{2}(\boldsymbol{x}_3 - \boldsymbol{x}_4), \\ &\boldsymbol{\xi}_3 = \tfrac{1}{2}(\boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2 - \boldsymbol{\zeta}_3 - \boldsymbol{\zeta}_4) = \tfrac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 - \boldsymbol{x}_4).\end{aligned} \tag{4.6}$$

We shall assume for χ an expression of the form

$$\chi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, s_1, s_2, s_3, s_4) = N\,\Xi\,(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)\,\Sigma\,(s_1, s_2, s_3, s_4), \tag{4.7}$$

with

$$\Xi(\xi_1, \xi_2, \xi_3) = \exp\,[-\,\textstyle\sum_{i,k=1}^{4} (\boldsymbol{x}_i - \boldsymbol{x}_k)^2/(16\beta_\alpha^2)] =$$
$$= \exp\,[-\,\textstyle\sum_{i=1}^{4} \zeta_i^2/(2\beta_\alpha^2)] = \exp\,[-\,\sum_{j=1}^{3} \xi_j^2/(2\beta_\alpha^2)] \tag{4.8}$$

and

$$\Sigma(s_1, s_2, s_3, s_4) = \tfrac{1}{2}\sqrt{2}[\alpha(1)\,\beta(2)-\alpha(2)\,\beta(1)]\,.\,\tfrac{1}{2}\sqrt{2}[\alpha(3)\,\beta(4)-\alpha(4)\,\beta(3)]. \tag{4.9}$$

We shall fix the normalization factor N by the requirement

$$\textstyle\sum_{spin} \int |g(\boldsymbol{Z})\,\chi(\xi_1, \xi_2, \xi_3, s_1, s_2, s_3, s_4)|^2\, \mathrm{d}\boldsymbol{x}_1\, \mathrm{d}\boldsymbol{x}_2\, \mathrm{d}\boldsymbol{x}_3\, \mathrm{d}\boldsymbol{x}_4 = \int |g(\boldsymbol{Z})|^2\, \mathrm{d}\boldsymbol{Z}. \tag{4.10}$$

Using (4.6), (4.7), (4.8), (4.9) and (4.10), we calculate

$$\textstyle\sum_{spin} \int |g(\boldsymbol{Z})|^2\, |\chi(\xi_1, \xi_2, \xi_3, s_1, s_2, s_3, s_4)|^2\, \dfrac{\partial(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)}{\partial(\xi_1, \xi_2, \xi_3, \boldsymbol{Z})}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3\, \mathrm{d}\boldsymbol{Z} =$$
$$= \int |g(\boldsymbol{Z})|^2\, \mathrm{d}\boldsymbol{Z}\,.\,\textstyle\sum_{spin} \int |\,\chi(\xi_1, \xi_2, \xi_3, s_1, s_2, s_3, s_4)|^2\, 2^3\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3 =$$
$$= \int |g(\boldsymbol{Z})|^2\, \mathrm{d}\boldsymbol{Z}\,.\,N^2 \beta_\alpha^9\, \pi^{9/2}\, 2^3 \tag{4.11}$$

or

$$N = 2^{-3/2}\, \pi^{-9/4}\, \beta_\alpha^{-9/2}. \tag{4.12}$$

The internal problem of alpha particle formation requires a derivation of the value of the function $g(\boldsymbol{Z})$ near the nuclear surface, which can be used as a boundary condition for a solution of the problem of the external propagation of the alpha particle. This can be regarded as a problem of fitting an alpha particle wave function $\Psi_\alpha(\boldsymbol{x}_i, s_i)$ (4.4) in a region $|\boldsymbol{Z}| > R$ to the shell model wave function $\Psi_{sh}(\boldsymbol{x}_i, s_i)$ (4.3) for the four nucleons in individual states in the internal region of the nucleus. (As to the value, which should be chosen for the "nuclear radius" R, some further discussion is needed, cf. below in this section.) It may seem that there is no need for any change in the function $\Psi_{sh}(\boldsymbol{x}_i, s_i)$ as an approximate solution in any region of space, as no exceptional regions are assumed to be present for such a shell model wave function. However, the wave function $\Psi_{sh}(\boldsymbol{x}_i, s_i)$ does not describe a possible emission of an alpha particle at all, as all functions describing the spatial behavior of the nucleons $i = 1,2,3,4$ are simply exponentially decreasing for $|\boldsymbol{x}_i| > R$. We therefore propose the following wave function $\Psi(\boldsymbol{x}_i, s_i)$ for the system as a whole in an attempt to describe both the shell model characteristics of the nuclear structure and the feature of alpha emission

$$\Psi(\boldsymbol{x}_i, s_i) = \Psi_{sh}(\boldsymbol{x}_i, s_i), \text{ if } |\boldsymbol{Z}| < R, \tag{a}$$
$$\Psi(\boldsymbol{x}_i, s_i) = [1 - \Xi(\xi_1, \xi_2, \xi_3)\, \mathrm{P}_{sing}]\, \Psi_{sh}(\boldsymbol{x}_i, s_i) + \tag{4.13}$$
$$+ \Psi_\alpha(\boldsymbol{x}_i, s_i), \text{ if } |\boldsymbol{Z}| > R, \tag{b}$$

where $\Psi_\alpha(\boldsymbol{x}_i, s_i)$ (4.4) is fixed in such a way that

$$[\Psi_\alpha(\boldsymbol{x}_i, s_i)]_{x_1=x_2=x_3=x_4=R} = [\Psi_{sh}(\boldsymbol{x}_i, s_i)]_{x_1=x_2=x_3=x_4=R}. \tag{4.14}$$

($\boldsymbol{R}$ may be a vector of any direction, but of magnitude R.) P_{sing} is the projection operator which singles out the singlet states for neutrons and protons

$$P_{sing} = (1/16)[1 - \boldsymbol{\sigma}(1).\boldsymbol{\sigma}(2)][1 - \boldsymbol{\sigma}(3).\boldsymbol{\sigma}(4)]. \qquad (4.15)$$

If we evaluate the right-hand member of (4.14) according to (4.2) and (4.3) we obtain

$$[\Psi_{sh}(\boldsymbol{x}_i, s_i)]_{\boldsymbol{x}_1=\boldsymbol{x}_2=\boldsymbol{x}_3=\boldsymbol{x}_4=\boldsymbol{R}} = [A_1(\boldsymbol{R})B_2(\boldsymbol{R}) - A_2(\boldsymbol{R})B_1(\boldsymbol{R})] \,. \\ .\, [A_3(\boldsymbol{R})B_4(\boldsymbol{R}) - A_4(\boldsymbol{R})B_3(\boldsymbol{R})]\, \Sigma(s_1, s_2, s_3, s_4). \qquad (4.16)$$

Hence the condition (4.14) reduces, according to (4.4), (4.7), (4.12) and (4.16), to

$$g(\boldsymbol{Z} = \boldsymbol{R}) = 2^{3/2}\pi^{9/4}\beta_\alpha^{9/2} \,. \\ .\, [A_1(\boldsymbol{R})B_2(\boldsymbol{R}) - A_2(\boldsymbol{R})\, B_1(\boldsymbol{R})][A_3(\boldsymbol{R})\, B_4(\boldsymbol{R}) - A_4(\boldsymbol{R})B_3(\boldsymbol{R})], \qquad (4.17)$$

providing a boundary condition for $g(\boldsymbol{Z})$ at any point (R, ϑ, φ) of the nuclear surface. We further require for $g(\boldsymbol{Z})$ that this function should represent outgoing alpha particles in its asymptotic behavior for $r \to \infty$. In contrast to the functions $\psi_k(\boldsymbol{x}_i, s_i)$, which behave exponentially for $r \to \infty$, $g(\boldsymbol{Z})$ becomes oscillatory again for $r \to \infty$ if alpha emission is possible. Equations (4.13) ... (4.17) were formulated for a spherical nuclear surface; it is quite easy to reformulate them for a spheroidal nuclear surface: the condition $|\boldsymbol{Z}| < R$ or $|\boldsymbol{Z}| > R$ should then be replaced by the condition of being interior or exterior to the nuclear surface, etc.

The wave function $\Psi(\boldsymbol{x}_i, s_i)$ according to (4.13) is written down on a somewhat intuitive basis. However, we note the following merits of this function:

(1) inside the surface $|\boldsymbol{Z}| = R$ the wave function is simply a shell model function,

(2) outside the surface $|\boldsymbol{Z}| = R$ the same holds, unless the four nucleons are in each others neighborhood (within a sphere of a radius of about β_α) and the two protons as well as the two neutrons are in singlet states,

(3) if the four nucleons are together in this way for $|\boldsymbol{Z}| >$ R, they are travelling together as an alpha particle, with a center-of-mass motion $g(\boldsymbol{Z})$ describing the possibility of emission,

(4) according to (4.13) and (4.14) the functions in the regions $|\boldsymbol{Z}| < R$ and $|\boldsymbol{Z}| > R$ fit continuously in points where not all nucleons are close together, as well as in the case that all four nucleons are in the same point of the surface $|\boldsymbol{x}| = R$. Also it seems very plausible that the functions fit almost continuously if the nucleons are close to the same point $\boldsymbol{R}$ (within a distance of about β_α), at least if the function in the right-hand member (product of $A_i(\boldsymbol{R})$ and $B_k(\boldsymbol{R})$) does not vary much over a distance β_α,

(5) in the preceding equations no special parameters are left for fitting

the derivatives of the wave function at the surface $|\boldsymbol{Z}| = R$. However, one might expect that the function $g(\boldsymbol{Z})$ decreases exponentially for increasing $|\boldsymbol{Z}|$ in a way, which is not too different from the variation of the right-hand member of (4.17), when calculated for increasing values of $|\boldsymbol{R}|$. Hence one can hope that the requirement of fitting the derivatives will change the boundary condition (4.17) only slightly. Furthermore one should then expect that the final result is not too sensitive to the place of the surface $|\boldsymbol{Z}| = R$, where the different parts of the wave function are joined.

From the preceding considerations it is seen that the proposed wave function represents correctly a number of features, which should be expected for the given physical situation. The result (4.17) for the boundary condition for $g(\boldsymbol{Z})$ further has the advantage that it is sufficiently simple, so that detailed calculations can be carried out and be compared with experiment. An alternative to (4.17) would be to put (the integration should be performed over 3 of the 4 space coordinates, and includes also summation over all spin variables)

$$g(\boldsymbol{Z} = \boldsymbol{R}) \propto \int [\Psi^*_{sh}(\boldsymbol{x}_i, s_i)\, \chi(\boldsymbol{x}_i, s_i)]_{\boldsymbol{Z}=\boldsymbol{R}}\, \mathrm{d}\mathfrak{S}, \tag{4.18}$$

which leads back to (4.17) if the functions $A_k(\boldsymbol{x}_i)$, $B_k(\boldsymbol{x}_i)$ are slowly varying over a distance β_α, but which becomes different from (4.17) if these functions vary more rapidly. In general a detailed calculation of (4.18) will be very difficult (however, it is feasible for harmonic oscillator wave functions, cf. [16]).

An adequate general framework for formulating the theory of alpha disintegration is also provided by the R-matrix formalism of Wigner and collaborators [34] [35] and was developed to some extent by Thomas [9]. However, it does not eliminate the necessity of making more specific physical assumptions for the wave function in the way explained above. The results of such assumptions can of course be described within this general formal theory.

The boundary condition (4.17) allows the calculation of absolute transition probabilities for alpha disintegration, as well as of relative transition probabilities in alpha decay structure *). For a first estimate of the absolute transition probability one may assume that the functions $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ are constant throughout the nuclear volume (hence have a value of the order $(\frac{4}{3}\pi R^3)^{-\frac{1}{2}}$). From the absolute value of the boundary condition, obtained in this way and the experimental data a radius of the nuclear potential well for alpha particles can be calculated. Using the lifetime and energy of ^{214}Po one finds in this way for the radius at which the nuclear potential has a depth of 20 MeV the value of $(9.4 \pm 0.1) \times 10^{-13}$ cm for this nucleus (we

*) The value of $g(\boldsymbol{Z} = \boldsymbol{R})$ according to (4.17) has in common with an earlier analysis [13] (for larger values of l_α) that the formation probability of an alpha particle is very small, differing roughly by a factor $(\beta_\alpha/R)^9$ from the old model with one alpha particle in a box.

calculated this value using methods of calculation of [13]) § 6 and assuming the same value for the surface thickness s_p of the potential well as found for the experimental potential well for alpha particle scattering according to [17]) and [18]); a value $\beta_\alpha = 1.6 \times 10^{-13}$ cm was used [19])). If this radius (at a depth of 20 MeV) is calculated for the nuclear potential wells found for alpha particle scattering [17]) [18]) a value of $(9.40 \pm 0.05) \times 10^{-13}$ cm is found (when the measured data are extrapolated in a plausible way to ^{214}Po). The close agreement between both radii supports as well the theoretical analysis based on the boundary condition (4.17) as the optical model analysis of alpha particle scattering (if one assumes that the real part of the alpha particle potential well remains about the same for external alpha particle energies from 5 to 40 MeV).

The preceding proposals (4.13) and (4.14) for the wave function of the complete system may be changed somewhat so that they become more general and contain more parameters which might be fitted. The wave function which can be taken as a generalization of (4.13) is

$$\Psi(\boldsymbol{x}_i, s_i) = [1 - \eta(\boldsymbol{Z})\, \Xi\, (\xi_1, \xi_2, \xi_3)\, \mathrm{P}_{sing}]\, \Psi_{sh}(\boldsymbol{x}_i, s_i) + \eta(\boldsymbol{Z})\, \Psi_\alpha(\boldsymbol{x}_i, s_i). \quad (4.19)$$

(4.13) can be considered as the special case of (4.19), obtained if $\eta(\boldsymbol{Z})$ is the step function

$$\eta(\boldsymbol{Z}) = \begin{cases} 0, & \text{if } |\boldsymbol{Z}| < R, \\ 1, & \text{if } |\boldsymbol{Z}| > R. \end{cases} \quad (4.20)$$

Other functions might be considered as well for $\eta(\boldsymbol{Z})$, e.g. other monotonous functions rising from 0 to 1, rounded off step functions etc.. The advantages of using such functions for $\eta(\boldsymbol{Z})$ could be that the transition from shell model wave function $\Psi_{sh}(\boldsymbol{x}_i, s_i)$ to alpha particle wave function is not located at a surface $|\boldsymbol{Z}| = R$ but occurs in a certain transition region, for which a depth may be chosen corresponding to the mean free path l_α. In this way (4.19) might be appropriate to represent even both extreme cases, taking (4.20) for $\eta(\boldsymbol{Z})$ in case of small l_α ($\to 0$), and taking a value for $\eta (0 < \eta < 1)$ which is constant throughout all space for large l_α ($\to \infty$) [13]). One would expect (for $l_\alpha \ll R$) that the region where both parts of (4.19) must be fitted, should be situated near the most outward (radial) maxima for the nucleon states of the shell model wave function.

Other possibilities for generalizing (4.13) are: (a) introduction of a dynamical correlation amongst the nucleons in the shell model wave function (favoring alpha "clustering") by writing $[1 + a\, \Xi\, (\xi_1, \xi_2, \xi_3)\, \mathrm{P}_{sing}]\, \Psi_{sh}(\boldsymbol{x}_i, s_i)$ instead of $\Psi_{sh}(\boldsymbol{x}_i, s_i)$ (a may be taken as a constant); (b) taking the radius β_α of the alpha particle dependent on the center-of-mass coordinate:

$$\beta_\alpha = \beta_\alpha(\boldsymbol{Z}).$$

The various forms which are proposed, can be considered as different trial functions for the problem of alpha disintegration, which is a complex many body problem. Further investigations will also need a better understanding of the physical dynamical correlations of nucleons which may be implicitly contained but not explicitly expressed in the formal shell model solutions for the nucleus. Much work remains to be done for deciding which functions provide the best approximate solution of the Hamiltonian for the many nucleon problem. However, we expect (4.14) to be at least sufficiently realistic to provide reasonably accurate values for the radius of the alpha particle potential well.

§ 5. *Summary of experimental methods which may serve as a test for the picture of alpha particle formation.* It is clear from the developments in § 2 and § 4 that the problem of making the picture of alpha disintegration more precise is a very complex one and one certainly cannot find a solution from first principles without several approximations. This applies particularly to the problem of the mean free path of an alpha particle in nuclear matter and to the problem of joining the exterior alpha particle wave function to the internal shell model wave function. It would therefore be very valuable to try to test the picture of alpha emission explained in § 2 and § 4 by means of experiments. We want to summarize in this section what experimental data can be used for this purpose:

(*a*) Relative intensities in the alpha emission spectrum of spherical nuclei. If one knows the shell model states from which the alpha particle is formed, the intensities with which alpha particles are emitted can be calculated starting from a boundary condition on the nuclear surface, (4.17) or (4.18), derived from the shell model states of the nucleons. For an experimental test of such calculations, one will consider in particular the relative intensities of different alpha transitions (to different final states where the nucleons are in given states of a spherical potential well) and the relative intensities of the alpha groups with different orbital momenta l. The absolute transition probability should be considered separately, as it is very sensitive to the radius and shape of the alpha particle nuclear potential well (cf. (*e*)). Interesting calculations on this topic were made by Mang [16]) for ^{211}Po and ^{212}Po with the aid of oscillator wave functions for the nucleon states, and starting from boundary conditions similar to (4.18).

(*b*) Relative intensities in the alpha emission spectrum of spheroidally deformed nuclei. Similar calculations as for spherical nuclei could be made here by starting from a boundary condition (4.17) or (4.18). However, one should take here the nucleon states of a spheroidal potential well [21]) [22]) [23]). One should keep in mind that more consideration will be necessary to decide in which states the nucleons are and whether these states are reasonably pure. The favored transitions to a rotational band of the daughter nucleus have a particular interest. The theoretical relative intensities of the transitions to the rotational levels and with

$l = 0,2,4, \ldots$ can be compared with experiment. The external problem for deformed nuclei was worked out by Fröman [15]) and others (cf. also [14])). It is shown that the boundary condition on the spheroidal nuclear surface $\psi_0(\vartheta)$ can be substituted by an effective boundary condition on a sphere $\psi_1(\vartheta)$, from which one may take the propagation to proceed as from a spherical nucleus (ϑ: azimuthal angle indicating a point on the nuclear surface, referred to the symmetry axis of the spheroidal nucleus). $\psi_1(\vartheta)$ is obtained from $\psi_0(\vartheta)$ by multiplication with a factor $T(\vartheta)$ containing a differential barrier penetrability, decreasing from $\vartheta = 0$ to $\vartheta = \pi/2$ for prolate nuclei (cf. II (3.28)). Several authors have considered the relative intensities of the $l = 2,4,6$ contributions with respect to $l=0$ as they follow from the experiments and have tried to formulate rules for the variation of c_2'/c_0' and c_4'/c_0' (indicating the intensity ratios of $l = 2$ and 4 to $l = 0$ waves) with the mass number A. Although indications for a gradual change with A were found, one should be quite careful in accepting this as a strict law. The theoretical expectation on the basis of a boundary condition (4.17) would be that the nuclei should show substantial individual variations according to the (spheroidal) shell model states occupied by the component nucleons. The substantial probability c_4'/c_0' for $l = 4$ which is obtained from the experiments does not seem surprising in view of the fact that $\psi_1(\vartheta)$, as a product of two factors ($\psi_0(\vartheta)$ and $T(\vartheta)$) which may decrease in opposite directions, can easily obtain a rather sharply peaked behavior. For further work on these relative intensities one should keep in mind in addition that admixtures to the nucleon states and higher order deformations of the nuclear surface might have an appreciable influence on the effective boundary condition.

(*c*) The alpha directional distribution of oriented nuclei. In addition to possible spin and parity assignments in a disintegration scheme (for which also alpha-gamma directional correlations can be useful), these directional distributions can give useful information on the $l = 0$ and $l = 2$ interference in ground state to ground state favored alpha transitions of deformed nuclei. Such observations can therefore form an additional experimental test for a boundary condition (4.17). This is explained in detail in chapter II. For spherical nuclei similar information on the interference of contributions with different l to a certain alpha transition, might be obtained.

(*d*) Histogram of $\log F$-values. The F-values (defined by (2.4)) give a measure for the intrinsic probability of alpha particle emission. One should expect that the relative F-values for different alpha transitions (and for different nuclei) reflect the different boundary conditions in different cases and therefore should show individual variations. These should show up in a statistical way in a histogram of $\log F$-values (analogous to a histogram of $\log ft$-values in beta decay). In such a histogram one may hope to recognize the favored alpha transitions as a more or less distinct group.

Fig. 3 shows such a histogram; the log F-values from -1.5 to 0.5 should probably be considered as favored alpha transitions. The spread in these values may reflect the expected individual variations although other causes of this spread could also be present.

