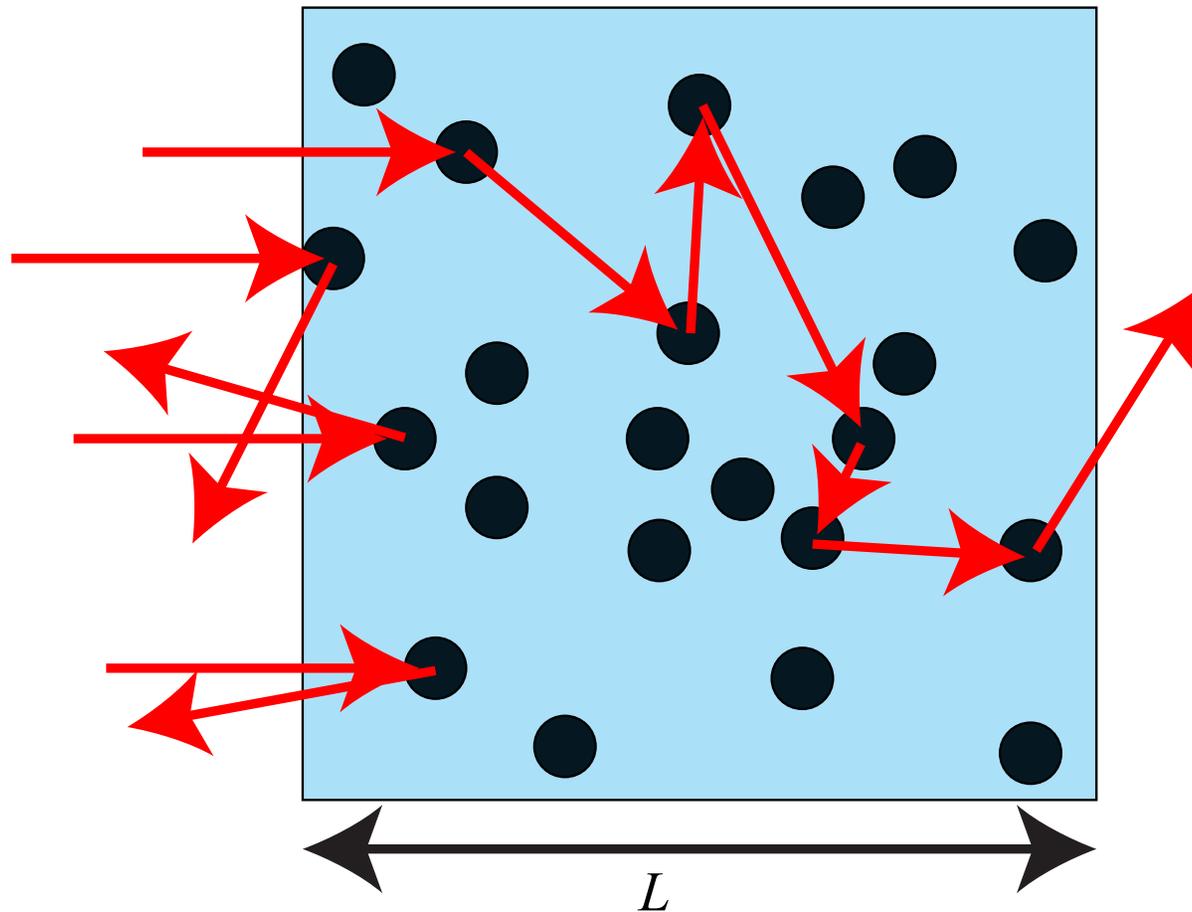


Waves in strongly disordered systems

JB Pendry
Imperial College London

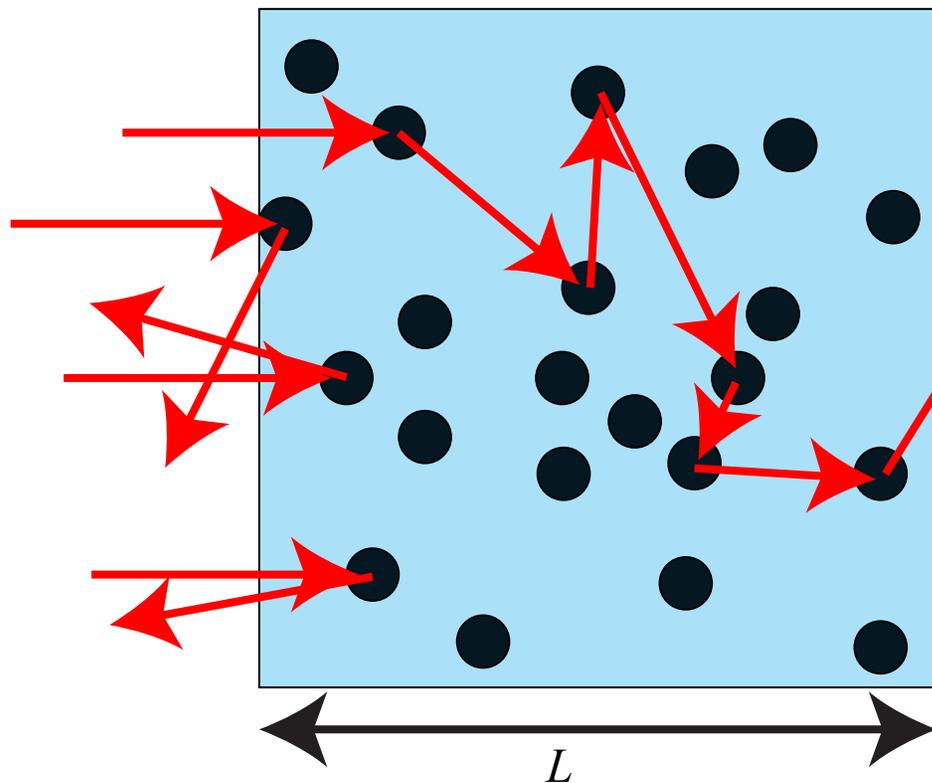


Waves in disordered systems – the classical picture

Conventionally waves are treated like particles randomly diffusing through a medium – their phase does not matter much.

Most incident waves are reflected, but a few get through the maze:

$$\text{transmission} \propto L^{-1}$$



This law holds if the disorder is weak:
e.g. in a metal the resistance is $\propto L$.

However in the presence of strong disorder this is no longer true ...

Anderson localisation

PHYSICAL REVIEW

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MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON

Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the “impurity band.” These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

Long ago in 1958 PW Anderson pointed out that in the presence of strong disorder diffusion is completely eliminated, electrons are localised on specific sites, and transmission drops dramatically, only happening when the frequency of the wave coincides with that of a localised state.

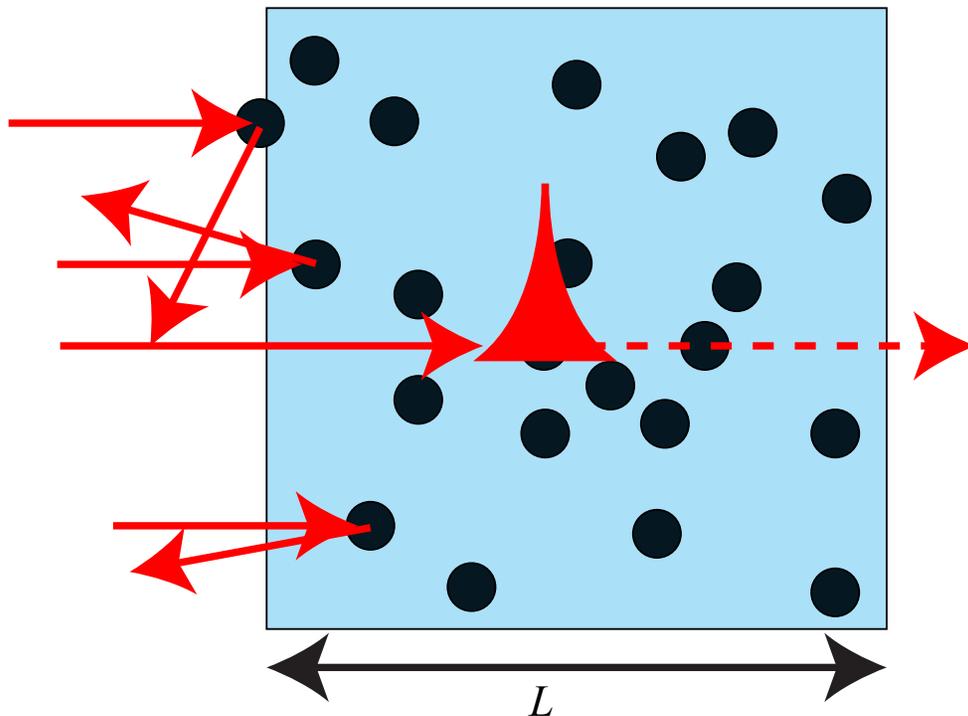


Anderson localisation

Strong disorder ‘localises’ the states of the system and transport is by tunnelling through these states which act as stepping stones. The tunnelling events are rare and transmission drops dramatically with sample thickness:

$$\text{transmission} \propto \exp(-L/\ell_0)$$

where ℓ_0 is the localisation length.



Despite intense theoretical activity, the nature of the transition from diffusion to localisation is not understood.

However recent advances enable the transition to be studied using optical experiments and this has brought new urgency to unravelling the mystery.

Anderson localisation

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What is not known:

- no theory has correctly predicted the critical exponent
- still less, no theory has correctly predicted the critical disorder, $\delta_c'^2$

Disorder in 1D systems

Although 3D systems and their transition for diffusion to localisation are still a mystery, 1D systems which are always localised are well understood.

We can make a good start on understanding localisation by studying the 1D case.