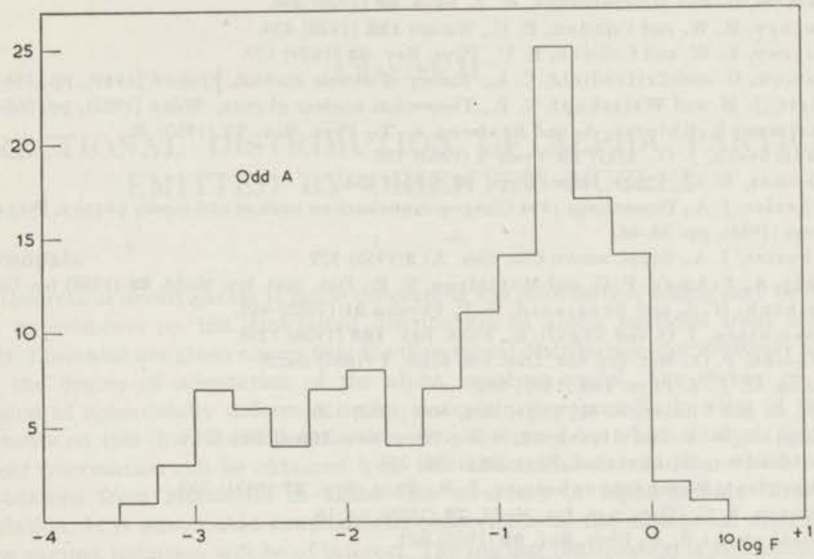

Fig. 3. Histogram of log F-values for odd-A nuclei. The favored factor F gives a kind of intrinsic formation probability for the alpha particle. The group of alpha transitions with log F-values between -1.5 and 0.5 can possibly be identified as favored alpha transitions in the sense of fig. 1. The spread in log F-values for this group might provide information on alpha particle formation (the data for this histogram were taken from the tables XII and XIII of [24])).

(*e*) Absolute transition probabilities for alpha disintegration and comparison with alpha particle scattering. Our considerations of § 2, § 3 and § 4 show that both the absolute transition probability for alpha disintegration and alpha particle scattering depend sensitively on the shape and radius of the nuclear potential well for alpha particles. In view of the possibility that the potential well parameters vary with the energy, a study of such a possible variation for alpha particle scattering of different energies would be valuable. In this way one may make a good check as to whether one really finds the same nuclear potential wells for both phenomena. In such a check one also needs the absolute value of the boundary condition (4.17) in order to arrive at a value characterizing the radius of the nuclear potential well. According to § 4 the present experimental data give good agreement for this radius from both phenomena. A further point of considerable interest would be a comparison of the (outer part) of the alpha particle nuclear potential well with the potential wells for neutrons and protons; however, the experimental results are rather inaccurate at the moment for this comparison.

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Chapter II

DIRECTIONAL DISTRIBUTION OF ALPHA PARTICLES EMITTED BY ORIENTED NUCLEI

Synopsis

A theoretical investigation is made concerning the information which may be drawn from experiments on the directional distribution of alpha particles from oriented nuclei. Formulae are given connecting the directional distribution (for arbitrary nuclei) with the degree of orientation of the alpha emitting nuclei. The theory of alpha emission of spheroidally deformed nuclei is explained, especially in view of the experiments on this directional distribution. It is discussed by which analysis typically nuclear information can be obtained from this directional distribution, which cannot be obtained from intensities in alpha fine structure or alpha-gamma directional correlation. It is shown that simultaneous observation of the directional distribution of the gamma radiation will be of interest. The nuclear information is connected with a preferential emission of the alpha particle from the poles or from the equator of the surface of the spheroidally deformed nucleus, depending on the internal problem of alpha particle formation from nucleon states. The relation between directional distributions in the laboratory system and the body fixed system can be visualized for the classical limit by a simple geometrical averaging over the precession of the nucleus.

§ 1. *Introduction.* During the last few years the available data on alpha disintegration have accumulated considerably [1] [2] [3] [4] [5] [6] [7] (and references cited there). It was explained in chapter I that for a long time the theory was practically confined to the lifetime-energy relation following from the quantum mechanical Coulomb barrier penetration formula [8] [9] [10] [11] [12] [13] [14]. In the first chapter a number of considerations concerning the internal problem of the formation of the alpha particle from the nucleons in the nucleus were given.

We shall study in this chapter the external dynamical problem of alpha emission from a spheroidally deformed nucleus. Some papers on these problems have appeared [15] [16] [17] [17a] [18] [19] [20]. It is furthermore tempting to make a comparison with the optical model interpretation (in terms of a complex potential) of scattering of energetic alpha particles by nuclei [21] [22] [23] [24].

This paper is concerned with the information, which may be obtained from experiments on the directional distribution of alpha particles emitted by oriented nuclei, in particular for nuclei with spheroidal deformation. As was first emphasized by Hill and Wheeler [25]), the barrier penetrability will be substantially increased at the poles of a nucleus with prolate spheroidal deformation, so that an increased intensity in the direction of the poles might be expected for an oriented nucleus of this intrinsic shape. However, the result could be the opposite if this effect would be overcompensated by a strong preference (due to the internal nuclear structure) for formation of alpha particles near the equator of the nucleus. Furthermore one must take correctly into account how the directional distribution of alpha particles in the laboratory system is obtained by quantum mechanical averaging procedures from the alpha particle amplitudes near the surface of the oriented nucleus. This can be compared with the more intuitive classical picture of the directional distribution in the laboratory system as a result of a slow precession of a directional distribution associated with the intrinsic shape of the nucleus (such a picture would be valid for a very heavy nucleus with high nuclear spin).

The first positive experimental results showing an anisotropic emission of alpha particles from oriented nuclei were reported by Roberts et al. [26]) [27]). A concise discussion of some points concerning the directional distribution of alpha particles from deformed oriented nuclei was given by Fröman [20]).

In § 2 of this paper explicit formulae are given, relating the directional distribution of alpha particles to the degree of orientation of the parent nuclei (these formulae do not depend explicitly on a possible deformation of the nuclei). The dynamics and geometry which are involved in the case of alpha emission from oriented nuclei, which have an intrinsic deformation, are explained in § 3. In § 4 it is examined in which way the quantum mechanical averaging of the directional distribution is related to the classical averaging, which would be valid for a very heavy nucleus (in this section the results of chapter III on the classical limits of Clebsch-Gordan coefficients and Racah coefficients are used). A discussion of the information which can be obtained from experiments on oriented alpha emitters is given in § 5.

§ 2. *Directional distribution of alpha particles emitted by oriented nuclei.* In this section we want to derive formulae for directional distributions, which do not yet contain any explicit reference to the internal nuclear structure (e.g. intrinsic nuclear deformation). In the method of calculation we follow closely the notations of some earlier papers [29]) [30]) [31]) [32]). The calculation for alpha particles is particularly simple to the extent that alpha particles, having spin 0, cannot show polarization phenomena. However, one should note a difference as to the formulation of the starting point with the emission of beta- and gamma radiation, for which the calculation of the transition probabilities is always formulated with the aid of perturbation theory. But alpha emission is a direct consequence of the nuclear forces and the penetrability of the Coulomb barrier and one does not see that the transition probabilities could be obtained by splitting off a part of the Hamiltonian as a perturbation part.

One may use the following formula as a starting point for calculating the directional distribution of radiation, in case perturbation theory can be applied (e.g. beta- and gamma emission),

$$W(\boldsymbol{k}) \propto |\langle f, \boldsymbol{k}\, |H_1|\, i\rangle|^2 = |\langle f|\mathscr{H}(\boldsymbol{k})|\, i\rangle|^2, \tag{2.1}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states of the nucleus (*stationary* states of the unperturbed Hamiltonian), and H_1 is the perturbing part of the Hamiltonian. $|f, \boldsymbol{k}\rangle$ represents the final state which can be considered as the product state of the final state $|f\rangle$ of the nucleus and the state of the emitted radiation of direction $\boldsymbol{k}$. The second and third member of (2.1) show two equivalent but slightly different notations, which are in use. In the third member the state of the emitted radiation is multiplied into H_1, which is not done in the second member.

Without the use of perturbation theory, an equation analogous to (2.1) can be written down for alpha emission

$$W(\boldsymbol{k}) \propto |\langle f, e^{i\boldsymbol{k}.\boldsymbol{r}}\ |\ i, \alpha''\rangle|^2 = |\langle f|\ \mathscr{E}(\boldsymbol{k})|i, \alpha''\rangle|^2, \tag{2.2}$$

where $|f\rangle$ represents a state of the final nucleus, $e^{i\boldsymbol{k}.\boldsymbol{r}}$ specifies a plane wave with wave vector $\boldsymbol{k}$ for the alpha particle. However, $|i, \alpha''\rangle$ does not represent the state of the initial nucleus but a state of the final nucleus together with the emitted radiation, having those geometrical characteristics, which such a state possesses if it develops from the initial state $|i, \alpha'\rangle$. This initial state may be, e.g., a state of the initial nucleus specified by the quantum numbers I_i, M_i (for initial nuclear spin and its z-component). The state $|i, \alpha''\rangle$ should then also be characterized by the quantum numbers I_i, M_i, if we assume invariance of the total Hamiltonian for Lorentz transformations, hence in particular for spatial rotations. (2.1) and (2.2) can be given in a very similar form, if we write (2.2) also as a matrix element between states $|i, \alpha''\rangle$ and $|f\rangle$ with an expression $\mathscr{E}(\boldsymbol{k})$ in between. Equation (2.2) contains only the fundamental probability assumption of quantum mechanics. However, it is sufficient to derive general formulae for directional distributions (these are mainly geometrical formulae, containing no detailed physics). Of course (2.2) does not allow to calculate the absolute transition probability, which is provided by the perturbation theory formula (2.1), when correctly normalized.

For the actual calculation we write (2.2) in the following more explicit form

$$W(\boldsymbol{k}) \propto \Sigma_{M_f}\, |\Sigma_{M_i}\, b_{M_i} \langle I_f, M_f\ |\ \mathscr{E}(\boldsymbol{k})\ |I_i, M_i, \alpha''\rangle|^2. \tag{2.3}$$

If the initial state of the nucleus is $|i, \alpha'\rangle = \Sigma_{M_i} b_{M_i}|I_i, M_i, \alpha'\rangle$, then the state $|i, \alpha''\rangle$ can be written as $|i, \alpha''\rangle = \Sigma_{M_i} b_{M_i}|I_i, M_i, \alpha''\rangle$. For an ensemble of oriented nuclei, the state of orientation can be characterized by the density matrix

$$\rho_{M_i M_{i'}} = \overline{\overline{b_{M_i} b_{M_{i'}}{}^*}}, \tag{2.4}$$

where the double bar indicates the ensemble averaging.

As to the formulation of the decaying alpha emitting states, it may be remarked that one does not obtain strictly stationary states, but solutions (of the time dependent problem) which have a time dependence contained in the factor $\exp[-i(E-i\varepsilon)t/\hbar]$ (with $\varepsilon \ll E$). The square of the absolute value of this factor gives

$$|\exp[-i(E-i\varepsilon)t/\hbar]|^2 = \exp[-2\varepsilon t/\hbar] = \exp[-\lambda t], \tag{2.5}$$

relating the decay constant λ to ε. In the following we are only interested in directional distributions (which we shall normalize such that $\int W(\vartheta)\,\mathrm{d}\Omega = 4\pi$) and we shall omit the time dependent factor altogether.

The evaluation of directional distributions can proceed from (2.3), if one can write down an expansion of $\mathscr{E}(\boldsymbol{k})$ (the formal development starting from (2.1) is identical if such an expansion can be written down for $\mathscr{H}(\boldsymbol{k})$) of the form

$$\mathscr{E}^*(\boldsymbol{k}) = \sum_{lmm'} \alpha_{lm'}\, T^l_m(\boldsymbol{r})\, D^l_{mm'}(R). \tag{2.6}$$

R is the rotation that transforms the laboratory system into the coordinate system where $\boldsymbol{k}$ is directed along the z-axis. The $\alpha_{lm'}$ are variables characterizing the emitted radiation ((2.6) being the expansion for the radiation with $\boldsymbol{k}$ directed along the z-axis, if R is the identity). The expansion (2.6) can be written down for an arbitrary radiation; we shall specify later to alpha emission. For an arbitrary radiation variables specifying the polarization (e.g. $\boldsymbol{c}$) may occur, which are here not written explicitly (one should then write, e.g., $\mathscr{E}(\boldsymbol{k}, \boldsymbol{c})$ and $\alpha_{lm}(\boldsymbol{c})$). The coefficients $D^l_{mm'}(R)$ are the matrix elements in the l-representation of the rotation R.

The expression (2.3) for the directional distribution can now be reduced with the use of Racah algebra [29] [30] [31] [33] [34] [34a]

$$W(\vartheta) = \sum_{l\bar{l}} a_l a_{\bar{l}}^* \sum_{k\rho\sigma} (-1)^k C_{k\sigma}(l\bar{l})\, W(I_f \bar{l} I_i k; I_i l) \langle |(I_i I_i) k\rho\rangle\, D^k_{\rho\sigma}(R). \tag{2.7}$$

The reduced matrix elements

$$a_l = \langle I_f \,\|\, T^l \,\|\, I_i \rangle \tag{2.8}$$

are defined by (Wigner-Eckart theorem) *)

$$\begin{aligned} \langle I_f M_f | T^{l}_m{}^* | I_i M_i \rangle &= (-1)^m \langle I_f M_f | T^l_{-m} | I_i M_i \rangle = \\ &= (-1)^{I_f+M_i} \langle I_f \| T^l \| I_i \rangle\, V(I_f I_i l; -M_f M_i - m), \end{aligned} \tag{2.9}$$

where the V-coefficient, introduced by Racah [33], is related to our notation

*) This definition of reduced matrix elements is identical with the definition by Racah [33].

for the Clebsch-Gordan coefficients according to

$$C^{c\gamma}_{a\alpha b\beta} = (-1)^{c+\gamma}\,(2c+1)^{\frac{1}{2}}\,V(abc;\alpha\beta-\gamma). \tag{2.10}$$

The statistical tensor $\langle|(I_iI_i)k\rho\rangle$ for states with a definite I_i, describing the initial ensemble of nuclei, has been introduced by Fano [34] (cf. (2.4))

$$\langle|(I_iI_i)k\rho\rangle = \Sigma_{M_iM_i'}\,\rho_{M_iM_i'}(-1)^{I_i-M_i}\,C^{k\rho}_{I_iM_i'I_i-M_i}. \tag{2.11}$$

The abbreviation $C_{k\sigma}(l\bar{l})$ for the radiation is defined by [29] [35]

$$C_{k\sigma}(l\bar{l}) = \Sigma_{m\bar{m}}\,\alpha_{lm}{}^*\,\alpha_{\bar{l}\bar{m}}(-1)^{l-m}C^{k\sigma}_{l-m\,\bar{l}\bar{m}}. \tag{2.12}$$

This coefficient can be considered as a statistical tensor of the emitted radiation. The coefficient $W(I_f\bar{l}I_ik;I_il)$ for the recoupling of angular momenta is the usual Racah coefficient [33].

We shall now specialize the general formula (2.7) for the case of *alpha radiation*. Since we can observe only the direction of motion of the alpha particle (no polarization) only the α_{l0} can be different from zero (so $C_{k\sigma}(l\bar{l}) = 0$ if $\sigma \neq 0$). The m-dependence of the α_{lm} is characteristic of the polarization properties of the emitted radiation [29] [35]. We shall assume further that the ensemble of oriented nuclei has rotational symmetry about the z-axis of the laboratory coordinate system. This implies that only the statistical tensors $\langle|(I_iI_i)k0\rangle$ can be different from zero [34]. It is then convenient to use the orientation parameters $f_k(I_i)$ introduced by Tolhoek and Cox [30] [32] [36]

$$f_k(I_i) = \langle|(I_iI_i)\,k0\rangle\,w_k(I_i) =$$
$$= \langle|(I_iI_i)k0\rangle \binom{2k}{k}^{-1} I_i^{-k}\left[\frac{(2I_i+k+1)!}{(2k+1)(2I_i-k)!}\right]^{\frac{1}{2}}. \tag{2.13}$$

The expansion (2.6) for the case of alpha radiation is nothing but the Rayleigh expansion [37] (p. 1466) of a plane wave

$$e^{i\mathbf{k}.\mathbf{r}} = \Sigma_{lm}\,\{4\pi(2l+1)\}^{\frac{1}{2}}\,i^l\,j_l(kr)\,Y_{lm}(\vartheta_r,\varphi_r)\,D^l_{m0}(R), \tag{2.14}$$

with

$$D^l_{m0}\,(R) = \{4\pi/(2l+1)\}^{\frac{1}{2}}\,Y_{lm}{}^*(\vartheta,\varphi). \tag{2.14a}$$

(The arguments ϑ_r, φ_r and ϑ, φ of the spherical harmonics represent the directions of $\boldsymbol{r}$ and $\boldsymbol{k}$ respectively, measured in the laboratory coordinate system.) The spherical Bessel function $j_l(kr)$ shows the following asymptotic behavior [37] (p. 622)

$$j_l(kr) = \{\pi/(2kr)\}^{\frac{1}{2}}\,J_{l+\frac{1}{2}}(kr) \rightarrow \frac{\sin\,[kr - l\pi/2]}{kr}\,(kr \gg l). \tag{2.15}$$

By comparison of the expansions (2.6) and (2.14) we find

$$\alpha_{lm} = \delta_{m0}\,\{4\pi(2l+1)\}^{\frac{1}{2}}\,i^l, \tag{2.16}$$

$$T^l_m(\boldsymbol{r}) = j_l(kr)\,Y_{lm}(\vartheta_r,\varphi_r). \tag{2.17}$$

Thus we conclude from (2.12)

$$C_{k\sigma}(l\bar{l}) = \delta_{\sigma 0}\, 4\pi\{(2l+1)(2\bar{l}+1)\}^{\frac{1}{2}}\, i^{l+\bar{l}}\, C^{k0}_{l0\,\bar{l}0}. \tag{2.18}$$

Substitution of (2.8), (2.9), (2.10), (2.13), (2.17) and (2.18) into (2.7) now yields, use being made of the property

$$C^{k0}_{l0\,\bar{l}0} = 0 \text{ if } l + \bar{l} + k = \text{odd}, \tag{2.19}$$

that

$$W(\vartheta) = \Sigma_{l\bar{l}k}\, a_l a_{\bar{l}}^*\, i^{l+\bar{l}}\, \{(2l+1)(2\bar{l}+1)\}^{\frac{1}{2}}\, C^{k0}_{l0\,\bar{l}0}\cdot$$
$$\cdot W(I_f\bar{l}I_ik;\, I_il) f_k(I_i)\{w_k(I_i)\}^{-1}\, P_k(\cos\vartheta)$$

or

$$W(\vartheta) = \Sigma_{l\bar{l}(l+\bar{l}=\text{even})}\, |a_l||a_{\bar{l}}|\cos(\alpha_l - \alpha_{\bar{l}})\, (-1)^{(l+\bar{l})/2}\{(2l+1)(2\bar{l}+1)\}^{\frac{1}{2}}\,\cdot$$
$$\cdot\, \Sigma_{k=\text{even}}\, C^{k0}_{l0\,\bar{l}0}\, W(I_f\bar{l}I_i\,k;\, I_il)\, f_k(I_i)\, \{w_k(I_i)\}^{-1} P_k(\cos\vartheta). \tag{2.20}$$

We take into account that, because of the conservation of parity in alpha decay, $|l - \bar{l}|$ = even for transitions between states of initial and final nucleus with definite parities. The phase angles α_l are defined by

$$a_l = |a_l|\, e^{i\alpha_l}, \tag{2.21}$$

where the reduced matrix elements a_l are given by (2.8) and (2.9). (A factor 4π is dropped during the reductions.) Concluding this section we list a number of special cases of (2.20) in an entirely explicit form, applicable to most cases of practical interest. In case of $l = 0{,}2$ interference, a_0 and a_2 are normalized according to

$$|a_0|^2 + |a_2|^2 = 1. \tag{2.22}$$

The distribution functions $W(\vartheta)$ are all normalized according to

$$\int W(\vartheta)\, \mathrm{d}\Omega = 4\pi. \tag{2.23}$$

List of formulae for the directional distribution of alpha particles from oriented nuclei.