Transmission of waves through disordered systems

PRL **94**, 113903 (2005) PHYSICAL REVIEW LETTERS

week ending
25 MARCH 2005

Optical Necklace States in Anderson Localized 1D Systems

Jacopo Bertolotti, Stefano Gottardo, and Diederik S. Wiersma

European Laboratory for Nonlinear Spectroscopy and INFN, 50019 Sesto Fiorentino (Florence), Italy

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(Received 8 November 2004; published 22 March 2005)

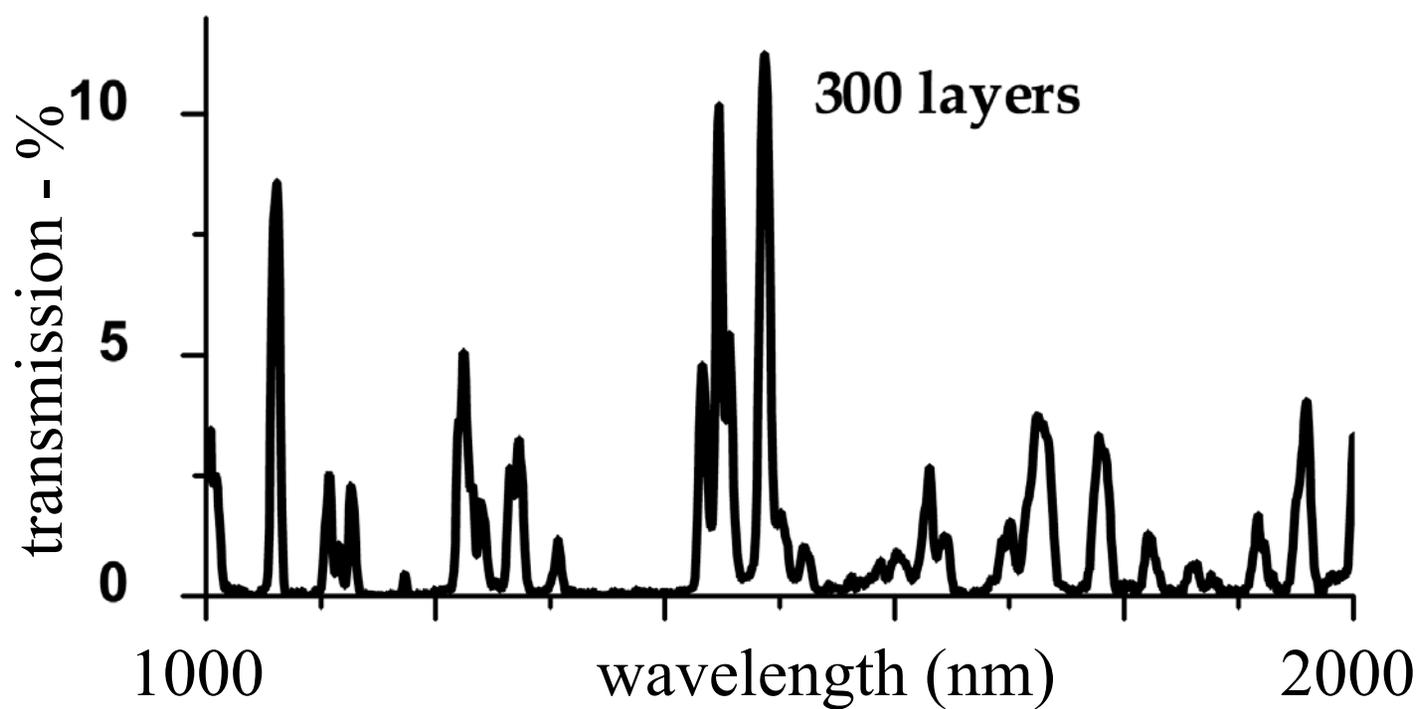
We report on the observation of nonlocalized modes or necklace states of light waves in disordered systems in the Anderson localized regime. The samples consist of positional-disordered binary multilayer systems. Anderson localized modes manifest themselves as narrow high-transmission peaks in the transmission spectrum, whereas the average of the logarithm of the transmission coefficient decreases linearly with thickness. Optical necklace states are observed as modes with a characteristic multi-resonance time response and relatively fast decay time.

DOI: 10.1103/PhysRevLett.94.113903

PACS numbers: 42.25.Dd, 05.40.Fb, 42.25.Hz, 42.55.Zz

Transmission of waves through disordered systems

1D disordered arrangement of layers



Transmission of waves through disordered systems

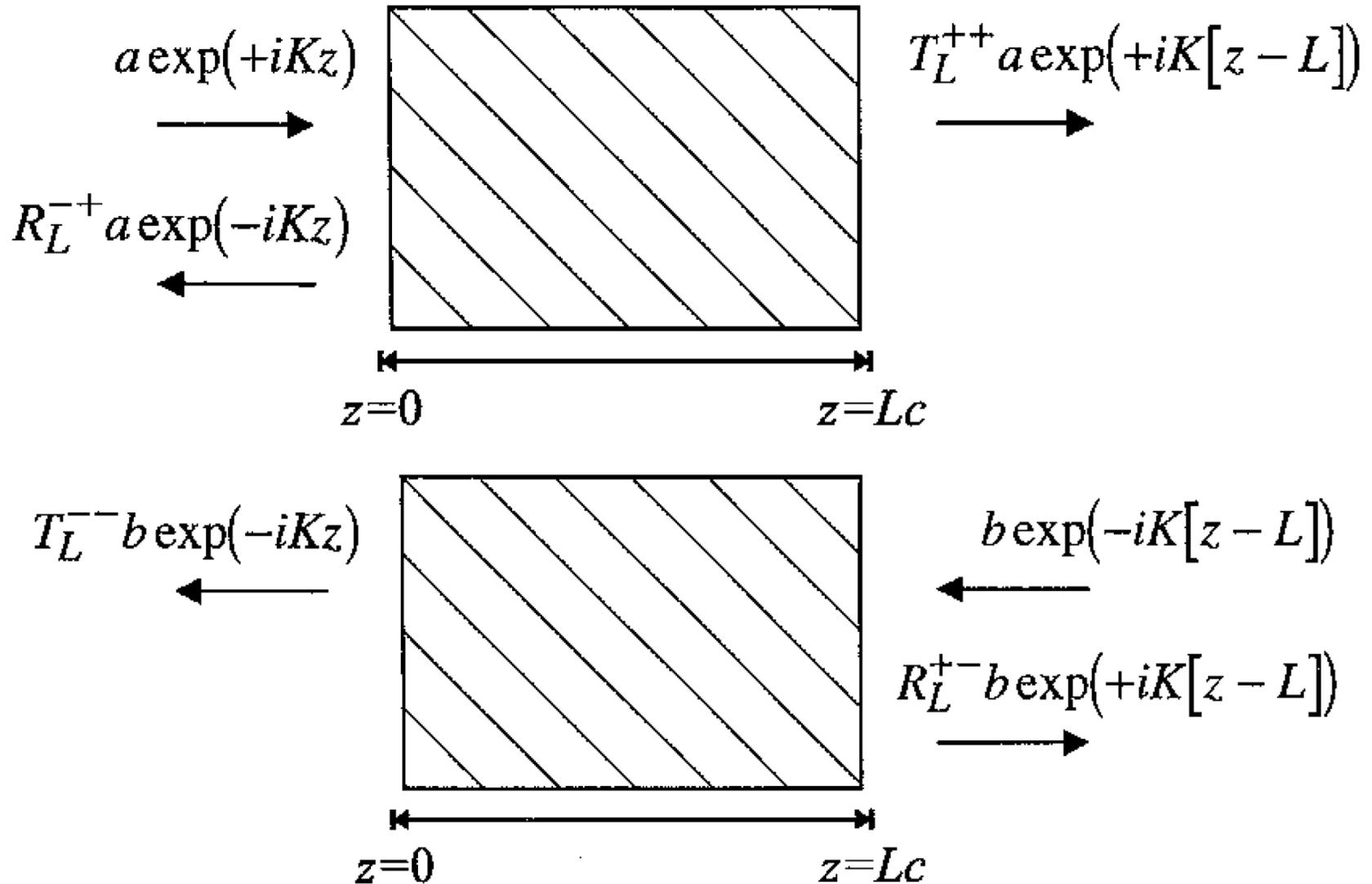
Strongly disordered systems, and all 1D disordered systems, trap electrons on localised sites.

When the energy of an electron hits one of these resonances we see transmission. The intensity of the transmission depends on how far the resonance is from the ends – in the middle works best.

How do we study the wild statistics of transmission through disordered systems?