$I_f = I_i - 2;\ l = 2$

$$W(\vartheta) = 1 - \tfrac{30}{7} N_2 f_2 P_2(\cos\vartheta) + \tfrac{15}{2} N_4 f_4 P_4(\cos\vartheta), \tag{2.24}$$

$I_f = I_i - 1;\ l = 2$

$$W(\vartheta) = 1 + \tfrac{15}{7} N_2 f_2 P_2(\cos\vartheta) - 15\frac{2I_i - 3}{I_i + 1} N_4 f_4 P_4(\cos\vartheta), \tag{2.25}$$

$I_f = I_i;\ l = 0{,}2$ *)

$$W(\vartheta) = 1 + \Bigg[-|a_0|\,|a_2|\cos(\alpha_0 - \alpha_2)\, 6\sqrt{5}\left\{\frac{I_i(I_i+1)}{(2I_i-1)(2I_i+3)}\right\}^{\frac{1}{2}} +$$

*) The formula (2.26) has been quoted in [38] (p. 300), the factor cos $(\alpha_0 - \alpha_2)$ and the absolute value bars being omitted. This means that it was assumed that for the favored alpha emission the coefficients $b_{l0}^{(0)}$ (cf. (3.29)) and the coefficients a_0 and a_2 are real (cf. (3.59) and (3.60); the value $\cos(\alpha_0 - \alpha_2) \approx \pm 0.993$ for a typical case, being very close to ± 1).

$$+ |a_2|^2 \tfrac{15}{7} \frac{(2I_i - 3)(2I_i + 5)}{(2I_i - 1)(2I_i + 3)} \Big] K_2 f_2 P_2 (\cos \vartheta) +$$

$$+ |a_2|^2 \frac{45}{(2I_i - 1)(2I_i + 3)} K_4 f_4 P_4 (\cos \vartheta), \quad (2.26)$$

$I_f = I_i + 1;\ l = 2$

$$W(\vartheta) = 1 + \tfrac{15}{7} \frac{I_i + 6}{I_i} M_2 f_2 P_2 (\cos \vartheta) - 15 \frac{2I_i + 5}{I_i} M_4 f_4 P_4 (\cos \vartheta), \quad (2.27)$$

$I_f = I_i + 2;\ l = 2$

$$W(\vartheta) = 1 - \tfrac{30}{7} M_2 f_2 P_2 (\cos \vartheta) + \tfrac{15}{2} M_4 f_4 P_4 (\cos \vartheta). \quad (2.28)$$

N_k, K_k, M_k and f_k are functions of the initial spin [30] [32]). The parameters f_k have been defined by (2.13) (for explicit expressions see [30] [32]) and for graphs see [39])).

$$N_2 = \frac{I_i}{2I_i - 1} \quad \text{and} \quad N_4 = \frac{I_i^3}{(I_i - 1)(2I_i - 1)(2I_i - 3)}. \quad (2.30)$$

$$K_2 = \frac{I_i}{I_i + 1} \quad \text{and} \quad K_4 = \frac{I_i^3}{I_i + 1}. \quad (2.31)$$

$$M_2 = \frac{I_i^2}{(I_i + 1)(2I_i + 3)} \quad \text{and} \quad M_4 = \frac{I_i^4}{(I_i+1)(I_i+2)(2I_i+3)(2I_i+5)}. \quad (2.32)$$

§ 3. *Directional distribution of alpha particles emitted by oriented, spheroidally deformed nuclei.* We now want to specialize the general problem of § 2 to the case of spheroidally deformed nuclei. Our purpose will be to give a discussion of the information, which can be obtained from experiments on this directional distribution. We shall explain in this section the theory of alpha emission from spheroidally deformed nuclei aiming at this purpose. The development of the theory of alpha decay of deformed nuclei was started by Rasmussen [40] [41]), Bohr, Fröman and Mottelson [15]), Rasmussen and Segall [18]), Strutinsky [19]) and Fröman [20]), who mainly concentrated on the intensity ratios in alpha decay fine structure. We shall indicate the main lines of the theory and refer to these authors for certain details.

§ 3.1. *The expression for the wave function of the system: spheroidally deformed nucleus with emitted alpha particle.* Let, as in § 2, the initial state of the nucleus be characterized by I_i and M_i, the nuclear spin and its projection on the laboratory z-axis respectively and let I_f and M_f represent the same quantities for the daughter nucleus. The requirement of conservation of angular momentum during the alpha decay process then determines the transformation character of the wave function of the system: daughter nucleus with emitted alpha particle, which we can write as (in this and other wave functions below we shall drop the time dependent factor)

$$\Phi^{(f)}_{I_iM_i} = \Sigma_{lI_f}(1/r)\, v_{lI_f}(r)\, \Sigma_{mM_f}\, G_{I_fM_f}\, C^{I_iM_i}_{I_fM_flm}\, Y_{lm}(\vartheta, \varphi). \tag{3.1}$$

$G_{I_fM_f}$ represents the normalized wave function of the daughter nucleus, transforming under rotations as the wave function of a single particle with angular momentum I_f and z-component of the angular momentum M_f. The angular part of the motion of the alpha particle is given by $Y_{lm}(\vartheta, \varphi)$, the radial part by $(1/r)\, v_{lI_f}(r)$. The equation (3.1) holds if the alpha particle is situated outside the nucleus. Therefore the function $\Phi^{(f)}_{I_iM_i}$ is defined only for values of r larger than the nuclear radius. The same restriction holds for the wave functions (3.3), (3.4), (3.5) and (3.9). The wave function (3.1) is still quite general, i.e. nothing has been supposed, e.g., about axial symmetry of the nucleus. However, we shall specialize now to spheroidally deformed nuclei, expressing that $G_{I_fM_f}$ is the wave function of a rotating spheroidally deformed nucleus

$$G_{I_fM_f} = \{(2I_f + 1)/(4\pi)\}^{\frac{1}{2}}\, \chi_{K_f} D^{I_f*}_{M_fK_f}(\Theta_i). \tag{3.2}$$

The normalization factor of $D^{I_f}_{M_fK_f}(\Theta_i)$ has been taken $\{(2I_f + 1)/(4\pi)\}^{\frac{1}{2}}$ as the position of a *symmetric* rotator in space is determined by *two* Eulerian angles (so the integration $\int_0^{2\pi} d\psi$ is not to be performed for the normalization, cf. III (A. 3.8)). Hence the wave function of the daughter nucleus with emitted alpha particle can be written as [18] [20]

$$\Phi^{(f)}_{I_iM_i} = \Sigma_{lI_f}(1/r)\, v_{lI_f}(r)\, \Sigma_{mM_f}\, \{(2I_f + 1)/(4\pi)\}^{\frac{1}{2}}\, \chi_{K_f}\, D^{I_f*}_{M_fK_f}(\Theta_i)\, \cdot \\ \cdot\, C^{I_iM_i}_{I_fM_flm}\, Y_{lm}(\vartheta, \varphi). \tag{3.3}$$

The motion of the daughter nucleus has been split [42] [43] into an intrinsic part, represented by the normalized wave function χ_{K_f} depending on the internal coordinates of the daughter nucleus and into a rotational part represented by the normalized wave function $\{(2I_f + 1)/(4\pi)\}^{\frac{1}{2}}\, D^{I_f*}_{M_fK_f}(\Theta_i)$. The coefficients $D^l_{mn}(\Theta_i)$ of the l-representation of the rotational group are also eigenfunctions of the symmetric rotator [44]. The Eulerian angles Θ_i describe the position of the daughter nucleus and the angles ϑ, φ represent the angular coordinates of the alpha particle in the laboratory system. The quantum number K_f represents the projection of the nuclear spin I_f along the symmetry axis of the nucleus.

We shall also make use of a body fixed coordinate system, having the axis of nuclear spheroidal symmetry as z'-axis (the coordinates in this system will be indicated by a prime). The wave function has to be symmetrized appropriately [42] [43] to possess a definite parity. However, we shall not carry out this symmetrization explicitly in our formulae (it is easily performed and mainly consists of a kind of doubling of the notation; cf.[20]) for this complete notation of some of the formulae).

A way, alternative to (3.3), to write down the wave function of the spheroidally deformed nucleus and emitted alpha particle is

$$\Phi^{(f)} = \sum_{l\bar{I}_f m' \bar{M}_f} (1/r)\, w_{lm'}(r)\{(2\bar{I}_f + 1)/(4\pi)\}^{\frac{1}{2}}\, \chi_{K_f} \cdot$$
$$\cdot\, D^{\bar{I}_f *}_{\bar{M}_f, K_f+m'}(\Theta_i)\; Y_{lm'}(\vartheta', \varphi')\; g(\bar{I}_f, \bar{M}_f, l, m'), \tag{3.4}$$

where the motion of the alpha particle is expressed with the aid of the body fixed coordinates r, ϑ', φ'. The radial part of the motion of the alpha particle is described by $(1/r)\, w_{lm'}(r)$. In (3.4) we have left the function $g(\bar{I}_f, \bar{M}_f, l, m')$ unspecified. In order to determine the wave function of the system we may use: (a) the requirement of certain geometrical characteristics, e.g. given by (I_i, M_i), as satisfied by (3.3), (b) the requirement of certain boundary conditions at the nuclear surface; these will be specified in the body fixed system, so that (3.4) is the natural starting point (e.g., it could be required that only one value of l and m' in (3.4) contribute).

Furthermore we shall make use of the form (3.3) for deriving the directional distribution of the alpha particles in the laboratory system from the wave function. Hence we want to make use of both forms (3.3) and (3.4) for the wave function of the same system and we investigate under what conditions both forms are equivalent. We can reduce (3.4) by inserting $Y_{lm'}(\vartheta', \varphi') = \sum_m Y_{lm}(\vartheta, \varphi)\, D^l_{mm'}(\Theta_i)$ and by using the expansion of the product of two $D^l_{mn}(\varphi, \vartheta, \psi)$-functions and the familiar properties of Clebsch-Gordan coefficients to

$$\Phi^{(f)} = \sum_{l\bar{I}_f m' \bar{M}_f m I_f} (1/r)\, w_{lm'}(r)\{(2\bar{I}_f + 1)/(4\pi)\}^{\frac{1}{2}}\, \chi_{K_f}\, C^{I_f, \bar{M}_f - m}_{\bar{I}_f, \bar{M}_f, l, -m} \cdot$$
$$\cdot\, C^{I_f, K_f}_{\bar{I}_f, K_f+m', l, -m'}\, D^{I_f *}_{\bar{M}_f - m, K_f}(\Theta_i)(-1)^{m-m'} Y_{lm}(\vartheta, \varphi)\, g(\bar{I}_f, \bar{M}_f, l, m') =$$
$$= \sum_{l\bar{I}_f m' \bar{M}_f I_f} (1/r)\, w_{lm'}(r)\, C^{I_f, K_f}_{\bar{I}_f, K_f+m', l, -m'}\, (-1)^{I_f - \bar{I}_f + m'} \,.$$
$$\cdot \sum_{m M_f} \{(2I_f + 1)/(4\pi)\}^{\frac{1}{2}}\, \chi_{K_f} D^{I_f *}_{M_f K_f}(\Theta_i) C^{\bar{I}_f \bar{M}_f}_{I_f M_f l m} Y_{lm}(\vartheta, \varphi)\, g(\bar{I}_f, \bar{M}_f, l, m') \tag{3.5}$$

(we have introduced $M_f = \bar{M}_f - m$). Hence, if we want (3.4) to represent the same function as (3.3), we find, comparing (3.5) and (3.3),

$$g(\bar{I}_f, \bar{M}_f, l, m') = \delta_{\bar{I}_f I_i}\, \delta_{\bar{M}_f M_i}\, h(l, m'). \tag{3.6}$$

The l, m'-dependent function $h(l, m')$ may be put equal to unity, since it can be absorbed into $w_{lm'}(r)$. We then obtain the following relation between $w_{lm'}(r)$ and $v_{lI_f}(r)$ [18] [20])

$$v_{lI_f}(r) = \sum_{m'} (-1)^{I_f - I_i + m'}\, C^{I_f, K_f}_{I_i, K_f+m', l, -m'}\, w_{lm'}(r), \tag{3.7}$$

which may be inverted, if necessary

$$w_{lm'}(r) = \sum_{I_f} (-1)^{I_f - I_i + m'}\, C^{I_f, K_f}_{I_i, K_f+m', l, -m'}\, v_{lI_f}(r). \tag{3.8}$$

According to (3.6) the wave function equivalent to (3.3), expressing the alpha particle motion in the body fixed system, is (if we suppose that (3.7), (3.8) are satisfied) [18] [20])

$$\Phi^{(f)}_{I_iM_i} = \Sigma_{lm'} (1/r)\, w_{lm'}(r)\, \{(2I_i + 1)/(4\pi)\}^{\frac{1}{2}}\, \chi_{K_f} D^{I_i*}_{M_i, K_f+m'}(\Theta_i)\, Y_{lm'}(\vartheta', \varphi'). \quad (3.9)$$

Because of the symmetrization of the wave function, the summation over m' in (3.4), (3.5), (3.7) and (3.9) has to be extended over values of m', such that

$$m' = \pm K_i - K_f, \quad (3.10)$$

where K_i represents the projection of the nuclear spin I_i of the parent nucleus along the symmetry axis (see Fröman [20]) for a detailed discussion of this point).

§ 3.2. *Directional distribution of alpha particles in relation to the asymptotic expressions for $v_{lI_f}(r)$ and $w_{lm'}(r)$.* In order to relate the reduced matrix elements a_l, defined by (2.8), to the radial part $(1/r)\, v_{lI_f}(r)$ of the wave function (3.3), we can substitute (3.3) into (2.9). From (2.8), (2.9), (2.10), (2.17) and (3.3) we then derive

$$\langle G_{I_fM_f} | T^l_m{}^* | \Phi^{(f)}_{I_iM_i} \rangle = (-1)^{I_f+M_i}\, a_l\, V(I_f I_i l; -M_f M_i - m) \quad (3.11)$$

or

$$a_l = (-1)^{I_i - I_f}\, (2I_i + 1)^{-\frac{1}{2}} \int_{R'}^{R_N} j_l(kr)\, v_{lI_f}(r)\, r\, \mathrm{d}r. \quad (3.12)$$

We now introduce the abbreviation

$$A_{lI_f} = \int_{R'}^{R_N} j_l(kr)\, v_{lI_f}(r)\, r\, \mathrm{d}r. \quad (3.13)$$

Here $r = R'$ is a sphere (closely) surrounding the nucleus. The integration in (3.13) has to be extended to a radius R_N of a large "normalization sphere", the same boundary as has been taken for the matrix element in (2.9). We can not put $R_N = \infty$ since the integral then diverges. This does not cause any difficulty as only relative values of the a_l and A_{lI_f} are of importance for the directional distribution. Since R_N is supposed to be large the value of A_{lI_f} is determined by the contribution of the asymptotic region of $j_l(kr) v_{lI_f}(r)$ to the integral. Equation (3.3) represents an outgoing alpha particle and the asymptotic behavior of $v_{lI_f}(r)$ can be written (assuming that the Coulomb field is screened for $r \to \infty$)

$$v_{lI_f}(r) \to \hat{v}_{lI_f} \exp\,[i(kr - l\pi/2)] \text{ for } r \to \infty. \quad (3.14)$$

From (2.15), (3.13) and (3.14) we deduce

$$A_{lI_f} = Q(k, R', R_N)\, \hat{v}_{lI_f}, \quad (3.15)$$

where the factor $Q(k, R', R_N)$ does not depend on l and I_f, hence is a constant for our purposes. (The dependence of Q on R' and R_N is not required in our formulae and the values chosen for R' and R_N do not figure in our results.) Analogously the asymptotic behavior of $w_{lm'}(r)$ can be written

$$w_{lm'}(r) \to \hat{w}_{lm'} \exp\,[i(kr - l\pi/2)] \text{ for } r \to \infty, \quad (3.16)$$

so that we find, using (3.7), (3.14), (3.15) and (3.16)

$$A_{lI_f} = (-1)^{I_f - I_i}\, Q(k, R', R_N)\, \textstyle\sum_{m'} (-1)^{m'}\, C^{I_f, K_f}_{I_i, K_f + m', l, -m'}\, \hat{w}_{lm'}. \tag{3.17}$$

The equations (3.12), (3.13) and (3.17) relate the reduced matrix elements a_l to the amplitude and the phase of the wave function of the alpha particle in the body fixed system (cf. § 4)

$$a_l = Q(k, R', R_N)(2I_i + 1)^{-\frac{1}{2}} \textstyle\sum_{m'} (-1)^{m'}\, C^{I_f, K_f}_{I_i, K_f + m', l, -m'}\, \hat{w}_{lm'}. \tag{3.18}$$

§ 3.3. *Solution of differential equations for boundary conditions at the nuclear surface.* The wave functions (3.3) and (3.9) represent the system when an alpha particle is emitted (i.e. an alpha particle being outside the region of the daughter nucleus). They must satisfy a Schrödinger equation, with a Hamiltonian, which may be written as

$$H = H_{part} - \{\hbar^2/(2M)\}\, \Delta_r + V(\boldsymbol{r}') + T_{rot}, \tag{3.19}$$

where H_{part} is the Hamiltonian for the internal motion of the nucleons within the daughter nucleus, M is the reduced mass of alpha particle and daughter nucleus, T_{rot} is the part of the Hamiltonian related to the collective rotational energy of the daughter nucleus, $\boldsymbol{r}$ and $\boldsymbol{r}'$ indicate (as before) the place of the alpha particle in the laboratory and body fixed coordinate system respectively. The potential $V(\boldsymbol{r}')$ of the alpha particle in the body fixed system consists of the Coulomb potential (which may be developed in a spherically symmetric term, quadrupole term, etc.) and a nuclear potential, if the alpha particle comes in the nuclear surface region. If we consider only the first two terms of the Coulomb potential, we can write

$$V(\boldsymbol{r}') = V_0(\boldsymbol{r}') + V_2(\boldsymbol{r}'), \tag{3.20}$$

with

$$V_0(\boldsymbol{r}') = 2(Z-2)\, e^2/r, \tag{3.21}$$

$$V_2(\boldsymbol{r}') = \{e^2 Q_0/r^3\} P_2(\cos\vartheta'), \tag{3.22}$$

where Q_0 represents the intrinsic quadrupole moment of the nucleus, defined by

$$eQ_0 = \textstyle\int \rho_e(\boldsymbol{r}')(3z'^2 - r^2)\, \mathrm{d}\boldsymbol{r}' \tag{3.23}$$

($\rho_e(\boldsymbol{r}')$ is the nuclear charge distribution).

A spheroidal nuclear surface may be given by the equation

$$r = R(\vartheta') = R_0\, [1 + \beta P_2 (\cos\vartheta')]. \tag{3.24}$$

If the charge distribution inside the nuclear surface is uniform, the relation between Q_0 and β is given by

$$Q_0 = (6/5)(Z-2)\, R_0^2 \beta. \tag{3.25}$$

In order to solve the Schrödinger equation with the Hamiltonian (3.19) by an expression (3.3) or (3.9) one needs *boundary conditions*, for which we shall take the following conditions: (a) for $r \to \infty$ (or $r' \to \infty$) – alpha particle at large distance from the nucleus – we require that the wave function represents outgoing alpha particles only, (b) at the nuclear surface a boundary condition is given, specifying the value of the radial functions $w_{lm'}(r)$ of the alpha particle; a more general boundary condition would be a relation between these functions and their derivatives. Such a boundary condition must be derived by joining the wave function for the internal motion of the nucleons to the wave function representing an alpha particle leaving the nucleus. We shall try to solve the problem of the external alpha particle motion for arbitrary boundary conditions (b), prescribing the values of the functions $w_{lm'}(r)$ at the nuclear surface.

Even then the solution of the external problem is difficult because the Hamiltonian (3.19) contains the terms $V(\boldsymbol{r}')$ and T_{rot} which are diagonal in the body fixed (in m') and in the laboratory system (in m) respectively, but which are not both diagonal in one and the same coordinate system. This implies that the exact treatment of the Schrödinger equation with the Hamiltonian (3.19) leads to coupled differential equations for the radial part of the wave function of the alpha particle, $w_{lm'}(r)$ or $v_{lI_f}(r)$. These coupled differential equations have been treated and integrated numerically for a few cases by Rasmussen and Segall [18]. Neglecting T_{rot} in (3.19) altogether would mean that the nucleus was considered as being at rest. The term T_{rot} is comparatively unimportant for not too large distances from the nucleus [18] [20]. For larger distances from the nucleus the quadrupole field becomes negligible and we may consider $V(\boldsymbol{r}')$ as a spherically symmetric potential. These remarks lead to an attempt to determine an approximate solution (avoiding the above mentioned numerical integrations at least for a first analysis) in the following way: we divide the external region (outside the nucleus) into two parts:

region I: the inner part of the external region ($R(\vartheta') < r < R_1$), where in first approximation the rotational energy T_{rot} of the daughter nucleus may be neglected,

region II: the outer part of the external region ($R_1 < r$), where the deviation of $V(\boldsymbol{r}')$ from spherical symmetry may be neglected.

Assuming that these two regions overlap (for $r \approx R_1$) it is attractive to use the body fixed coordinate system in region I, and the laboratory system in region II, so that the solutions can be joined in their common region.

§ 3.3*a*. *Solution in region I with the aid of a three-dimensional WKB-method* (*Fröman*). Fröman [20] has applied a three-dimensional extension (initiated by Christy [45]) of the WKB-method to the problem of propagation of an alpha particle in the inner region I (neglecting the term T_{rot}). By means of this method Fröman has derived that in a certain approximation the influence of (a) the quadrupole field and (b) the spheroidal shape of the nuclear surface, can be accounted for by replacing the boundary condition, giving the alpha particle wave function at the nuclear surface, by a different one, which will be written down below, and by calculating the propagation of the alpha particle as if only the spherically symmetric part of the potential were acting. We introduce an alpha particle wave function $\psi = (1/r) \sum_{lm'} w_{lm'}(r)\, Y_{lm'}(\vartheta', \varphi')$ and give the formulae, which express these statements. Let

$$K(\boldsymbol{r}') = \{(2M/\hbar^2)[V(\boldsymbol{r}') - E_0]\}^{\frac{1}{2}} = K_0(r) + \Delta K(r, \vartheta'), \tag{3.26}$$

where $K_0(r)$ is the function which we obtain from $K(\boldsymbol{r}')$ by replacing $V(\boldsymbol{r}')$ by its spherically symmetric part $V_0(\boldsymbol{r}')$, so that $\Delta K(r, \vartheta')$ is the anisotropic part of $K(\boldsymbol{r}')$. M is the reduced mass of alpha particle and daughter nucleus; the energy E_0 is the sum of the kinetic energy of the alpha particle at large distance from the nucleus, the recoil energy of the daughter nucleus and the rotational energy of the level of the daughter nucleus we consider (T_{rot} has only a small influence on the solution in region I). Now let the boundary condition at the nuclear surface (3.24) be given as

$$[\psi]_{r=R(\vartheta')} = \psi_0(\vartheta', \varphi') \tag{3.27}$$

and introduce the boundary condition

$$[\psi]_{r=R_0} = \psi_1(\vartheta', \varphi') = \psi_0(\vartheta', \varphi') \exp\left[-\int_{R(\vartheta')}^{R_0} K(\boldsymbol{r}')\, \mathrm{d}r - \int_{R_0}^{r_b} \Delta K(r, \vartheta')\, \mathrm{d}r\right], \tag{3.28}$$

where r_b represents the (outer) turning point of the alpha particle $V(\boldsymbol{r}_b) - E_0 = 0$. According to F r ö m a n [20]) we may now use the boundary condition (3.28) on the sphere $r = R_0$ and calculate the propagation as if only the spherically symmetric part $V_0(\boldsymbol{r}')$ of the potential counted.

Sometimes it is useful to expand the functions in spherical harmonics

$$\psi_0(\vartheta', \varphi') = \sum_{lm'} b_{lm'}^{(0)}\, Y_{lm'}(\vartheta', \varphi') \tag{3.29}$$

and

$$\psi_1(\vartheta', \varphi') = \sum_{\bar{l}\,\bar{m}'} b_{\bar{l}\bar{m}'}^{(1)}\, Y_{\bar{l}\bar{m}'}(\vartheta', \varphi'); \tag{3.30}$$

then the coefficients $b_{lm'}^{(0)}$ and $b_{\bar{l}\bar{m}'}^{(1)}$ are connected by a real and symmetric matrix $k_{\bar{l}l}^{m'}(\beta)\, \delta_{\bar{m}'m'}$, diagonal in m' (cf. F r ö m a n [20]))

$$k_{\bar{l}l}^{m'}(\beta) = \int_0^\pi \overline{P}_{\bar{l}\bar{m}'}(\cos\vartheta') \exp\left[-\int_{R(\vartheta')}^{R_0} K(\boldsymbol{r}')\, \mathrm{d}r - \int_{R_0}^{r_b} \Delta K(r, \vartheta')\, \mathrm{d}r\right] . \overline{P}_{lm'}(\cos\vartheta') \sin\vartheta'\, \mathrm{d}\vartheta', \tag{3.31}$$

where the $\overline{P}_{lm}(\cos\vartheta)$ are associated Legendre functions, normalized according to $\int_0^\pi \{\overline{P}_{lm}(\cos\vartheta)\}^2 \sin\vartheta\, \mathrm{d}\vartheta = 1$. An equivalent formulation of (3.28) is now

$$b_{\bar{l}\bar{m}'}^{(1)} = \sum_{lm'} k_{\bar{l}l}^{m'}(\beta)\, \delta_{\bar{m}'m'}\, b_{lm'}^{(0)}. \tag{3.32}$$

The propagation of the parts with a definite angular dependence $Y_{lm'}(\vartheta', \varphi')$

in a spherically symmetric potential, to which the problem is now reduced, has been investigated thoroughly (cf. [20]) and § 3.3*c*).

§ 3.3*b*. *Solution in region I with the aid of ellipsoidal coordinates.* We want to mention an alternative to the method of solution in region I discussed in § 3.3*a*. This consists in introducing ellipsoidal coordinates ξ, η, φ' in such a way that the problem is separable in these coordinates. We relate these coordinates to the cartesian coordinates x', y', z' in the body fixed system according to (the constant q will be specified later) [37] [46])

$$\left.\begin{aligned} x' &= q\,\{(\xi^2-1)(1-\eta^2)\}^{\frac{1}{2}}\cos\varphi', \\ y' &= q\,\{(\xi^2-1)(1-\eta^2)\}^{\frac{1}{2}}\sin\varphi', \\ z' &= q\xi\eta. \end{aligned}\right\} \tag{3.33}$$

(This implies that the surfaces ξ = constant and η = constant are prolate ellipsoids of revolution and two sheeted hyperboloids of revolution, respectively. Comparing with the spherical coordinates r, ϑ', φ', one has for the limit $r \to \infty$: $q\xi \to r$, $\eta \to \cos\vartheta'$, $\varphi' \to \varphi'$.) The Laplacian is in these coordinates [37] [46])

$$\begin{aligned} \Delta \equiv \frac{1}{q^2(\xi^2-\eta^2)}\Big\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right] &+ \\ + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right] &+ \frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\varphi'^2}\Big\}. \end{aligned} \tag{3.34}$$

If $V(\boldsymbol{r}')$ is given by (3.20), the Schrödinger problem, determined by the Hamiltonian $H \equiv -(\hbar^2/2M)\Delta + V(\boldsymbol{r}')$ becomes separable if we put

$$q^2 = \tfrac{1}{2}Q_0/(Z-2), \tag{3.35}$$

so that $V(\boldsymbol{r}')$ can be written as (strictly speaking equations (3.36) and (3.43) are only correct up to a certain power of q^2/r^2, neglecting higher powers of this quantity)

$$V(\boldsymbol{r}') = [2(Z-2)/Q_0]^{\frac{1}{2}}\, 2(Z-2)\, e^2\xi/(\xi^2-\eta^2). \tag{3.36}$$

Putting the solution $\psi(\boldsymbol{r}')$ of the Schödinger problem equal to

$$\psi(\boldsymbol{r}') = F^{(1)}(\xi)\, F^{(2)}(\eta)\, F^{(3)}(\varphi'), \tag{3.37}$$

we obtain a separation of the Schrödinger equation into the following three equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[(\xi^2-1)\frac{\mathrm{d}F^{(1)}}{\mathrm{d}\xi}\right] - \frac{m'^2}{\xi^2-1}F^{(1)} &+ \\ + \frac{2Mq^2}{\hbar^2}\left[E\xi^2 - \frac{2(Z-2)e^2\xi}{q}\right]F^{(1)} &= \Lambda\, F^{(1)}, \end{aligned} \tag{3.38}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[(1-\eta^2)\frac{\mathrm{d}F^{(2)}}{\mathrm{d}\eta}\right] - \frac{m'^2}{1-\eta^2}F^{(2)} - \frac{2Mq^2}{\hbar^2}E\eta^2F^{(2)} = \Lambda F^{(2)}, \qquad (3.39)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi'^2}F^{(3)} = -\,m'^2\,F^{(3)}. \qquad (3.40)$$

The solution of (3.40) is trivial: $F_{m'}{}^{(3)} = e^{im'\varphi'}$, where m' must be an integer in order that $F_{m'}{}^{(3)}$ be a single-valued function. Equation (3.39) corresponds for $q \to 0$ to the equation providing associated Legendre polynomials P_{lm} for eigenvalues $\Lambda \to l(l+1)$ in the limit of the spherical case. However, for cases of interest here we have $(2Mq^2/\hbar^2)\,E > 1$ and equation (3.39) cannot be treated by perturbation methods (as has been done by Strutt [47]). The solutions $F_{m'\Lambda}{}^{(2)}(\eta)$ have been discussed in detail by Stratton, Morse, Chu, Little and Corbató [48]) and Flammer [49]), who give series expansions of $F_{m'\Lambda}{}^{(2)}(\eta)$ in terms of Legendre functions $P_{lm}(\eta)$ and who give the separation constant Λ numerically for a great variety of the parameters. Finally, we want to find a solution of equation (3.38) for $F_{m'\Lambda}{}^{(1)}(\xi)$, corresponding to the equation for the radial dependence in the spherical case. We can obtain an approximate solution for $F_{m'\Lambda}{}^{(1)}(\xi)$ by applying a one-dimensional WKB-method to (3.38). This gives, for example, in the exponential region

$$F_{m'\Lambda}{}^{(1)}(\xi) =$$
$$=(E/q^2)^{\frac{1}{4}}|(\xi^2-1)\,X(\xi)|^{-\frac{1}{4}}\exp\left[\pm\frac{(2M)^{\frac{1}{2}}}{\hbar}\int^{\xi}\left\{\frac{|X(\xi)|}{\xi^2-1}\right\}^{\frac{1}{2}}\mathrm{d}\xi\right] \text{ if } X(\xi)<0, \qquad (3.41)$$

with

$$X(\xi) = q^2\xi^2\,[E-2(Z-2)\,e^2/q\xi-\hbar^2\Lambda/(2Mq^2\xi^2)]-m'^2/(\xi^2-1). \qquad (3.42)$$

Hence the separation of the problem in the ξ, η, φ' coordinates makes the use of a three-dimensional WKB-method unnecessary. The boundary conditions determining the solutions $F^{(1)}(\xi)$ are: (a) the condition that one has outgoing waves for $r \to \infty$, (b) the boundary condition at the nuclear surface.

A complication, which should be noted, is that the nuclear surface is not a surface $\xi =$ constant. If we have for example a homogeneous charge distribution within the nuclear surface (3.24) (so that Q_0 and q are given by (3.25) and (3.35) respectively), we derive that the nuclear surface is represented by

$$\xi = \xi_{surf}(\eta) = (R_0/q)[1 + (\beta/5)(6\eta^2-1)]. \qquad (3.43)$$

In order to use the boundary condition at the nuclear surface for the problem separated in ξ, η, φ' coordinates, we must propagate from the nuclear surface (3.43) to a surface $\xi = \xi_0$ (e.g. $\xi_0 = (R_0/q)(1+\beta)$, if $\beta > 0$). An approximate WKB-solution for this small distance can easily be given, if (3.41) is used.

We shall not at this place give a detailed comparison of this separation in ellipsoidal coordinates with the three-dimensional WKB-method, which is summarized in § 3.3*a* and which seems to lead to simpler formulae. Such a comparison could be useful in order to get an idea of the accuracy of both ways of calculation, which was the reason for mentioning this second method here.

§ 3.3*c*. *Solution with the wave functions of the Coulomb field.* In order to relate the reduced matrixelements a_l to the boundary condition at the nuclear surface, it is necessary to consider the propagation of the alpha particle from the nuclear surface through the regions I and II to the asymptotic region. We shall use the boundary condition (3.28) (or its equivalent formulation (3.32)) so that the quadrupole field is accounted for according to Fröman's method and the problem may be treated as being spherically symmetric. For simplicity we shall confine ourselves to the transitions to one definite level (K_f, I_f), which does not mean any loss of generality. The Coulomb functions $F_l(\rho)$ and $G_l(\rho)$, being regular and irregular at $\rho = 0$ respectively and being both real functions, are solutions of the differential equation [50] [51])

$$d^2f(\rho)/d\rho^2 + \{1-2\eta/\rho - l(l+1)/\rho^2\}f(\rho) = 0. \tag{3.44}$$

This is the radial part of the Schrödinger equation, i.e. the differential equation $v_{II_f}(r)$ has to satisfy in region II, if we put

$$\rho = kr,\ \eta = 1/(ka),\ \hbar^2k^2 = 2ME,\ a = \hbar^2/\{2(Z-2)\,Me^2\}, \tag{3.45}$$
Z = charge number of the parent nucleus.

The functions are normalized according to

$$G_l(\rho) + iF_l(\rho) \to \exp\,[i\{\rho - \eta \ln(2\rho) - l\pi/2 + \sigma_l(\eta)\}] \text{ if } \rho \to \infty, \tag{3.46}$$

so that the phases are defined by giving

$$\sigma_l(\eta) = \arg \Gamma(l+1+i\eta). \tag{3.47}$$

Comparing (3.14) and (3.46), we see that $v_{II_f}(r)$ can be written in region II as

$$v_{II_f}(r) = \hat{v}_{II_f}[G_l(kr) + iF_l(kr)]\ \exp\,[-i\{\sigma_l(\eta) - \eta \ln(2kr)\}]. \tag{3.48}$$

The differential equation for $w_{lm'}(r)$ in region I can be brought into the form (3.44) if the non-diagonal elements of T_{rot} are neglected. However, the energy which must be taken, differs slightly from the energy occurring for $v_{II_f}(r)$ (by an amount of the order of T_{rot} (K_fI_f), the rotational energy of the final state). Since in the barrier region $G_l(kr)$ and $F_l(kr)$ are exponentially decreasing and increasing functions of r respectively, their amplitudes being equal at the (outer) turning point of the alpha particle, the following inequality holds

$$F_l(kr) \ll G_l(kr) \quad (R_0 < r < R_1). \tag{3.49}$$

so that we have in region I, using boundary condition (3.30)

$$w_{lm'}(r) = b_{lm'}{}^{(1)} G_l(kr)/G_l(kR_0). \tag{3.50}$$

Neglecting the just mentioned energy difference, i.e. putting in both regions I and II

$$\left.\begin{array}{l} \hbar^2 k^2 = 2ME_Q, \\ E_Q = \text{sum of the kinetic energy of the alpha particle at large distance} \\ \qquad \text{from the nucleus and the recoil energy of the daughter nucleus,} \end{array}\right\} \tag{3.51}$$

implies that we commit an error equivalent to multiplying $G_l(kr)$ at $r = R_1$ by a factor close to one, approximately equal to

$$\left[\frac{G_l(kR_1)}{G_l(kR_0)}\right]_{E_Q} \Bigg/ \left[\frac{G_l(kR_1)}{G_l(kR_0)}\right]_{E_Q+T_{rot}(K_f I_f)} =$$

$$= \frac{[G_l(kR_0)]_{E_Q+T_{rot}(K_f I_f)}}{[G_l(kR_0)]_{E_Q}} \Bigg/ \frac{[G_l(kR_1)]_{E_Q+T_{rot}(K_f I_f)}}{[G_l(kR_1)]_{E_Q}} \tag{3.52}$$

(T_{rot} $(K_f I_f)$ represents the rotational energy of the level I_f of the daughter nucleus we consider, with respect to its ground state $I = K_f$). As has been pointed out by Fröman [20] (p. 60) the quotient (3.52) is less dependent on $T_{rot}(K_f I_f)(\ll E_Q)$ then might be expected from the energy dependence of the function $G_l(kr)$. The dependence of (3.52) on T_{rot} and R_1 was derived by Fröman with a WKB-method. This dependence can be obtained in a similar way from the series expansion of $G_l(\rho)$ determined with the Riccati method [52] [53]. It is found that the error committed by assuming (3.51) increases with increasing values of R_1. But in order to justify neglecting the quadrupole field in region II we want to choose R_1 as large as possible. Therefore, one must take a compromise between larger and smaller values for the actual choise of R_1. It seems reasonable to choose R_1 about half-way through the potential barrier, so that, according to (3.52), we commit an error of about 8% and the quadrupole field has decreased to 1% of its value at the nuclear surface (where the field strength of the quadrupole field is about 15% of the Coulomb field strength). According to (3.7), (3.12), (3.13), (3.15), (3.48), (3.49) and (3.50) we find, if we take the Coulomb field to be screened at some large distance from the nucleus, that

$$\frac{a_l}{a_{\bar{l}}} = \frac{\sum_{m'} b_{lm'}{}^{(1)} C^{I_f,K_f}_{I_i,K_f+m',\,l,-m'} (-1)^{m'}}{\sum_{\bar{m}'} b_{\bar{l}\bar{m}'}{}^{(1)} C^{I_f,K_f}_{I_i,K_f+\bar{m}',\,\bar{l},-\bar{m}'} (-1)^{\bar{m}'}} \cdot \frac{G_{\bar{l}}(kR_0)}{G_l(kR_0)} . \exp[i\{\sigma_l(\eta) - \sigma_{\bar{l}}(\eta)\}] \tag{3.53}$$

or, specializing to favored transitions (only $m' = 0$)

$$\frac{b_{l0}{}^{(1)}}{b_{\bar{l}0}{}^{(1)}} = \frac{a_l}{a_{\bar{l}}} \cdot \frac{C^{I_f K_f}_{I_i K_f \bar{l} 0}}{C^{I_f K_f}_{I_i K_f l 0}} \cdot \frac{G_l(kR_0)}{G_{\bar{l}}(kR_0)} . \exp[i\{\sigma_{\bar{l}}(\eta) - \sigma_l(\eta)\}]. \tag{3.54}$$

The quotient $G_l(kR_0)/G_{\bar{l}}(kR_0)$ is easily calculated with the Riccati method ($kR_0 < 2\eta$), considered in detail by Abramowitz [52]) (Eq. (4.5) *)) and Fröberg [53]) (Eq. (9.3)). We use (in their notation)

$$g_2 = -\{\rho/(2\eta-\rho)\}^{\frac{1}{2}}[(2\rho^2 - 6\eta\rho + 9\eta^2)/\{12\rho(\eta-\rho)\}] - \\ - l(l+1)\{(2\eta-\rho)/\rho\}^{\frac{1}{2}}, \qquad (3.55)$$

sothat we have, to the first order in $(2\eta)^{-1}$ in the exponent

$$G_l(kR_0)/G_{\bar{l}}(kR_0) = \exp\,[\{l(l+1) - \bar{l}(\bar{l}+1)\}\{(2\eta)/(kR_0) - 1\}^{\frac{1}{2}}/(2\eta)]. \qquad (3.56)$$

The differences of the Coulomb phases $\sigma_l(\eta)$, defined by (3.47) can be approximated for

$$l, \bar{l} \ll \eta \qquad (3.57)$$

by

$$\sigma_{\bar{l}}(\eta) - \sigma_l(\eta) = \arg\,[\Gamma(\bar{l}+1+i\eta)/\Gamma(l+1+i\eta)] \approx (\bar{l}-l)\,\pi/2 + \\ + \{l(l+1) - \bar{l}(\bar{l}+1)\}/(2\eta). \qquad (3.58)$$

Summarizing we find from (3.54), if (3.57) is satisfied,

$$\frac{b_{l0}{}^{(1)}}{b_{\bar{l}0}{}^{(1)}} = \frac{a_l}{a_{\bar{l}}} \cdot \frac{C^{I_fK_f}_{I_iK_i\bar{l}0}}{C^{I_fK_f}_{I_iK_il0}} \cdot (-1)^{(l-\bar{l})/2} . \exp\left[\frac{l(l+1) - \bar{l}(\bar{l}+1)}{2\eta}\left\{\frac{2\eta}{kR_0} - 1\right\}^{\frac{1}{2}}\right]. \\ . \exp\left[i\,\frac{l(l+1) - \bar{l}(\bar{l}+1)}{2\eta}\right]. \qquad (3.59)$$

The matrix $k_{\bar{l}l}^{m'}(\beta)$, defined by (3.31), is real. Hence the relation (3.59) can be used to derive whether the expansion coefficients $b_{l0}{}^{(0)}$ (cf. (3.29)) on the actual nuclear surface have a real or complex ratio, once the relative values of the a_l are obtained from experiments. If the coefficients $b_{l0}{}^{(0)}$ are real, as might be expected from certain considerations for favored alpha transitions (cf. chapter I), we find for the factor $\cos(\alpha_0 - \alpha_2)$ in the explicit formula (2.26), using (3.59) and taking $\eta \approx 25$, that

$$\cos(\alpha_0 - \alpha_2) \approx \cos(n\pi + 0{,}12) = \pm\, 0.993 \qquad (3.60)$$

(the integer n depends on the relative sign of $b_{00}{}^{(1)}$ and $b_{20}{}^{(1)}$). This implies that we can take the coefficients a_0 and a_2 real in (2.26) to a good approximation and that we can omit the factor $\cos(\alpha_0 - \alpha_2)$ and the absolute value bars.

§ 3.3*d*. *Comparison with the solution according to the WKB-method*. The same result (3.59) is obtained if one makes use of the WKB-solution of the wave function in the Coulomb field, as has been done by Fröman [20]).

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*) In [52]) Eqs. (4.5) and (4.6) the denominator of g_2 should read $12\varrho(\eta - \varrho)$ instead of $12\eta\,(\eta - \varrho)$.

However, since $kR_0 < 2\eta$ in alpha disintegration we can also use the Riccati method [52]) [53]), which gives a rapidly converging series expansion in powers of $(2\eta)^{-1}$. This method has the advantage that it is easier to determine the error one commits; furthermore the l-dependence of the transparency of the potential barrier is easily recognized. These features led us to present this treatment in § 3.3*c* as an alternative to the WKB-method. The condition (3.49) corresponds in the WKB-approximation to the neglecting of one of the two solutions inside the barrier (which is valid at some distance from the turning point where the solution inside the barrier is connected to the oscillatory solution outside the barrier); the phase of the oscillatory solution is provided by (3.47).

However, it should be realized that the use of the Riccati method according to § 3.3*c* does not avoid the use of Fröman's three-dimensional WKB-method for taking quadrupole field and nuclear deformation into account. Hence we keep an approximate treatment. For an exact treatment of the problem, one comes always back to the numerical solution of a system of coupled differential equations.

§ 4. *Relation between the directional distributions in the body fixed system of the nucleus and the laboratory system for the classical limit of a heavy nucleus.* The understanding of the relation between the descriptions of alpha emission in the body fixed and laboratory system can be deepened by studying the classical limit of alpha emission from a nucleus which is considered as a classical heavy body. Although this limit is not yet attained for actual nuclei, the study of this limit provides a better insight in the meaning of geometrical averaging procedures used in connection with the alpha particle directional distribution.

In order to approach this limit we shall assume the nucleus to be heavy, so that it will behave as a classical top. We further assume I_i to be large, permitting the use of the asymptotic expressions for Clebsch-Gordan coefficients etc. for large I_i [28]). Finally we suppose that

$$\left.\begin{array}{l} I_i = I_f = K_i = K_f \\ \text{and that } m' = 0 \text{ (i.e., } b_{lm'}^{(0)} \neq 0 \text{ only for } m' = 0\text{; cf. (3.29))} \end{array}\right\} \quad (4.1)$$

(favored transition from the ground state $I_i = K_i$ of the parent nucleus to the ground state $I_f = K_f$ of the daughter nucleus).