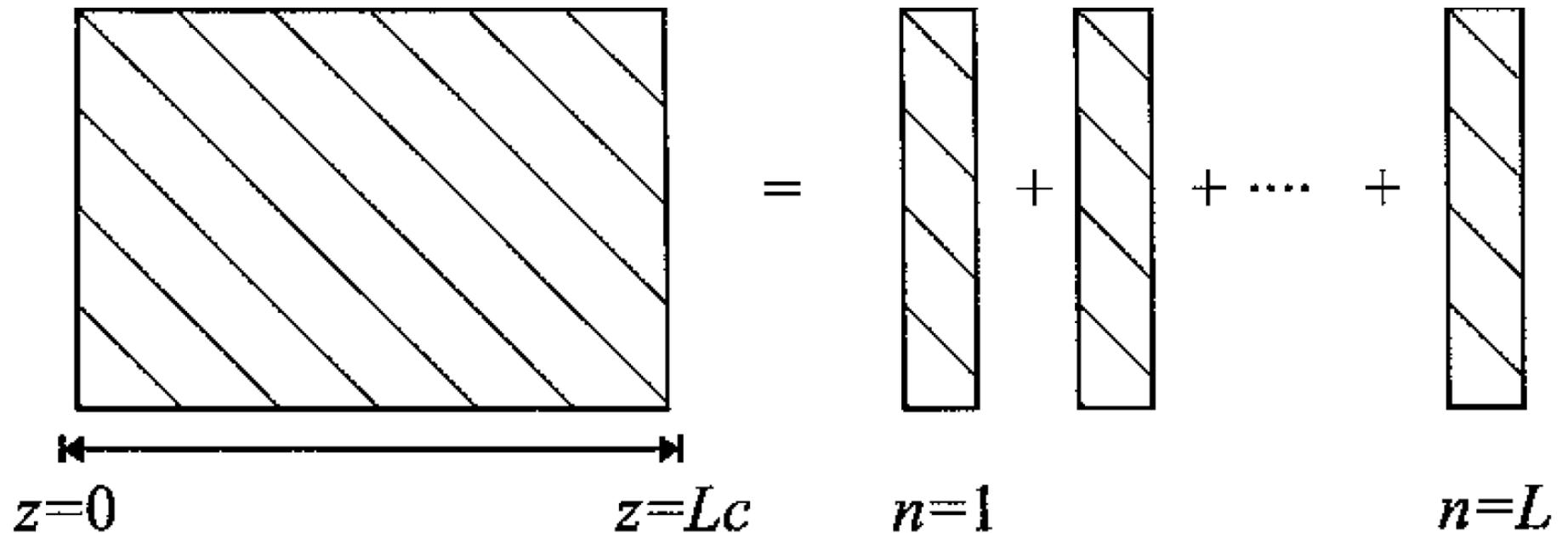
The problem

We wish to calculate T_L^{++} , T_L^{--} , R_L^{+-} , R_L^{-+} defined by,



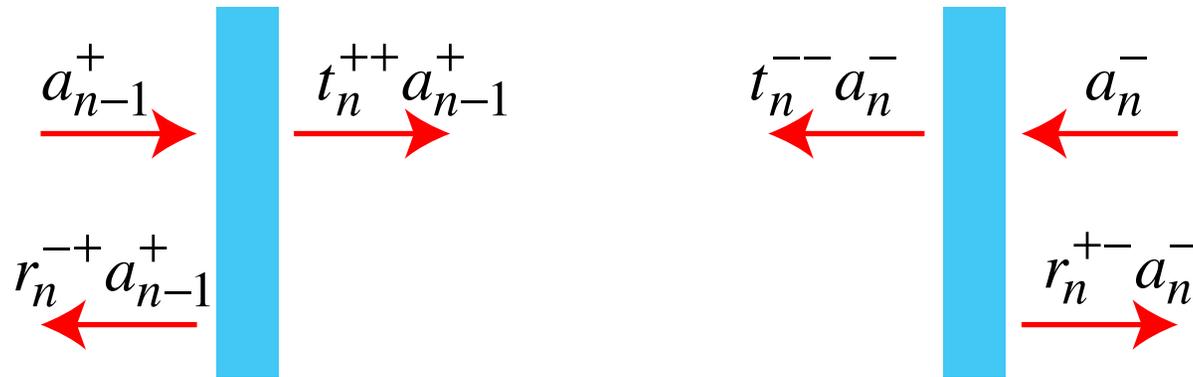
Transfer Matrices

We divide the system into statistically independent slices,

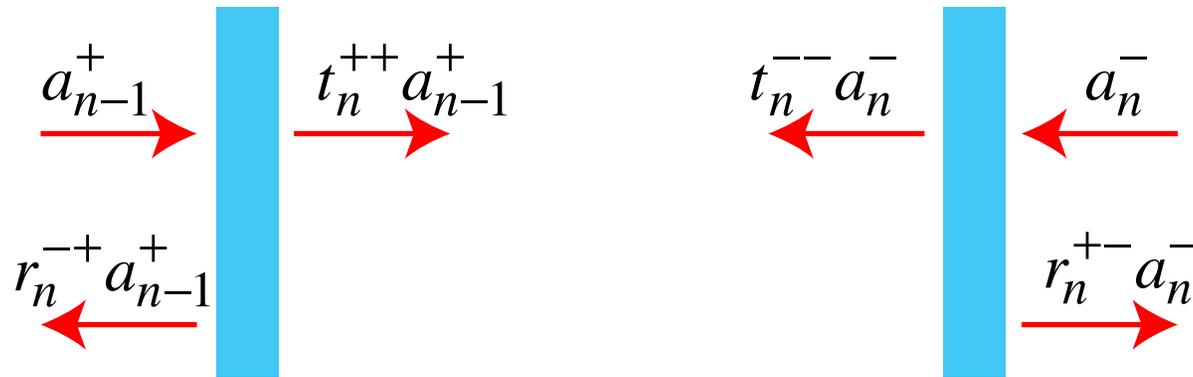


whose properties can easily be calculated. For example each slice might contain a single scattering centre.

Scattering by a single slice



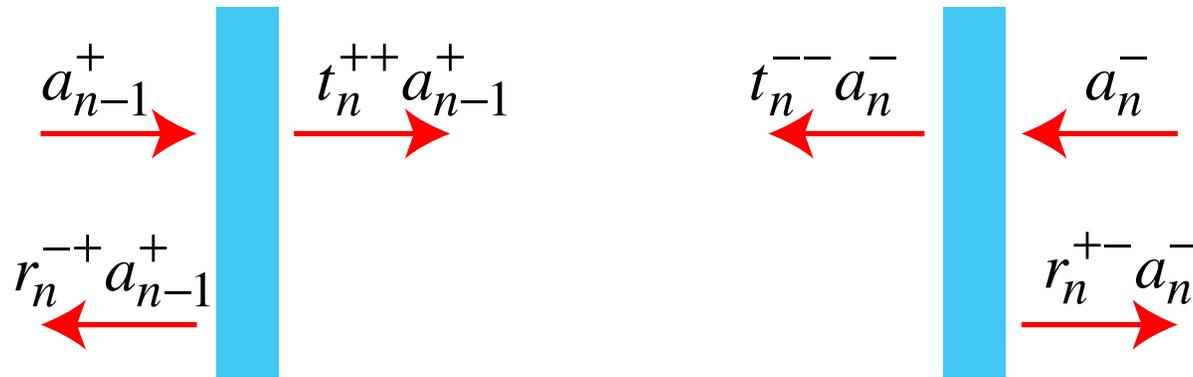
Scattering by a single slice



$$a_n^+ = t_n^{++} a_{n-1}^+ + r_n^{+-} a_n^-$$

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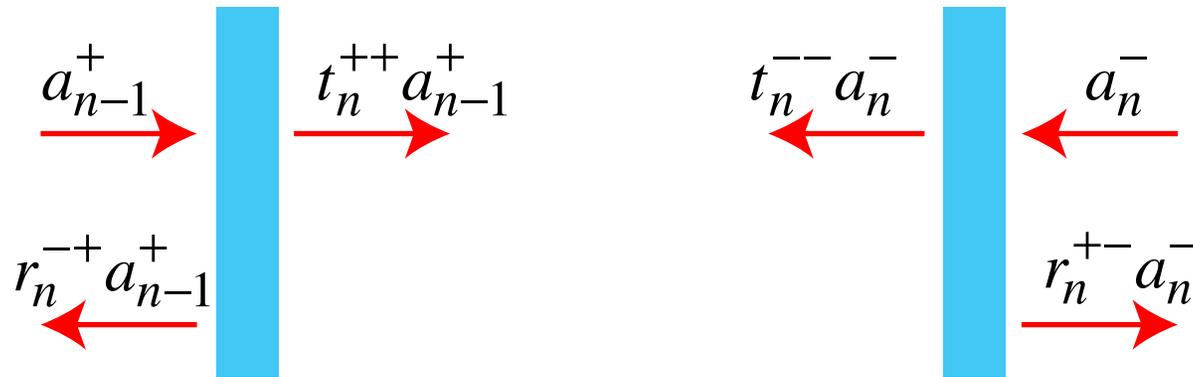
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rearranging,

$$\begin{bmatrix} 1 & -r_n^{+-} \\ 0 & t_n^{--} \end{bmatrix} \begin{bmatrix} a_n^+ \\ a_n^- \end{bmatrix} = \begin{bmatrix} t_n^{++} & 0 \\ r_n^{-+} & 1 \end{bmatrix} \begin{bmatrix} a_{n-1}^+ \\ a_{n-1}^- \end{bmatrix}$$

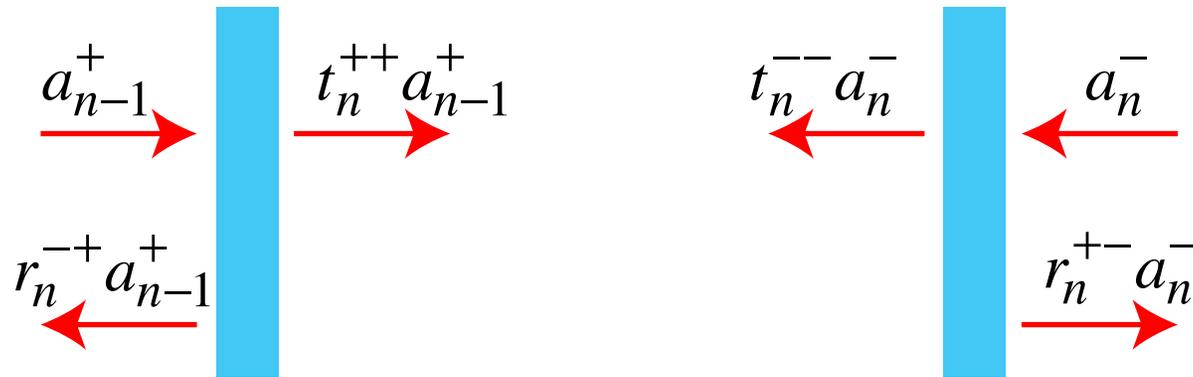
Scattering by a single slice



and inverting,

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Scattering by a single slice



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$$\begin{bmatrix} a_n^+ \\ a_n^- \end{bmatrix} = \mathbf{X}_n \begin{bmatrix} a_{n-1}^+ \\ a_{n-1}^- \end{bmatrix}$$

where \mathbf{X}_n is the transfer matrix,

$$\mathbf{X}_n = \begin{bmatrix} t_n^{++} - r_n^{+-} (t_n^{--})^{-1} r_n^{-+} & r_n^{+-} (t_n^{--})^{-1} \\ - (t_n^{--})^{-1} r_n^{-+} & (t_n^{--})^{-1} \end{bmatrix} = \begin{bmatrix} (t_n^{--*})^{-1} & r_n^{+-} (t_n^{--})^{-1} \\ r_n^{+-*} (t_n^{--*})^{-1} & (t_n^{--})^{-1} \end{bmatrix}$$

The fundamental theorem

Evidently we can repeatedly apply **transfer matrices** to give,

$$\begin{bmatrix} a_L^+ \\ a_L^- \end{bmatrix} = \mathbf{Z}_L \begin{bmatrix} a_0^+ \\ a_0^- \end{bmatrix} = \prod_{n=1}^L \mathbf{X}_n \begin{bmatrix} a_0^+ \\ a_0^- \end{bmatrix}$$

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\mathbf{Z}_L has the same functional form as \mathbf{X}_n providing a means of calculating transmission and reflection coefficients for a slab from the easy-to-obtain transmission and reflection coefficients for a thin slice.

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\mathbf{Z}_L has the same functional form as \mathbf{X}_n providing a means of calculating transmission and reflection coefficients for a slab from the easy-to-obtain transmission and reflection coefficients for a thin slice.

Also since the slices are independent we can easily average \mathbf{Z}_L :

$$\overline{\mathbf{Z}}_L = \prod_{n=1}^L \overline{\mathbf{X}}_n$$

More transfer matrices

We know how to average \mathbf{Z}_L , but this average is of little interest as $\overline{\left(T_L^{-}\right)^{-1}}$ tells is nothing about $\overline{\left|T_L^{-}\right|^2}$, the quantity we really want to know.

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We know how to average \mathbf{Z}_L , but this average is of little interest as $\overline{(T_L^-)^{-1}}$ tells is nothing about $\overline{|T_L^-|^2}$, the quantity we really want to know.

However there are other transfer matrices obtained by taking direct products,

$$\begin{bmatrix} \mathbf{Z}_L \\ \otimes \\ \mathbf{Z}_L \end{bmatrix} = \prod_{n=1}^L \begin{bmatrix} \mathbf{X}_n \\ \otimes \\ \mathbf{X}_n \end{bmatrix}$$

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We can think of each component matrix of the direct product as a particle evolving as the length of the system increases, much as real particles evolve in time. Since each of the N components of $\mathbf{X}_n^{\otimes N}$ is identical we can ask what is the symmetry of these particles? It turns out that **they are Bosons**.