We now want to consider what may be called the directional distribution in the body fixed system. For the wave function of the alpha particle in the body fixed system we should take

$$\psi(\boldsymbol{r}') = \Sigma_l (1/r)\, w_{l0}(r)\; Y_{l0}(\vartheta', \varphi'). \quad (4.2)$$

The expression (4.2) agrees with the form (3.9), where one may note, however, that the symmetrization of the wave function does not have a strict classical analogue. Using (3.16) we find for the directional distribution in the body fixed coordinate system

$$W_B(\vartheta') = 4\pi\, |\Sigma_l\, \hat{w}_{l0} \exp\,[i(kr - l\pi/2)]\; Y_{l0}(\vartheta', \varphi')|^2 =$$
$$= \Sigma_{l\bar{l}}\, \hat{w}_{l0}\hat{w}_{\bar{l}0}{}^*\{(2l+1)(2\bar{l}+1)\}^{\frac{1}{2}}\, i^{-l+\bar{l}}\, |C^{k0}_{l0\bar{l}0}|^2\, P_k\,(\cos\vartheta'). \qquad (4.3)$$

If the nucleus may be considered as a classical body, i.e. a heavy and slowly rotating top, a directional distribution in the body fixed system can clearly be defined. The coefficients $\hat{w}_{l0}$ are always defined by (3.16) and one could define in any case a directional distribution $W_B(\vartheta')$ related to the coefficients $\hat{w}_{l0}$, according to equation (4.3). However, this directional distribution $W_B(\vartheta')$ can be said to represent really the *directional distribution in the body fixed system* only in the classical limit (otherwise one cannot measure simultaneously all variables specifying the direction of emission of an alpha particle and the position angles of the nuclear symmetry axis).

We shall now write down the directional distribution (2.20) for an oriented ensemble of nuclei for which the orientation is specified by the density matrix

$$\rho_{M_i''M_i'} = \delta_{M_i''M_i}\, \delta_{M_i'M_i}, \qquad (4.4)$$

or alternatively by the orientation parameters

$$f_k(I_i)\, \{w_k(I_i)\}^{-1} = \langle|(I_iI_i)k0\rangle = C^{k0}_{I_iM_iI_i-M_i}(-1)^{I_i-M_i}. \qquad (4.5)$$

Substitution of (4.5) into (2.20) yields, if we take into account the assumptions (4.1),

$$W_L(\vartheta) = \Sigma_{l\bar{l}k}\, a_l\, a_{\bar{l}}{}^*\, i^{l+\bar{l}}\, \{(2l+1)(2\bar{l}+1)\}^{\frac{1}{2}}\, C^{k0}_{l0\,\bar{l}0}\, \cdot$$
$$\cdot\; W(I_l\bar{l}I_ik;\, I_il)(-1)^{I_i-M_i}\, C^{k0}_{I_iM_iI_i-M_i}\, P_k\,(\cos\vartheta). \qquad (4.6)$$

The axis of rotational symmetry of the nuclei of this ensemble (4.4) will precess about the z-axis of the laboratory system and it will make an angle β with this z-axis, β being determined by

$$\cos\beta = M_i/I_i. \qquad (4.7)$$

If we write down the directional distribution (4.6) for $M_i = I_i$, the resulting distribution should coincide in the classical limit with the directional distribution in the body fixed system of the nucleus (4.3), since then the z-axis and the z′-axis coincide. The density matrix for this case is

$$\rho_{M_i''M_i'} = \delta_{M_i''I_i}\, \delta_{M_i'I_i} \qquad (4.8)$$

and the angular distribution (4.6) becomes

$$W_{LB}(\vartheta'') = \Sigma_{l\bar{l}k}\, a_l\, a_{\bar{l}}{}^*\, i^{l+\bar{l}}\, \{(2l+1)(2\bar{l}+1)\}^{\frac{1}{2}}\, C^{k0}_{l0\,\bar{l}0}\, \cdot$$
$$\cdot\; W(I_l\bar{l}I_ik;\, I_il)\, C^{k0}_{I_iI_iI_i-I_i}\, P_k\,(\cos\vartheta''). \qquad (4.9)$$

We shall prove now for the classical limit (4.1):

(a) that the distributions (4.3) and (4.9) are equivalent in the limit for large I_i,

(b) that the distribution (4.6) results if we "smear out" the distribution (4.9) over the precession of the angular momentum vector I_i about the laboratory z-axis.

Proof of statement (a).

We deduce from III (2.1) *)

$$C^{k0}_{I_iM_iI_i-M_i} \approx (-1)^{I_i-M_i} \{(2k+1)/(2I_i+1)\}^{\frac{1}{2}} P_k(\cos\beta = M_i/I_i) \quad (4.10)$$

if

$$I_i \gg k \ (I_i \text{ large}) \quad (4.10a)$$

and also

$$C^{I_iI_i}_{I_iI_l0} \approx 1 \quad (4.11)$$

if

$$I_i \gg l \ (I_i \text{ large}). \quad (4.11a)$$

Using (3.18), (4.1), (4.10), (4.11) and III (3.1), we see that the two distributions (4.3) and (4.9) are equal in this classical limit. We point out that there is no summation over m' to be performed in (3.18), since $w_{lm'}(r) = 0$ if $m' \neq 0$ on or just outside the nuclear surface (cf. (4.1)) implies $\hat{w}_{lm'} = 0$ if $m' \neq 0$ (in the classical limit).

Proof of statement (b).

The distribution function that results from "smearing out" the distribution (4.9) over the precession of z' about z (cf. (4.7)) is calculated according to (cf. fig. 1)

$$W_{Av}(\vartheta) = (2\pi)^{-1} \int_0^{2\pi} W_{LB}(\cos\vartheta'')\, \mathrm{d}\Phi, \quad (4.12)$$

where

$$\cos\vartheta'' = \cos\beta\cos\vartheta + \sin\beta\sin\vartheta\cos\Phi. \quad (4.13)$$

We shall use the addition theorem [46] (p. 74) for Legendre polynomials for a reduction of (4.12)

$$P_k(\cos\vartheta'') = P_k(\cos\beta)\, P_k(\cos\vartheta) + \\ + 2\sum_{m=1}^{k} [(k-m)!/(k+m)!] P_{km}(\cos\beta)\, P_{km}(\cos\vartheta)\cos(m\Phi). \quad (4.14)$$

It follows from (4.7) and (4.14) that

$$(2\pi)^{-1} \int_0^{2\pi} P_k(\cos\vartheta'')\, \mathrm{d}\Phi = P_k(\cos\beta = M_i/I_i)\, P_k(\cos\vartheta). \quad (4.15)$$

*) Using this equation we can easily write down the classical limit of the orientation parameters $f_k(I_i)$ for ensembles of nuclei given by (4.4). We derive from (2.13), (4.5) and (4.10), using the Stirling approximation III (A. 2.4) to reduce the factorials, that

$$f_k(I_i) \approx (k!)^2\{(2k)!\}^{-1}\, 2^k\, P_k(\cos\beta = M_i/I_i)$$

if $I_i \gg 1$ and the ensemble is given by (4.4).

Furthermore we derive from (4.10)

$$C^{k0}_{I_iM_iI_i-M_i} \approx (-1)^{I_i-M_i}\, C^{k0}_{I_iI_iI_i-I_i}\, P_k\,(\cos\beta = M_i/I_i) \tag{4.16}$$

$$\text{if } I_i \gg k \;(I_i \text{ large}). \tag{4.16a}$$

Using (4.15) and (4.16), we see that $W_{Av}(\vartheta)$, defined by (4.12) and representing the averaged distribution (4.9), equals $W_L(\vartheta)$ defined by (4.6); hence we have proved statement (b).

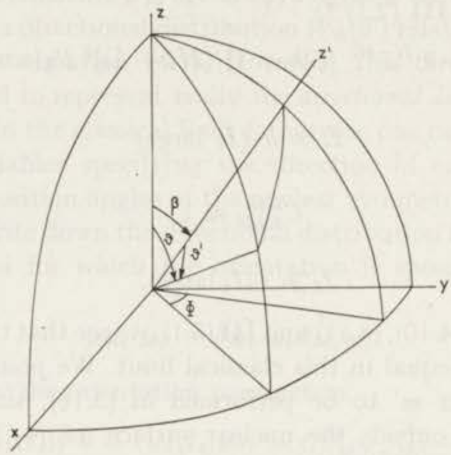

Fig. 1. Relation between the laboratory system (x, y, z-axes) and the body fixed system (z'-axis). The directional distribution in the laboratory system $W_L(\vartheta)$ (cf. (4.6)) is obtained (for a heavy nucleus, $I_i \gg 1$) by averaging the directional distribution in the body fixed coordinate system $W_B(\vartheta')$ (cf. (4.3)) over the precession of the z'-axis about the z-axis.

This latter statement expresses clearly that the averaging, which is carried out in (4.6) consists in the classical limit of a smearing out of the body fixed directional distribution by a precession at the angle $\beta(\cos\beta = M_i/I_i)$ so that the directional distribution in the laboratory system is obtained. We may point to the fact that the averaging by the precession of I_i around the z-axis is still quite general, i.e. not restricted to our special case of radiation from an ensemble of spheroidally deformed nuclei. This restriction comes in as soon as we relate (in the classical limit) the directional distribution (4.3) to the distributions (4.6) and (4.9). In order to establish this relation we made use of equation (3.18) that we derived after introducing the description of the motion of the alpha particle in the body fixed coordinate system.

§ 5. *The analysis of experiments on the directional distribution of alpha particles from oriented nuclei and the information which it can provide.*

§ 5*a*. *The experimental situation.* Alignment of alpha emitters was attained as well by the magnetic as electric h.f.s. alignment methods [38]), i.e. by means of the interactions of the magnetic dipole moment and of the electric quadrupole moment in crystals at liquid helium temperatures. This means that the aligned nuclei must be contained in a cryostat, which requires that also the alpha particle detection must take place within the cryostat. The procedure which was followed consisted in the use of a scintillator within the cryostat and a light pipe carrying the light pulse to a photomultiplier outside the cryostat.

The character of the disintegration scheme of a strongly deformed nucleus emitting alpha particles is shown in fig. 2. One should note that the alpha transitions show a fine structure due to the existence of the rotational band for the final nucleus (energy differences of the order of 50 keV at a total energy of the order of 5 MeV). Between the rotational states gamma transitions occur with energies of the order of 50 keV. For non-oriented nuclei one may observe: (a) the intensity ratios of the fine structure components, (b) alpha-gamma directional correlations, (c) conversion coefficients and lifetimes of the gamma transitions. For aligned nuclei one may observe in addition: (1) the directional distribution of the alpha particles, (2) the directional distribution of the gammas, (3) the influence of the alignment on alpha-gamma coincidence measurements.

We want to consider in detail the measurements on the alpha particle directional distribution and we shall discuss the other possible observations only in so far as they have a direct bearing on the first topic. We shall give the discussion for favored alpha emission, for which we assume $K_i = K_f$, $m' = 0$, so that $I_i = K_i = I_f = K_f$ for the transition to the ground state $I_f = K_f$ of the rotational band. Furthermore the parity of the nucleus does not change in these transitions and the alpha particle can only be emitted with $l = 0, 2, 4, \ldots$. (The discussion for unfavored alpha transitions would be analogous in many respects).

§ 5*b*. *The directional distributions of the partial transitions.* In the analysis of the experiments on the alpha particle directional distribution, we start from the formulae (2.26) (2.27) and (2.28), which express the directional distribution $W(\vartheta)$ as a function of the coefficients a_0, a_2, $a_4 \ldots$ and the orientation parameters $f_2, f_4, \ldots$. The values of the orbital angular momenta of the alpha particles, which contribute appreciably to the partial transitions, are indicated in fig. 2 (the $l = 4$ wave could contribute to α_1, α_2, α_3 in so far as angular momentum conservation is concerned, but is still very weak [7]) [20])). If only one value of the angular momentum l contributes to a partial transition, and the initial and final nuclear spins are known, no nuclear parameters are left undetermined and the directional distribution is fixed for given orientation parameters. This holds for the partial transitions α_2, α_3, α_4. However, the partial transition α_1 is caused in general by interfering $l = 0$ and $l = 2$ waves, and the directional distribution $W(\vartheta)$ depends on the typically nuclear parameter a_2/a_0 (see formula (2.26)). Therefore experiments on the directional distributions of α_2, α_3, α_4 can only give a check of the theory (as soon as the spin and parity assignments in the scheme are known), but the distribution of α_1 gives really new information. Similarly it should be noted that the directional distributions of α_2, α_3, α_4 for oriented nuclei (being of a geometrical nature) do not provide more information on the alpha transition than could also be obtained by observing the alpha-gamma directional correlation (of non-oriented nuclei) for these partial alpha transitions with the succeeding gammas. However, no such observation is possible for α_1, as this partial transition is not succeeded by a gamma.

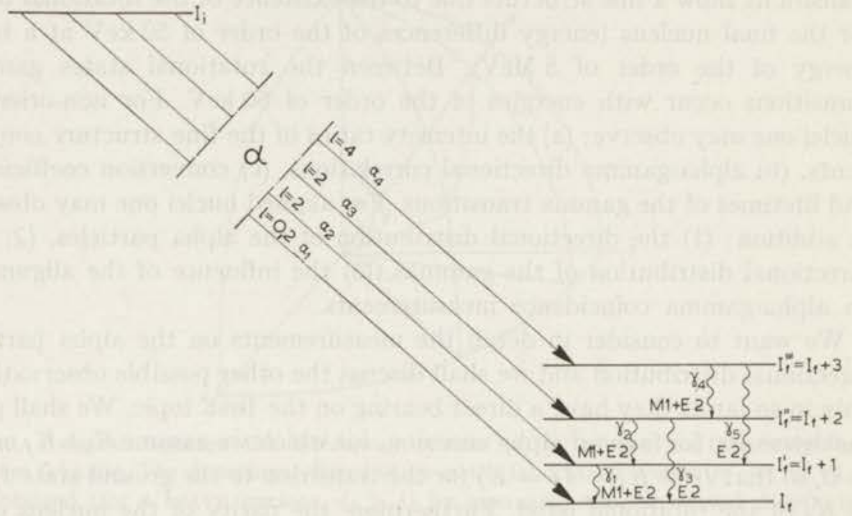

Fig. 2. Disintegration scheme for a spheroidally deformed alpha emitting nucleus. Several states of the rotational band of the final nucleus can be reached. For these favored transitions it is assumed that $I_f = I_i$, so that $l = 0$ transitions are possible to the ground state.

Thus the directional distribution of α_1 for oriented nuclei is the thing of most interest. In the above mentioned experimental situation it will be very difficult to measure the distributions of α_1, α_2, α_3, and α_4 separately (because a scintillator cannot distinguish the small energy differences). However, this is only a small complication for the analysis which can be carried out in the following way: we shall assume that

(a) the orientation parameters $f_2(I_i)$ and $f_4(I_i)$ for the initial nuclei are known,

(b) the spin assignments in the disingration scheme are known,

(c) the ratios of the partial intensities $P_{\alpha_1} : P_{\alpha_2} : P_{\alpha_3} : P_{\alpha_4}$ for the transitions α_1, α_2, α_3 and α_4 are measured for non-oriented nuclei.

The directional distributions for α_2, α_3 and α_4 can then be calculated according to (2.27), (2.28) and (2.20) from the values $f_2(I_i)$ and $f_4(I_i)$ and can be subtracted (taking the intensity ratios $P_{\alpha_1} : P_{\alpha_2} : P_{\alpha_3} : P_{\alpha_4}$ into account) from the total observed directional distribution, providing the directional distribution of α_1 separately.

Once the directional distribution of α_1 is obtained separately, one may proceed in two ways:

I. One may assume (from theoretical arguments (cf. § 3.3*c*) that the ratio a_2/a_0 is real and determine the ratio a_2/a_0 in sign and magnitude from $W(\vartheta)$, using (2.26).

II. If the absolute value of the ratio $|a_2/a_0|$ can be derived with sufficient precision from $P_{\alpha_2}/P_{\alpha_1}$ and $P_{\alpha_3}/P_{\alpha_1}$ according to § 5*d*, one may check with the aid of (2.26) whether the ratio a_2/a_0 determined from $W(\vartheta)$ is real and determine its sign.

When proceeding according to I. one can check whether the value obtained for a_2/a_0 from $W(\vartheta)$ is in accordance with $|a_2/a_0|$ determined from $P_{\alpha_2}/P_{\alpha_1}$ and $P_{\alpha_3}/P_{\alpha_1}$ (§ 5*d*).

The sign of a_2/a_0 forms the typically nuclear information which is obtained from this type of experiment. How this information can be utilized will be discussed below.

§ 5*c*. *The determination of the orientation parameters* $f_2(I_i)$ *and* $f_4(I_i)$ presents a certain problem which one may try to solve in both following ways:

I. *The hyperfine structure splittings of the aligned nuclei in the crystal* (due to the magnetic moment μ and (or) the electric quadrupole moment Q are known from other experiments (nuclear resonance or hyperfine structure of paramagnetic resonance). If these splittings are known and the temperature can be determined, the orientation parameters $f_2(I_i)$ and $f_4(I_i)$ can be calculated immediately from the populations according to Boltzmann's law. This method requires that either the h.f.s. measurements are performed with radioactive nuclei or that they are done for stable nuclei and that the ratios of μ and Q for the stable and radioactive nuclei are known as well as the way in which the hyperfine structure is composed from the electric and magnetic interactions.

II. A method, which avoids some difficulties of method I. consists in measuring the directional distribution of the gamma radiation at the same time as for the alpha particles for oriented nuclei (at the same time but without alpha-gamma coincidences). When comparing the different gamma transitions that occur (cf. fig. 2), the transition $\gamma_3(I_f + 2 \rightarrow I_f)$ is most advantageous for determining the orientation in the following respects:

(1) it has a higher energy (of about 100 keV) than the $\gamma_2(I_f + 2 \rightarrow I_f + 1)$ and $\gamma_1(I_f + 1 \rightarrow I_f)$ transitions (of about 50 keV), so that it should give a better separated photopeak in a scintillation counter and penetrate the cryostat walls more easily.

From the observed directional distribution of γ_3, one can immediately derive the orientation parameters $f_2(I_f'')$ and $f_4(I_f'')$ of the second excited level of the final nucleus (see [30]) [32]) for the formulae). However we need the orientation parameters $f_2(I_i)$ and $f_4(I_i)$ of the initial nuclei. The relation between f_2 and f_4 before and after the alpha transition α_3 can be taken from the discussion by Cox and Tolhoek [54]) (cf. formulae (6), (12), (16) and (17); the character of the transition α_3 is specified by $l = 2$) and is expressed by (in notation of [54]); cf (2.32))

$$M_k(I_i)\, f_k(I_i) = M_k(I_f'')\, f_k(I_f''), \tag{5.1}$$

or written explicitly for $k = 2,4$ if $I_f'' = I_f + 2$ and $I_i = I_f = I$

$$f_2(I_i) = \frac{(I+1)(I+2)^2\,(2I+3)}{I^2(I+3)(2I+7)}\, f_2(I_f''), \tag{5.2a}$$

$$f_4(I_i) = \frac{(I+1)(I+2)^5(2I+3)(2I+5)}{I^4(I+3)(I+4)(2I+7)(2I+9)}\, f_4(I_f''). \tag{5.2b}$$

Hence we see that the determination of $f_2(I_i)$ and $f_4(I_i)$ can be performed with the aid of the γ_3 directional distribution avoiding the difficult h.f.s. and temperature measurements.

§ 5*d*. *The relation between* $|a_2/a_0|$ *and the intensity ratios in the alpha fine structure.* The ratio $|a_2/a_0|$ is closely related to the ratio c_2'/c_0' of the (positive) coefficients c_l', introduced by Bohr, Fröman and Mottelson [15]) and Fröman [20]) (Eq. (VIII–19)) and defined by (cf. (3.30))

$$c_l' = \left| b_{l0}{}^{(1)} \exp\left[-\frac{l(l+1)}{2\eta} \left\{ \frac{2\eta}{kR_0} - 1 \right\}^{\frac{1}{2}} \right] \right|^2 \tag{5.3}$$

so that we find, using (3.59) and (5.3)

$$|a_2/a_0|^2 = |C^{I_iI_i}_{I_iI_i20}|^2\, c_2'/c_0', \tag{5.4}$$

where we took into account the equality

$$I_i = K_i = I_f = K_f, \tag{5.5}$$

valid for the favored ground state to ground state transition. The coefficients c_l' can be determined from the fine structure measurements in alpha decay by the use of [20]) (Eq. (VIII–18))

$$\frac{F_{I_f}}{F_{I_f=I_i}} = \frac{\sum'_l c_l'\, |C^{I_fK_f}_{I_iK_fl0}|^2}{\sum'_l c_l'\, |C^{I_iK_f}_{I_iK_fl0}|^2} \tag{5.6}$$

(Σ' means summation over only even values of l, as we assumed that the parities of parent and daughter nucleus are the same). The intrinsic transition probability F_{I_f} represents the F-value [5]) [7]) [15]) [20]) of the alpha transition $I_i \to I_f$, defined as the quotient of the experimentally observed transition probability P_α and the "smoothed out" transition probability $P_0(Z, E)$, given by the Geiger-Nuttall law. Thus we are able to determine the value of the ratio c_2'/c_0' from the experimental data, using (5.6). Fröman derived the coefficients c_l' from interpolation between the neighboring even-even nuclei; then checked the formula (5.6). However, it is then necessary to assume that the alpha particle wave function on the nuclear surface for favored odd-A alpha decay can be determined by interpolation of the corresponding wave functions of the neighboring even-even alpha emitters, which is an additional assumption.

§ 5e. *Relation between different quantities, figuring in the analysis.* The relative sign (and phase) of a_0 and a_2, as could be obtained from experiments according to the preceding analysis, provides a datum on the boundary condition for the alpha particle wave function at the nuclear surface. According to the analysis of § 3 (equations (3.54), (3.56), (3.58), (3.59)) the quantitative relation between the ratio $a_l/a_{\bar{l}}$ and $b_{l0}^{(1)}/b_{\bar{l}0}^{(1)}$ allows to specify what a result for a_2/a_0 means for the effective boundary condition $\psi^{(1)}(\vartheta)$ at the sphere $r = R_0$, or through (3.32) also for the boundary condition at the real nuclear surface (3.24). It should be noted that the phase (up to a sign) of a_2/a_0 almost equals the phase of $b_{20}^{(1)}/b_{00}^{(1)}$. It is reasonable to suppose that the coefficients $b_{l0}^{(1)}$ can be chosen real for favored alpha transitions (cf. chapter I). It follows that the coefficients a_0 and a_2 should also be (approximately) real (cf. (3.59) and (3.60)).

We note that we have

$$C^{I_f K_f}_{I_i K_i l0} > 0 \tag{5.7}$$

for all values of l, if we have a ground state to ground state transition (5.5). It then follows from (3.59) that the relative sign of $b_{00}^{(1)}$ and $b_{20}^{(1)}$ is opposite to the relative sign of a_0 and a_2. Hence one sees that the result that a_0 has the opposite (same) sign as a_2 determines whether the absolute value of the function (say $\psi_{0,2}^{(1)}(\vartheta')$) obtained from $\psi^{(1)}(\vartheta')$ by taking only its $l = 0$ and $l = 2$ components, has its maximum at the poles (equator) of the spheroidally deformed nucleus (i.e. for $\vartheta' = 0$ or $\vartheta' = \pi/2$ respectively). Neglecting in (2.26) the influence of the term P_4 (cos ϑ) (which is multiplied by the orientation parameter f_4 which is still small at liquid helium temperatures) on the position of the maximum of the directional distribution, we see that the sign of the coefficient P_2 (cos ϑ)

$$\left[-|a_0|\,|a_2|\cos(\alpha_0-\alpha_2)\,6\sqrt{5}\left\{\frac{I_i(I_i+1)}{(2I_i-1)(2I_i+3)}\right\}^{\frac{1}{2}} + \right.$$
$$\left. + |a_2|^2\,\tfrac{15}{7}\,\frac{(2I_i-3)(2I_i+5)}{(2I_i-1)(2I_i+3)}\right]K_2 f_2 \tag{5.8}$$

determines the position of the maximum and conversely. From (5.8) and the preceding discussion it is clear that, if the condition

$$\left|\frac{a_2}{a_0}\right| < \frac{14}{\sqrt{5}}\cdot\frac{\{I_i(I_i+1)(2I_i-1)(2I_i+3)\}^{\frac{1}{2}}}{(2I_i-3)(2I_i+5)} \tag{5.9}$$

is satisfied, the following statements are equivalent:

(1) the directional distribution $W(\vartheta)$ has its maximum along the laboratory z-axis, if $f_2 > 0$,

(2) a_0 and a_2 have opposite sign,

(3) $b_{00}^{(1)}$ and $b_{20}^{(1)}$ have the same sign, so that there is a maximum in the absolute value of $\psi_{0,2}^{(1)}(\vartheta')$ (the $l = 0,2$ part of $\psi^{(1)}(\vartheta')$) along the body fixed z'-axis.

It seems that the inequality (5.9) will be satisfied for practically all nuclei: the right hand member has a value of at least 3.1 for $I_i \geqslant \frac{3}{2}$ and the values of $|a_2/a_0|$ obtained from the c_2', c_0'-values (see (5.4)) (determined by Fröman for a number of nuclei from intensity ratios in the alpha fine structure) are all smaller than 0.6.

We can furthermore consider a directional distribution $W_B(\vartheta')$ defined by (4.3) (and depending on $\hat{w}_{00}$ and $\hat{w}_{20}$), although only in the classical limit $W_B(\vartheta')$ can strictly be considered as the body fixed directional distribution. According to (3.18) the relative sign of a_0 and a_2 is the same as for $\hat{w}_{00}$ and $\hat{w}_{20}$, assuming that (5.5) is satisfied so that (5.7) holds. It follows from (4.3) that a negative sign of $\hat{w}_{20}/\hat{w}_{00}$ (or of a_2/a_0) and a maximum of $W_B(\vartheta')$ along the z'-axis, will occur at the same time. Furthermore one sees that $W_B(\vartheta')$ and $|\psi_{0,2}^{(1)}(\vartheta')|$ will also have their maxima along the z'-axis simultaneously, if (5.9) is astisfied.

We have included this discussion of $W_B(\vartheta')$ merely as an illustration; it is not necessary for the analysis of the experiments, in which one may pass directly from a_0 and a_2 to $b_{00}^{(1)}$ and $b_{20}^{(1)}$. The purpose of the preceding paragraphs has been to show the relations between the different quantities figuring in the analysis ($W(\vartheta)$; a_0, a_2; $\hat{w}_{00}$, $\hat{w}_{20}$; $W_B(\vartheta')$; $b_{00}^{(1)}$, $b_{20}^{(1)}$; $\psi^{(1)}(\vartheta')$, $\psi^{(0)}(\vartheta')$) and to visualize these relations to some extent, especially under the conditions of a feasible experiment.

There are very few experimental data [26]) [27]) available to which this theory can be applied. An analysis of the data for ^{237}Np cannot be given because the interpretation of the disintegration scheme is not clear. For ^{233}U one has favored alpha transitions to a rotational band. However, the orientation parameter f_2 was not determined in this experiment. A preliminary analysis along the lines of this paper, shows that the ^{233}U data [27]) could be fitted with $f_2 \approx 0{,}05$ and $a_2/a_0 \approx +0{,}5$ (preferential emission from the equator).

§ 5*f*. *A summary of our investigation of the directional distribution of alpha particles* from oriented nuclei (in favored alpha transitions) can be given in the following points:

(1) The measurement of this directional distribution provides independent information which cannot be obtained from other measurements as alpha-gamma directional correlation (for non-oriented nuclei) or relative intensities in the alpha fine structure.

(2) A useful method of determining the degree of orientation of the

initial nuclei consists in the determination of the directional distribution of the $I_f'' = I_f + 2 \rightarrow I_f$ gamma transition of the final nucleus (simultaneously with the alpha measurement, but without alpha-gamma coincidences).

(3) The independent information, which is provided by this type of experiment, can be expressed as the relative sign (and phase) of the amplitudes a_0 and a_2 (cf. (2.26)). Except for a sign the theoretical calculation provides the relative phase, which can therefore be checked. The relative sign is connected with a preference of emission of the alpha particles from the poles or from the equator of the spheroidally deformed nucleus. This depends on the way the alpha particle is formed from shell model states of nucleons in the parent nucleus; hence it is seen that information is obtained concerning the *internal* problem of alpha particle formation.

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Chapter III

CLASSICAL LIMITS OF CLEBSCH-GORDAN COEFFICIENTS, RACAH COEFFICIENTS AND $D^{l}_{mn}(\varphi, \vartheta, \psi)$-FUNCTIONS

Synopsis

The "classical limits" (asymptotic expressions for large angular momenta) are investigated for a number of quantities. Some relations for Clebsch-Gordan coefficients and Racah coefficients for large angular momenta are derived. Classical analogues for the square of Clebsch-Gordan coefficients and the square of the $D^{l}_{mn}(0, \vartheta, 0)$-functions are proposed on the basis of their geometrical meaning. WKB-expressions are derived for $D^{l}_{mn}(\varphi, \vartheta, \psi)$ starting from the Schrödinger equation for the symmetric top. These expressions include WKB-expressions for the spherical harmonics $Y_{lm}(\vartheta, \varphi)$. It is shown in which way these WKB-expressions are related to the classical analogues for the Clebsch-Gordan coefficients and $D^{l}_{mn}(\varphi, \vartheta, \psi)$-functions.

§ 1. *Introduction.* In recent years the vector addition and recoupling of angular momenta in quantum mechanics has been used in a wide variety of applications. In the calculations Clebsch-Gordan coefficients [1]) [2]), spherical harmonics, Racah coefficients [2]) and the $D^{l}_{mn}(\varphi, \vartheta, \psi)$-functions [3]) are constantly used. The study of the behavior of these quantities for high values of the angular momenta ("classical limit") is of interest in connection with certain applications, as well as from the mathematical point of view. In this paper some new mathematical results are derived for this "classical limit"; they are mutually compared and are also discussed in connection with some older results of this kind.

In § 2 an expression for the Clebsch-Gordan coefficients with one small and two large l-values, given before by Edmonds [4]), is discussed. This expression is used in § 3 for deriving the asymptotic value of the Racah coefficient, if certain angular momenta are equal and large. A classical analogue of (the square of) Clebsch-Gordan coefficients is given in § 4. WKB-expressions for $D^{l}_{mn}(\varphi, \vartheta, \psi)$ and $Y_{lm}(\vartheta, \varphi)$ are derived in § 5. In § 6 it is shown which correspondences between the results of § 4 and § 5 exist. The appendices list the conventions used in this paper and contain a number of formulae and derivations, to which we referred in the main text.

§ 2. *Limit of Clebsch-Gordan coefficients* [1] [2] *with one small and two large angular momenta.* The following approximate formula relates the behavior of Clebsch-Gordan coefficients *) with one small and two large angular momenta to a value of the $D^{\kappa}_{\mu\tau}(\varphi, \vartheta, \psi)$-function (cf. (A. 1.3))

$$C^{l,m}_{\kappa,\mu,l-\tau,m-\mu} \approx (-1)^{\kappa-\tau} D^{\kappa}_{\mu\tau}(0, \vartheta, 0), \tag{2.1}$$

if

$$l \gg 1, \quad \kappa \ll l, \tag{2.1a}$$

$$\cos\vartheta = m/l. \tag{2.1b}$$

This formula is mentioned by Edmonds [4], who also gave an indication for the proof. We have written down a derivation of (2.1) in Appendix 2 in order to show the limits of validity of the formula (an assumption $m \gg 1$, given by Edmonds for the validity of (2.1) is not necessary for this derivation).

The condition (2.1a) $\kappa \ll l$ implies $\mu \ll l$ and $\tau \ll l$.

§ 3. *Expression for the Racah coefficient* [2] $W(I_i l' I_i k; I_i l)$ *in the limiting case* $I_i \gg 1$ *and* $l', k, l \ll I_i$. For this limiting case the following expression is valid

$$W(I_i l' I_i k; I_i l) \approx (-1)^{l'} \{(2k+1)(2I_i+1)\}^{-\frac{1}{2}} C^{ko}_{lol'o}, \tag{3.1}$$

if

$$I_i \gg 1 \text{ and } l', k, l \ll I_i. \tag{3.1a}$$

If not all three angular momenta l', k and l are integral, both the Racah coefficient and the Clebsch-Gordan coefficient in (3.1) vanish.

A proof of (3.1) may be given with the aid of (2.1). We derive from the definition of the Racah coefficient [2] and the symmetry properties of the Clebsch-Gordan coefficients

$$C^{ko}_{l'olo} W(I_i l' I_i k; I_i l) \{(2I_i+1)(2l+1)\}^{\frac{1}{2}} =$$
$$= \sum_\sigma C^{I_i\sigma}_{l'oI_i\sigma} C^{ko}_{I_i\sigma I_i-\sigma} C^{lo}_{I_i\sigma I_i-\sigma} =$$
$$= \frac{\{(2k+1)(2l+1)\}^{\frac{1}{2}}}{2I_i+1} (-1)^{l+k} \sum_\sigma C^{I_i\sigma}_{l'oI_i\sigma} C^{I_i\sigma}_{koI_i\sigma} C^{I_i\sigma}_{loI_i\sigma}. \tag{3.2}$$

In the limit $I_i \gg 1$ we write, using (2.1)

$$\sum_{\sigma=-I_i}^{+I_i} C^{I_i\sigma}_{l'oI_i\sigma} C^{I_i\sigma}_{koI_i\sigma} C^{I_i\sigma}_{loI_i\sigma} \approx$$
$$\approx I_i \int_{\sigma/I_i=-1}^{+1} (-1)^{l'+k+l} D^{l'}_{oo}(0, \arc\cos(\sigma/I_i), 0) D^{k}_{oo}(0, \arc\cos(\sigma/I_i), 0) \,.$$
$$. D^{l}_{oo}(0, \arc\cos(\sigma/I_i), 0) \, \mathrm{d}(\sigma/I_i) = \{2I_i/(2k+1)\} C^{ko}_{l'olo} C^{ko}_{l'olo}. \tag{3.3}$$

For the derivation of (3.3) we made use of the expansion [3] [4] [5] of the product of two functions $D^{l}_{mn}(\varphi, \vartheta, \psi)$, (A. 1.7) and the orthogonality relations of spherical harmonics. Substitution of (3.3) into (3.2) gives the formula (3.1).

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*) $C^{lm}_{l_1m_1l_2m_2}$ denotes the Clebsch-Gordan coefficient for the composition of $|l_1m_1\rangle$ and $|l_2m_2\rangle$ to $|lm\rangle$.

The result (3.1) is also mentioned by Biedenharn [6]) but was probably derived by him in a different way *). Equation (3.1) agrees with another limiting case of the Racah coefficient, found independently by Biedenharn [6]) and Racah [7]) [8]), namely

$$W(I_f l I_i k; I_i l) \approx (-1)^k \{(2l+1)(2I_i+1)\}^{-\frac{1}{2}} P_k(\cos(\boldsymbol{l}, \boldsymbol{I}_i)), \tag{3.4}$$

if

$$l \gg 1 \text{ and } I_i \gg 1, \tag{3.4a}$$

where

$$2lI_i \cos(\boldsymbol{l}, \boldsymbol{I}_i) = I_f(I_f+1) - I_i(I_i+1) - l(l+1). \tag{3.5}$$

Using (2.1) and (A.1.7), one easily checks that (3.1) and (3.4) agree for those values of I_f, l and l', in which both (3.1) and (3.4) are valid, namely for $1 \ll l = l' \ll I_i = I_f$.

§ 4. *Classical analogues of the Clebsch-Gordan coefficients and the functions* $D^l_{mn}(0, \vartheta, 0)$. From the quantum mechanical formula

$$|LM\rangle = \Sigma_{m_1 m_2} C^{LM}_{l_1 m_1 l_2 m_2} |l_1 m_1\rangle |l_2 m_2\rangle \tag{4.1}$$

it is seen that the square of a Clebsch-Gordan coefficient

$$W^{LM}_{l_1 m_1 l_2 m_2} = \{C^{LM}_{l_1 m_1 l_2 m_2}\}^2 \tag{4.2}$$

can be interpreted as the probability of finding states $|l_1 m_1\rangle |l_2 m_2\rangle$ for different values of m_1 (hence $m_2 = M - m_1$) as components of the state $|LM\rangle$ for fixed values of l_1, l_2, L and M.

We shall now indicate a classical analogue for the square of the Clebsch-Gordan coefficients: the vector addition of $\boldsymbol{l}_1 + \boldsymbol{l}_2 = \boldsymbol{L}$ corresponds in the classical case (l_1, l_2, L large numbers) to a definite triangle formed by the vectors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $\boldsymbol{L}$ (see fig. 1 and fig. 2). If only the quantum numbers L and M are specified, the angle of $\boldsymbol{L}$ with the z-axis is fixed and two (classical) degrees of freedom are left unspecified: (a) $\boldsymbol{L}$ may rotate in a cone about the z-axis, (b) the plane of the triangle may rotate about $\boldsymbol{L}$ as an axis of rotation. The orientation of the plane determines the angles of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ with the z-axis (m_1 and m_2 in quantum mechanics). We shall denote by ϑ, β_1, β_2 the angles with the z-axis of $\boldsymbol{L}$, $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ respectively. In the classical limit of quantum mechanics these angles are related to M, m_1 and m_2 according to

$$\cos\vartheta = M/L,\ \cos\beta_1 = m_1/l_1,\ \cos\beta_2 = m_2/l_2. \tag{4.3}$$

We may now introduce the classical analogue of (4.2) (see fig. 1 and fig. 2)

$$W_{l_1 l_2 L L \cos\vartheta}(l_1 \cos\beta_1;\ l_2 \cos\beta_2)\ l_1\, \mathrm{d}(\cos\beta_1) \tag{4.4}$$

representing, for fixed values of l_1, l_2, L and $\cos\vartheta$, the probability of finding a value for m_1 such that

$$l_1 \cos\beta_1 \leqslant m_1 < l_1 (\cos\beta_1 + \mathrm{d}(\cos\beta_1)) \tag{4.5}$$

*) The formula (29) of [6]) contains a misprint. The factor $(-1)^{\bar{\nu}}$ should read $(-1)^{\nu}$.

and

$$l_1 \cos \beta_1 + l_2 \cos \beta_2 = L \cos \vartheta. \tag{4.6}$$

Let γ $(0 \leqslant \gamma < \pi)$ be the angle between the plane of $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{L}$ and the plane through $\boldsymbol{L}$ and the z-axis. The classical probability W follows if we

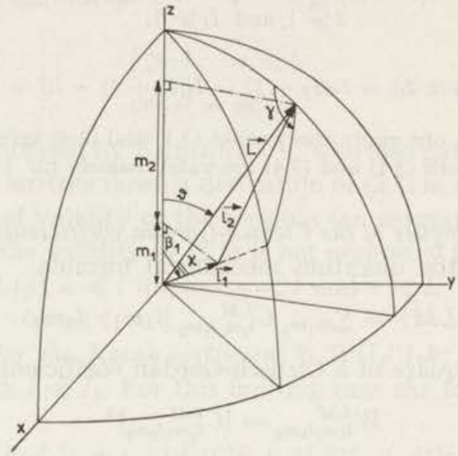

Fig. 1. Triangle of angular momenta $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{L}$, which may rotate about the direction of $\boldsymbol{L}$. The probability distribution of finding a value m_1 for the projection of $\boldsymbol{l}_1$ on the z-axis (and hence a value $m_2 = M - m_1$ for the projection of $\boldsymbol{l}_2$) can be calculated, if equal intervals $\mathrm{d}\gamma$ are taken to be equally probable (classical analogue of Clebsch-Gordan coefficients).

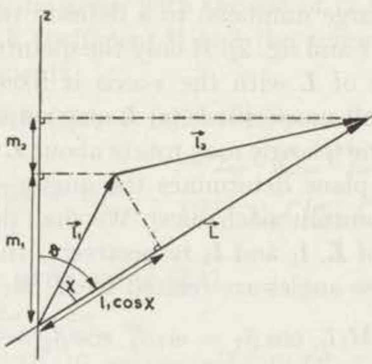

Fig. 2. Triangle of angular momenta $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{L}$.
ϑ: angle between $\boldsymbol{L}$ and the z-axis (plane of triangle chosen to be in a special position, $\gamma = 0$; cf. Fig. 1).

assume that all intervals $\mathrm{d}\gamma$ are equally probable, which is valid for any orientation of $\boldsymbol{L}$, if there are no preferred directions in space. This may be expressed as

$$W_{l_1 l_2 L L \cos\vartheta}\,(l_1 \cos \beta_1 ; l_2 \cos \beta_2) l_1 \,|\mathrm{d}(\cos \beta_1)| = (1/\pi)\, \mathrm{d}\gamma. \tag{4.7}$$

The relation between $d(\cos \beta_1)$ and $d\gamma$ is given by

$$d(\cos \beta_1) = - \sin \vartheta \sin \chi \sin \gamma \, d\gamma, \tag{4.8}$$

which follows from

$$\cos \beta_1 = \cos \vartheta \cos \chi + \sin \vartheta \sin \chi \cos \gamma, \tag{4.9}$$

where χ is the angle between $\boldsymbol{l}_1$ and $\boldsymbol{L}$.

Substitution of (4.8) into (4.7) yields, use being made of (4.9)

$$\begin{aligned} W_{l_1 l_2 L L \cos\vartheta} (l_1 \cos \beta_1 ; l_2 \cos \beta_2) &= \{\pi l_1 \sin \vartheta \sin \chi \sin \gamma\}^{-1} = \\ &= (\pi l_1)^{-1} \{(1 - \cos^2 \vartheta) \sin^2 \chi - (\cos \beta_1 - \cos \chi \cos \vartheta)^2\}^{-\frac{1}{2}} = \\ &= (\pi l_1)^{-1} \{(1 - \mu_1^2)(1 - \nu^2) - (\zeta - \mu_1 \nu)^2\}^{-\frac{1}{2}}, \end{aligned} \tag{4.10}$$

where we have introduced the abbreviations

$$\mu_1 = m_1/l_1 = \cos \beta_1, \quad \nu = \cos \chi = (l_1^2 - l_2^2 + L^2)/(2 l_1 L)$$

and

$$\zeta = M/L = \cos \vartheta. \tag{4.11}$$

In the limit

$$l_1 \ll l_2 \text{ and } l_1 \ll L \tag{4.12}$$

it follows from (4.11) that we have approximately

$$\nu \approx (L - l_2)/l_1. \tag{4.13}$$

Formula (4.10) is symmetric in $l_1 m_1$ and $l_2 m_2$, as could be expected from the analogy with the Clebsch-Gordan coefficients. In order to show this symmetry we write (4.10) as follows

$$\begin{aligned} W_{l_1 l_2 L L \cos\vartheta} (l_1 \cos \beta_1 ; l_2 \cos \beta_2) &= \\ = (2L/\pi) \, [-\{l_1^4 + l_2^4 + L^4\} &+ 2\{l_1^2 l_2^2 + l_2^2 L^2 + L^2 l_1^2\} + \\ &+ 4\{l_1^2 m_2 (-M) + l_2^2 (-M) m_1 + L^2 m_1 m_2\}]^{-\frac{1}{2}}. \end{aligned} \tag{4.14}$$

(4.14) shows also the complete symmetry in all three angular momenta of the classical analogue of the Wigner 3-j symbol [4) 9)], defined by

$$\begin{pmatrix} l_1 \, l_2 L \\ m_1 m_2 M \end{pmatrix} = (-1)^{l_1 - l_2 - M} (2L + 1)^{-\frac{1}{2}} \, C^{L-M}_{l_1 m_1 l_2 m_2}. \tag{4.15}$$

An alternative way for obtaining a classical analogue of the Clebsch-Gordan coefficients is by considering the inverse formula of (4.1)

$$|l_1 m_1\rangle \, |l_2 m_2\rangle = \textstyle\sum_{LM} C^{LM}_{l_1 m_1 l_2 m_2} \, |LM\rangle. \tag{4.16}$$

This suggests as the classical analogue of the square of the Clebsch-Gordan coefficient the probability of finding a resultant vector of length L when composing two vectors of lengths l_1 and l_2 with projections m_1 and m_2

$(m_1 + m_2 = M)$ on the z-axis (see fig. 3). Along these lines we again find the same result (4.10) as before.