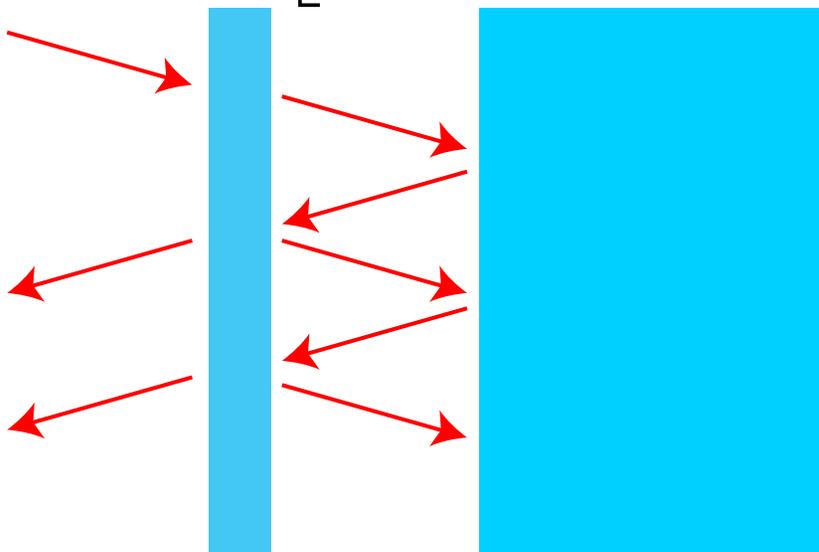
Even more transfer matrices

By various mathematical tricks we can extend the family of transfer matrices to negative and to fractional values of N . For example the $N = 0$ transfer matrix can be found as follows.

Suppose that we already know the reflection coefficient of $L - 1$ slices and use multiple scattering theory to add another slice,

$$R_L^{-+} = r^{-+} + t^{--} \left[R_{L-1}^{-+} + R_{L-1}^{-+} r^{+-} R_{L-1}^{-+} + \left(R_{L-1}^{-+} r^{+-} \right)^2 R_{L-1}^{-+} \dots \right] t^{++}$$

$$= \begin{bmatrix} r^{-+}, & t^{--} t^{++}, & t^{--} r^{+-} t^{++}, & t^{--} r^{+-2} t^{++}, & \dots \end{bmatrix} \begin{bmatrix} 1 \\ R_{L-1}^{-+} \\ R_{L-1}^{-+2} \\ R_{L-1}^{-+3} \\ \vdots \end{bmatrix}$$



taking direct products with itself,

$$\begin{bmatrix} 1 \\ R_L^{-+} \\ R_L^{-+2} \\ R_L^{-+3} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1, & 0, & 0, & 0, \dots \\ r^{-+}, & t^{--} t^{++}, & t^{--} r^{+-} t^{++}, & t^{--} r^{+-2} t^{++}, \dots \\ r^{-+} r^{-+}, & r^{-+} t^{--} t^{++}, & r^{-+} t^{--} r^{+-} t^{++} + (t^{--} t^{++})^2, & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ R_{L-1}^{-+} \\ R_{L-1}^{-+2} \\ R_{L-1}^{-+3} \\ \vdots \end{bmatrix}$$



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It follows that

$$\mathbf{Z}_L^{\otimes N=0} =$$

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Fractional and negative order transfer matrices are infinite.



Maximal fluctuations

$$\overline{\mathbf{Z}_L^{\otimes N=0}} = \prod_{n=1}^L \overline{\mathbf{X}_n^{\otimes N=0}} = \left[\overline{\mathbf{X}_n^{\otimes N=0}} \right]^L$$

is an infinite matrix and contains all possible products of $\overline{T_L T_L^*}$.

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In particular the averages of all powers of this quantity, $\overline{(T_L T_L^*)^M}$ can be found somewhere in the elements of $\overline{\mathbf{Z}_L^{\otimes N=0}}$.

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If the system is large, $L \gg 1$, then $\overline{\mathbf{Z}_L^{\otimes N=0}}$ is dominated by the largest eigenvalue of $\overline{\mathbf{X}_n^{\otimes N=0}}$,

$$\overline{(T_L T_L^*)^M} = \left[\overline{\mathbf{Z}_L^{\otimes N=0}} \right]_{i_M, j_M} = \left[\left[\overline{\mathbf{X}_n^{\otimes N}} \right]^L \right]_{i_M, j_M} \approx \langle i_M | \lambda_{\max} \rangle \langle \lambda_{\max} | j_M \rangle \lambda_{\max}^L$$



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Hence the ratio of the moments in the limit $L \rightarrow \infty$ is independent of L . This has implications for the probability distribution of $T_L T_L^*$.

$$\frac{\overline{(T_L T_L^*)^M}}{\overline{T_L T_L^*}} \approx C_M$$



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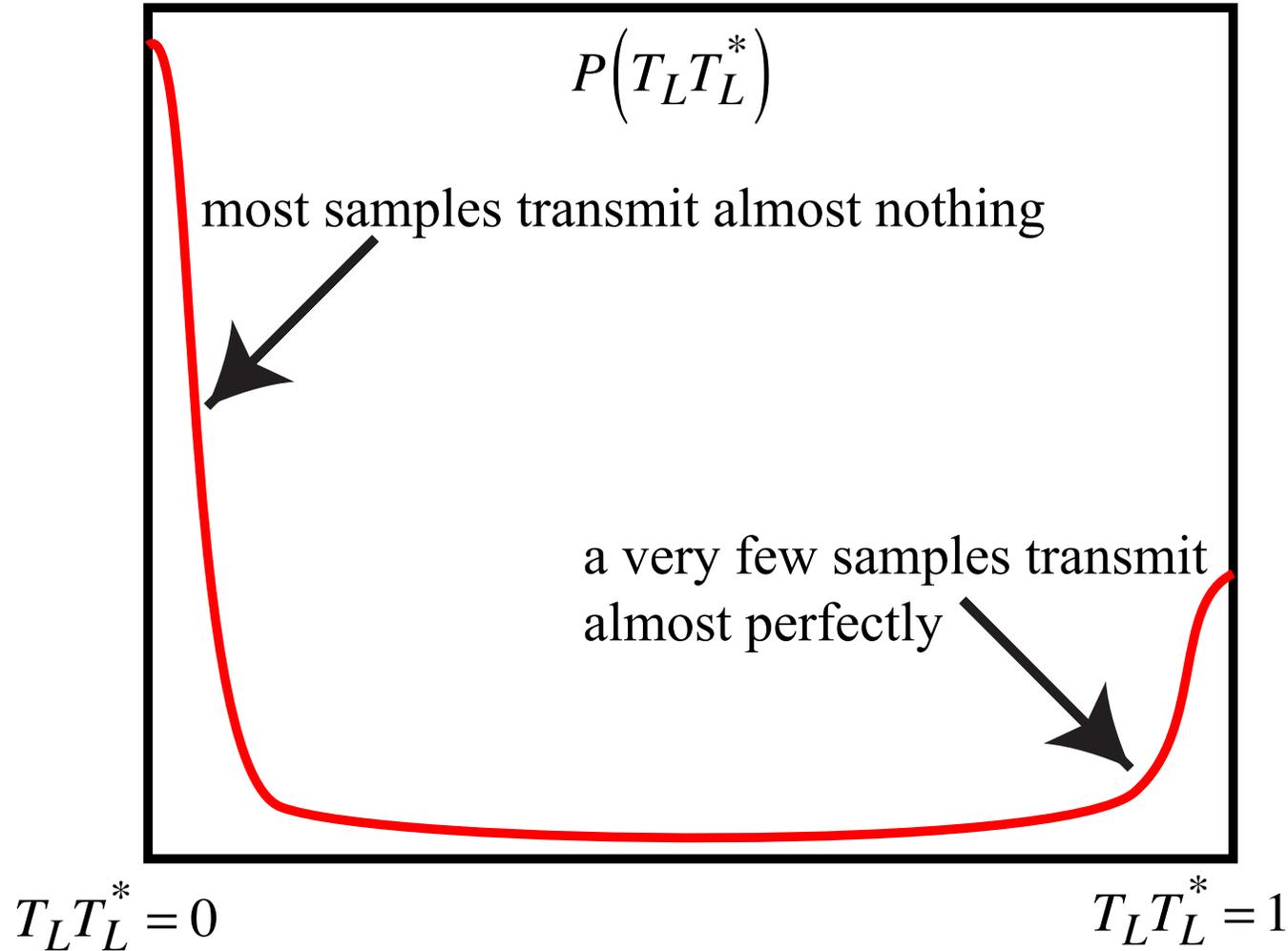
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$$\frac{\overline{(T_L T_L^*)^M}}{\overline{T_L T_L^*}} \approx C_M$$

The same result also applies in higher dimensions, whether or not the system is localised.

Maximal fluctuations

The distribution is 'bimodal' so that if you happen to catch the right frequency, the system can be almost perfectly transparent, but mostly it rejects the incident wave.



Physica **A168** 400 (1990), J.B. Pendry, A. Mackinnon & A.B. Prêtre.

Maximal fluctuations – 1D

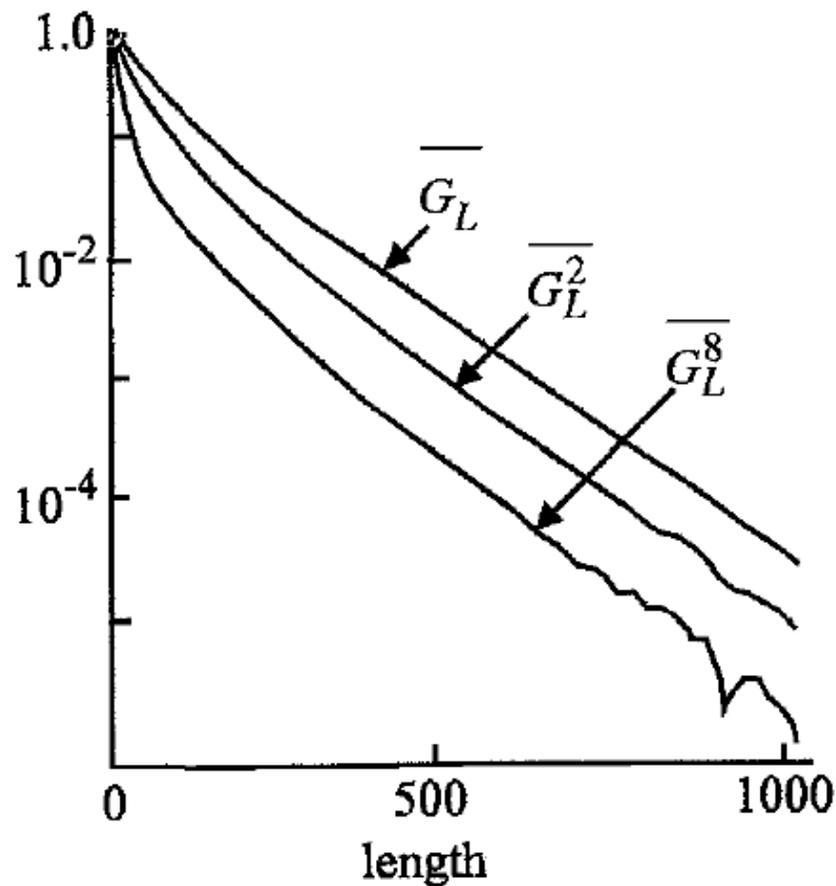
In 1D we can obtain an analytic expression for the moments,

$$\frac{\overline{(T_L T_L^*)^M}}{T_L T_L^*} \approx C_M = \frac{\Gamma^2(N - \frac{1}{2}) \Gamma^2(1)}{\Gamma^2(\frac{1}{2}) \Gamma^2(N)}$$