A classical analogue of the functions $D^l_{mn}(\varphi, \vartheta, \psi)$ (which specify the representations of the rotational group and also give the wave functions of the symmetric top) may be given in a quite similar way as for the Clebsch-Gordan coefficients: In view of the formula

$$\Psi_{lm'}(\boldsymbol{r}') = \sum_m \Psi_{lm}(\boldsymbol{r}) D^l_{mm'}(\varphi, \vartheta, \psi) \tag{4.17}$$

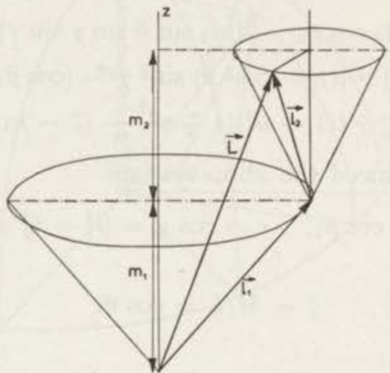

Fig. 3. Triangle of angular momenta $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{L}$, the positions of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ being specified by the projections on the z-axis, m_1 and m_2 respectively, so that the probability distribution of finding a length L for the sumvector $\boldsymbol{l}_1 + \boldsymbol{l}_2$ can be calculated (classical analogue of Clebsch-Gordan coefficients).

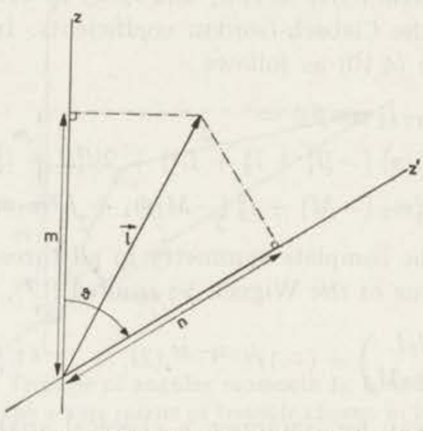

Fig. 4. A vector $\boldsymbol{l}$, its position in the primed coordinate frame specified by its projection n on the z'-axis, so that the probability of finding a projection m on the z-axis (ϑ: angle between the z-axis and the z'-axis) can be calculated (classical analogue of $D^l_{mn}(0, \vartheta, 0)$). Notice analogy with Fig. 2.

we may interpret $|D^l_{mn}(0, \vartheta, 0)|^2$ as the probability of finding a projection m of $\boldsymbol{l}$ on the z-axis if it is known that the projection of $\boldsymbol{l}$ on the z'-axis equals n and if the z-axis and z'-axis make an angle ϑ (see fig. 4). We therefore introduce, using the abbreviation $\mu = m/l$,

$$W_{ln\vartheta}\,(l\mu)l\,\mathrm{d}\mu \tag{4.18}$$

representing, for given values of l, n and ϑ (see fig. 4) the probability of finding a value for m such that

$$l\mu \leqslant m \leqslant l(\mu + \mathrm{d}\mu). \tag{4.19}$$

In an analogous way as for the classical analogue of the Clebsch-Gordan coefficients, one finds

$$W_{ln\vartheta}(l\mu) = (\pi l)^{-1}\,\{(1-\mu^2)(1-\nu^2) - (z-\mu\nu)^2\}^{-\frac{1}{2}} \tag{4.20}$$

(where the abbreviations $z = \cos\vartheta$, $\mu = m/l$ and $\nu = n/l$ are introduced) as a classical analogue of $|D^l_{mn}(0, \vartheta, 0)|^2$.

The formula (4.20) is symmetric in m and n, as could be expected from the analogy with $D^l_{mn}(0, \vartheta, 0)$.

§ 5. *Asymptotic expression for the function* $D^l_{mn}(\varphi, \vartheta, \psi)$. The functions $D^l_{mn}(\varphi, \vartheta, \psi)$ are the eigenfunctions of the symmetric rotator (cf., e.g., Wigner [3]). Hence a differential equation which they have to satisfy, can be deduced from the Schrödinger equation for the symmetric rotator, as has been done by Reiche and Rademacher [10], Kronig and Rabi [11], Manneback [12] and Van Vleck [13]:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}u}{\mathrm{d}z}\right] + \left[l(l+1) - \frac{m^2 - 2mnz + n^2}{1-z^2}\right]u = 0, \tag{5.1}$$

if

$$D^l_{mn}(\varphi, \vartheta, \psi) = u(z)\exp[-\,i(m\varphi + n\psi)] \tag{5.2}$$

and

$$z = \cos\vartheta. \tag{5.3}$$

The differential equation (5.1) is invariant for a) interchange of m and n (cf. (A. 1.5)), b) simultaneous inversion of sign of m and n (cf. (A. 1.4a)), c) simultaneous inversion of sign of z and either m or n (cf. (A. 1.6)).

We again introduce the abbreviations

$$\mu = m/l \quad \text{and} \quad \nu = n/l. \tag{5.4}$$

Substitution of (5.4) into (5.1) now yields, if $l \gg 1$,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}u}{\mathrm{d}z}\right] + (l+\tfrac{1}{2})^2(1-z^2)^{-1}[(1-\mu^2)(1-\nu^2) - (z-\mu\nu)^2]\,u = 0. \tag{5.5}$$

We shall use the WKB-method [14] [15] to solve equation (5.5) for large values of l. Therefore we put

$$u(z) = \exp\,[i(l+\tfrac{1}{2})\,\{S_0(z) + (l+\tfrac{1}{2})^{-1}\,S_1(z)\}]. \tag{5.6}$$

Equating to zero the coefficients of different powers of $(l + \frac{1}{2})^{-1}$, we obtain for the WKB-solution

$$u(z) = \frac{\exp\left[i(l+\frac{1}{2})\int^{z} \frac{\{(1-\mu^2)(1-\nu^2)-(x-\mu\nu)^2\}^{\frac{1}{2}}}{1-x^2}\,\mathrm{d}x\right]}{\{(1-\mu^2)(1-\nu^2)-(z-\mu\nu)^2\}^{\frac{1}{4}}}. \tag{5.7}$$

The real (imaginary) part of $u(z)$ is

$$v(z) = A\left\{\begin{matrix}\mathrm{Im}[u(z)]\\ \mathrm{Re}[u(z)]\end{matrix}\right\} = A\,\{(1-\mu^2)(1-\nu^2)-(z-\mu\nu)^2\}^{-\frac{1}{4}}\,.$$

$$\cdot\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(l+\tfrac{1}{2})\textstyle\int^{z}(1-x^2)^{-1}\{(1-\mu^2)(1-\nu^2)-(x-\mu\nu)^2\}^{\frac{1}{2}}\,\mathrm{d}x], \tag{5.8}$$

where we have introduced a normalization constant A. The WKB-solution (5.8) is the oscillatory solution valid for the region between the classical turning points (that is the classically allowed region) determined by

$$(z-\mu\nu)^2 < (1-\mu^2)(1-\nu^2). \tag{5.9}$$

Outside this region one might also write down the WKB-solution which then falls off very rapidly (exponentially) for high l, so that the normalization integral is only determined by the integral over the oscillatory part of the solution (in the limit of high l). We may first average $v^2(z)$ over a few periods of the (for large values of l) rapidly oscillating (co)sine. Since the other factor of $v(z)$ is a slowly varying function, we find

$$\overline{v^2(z)} = \tfrac{1}{2}A^2\{(1-\mu^2)(1-\nu^2)-(z-\mu\nu)^2\}^{-\frac{1}{2}}. \tag{5.10}$$

From the normalization of v we then find

$$A = \pm\,\{2/(\pi l)\}^{\frac{1}{2}}. \tag{5.11}$$

From (5.2), (5.7), (5.8) and (5.11) we now derive the expression

$$D^l_{mn}(0,\vartheta,0) \approx \{2/(\pi l)\}^{\frac{1}{2}}\,\{(1-\mu^2)(1-\nu^2)-(z-\mu\nu)^2\}^{-\frac{1}{4}}\,.$$

$$.\,\cos\,[(l+\tfrac{1}{2})\textstyle\int^{z}(1-x^2)^{-1}\{(1-\mu^2)(1-\nu^2)-(x-\mu\nu)^2\}^{\frac{1}{2}}\,\mathrm{d}x], \tag{5.12}$$

if

$$l \gg 1,\ z=\cos\vartheta,\ m=l\mu,\ n=l\nu \text{ and } (z-\mu\nu)^2 < (1-\mu^2)(1-\nu^2). \tag{5.12a}$$

A special case of (5.12) is the expression for a spherical harmonic. Making use of (A. 1.7) we derive from (5.12)

$$Y_{lm}(\vartheta,\varphi) \approx \pi^{-1}\exp\,(im\varphi)\,\{1-\mu^2-z^2\}^{-\frac{1}{4}}\,.$$

$$.\,\cos\,[(l+\tfrac{1}{2})\textstyle\int_0^{z}(1-x^2)^{-1}\{1-\mu^2-x^2\}^{\frac{1}{2}}\,\mathrm{d}x-(l+m)\pi/2]. \tag{5.13}$$

In (5.13) we also have specified the phase of the WKB-solution to express the symmetry character of the spherical harmonics for the transformation $\vartheta \to \pi - \vartheta$ (cf. (A. 1.6)) and to have the correct sign for $z = 0$ (cf. (A. 1.3)). Since there is no such symmetry for the functions $D^l_{mn}(0, \vartheta, 0)$ (except if $m = 0$ or $n = 0$), we did not specify the lower boundary of the integral in (5.12). If $m = 0$ we may reduce (5.13) even further to obtain an expression for the Legendre polynomials (cf.[16]) p. 92)

$$P_l (\cos \vartheta) = D^l_{oo}(0, \vartheta, 0) \approx \{2/(\pi l \sin \vartheta)\}^{\frac{1}{2}} \cos [(l + \tfrac{1}{2})\vartheta - \pi/4]. \quad (5.14)$$

§ 6. *Comparison of the results of* § 4 *and* § 5. The comparison of certain results of § 4 and § 5 throws some more light on the classical behavior of the Clebsch-Gordan coefficients and D-functions.

(*a*) The square of the function $D^l_{mn}(0, \vartheta, 0)$ has a rapidly oscillating behavior as a function of ϑ (for high l) (cf. (5.12)). Averaging the square of $D^l_{mn}(0, \vartheta, 0)$ over a few periods one obtains the value of the classical analogue (4.20).

(*b*) In § 4 we constructed a classical analogue of the square of Clebsch-Gordan coefficients. However, we did not give any rigorous mathematical derivation that the asymptotic value of the quantum-mechanical expression [1]) [2]) for the Clebsch-Gordan coefficients reduces to the classical analogue in the limit of high l.

By comparing (2.1) and the WKB-expression (5.12) for $D^l_{mn}(0, \vartheta, 0)$ one obtains an expression derived from quantum mechanics which shows the meaning of the classical analogue: The Clebsch-Gordan coefficient $C^{LM}_{l_1 m_1 l_2 m_2}$ (for the case $l_1 \ll l_2, L$) shows a rapidly oscillating character as a function of M/L. Averaging the square of the Clebsch-Gordan coefficient over a few oscillations one obtains the value of the classical analogue (4.10), taking into account (4.13).

(*c*) The foregoing remark does not yet provide a reduction of the quantum mechanical expressions $C^{LM}_{l_1 m_1 l_2 m_2}$ to its classical analogue for the limit of high l, except for the case $l_1 \ll l_2, L$. Because of the similarity between the derivations of (4.10) and (4.20) one might conjecture that a formula analogous to (2.1) might be valid for arbitrary values of l_1, l_2, L; such a formula should then have the form

$$C^{LM}_{l_1 m_1 l_2 m_2} \approx \omega D^{l_1}_{m_1 n_1} (0, \vartheta, 0), \quad (6.1)$$

where ω is a phase factor, $|\omega| = 1$,

$$\begin{aligned} n_1 &= l_1 \cos \chi = (l_1^2 - l_2^2 + L^2)/(2L), \\ \vartheta &= \arccos (M/L). \end{aligned} \quad (6.2)$$

However, we did not succeed in establishing a relation of this form. We may indicate two difficulties, which exist in the extension of (2.1) to a formula of the type (6.1) and which make this extension non-trivial if not impossible: (1) the value which $(l_1 - n_1)$ should have according to (6.2) is in general not an integer (if $l_1 \ll l_2, L$; $n_1 \approx L - l_2$), while the ordinary definition of D^l_{mn} is only given for integer values of $(l - n)$; (2) it may be that $l_1 \ll l_2, L$ is an exceptional case, as substitution of (5.12) into (6.1) leads to an oscillatory behavior of $C^{LM}_{l_1 m_1 l_2 m_2}$ as a function of M analogous to (co)sin $\{l_1 M/L\}$. It is only in the special case $l_1 \ll l_2, L$ that this (co)sine varies smoothly as a function of M, so that there may be an essential difference between this special case and the general case.

(*d*) For the purpose of indicating the meaning of the classical analogue of the Clebsch-Gordan coefficients in the general case, one may try to follow in detail the transition from quantum mechanics to the classical limit. Let $\Phi_{l_1 m_1}$, $\Phi_{l_2 m_2}$ be the wave functions of the component systems with the indicated values for the quantum numbers, and similarly Φ_{LM} the wave function of the resultant system

$$\Phi_{LM} = \sum_{m_1 m_2} \Phi_{l_1 m_1} \Phi_{l_2 m_2} C^{LM}_{l_1 m_1 l_2 m_2}. \tag{6.3}$$

The classical concept of a system with a definite $\boldsymbol{l}$ determined in magnitude and direction can be introduced in quantum mechanics only in the sense of a system with a definite l and a direction of the angular momentum confined to a certain small solid angle around a direction in space (as its x, y, z components do not commute), which we shall represent (for the component systems) as

$$\Phi(\boldsymbol{l}_1) = \sum_{m_1} a_{m_1} \Phi_{l_1 m_1}, \quad \Phi(\boldsymbol{l}_2) = \sum_{m_2} b_{m_2} \Phi_{l_2 m_2}. \tag{6.4}$$

The a_{m_1} and b_{m_2} are amplitudes chosen in such a way that the angular momenta have "roughly" some direction. For large l they differ from zero appreciably only in some interval Δm_1 or Δm_2 around the value m_1 or m_2, corresponding to this approximate definition of the direction.

We shall now specify a probability $P(L)$ in two ways:

I. establishing a simple "geometrical" calculation,

II. performing a quantum mechanical averaging procedure.

The comparison of I and II relates the classical analogue to the square of the quantum mechanical Clebsch-Gordan coefficients.

I. If we fix l_1, l_2, l_{1_z} $(= m_1)$ and l_{2_z} $(= m_2)$, then the remaining variables (azimuthal angles) to be specified in the classical picture may be denoted as φ_1, φ_2. Alternatively L, φ may be given

$$L = L(\varphi_1, \varphi_2), \quad \varphi = \varphi(\varphi_1, \varphi_2). \tag{6.5}$$

The probability distribution in L and φ is now determined by

$$P(L)\, \mathrm{d}L\, \mathrm{d}\varphi = C\, \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2. \tag{6.6}$$

(6.6) expresses that all intervals $\mathrm{d}\varphi_1$ and $\mathrm{d}\varphi_2$ are equally probable. Now introduce $\overline{P}(L)$

$$\overline{P}(L)\, \mathrm{d}L = \int_\varphi P(L)\, \mathrm{d}L\, \mathrm{d}\varphi \tag{6.7}$$

and normalize

$$\int_L \overline{P}(L)\, \mathrm{d}L = 1. \tag{6.8}$$

This $\overline{P}(L)$ is exactly the classical limit we discussed in § 4.

II. Quantum mechanically the probability that the resultant state $\Phi(\boldsymbol{l}_1)\,\Phi(\boldsymbol{l}_2)$ coincides with a state $\Phi(\boldsymbol{L})$ is given by

$$|\langle\Phi(\boldsymbol{L})\,|\Phi(\boldsymbol{l}_1)\Phi(\boldsymbol{l}_2)\rangle|^2. \tag{6.9}$$

We now want to perform the averaging process over φ_1, φ_2 in (6.6) according to quantum mechanics, This means that the orientations of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are given by ensembles (mixed states, no pure states), characterized by density matrices $\rho^{(1)}$ and $\rho^{(2)}$, obtained by performing an additional ensemble average (cf., e.g., Tolman [20]). $\rho^{(1)}$ and $\rho^{(2)}$ are diagonal in m_1 and m_2 because of the rotational symmetry about the z-axis.

$$\begin{aligned} \overline{\overline{a_{m_1}a^*_{m_1'}}} &= \rho^{(1)}_{m_1m_1'} = \rho^{(1)}_{m_1m_1}\,\delta_{m_1m_1'},\\ \overline{\overline{b_{m_2}b^*_{m_2'}}} &= \rho^{(2)}_{m_2m_2'} = \rho^{(2)}_{m_2m_2}\,\delta_{m_2m_2'}, \end{aligned} \tag{6.10}$$

where $\rho^{(1)}_{m_1m_1}=0$ (or $\rho^{(2)}_{m_2m_2}=0$) if m_1 (or m_2) is not in the interval Δm_1 (or Δm_2) which was introduced in (6.4). We may write

$$\Phi(\boldsymbol{l}_1)\Phi(\boldsymbol{l}_2) = \textstyle\sum_{m_1m_2} a_{m_1}\, b_{m_2}\, \Phi_{l_1m_1}\Phi_{l_2m_2} = \sum_{LMm_1m_2} a_{m_1}\, b_{m_2}\, C^{LM}_{l_1m_1l_2m_2}\Phi_{LM}, \tag{6.11}$$

so that the density matrix characterizing the resultant state (after the averaging procedure (6.10)) can be written as

$$\langle LM\,|\rho|\,L'M'\rangle = \textstyle\sum_{m_1m_1'm_2m_2'} \overline{\overline{a_{m_1}a^*_{m_1'}b_{m_2}b^*_{m_2'}}}\,C^{LM}_{l_1m_1l_2m_2}C^{L'M'}_{l_1m_1'l_2m_2'} \tag{6.12}$$

or

$$\langle LM\,|\rho|\,L'M'\rangle = \begin{cases} = \sum_{m_1m_2}\rho^{(1)}_{m_1m_1}\,\rho^{(2)}_{m_2m_2}\;C^{LM}_{l_1m_1l_2m_2}\;C^{L'M}_{l_1m_1l_2m_2}\;\delta_{MM'} \\ \quad\text{if } M \text{ is in the interval corresponding to } \Delta m_1 \text{ and } \Delta m_2,\\ = 0 \text{ otherwise.} \end{cases} \tag{6.13}$$

From (6.13) we deduce that the probability to find the resultant system in a state L is given by

$$\overline{P}(L) = \textstyle\sum_M \sum_{m_1m_2}\rho^{(1)}_{m_1m_1}\,\rho^{(2)}_{m_2m_2}\,|\,C^{LM}_{l_1m_1l_2m_2}|^2. \tag{6.14}$$

A summation over M should be carried out over an interval ΔM corresponding to Δm_1 and Δm_2 in order to cover the full width of the "wave packet" for the final state as a function of ϑ.

The comparison of the expressions under I (equations (6.6), (6.7), (6.8)) *and under II (equation* (6.14)) *shows that the classical analogue* $\overline{P}(L)$ *must be asymptotically equal (for large* l_1, l_2, L) *to an average of the square of the*

Clebsch-Gordan coefficients over small intervals of m_1 and m_2 (summation over all possible values of M).

For the coefficient $C^{Lo}_{l_1ol_2o}$ Racah [2]) has given a closed expression without summations. If one calculates, using the Stirling approximation (A.2.4), the limit for large values of l_1, l_2 and L, one finds

$$C^{Lo}_{l_1ol_2o} = 0 \text{ if } l_1 + l_2 - L = \text{odd}, \tag{6.15a}$$

$$C^{Lo}_{l_1ol_2o} = \sqrt{2}\,(-1)^{(l_1+l_2-L)/2}\,\{2L/\pi\}^{\frac{1}{2}}\,.$$
$$\cdot\,\{-(l_1{}^4 + l_2{}^4 + L^4) + 2(l_1{}^2l_2{}^2 + l_2{}^2L^2 + L^2l_1{}^2)\}^{-\frac{1}{4}} \text{ if } l_1 + l_2 - L = \text{even}. \tag{6.15b}$$

We note that these limiting values are consistent with the classical analogue (4.14), although the fact that only the value for $m_1 = m_2 = M = 0$ is given does not allow a real check of (6.14).