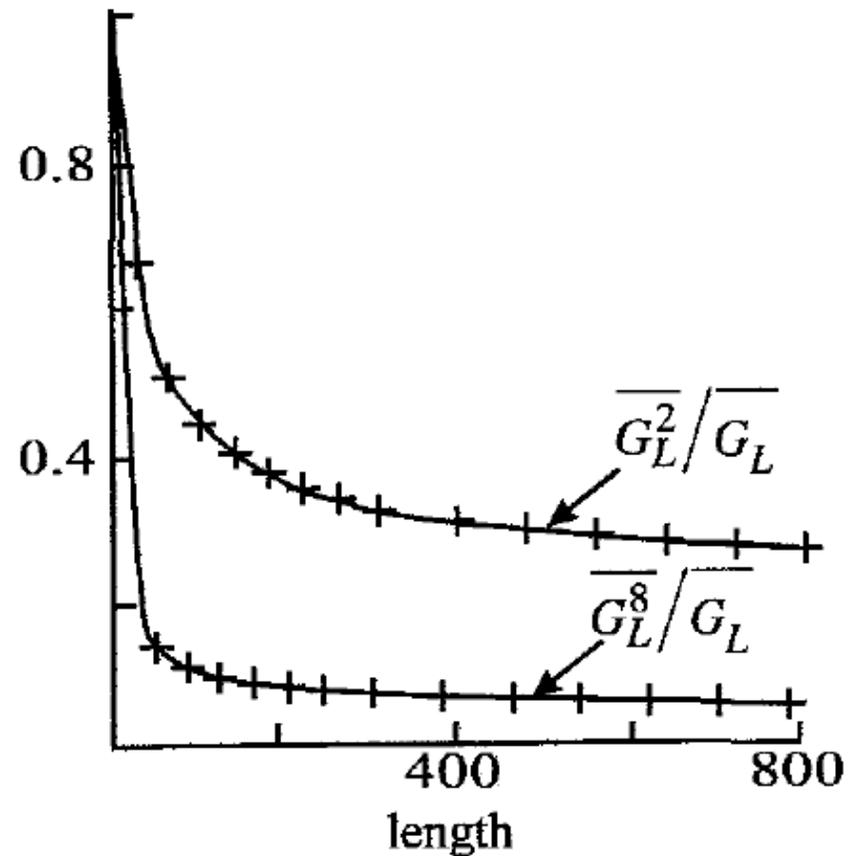
M	C_M (theory)	C_M (computed)
1	1.0	1.0
2	0.250	0.269
3	0.141	0.146
4	0.098	0.101
5	0.075	0.083
6	0.061	0.065

Comparison of values of C_M , as predicted by the theory of necklace states, and as calculated in simulation.

Maximal fluctuations – 1D computations



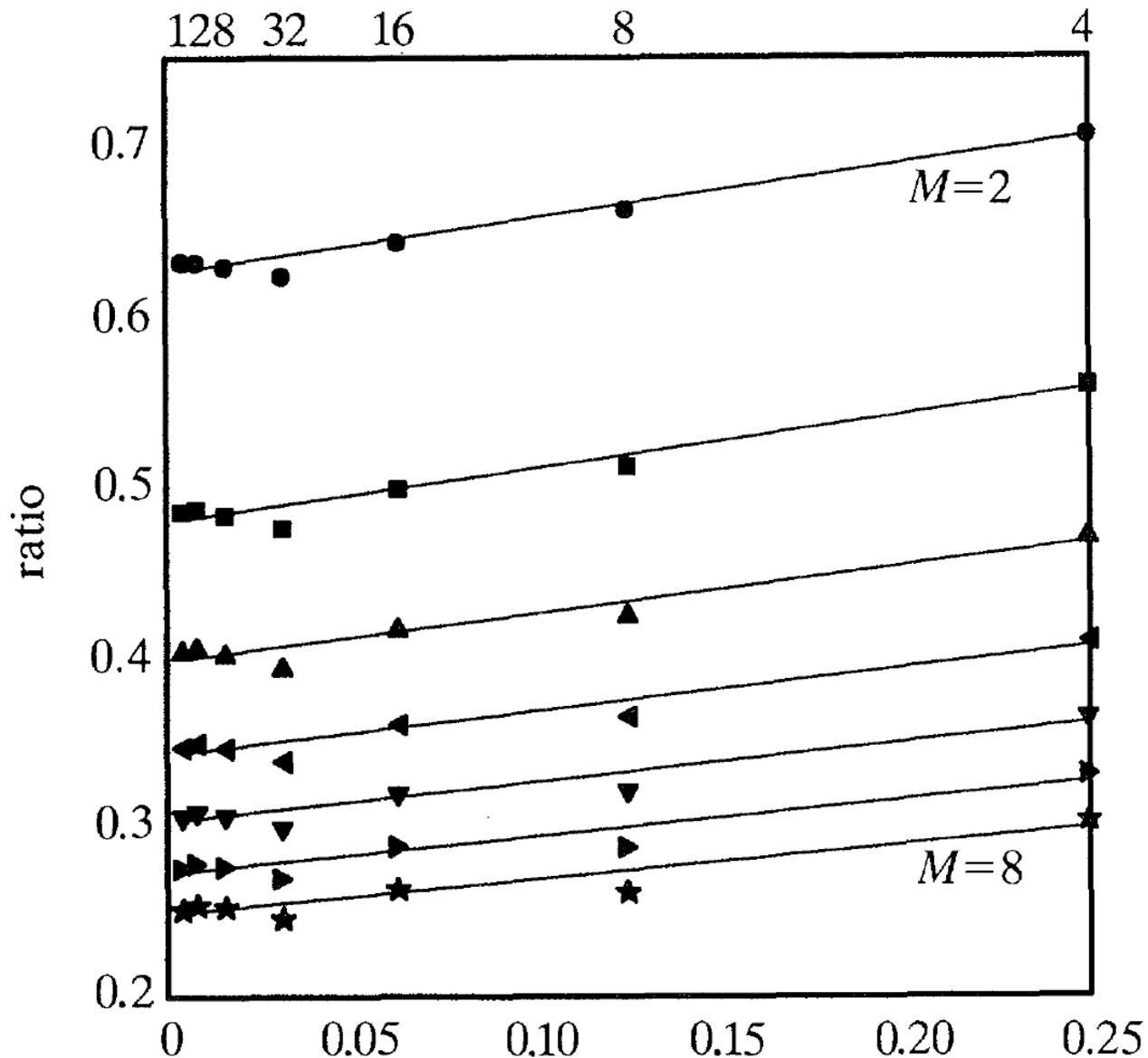
(a)



(b)

Moments of $G_L = T_L T_L^*$ computed by averaging over 9×10^6 1D samples of various lengths.

Maximal fluctuations – 2D computations



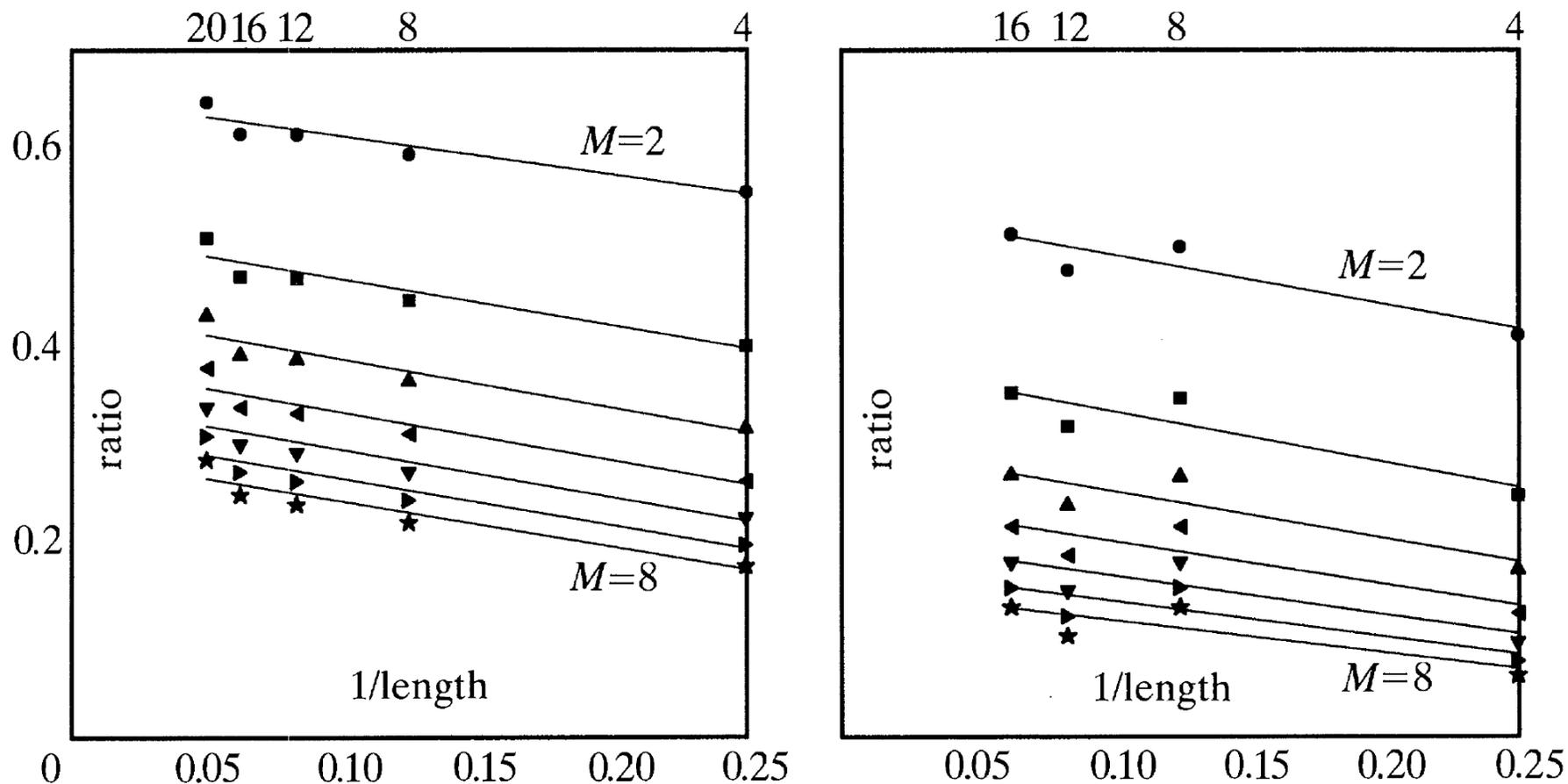
$$\frac{\overline{\text{trace}(T_L T_L^*)^M}}{\overline{\text{trace} T_L T_L^*}}$$

plotted against L^{-1} for squares of size $4 < L < 256$ averaged over 128 samples.

As in 1D, the ratios tend to a limit independent of L .

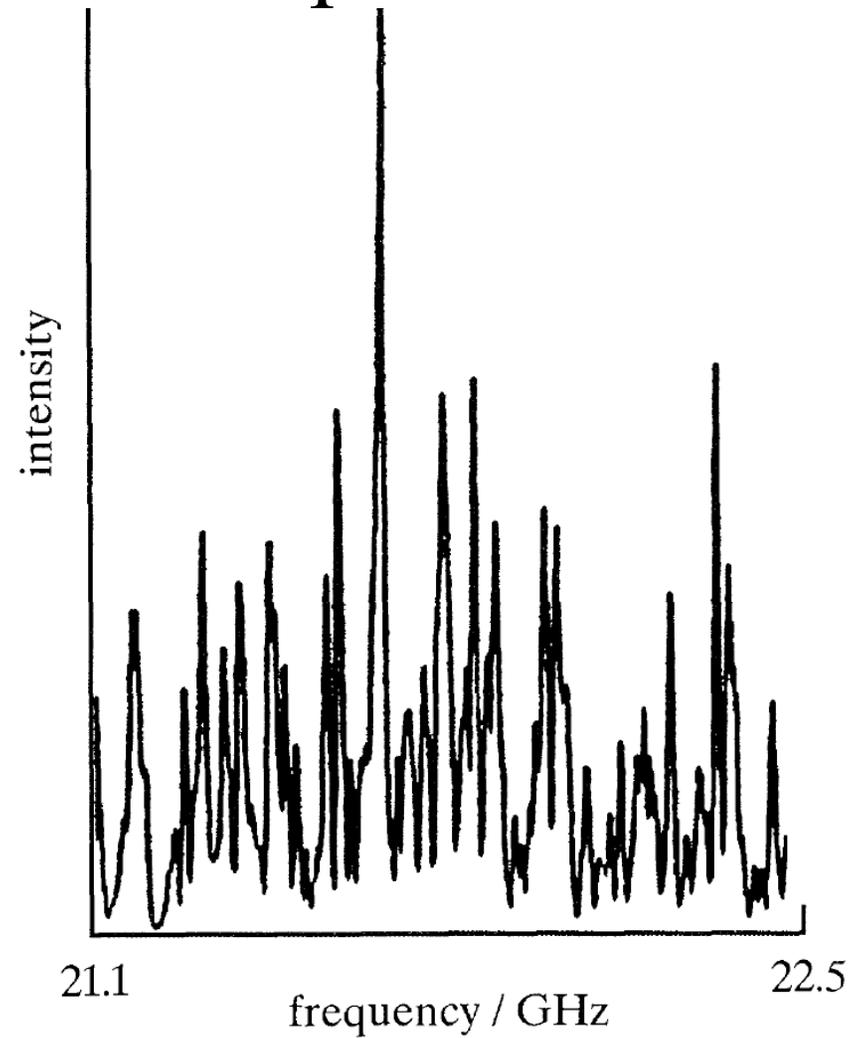
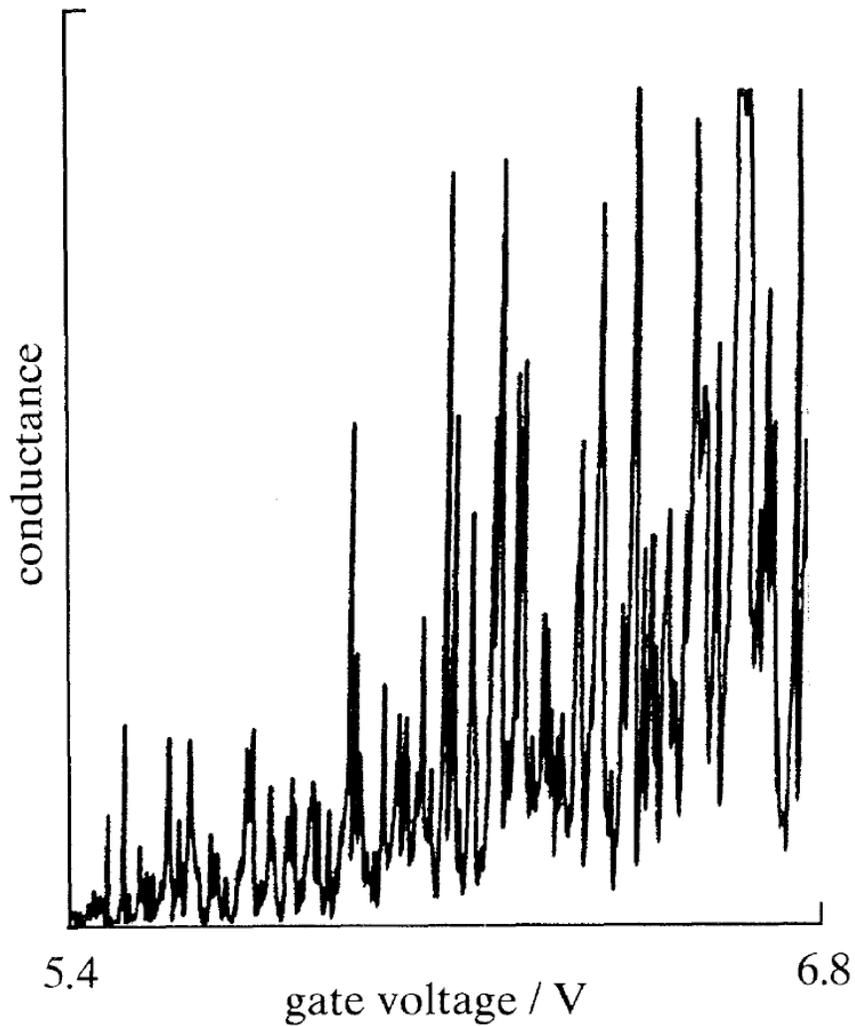


Maximal fluctuations – 3D computations



$\frac{\text{trace}(T_L T_L^*)^M}{\text{trace} T_L T_L^*}$ plotted against L^{-1} for cubes of size $4 < L < 20$ averaged over 128 samples. Left: disorder less than critical (diffusion); right: disorder greater than critical (localisation). As in 1D and 2D the ratios tend to a limit independent of L .

Maximal fluctuations – 3D experiment



Left: conductance of a silicon metal oxide field effect transistor measured at 50mK. Right: spectral intensity fluctuations for microwaves passing through a 140 cm length of tube filled with half-inch diameter polystyrene balls.

Maximal fluctuations

Our theorem predicts that, however small the average transmission, the average is dominated by a few highly transmitting eigenvalues of the transfer matrix.

If the incident waves had exactly the right form to select this dominant eigenvalue, then even a strongly disordered specimen would transmit most of the incident waves.

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Focusing coherent light through opaque strongly scattering media

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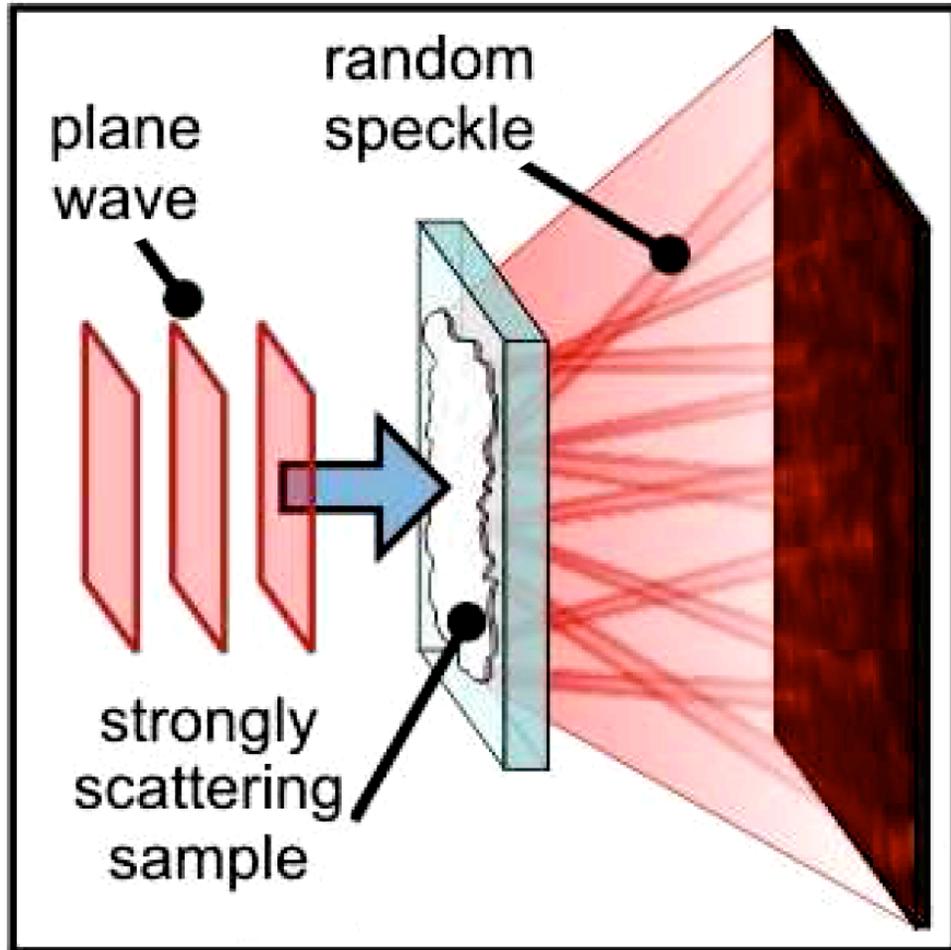
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We report focusing of coherent light through opaque scattering materials by control of the incident wavefront. The multiply scattered light forms a focus with a brightness that is up to a factor of 1000 higher than the brightness of the normal diffuse transmission. © 2007 Optical Society of America

OCIS codes: 290.1990, 290.4210, 030.6600.