(*e*) We may specialize the formulae (5.12) and (5.13) to the case of Legendre functions, using (A. 1.7). If the argument $z = \cos\vartheta$ satisfies (cf. (5.9))

$$-(1-\mu^2)^{\frac{1}{2}} < z < +(1-\mu^2)^{\frac{1}{2}}, \tag{6.16}$$

we have a rapidly oscillating function; if (6.16) is not satisfied, we have an exponential behavior. This is illustrated for the two cases $m = 0$ and $m \neq 0$ in the figures 5 and 6.

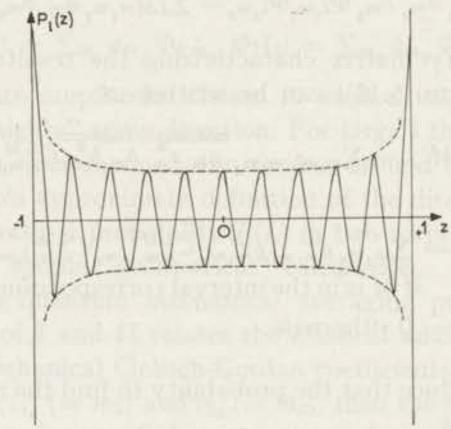

Fig. 5. Oscillatory behavior of the Legendre polynomial $P_l(z)$ in the interval $-1 < z < 1$ for large values of l; cf. formulae (5.13) and (5.14).

This behavior may be illustrated also by fig. 7, which represents the orbit of a particle having a total angular momentum l with projection m on the z-axis. Since only the projection m is specified, the angular momentum vector may "precess" about the z-axis, and it is seen that the exponential region of $P_{lm}(\cos\vartheta)$ is the region where the particle can not come according to the classical picture.

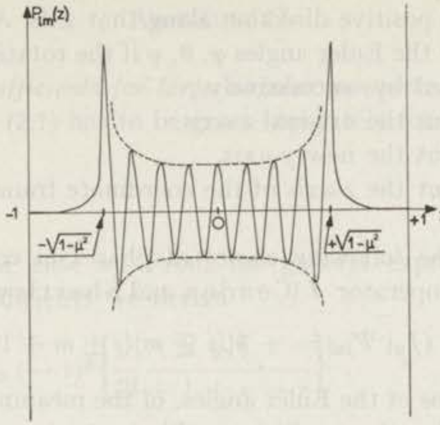

Fig. 6. Oscillatory behavior of the associated Legendre polynomial $P_{lm}(z)$ in the interval $-(1-\mu^2)^{\frac{1}{2}} < z < (1-\mu^2)^{\frac{1}{2}}$ and exponential behavior for $(1-\mu^2)^{\frac{1}{2}} < |z| < 1$ as obtained for large l; cf. formula (5.13); $\mu = m/l$.

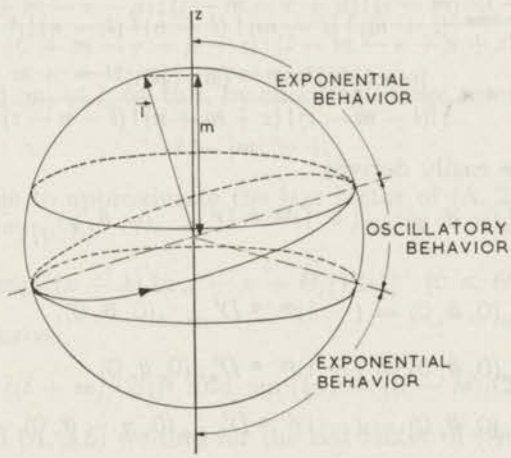

Fig. 7. The orbit of a particle specified by the quantum numbers l and m may "precess" about the z-axis. According to this picture the particle will move in the "classically allowed" region (oscillatory behavior) and will stay out of the region where $P_{lm}(z)$ behaves exponentially.

Appendix I

Definition of Euler angles and rotations. The representation of the rotational group is defined by [3] [4] [5])

$$\Psi_{lm'}(\boldsymbol{r}') = \mathrm{R}\Psi_{lm}(\boldsymbol{r}) = \sum_m \Psi_{lm}(\boldsymbol{r})\, D^l_{mm'}(R), \qquad \text{(A. 1.1)}$$

where R represents a rotation of the coordinate frame. $\boldsymbol{r}$ are the coordinates in the original frame, $\boldsymbol{r}'$ in the rotated frame. We use a right-handed frame of axes. A positive rotation about an axis is the rotation of a right-handed screw moving in the positive direction along that axis. A rotation R is said to be represented by the Euler angles φ, ϑ, ψ if the rotation R of the coordinate frame is effected by, successively

a) a rotation φ about the original z-axis,

b) a rotation ϑ about the new y-axis,

c) a rotation ψ about the z-axis of the coordinate frame that results after the rotation *b*).

We shall make the following choice of phases in connection with the angular momentum operator $\boldsymbol{J}$ (Condon and Shortley [1])).

$$J_{\pm}\Psi_{jm} \equiv (J_x \pm iJ_y)\,\Psi_{jm} = +\,\hbar\{(j \mp m)(j \pm m + 1)\}^{\frac{1}{2}}\,\Psi_{j,m\pm 1}. \quad \text{(A. 1.2)}$$

With these definitions of the Euler angles, of the meaning of the rotation R to be represented by the coefficients $D^l_{mn}(\varphi, \vartheta, \psi)$ and with the phase choice (A. 1.2), the phases of the $D^l_{mn}(\varphi, \vartheta, \psi)$ are determined. The coefficients $D^l_{mn}(\varphi, \vartheta, \psi)$ are [4]) *)

$$D^l_{mn}(\varphi, \vartheta, \psi) = e^{-im\varphi}\,[(l+m)!\,(l-m)!\,(l+n)!\,(l-n)!]^{\frac{1}{2}}\,.$$

$$\cdot\,\sum_z (-1)^{l-n-z}\,\frac{\{\cos(\frac{1}{2}\vartheta)\}^{2z+m+n}\,\{\sin(\frac{1}{2}\vartheta)\}^{2l-m-n-2z}}{z!(l-m-z)!\,(z+m+n)!\,(l-n-z)!}\,e^{-in\psi}. \quad \text{(A. 1.3)}$$

From (A. 1.3) one easily derives

$$D^{l*}_{mn}(\varphi, \vartheta, \psi) = (-1)^{m-n}\,D^l_{-m-n}(\varphi, \vartheta, \psi) \quad \text{(A. 1.4)}$$

or

$$D^l_{mn}(0, \vartheta, 0) = (-1)^{m-n}\,D^l_{-m-n}(0, \vartheta, 0), \quad \text{(A. 1.4a)}$$

$$D^l_{mn}(0, \vartheta, 0) = (-1)^{m-n}\,D^l_{nm}(0, \vartheta, 0), \quad \text{(A. 1.5)}$$

$$D^l_{mn}(0, \vartheta, 0) = (-1)^{l-n}\,D^l_{-mn}\,(0, \pi-\vartheta, 0) =$$

$$= (-1)^{l+m}\,D^l_{m-n}\,(0, \pi-\vartheta, 0). \quad \text{(A. 1.6)}$$

The spherical harmonics are defined by (corresponding to Condon and Shortley [1]))

$$D^{l*}_{mo}(\varphi, \vartheta, \psi) = \{4\pi/(2l+1)\}^{\frac{1}{2}}\,Y_{lm}(\vartheta, \varphi) =$$

$$= \{(l-m)!/(l+m)!\}^{\frac{1}{2}}\,P_{lm}(\cos\vartheta)\,e^{im\varphi}. \quad \text{(A. 1.7)}$$

In (A. 1.7) we have chosen the phase of the associated Lengendre polynomials $P_{lm}(\cos\vartheta)$ such that their sign equals that of the functions $\Theta(lm)$ of Condon and Shortley [1]). Our conventions are all in accordance with Edmonds [4]), except for $P_{lm}(\cos\vartheta)$.

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*) This formula, as given by Edmonds, differs from the form given by several other authors in the phase of $D^l_{mn}(0, \vartheta, 0)$ and/or in complex conjugation of $e^{-im\varphi}$ and $e^{-in\psi}$.

Clebsch-Gordan coefficients for large angular momenta. The proof of the asymptotic equality (2.1) has to be given separately for the two cases

a) $|m| \neq l,$ (A. 2.1a)

b) $|m| = l.$ (A. 2.1b)

We first treat the case *a)*. From the general expression [1] [2] for the Clebsch-Gordan coefficients we derive

$$C^{lm}_{\kappa,\mu,l-\tau,m-\mu} = \Sigma_z (-1)^z \left\{\frac{2l+1}{2l+1+\kappa-\tau}\right\}^{\frac{1}{2}} \cdot$$

$$\cdot \frac{\{(\kappa-\tau)!\,(\kappa+\tau)!\,(\kappa+\mu)!\,(\kappa-\mu)!\}^{\frac{1}{2}}}{z!(\kappa-\mu-z)!\,(\tau+\mu+z)!\,(\kappa-\tau-z)!} \cdot$$

$$\cdot \left\{\frac{(2l-\kappa-\tau)!\,(l+m-\tau-\mu)!\,(l-m-\tau+\mu)!\,(l+m)!\,(l-m)!}{(2l+\kappa-\tau)!\,\{(l+m-\tau-\mu-z)!\,(l-m-\kappa+\mu+z)!\}^2}\right\}^{\frac{1}{2}}. \quad \text{(A. 2.2)}$$

Since we supposed $|m| \neq l$, we can, by choosing l large enough, satisfy

$$l - |m| \gg 1, \quad \text{(A. 2.3)}$$

so that we are able to approximate the last factor of (A. 2.2) by using the Stirling formula [16] (p. 5)

$$x! = (2\pi)^{\frac{1}{2}} \exp\left[(x+\tfrac{1}{2}) \ln x - x + \Theta/(12x)\right] \quad (0 \leqslant \Theta \leqslant 1). \quad \text{(A. 2.4)}$$

From (2.1b) we derive

$$\cos(\tfrac{1}{2}\vartheta) = \{(l+m)/(2l)\}^{\frac{1}{2}} \text{ and } \sin(\tfrac{1}{2}\vartheta) = \{(l-m)/(2l)\}^{\frac{1}{2}}. \quad \text{(A. 2.5)}$$

Using (A. 2.4) and (A. 2.5) we find for the last factor of (A. 2.2)

$$\{(l+m)/(l-m)\}^{z+(\tau+\mu)/2} \{(l-m)/(2l)\}^{\kappa} =$$

$$= \{\cos(\tfrac{1}{2}\vartheta)\}^{2z+\mu+\tau} \{\sin(\tfrac{1}{2}\vartheta)\}^{2\kappa-\mu-\tau-2z}. \quad \text{(A. 2.6)}$$

Substituting (A. 2.6) into (A. 2.7), we find, using (A. 1.3)

$$C^{lm}_{\kappa,\mu,l-\tau,m-\mu} \approx (-1)^{\kappa-\tau} \{(\kappa-\tau)!\,(\kappa+\tau)!\,(\kappa+\mu)!\,(\kappa-\mu)!\}^{\frac{1}{2}} \cdot$$

$$\cdot \Sigma_z (-1)^{\kappa-\tau-z} \frac{\{\cos(\tfrac{1}{2}\vartheta)\}^{2z+\mu+\tau} \{\sin(\tfrac{1}{2}\vartheta)\}^{2\kappa-\mu-\tau-2z}}{z!\,(\kappa-\mu-z)!\,(z+\mu+\tau)!\,(\kappa-\tau-z)!} =$$

$$= (-1)^{\kappa-\tau} D^{\kappa}_{\mu\tau}(0, \vartheta, 0). \quad \text{(A. 2.7)}$$

In case *b)* we must prove (we give the proof for $m = +l$; the proof for the case $m = -l$ is entirely analogous)

$$C^{ll}_{\kappa,\mu,l-\tau,l-\mu} \approx (-1)^{\kappa-\tau} D^{\kappa}_{\mu\tau}(0, 0, 0) = (-1)^{\kappa-\tau} \delta_{\mu\tau}. \quad \text{(A. 2.8)}$$

From the general expression for the Clebsch-Gordan coefficients we derive

$$C^{\kappa\mu}_{l,l,l-\tau,-(l-\mu)} = \left\{\frac{2\kappa+1}{2l+1+\kappa-\tau}\right\}^{\frac{1}{2}} \left\{\frac{(\kappa-\tau)!\,(\kappa+\mu)!}{(\mu-\tau)!\,(\kappa+\tau)!\,(\kappa-\mu)!}\right\}^{\frac{1}{2}} \cdot$$

$$\cdot \left\{\frac{(2l)!\,(2l-\mu-\tau)!}{(2l+\kappa-\tau)!\,(2l-\tau-\kappa)!}\right\}^{\frac{1}{2}}. \quad \text{(A. 2.9)}$$

Using Stirling's approximation (A. 2.4) we deduce

$$C^{\kappa\mu}_{l,l,l-\tau,-(l-\mu)} \approx \left\{\frac{(\kappa-\tau)!\,(\kappa+\mu)!}{(\mu-\tau)!\,(\kappa+\tau)!\,(\kappa-\mu)!}\right\}^{\frac{1}{2}} \left\{\frac{2\kappa+1}{2l+1}\right\}^{\frac{1}{2}} \left\{\frac{1}{\sqrt{2l}}\right\}^{\mu-\tau}. \quad \text{(A. 2.10)}$$

Since $\mu \gg \tau$ we may conclude from (A. 2.10) for large values of l after interchange of indices of the Clebsch-Gordan coefficient that

$$C^{ll}_{\kappa,\mu,l-\tau,l-\mu} \approx (-1)^{\kappa-\tau}\,\delta_{\mu\tau}. \quad \text{(A. 2.11)}$$

(A. 2.7) and (A. 2.11) together are equivalent to (2.1).

Appendix III

Comparison of some expressions for $D^l_{mn}(\varphi, \vartheta, \psi)$. We want to compare some expressions of the form (5.2) for the coefficients of the representation of the rotational group. We introduce [10])

$$t = \frac{1-z}{2},\; s = |m+n|,$$

$$d = |m-n|,\; p = l - \frac{d+s}{2}. \quad \text{(A. 3.1)}$$

If we put

$$u(t) = w(t)\, t^{d/2}\,(1-t)^{s/2} \quad \text{(A. 3.2)}$$

then, as has been shown by Rademacher [10]), the differential equation (5.1) is reduced to Gauss' differential equation for the hypergeometric series. Thus we find

$$u(t) = t^{d/2}\,(1-t)^{s/2}\, F(-p, d+s+p+1; d+1; t). \quad \text{(A. 3.3)}$$

The terminating hypergeometric series in (A. 3.3) is a Jacobi polynomial. For convenience we also give the notations by Courant and Hilbert [17]) and Szegö [18])

$$F(-p, d+s+p+1; d+1; t) = G_p(d+s+1, d+1; t) =$$

$$= \binom{p+d}{p}^{-1} P^{(d,s)}_p(z) = \Sigma^p_{r=0}\,(-1)^r \binom{p}{r} \frac{(d+s+p+r)!\,d!}{(d+s+p)!\,(d+r)!} t^r =$$

$$= \frac{d!}{(d+p)!}\, t^{-d}\,(1-t)^{-s} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p [t^{d+p}\,(1-t)^{s+p}]. \quad \text{(A. 3.4)}$$

We now write

$$D^l_{mn}(\varphi, \vartheta, \psi) = Nu(t)\, e^{-i(m\varphi + n\psi)}. \tag{A. 3.5}$$

If we use the value given by Jacobi [10] [19] for $\int_0^1 \{u(t)\}^2 \,\mathrm{d}t$, we find for N, if we want the expression (A. 3.5) to be equal to (A. 1.3)

$$N = \frac{(-1)^d}{d!} \left[\frac{(d+s+p)!\,(d+p)!}{p!\,(s+p)!} \right]^{\frac{1}{2}}. \tag{A. 3.6}$$

The sign of (A. 3.6) holds for the case

$$m + n \geqslant 0 \text{ and } m - n \geqslant 0. \tag{A.3.7}$$

If the conditions (A. 3.7) are violated, we may use the relations (A. 1.4a) and (A. 1.5) to determine the value of $D^l_{mn}(\varphi, \vartheta, \psi)$. The normalization (A. 3.6) implies that

$$\int_0^{2\pi} \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\vartheta \sin\vartheta\, D^{l*}_{mn}(\varphi, \vartheta, \psi)\, D^l_{mn}(\varphi, \vartheta, \psi) = \frac{8\pi^2}{2l+1} \tag{A. 3.8}$$

and implies normalization of the $Y_{lm}(\vartheta, \varphi)$ according to

$$\int_0^{\pi} \int_0^{2\pi} Y^*_{lm}(\vartheta, \varphi)\, Y_{lm}(\vartheta, \varphi) \sin\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\varphi = 1, \tag{A. 3.9}$$

if the relation (A. 1.7) is assumed.

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SAMENVATTING

Het probleem van de alfadesintegratie kan gesplitst worden in twee gedeelten:

(1) het *inwendige probleem* betreffende de vorming van het alfadeeltje uit de nucleonen van de kern,

(2) het *uitwendige probleem* van de voortplanting van het alfadeeltje nadat het de kern heeft verlaten.

In hoofdstuk I van dit proefschrift wordt het inwendige probleem behandeld. De vorming (dissociatie) van alfadeeltjes uit (in) nucleonen is nauw verwant met de gemiddelde vrije weglengte van alfadeeltjes in kernmaterie. In geval van een korte vrije weglengte kan de vorming van alfadeeltjes bij benadering gekarakteriseerd worden door een randvoorwaarde op het kernoppervlak. Deze randvoorwaarde bepaalt vervolgens de voortplanting van het alfadeeltje in het gebied buiten de kern. Tevens wordt in hoofdstuk I een golffunctie voorgesteld die een kern (met schillenstructuur) representeert die een alfadeeltje uitzendt. Hiermee wordt de bovengenoemde randvoorwaarde afgeleid voor de berekening van absolute en relatieve overgangswaarschijnlijkheden in alfa-emissie. De waarde voor de straal van de potentiaalput voor het alfadeeltje, die zo verkregen wordt uit de levensduur van alfaradioactieve kernen, is in goede overeenstemming met de waarde die volgt uit de verstrooiing van alfadeeltjes door kernen. Het eerste hoofdstuk wordt besloten met een résumé van de experimentele informatie die van belang kan zijn voor verder onderzoek van het gegeven beeld van de vorming van alfadeeltjes.

In hoofdstuk II wordt het uitwendige probleem beschouwd en wel in het bijzonder de hoekverdeling van alfastraling van gerichte kernen. Er worden formules gegeven voor deze hoekverdeling (voor willekeurige kernen) in afhankelijkheid van de mate van gerichtheid van de emitterende kernen. Vervolgens wordt tot het speciale geval van sferoidaal gedeformeerde kernen overgegaan, vooral met het oog op experimenten betreffende de hoekverdeling van alfastraling en de nucleaire gegevens die hieruit kunnen worden verkregen en die niet volgen uit de intensiteiten in de fijnstructuur van alfastraling of uit de alfa-gamma-hoekcorrelaties. Het blijkt dat gelijktijdige waarneming (echter zonder alfa-gamma-coincidentiemetingen) van de hoekverdeling van de gammastraling van belang is. Met de verkregen nucleaire gegevens kan bepaald worden of de emissie van het alfadeeltje bij voorkeur geschiedt van een van de polen of van de equator van het oppervlak van de sferoidaal gedeformeerde kern; dit betreft dus direct het inwendige probleem van de vorming van alfadeeltjes uit de nucleonen in de kern. De relatie tussen de hoekverdeling in het laboratoriumcoordinatensysteem en die in het eigen coordinatensysteem van de kern, kan in de klassieke limiet duidelijk gemaakt worden door eenvoudige geometrische middeling over de precessie van de kern. Hierbij wordt gebruik gemaakt van enkele resultaten van hoofdstuk III. Tot slot wordt in het tweede hoofdstuk een analyse gegeven van experimenten over de hoekverdeling van alfastraling van gerichte kernen.

In hoofdstuk III worden de klassieke limieten (asymptotische uitdrukkingen voor grote impulsmomenten) van Clebsch-Gordancoefficienten, Racahcoefficienten en de functies $D^l_{mn}(\varphi, \vartheta, \psi)$ van de matrixrepresentatie van de rotatiegroep onderzocht. Klassieke analoga voor het kwadraat van Clebsch-Gordancoefficienten en het kwadraat van de $D^l_{mn}(0, \vartheta, 0)$-functies worden voorgesteld op grond van hun geometrische betekenis. Uit de Schrödingervergelijking voor de symmetrische rotator worden WKB-uitdrukkingen afgeleid voor de $D^l_{mn}(\varphi, \vartheta, \psi)$-functies en voor de bolfuncties $Y_{lm}(\vartheta, \varphi)$. Aangetoond wordt hoe deze WKB-uitdrukkingen samenhangen met de klassieke analoga voor de Clebsch-Gordancoefficienten en $D^l_{mn}(\varphi, \vartheta, \psi)$-functies.