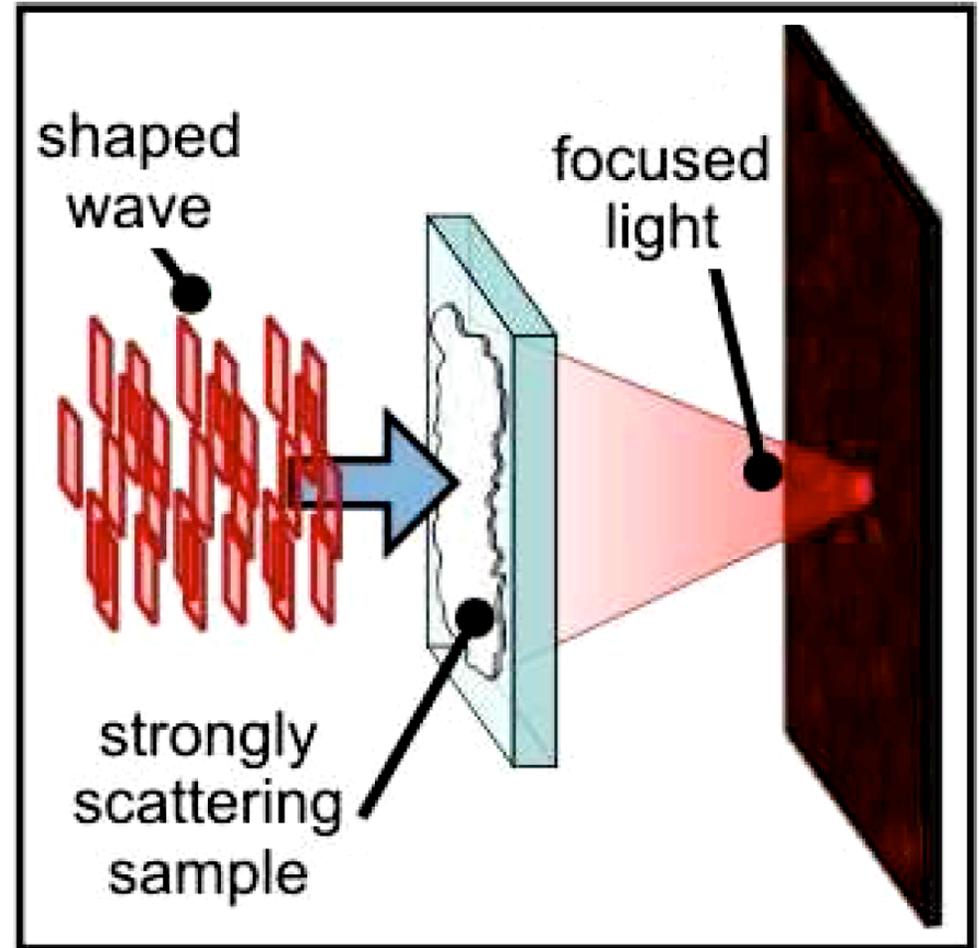
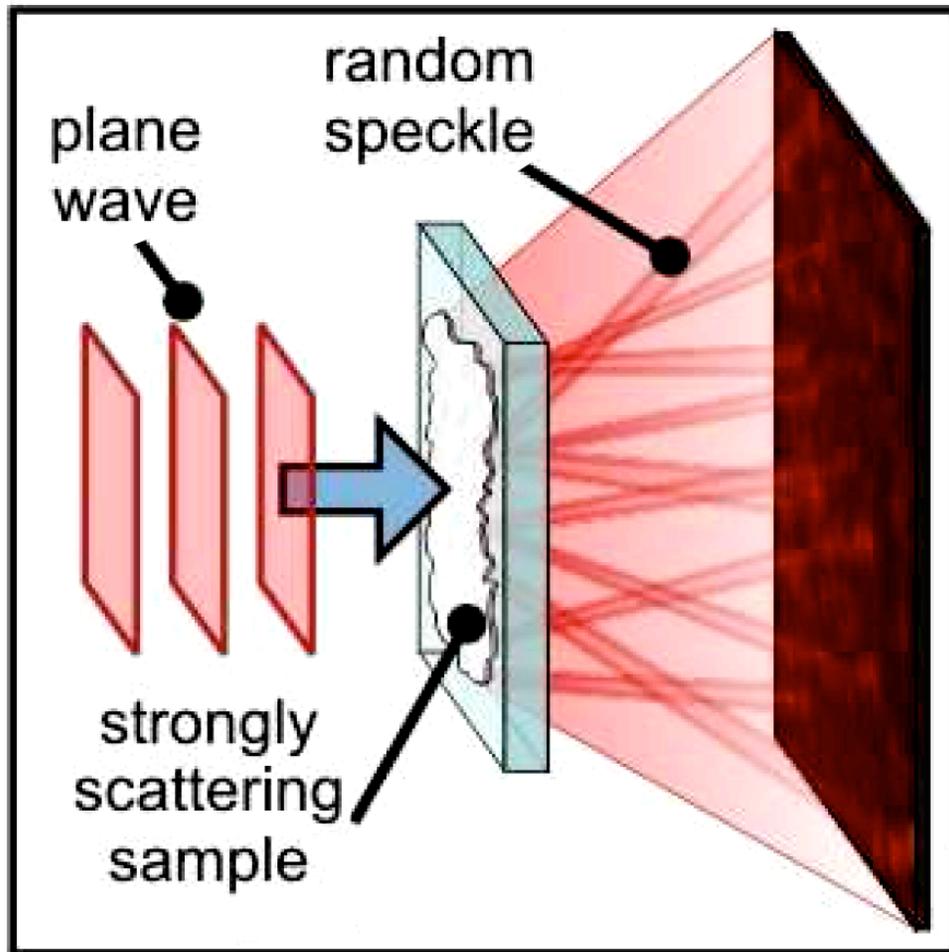
Vellekoop and Mosk

Optics Letters, **32**, 2309 (2007)

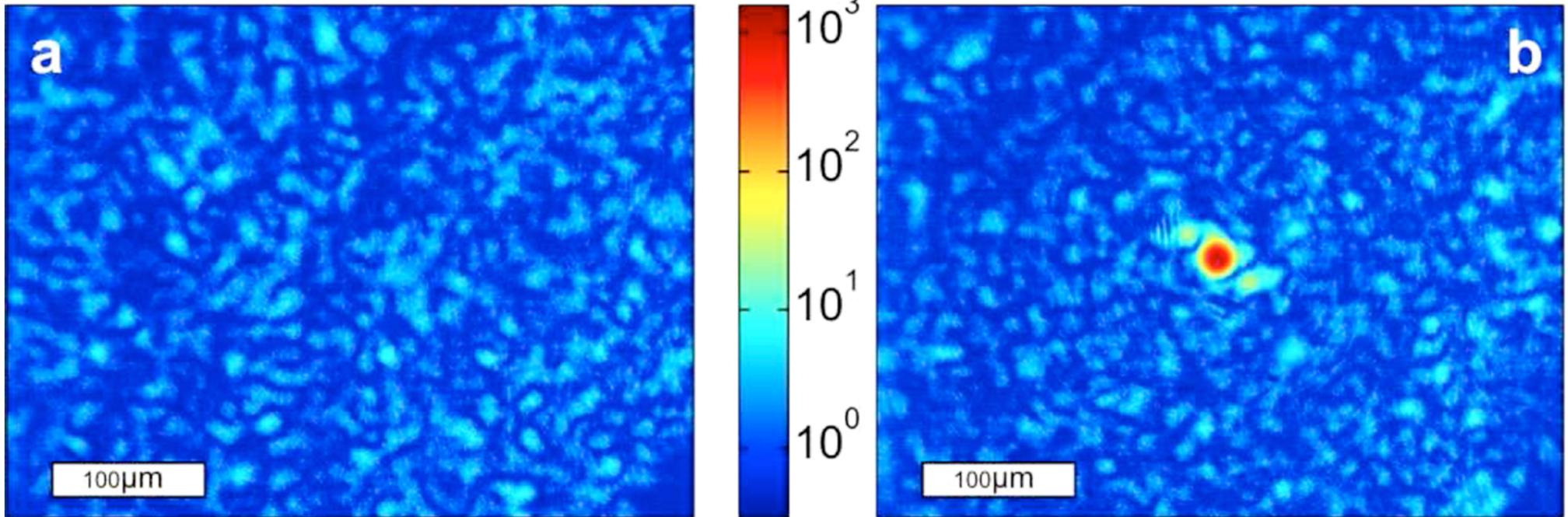


Vellekoop and Mosk

Optics Letters, **32**, 2309 (2007)



Vellekoop and Mosk *Optics Letters*, 32, 2309 (2007)



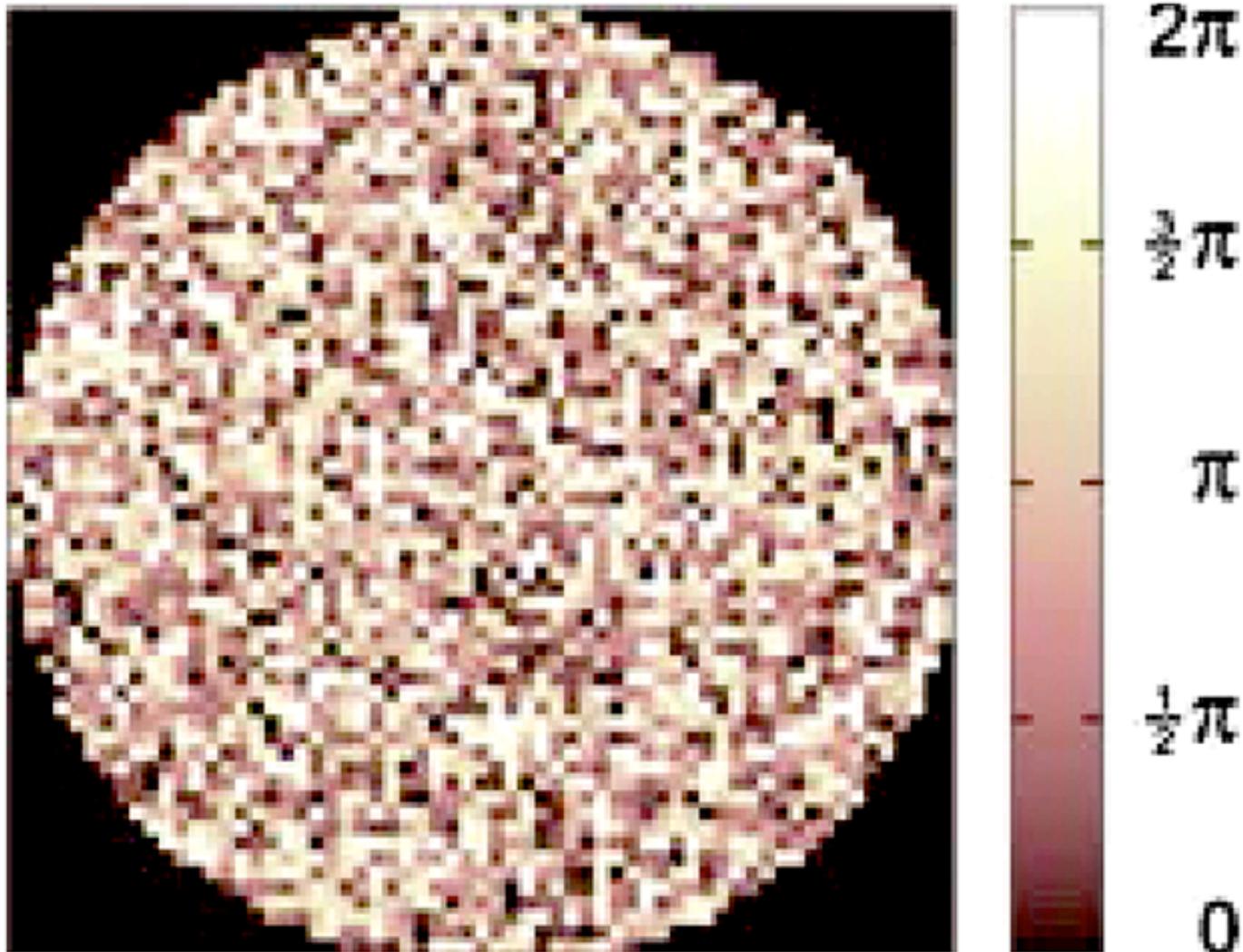
Transmission through a strongly scattering sample consisting of TiO_2 pigment (white paint).

(a) Transmission with an unshaped incident beam.

(b) Transmission after optimization

The scattered light is focused to a spot that is **1000 times brighter** than the original speckle pattern.

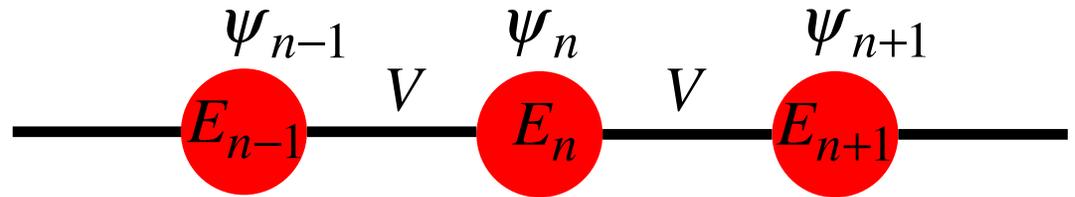
Vellekoop and Mosk *Optics Letters*, 32, 2309 (2007)



Phase of the optimized incident wavefront

Transfer Matrices – the model

The Anderson Model of disorder for electrons – 1D version



Consider a chain of atoms:

coupled together by ‘hopping integrals’, V , but with disordered energies,

$$E_n \psi_n + V(\psi_{n-1} + \psi_{n+1}) = E \psi_n$$

We can rearrange the Schrödinger equation to give,

$$\begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix} = \tilde{\mathbf{X}}_n \begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix}, \quad \tilde{\mathbf{X}}_n = \begin{bmatrix} (E - E_n)/V & -1 \\ +1 & 0 \end{bmatrix}$$

$\tilde{\mathbf{X}}_n$ is referred to as a *transfer matrix*: it transfers the wavefield down the chain of atoms and enables us to write,

$$\begin{bmatrix} \psi_{L+1} \\ \psi_L \end{bmatrix} = \prod_{n=1}^L \tilde{\mathbf{X}}_n \begin{bmatrix} \psi_1 \\ \psi_0 \end{bmatrix}$$



Transfer Matrices - continued

If there were no disorder in the system, the eigenstates would be Bloch waves,

$$\begin{bmatrix} (E - \overline{E}_n)/V & -1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} e^{\pm iKc} \\ 1 \end{bmatrix} = e^{\pm iKc} \begin{bmatrix} e^{\pm iKc} \\ 1 \end{bmatrix}$$

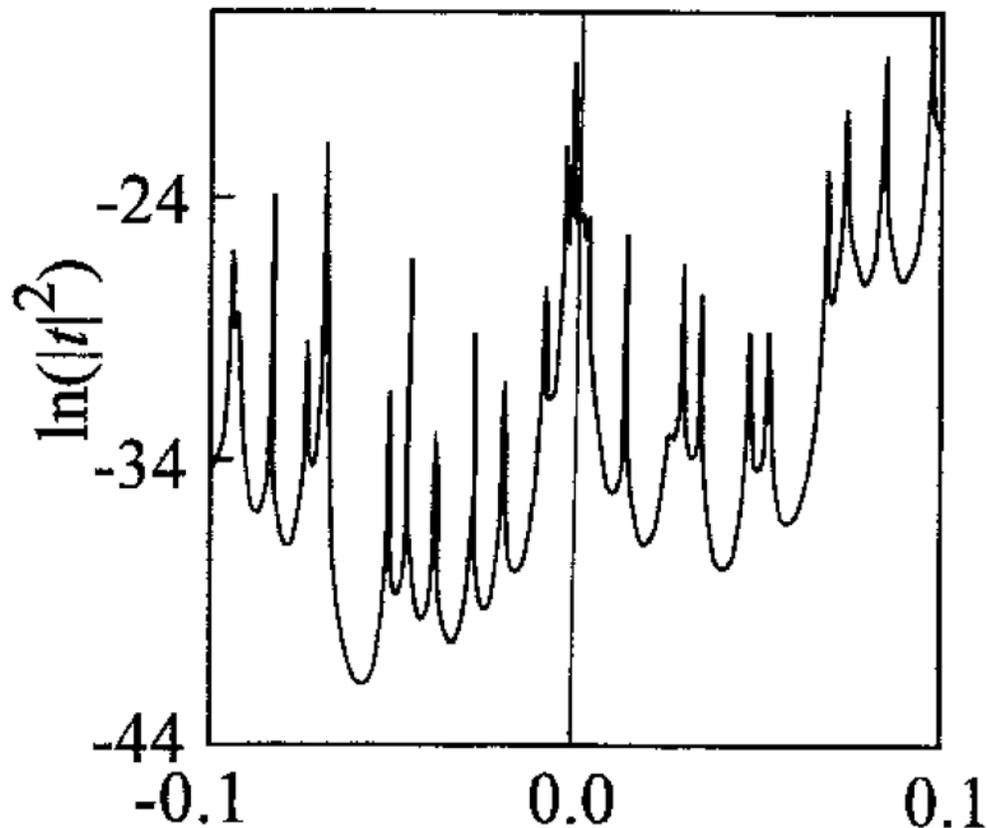
which we can use as a basis for $\tilde{\mathbf{X}}_n$,

$$\mathbf{X}_n = \begin{bmatrix} e^{+iKc} & 0 \\ 0 & e^{-iKc} \end{bmatrix} - i\delta'_n \begin{bmatrix} +e^{+iKc} & +e^{-iKc} \\ -e^{+iKc} & -e^{-iKc} \end{bmatrix}, \quad \delta'_n = \frac{\overline{E}_n - E}{2V \sin(Kc)}$$

where,

$$\delta'_n = \frac{\overline{E}_n - E}{2V \sin(Kc)}$$

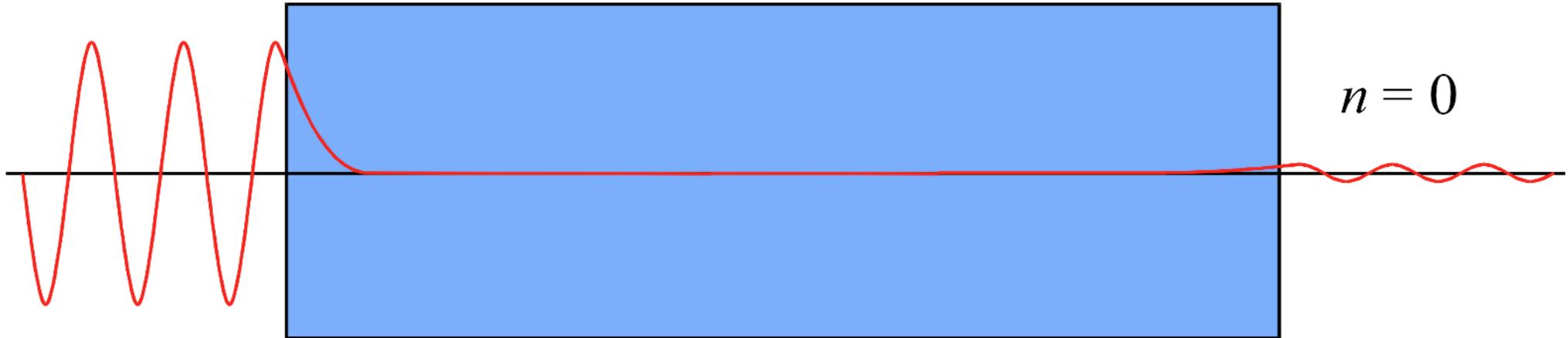
Computations using transfer matrices



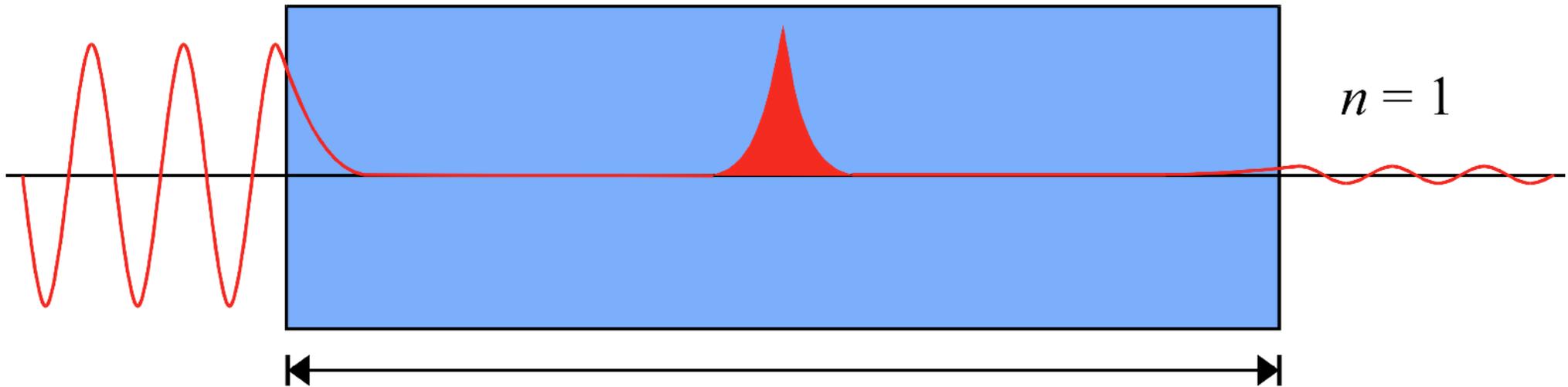
Transmission coefficient of a disordered 1D Anderson model ($L = 1000$) calculated using transfer matrices.

Necklace States

How does the wave pass through a disordered system?



or,



Necklace States

Argument for tunnelling via a resonance:

Rate at which waves *tunnel directly* given by the decay rate,

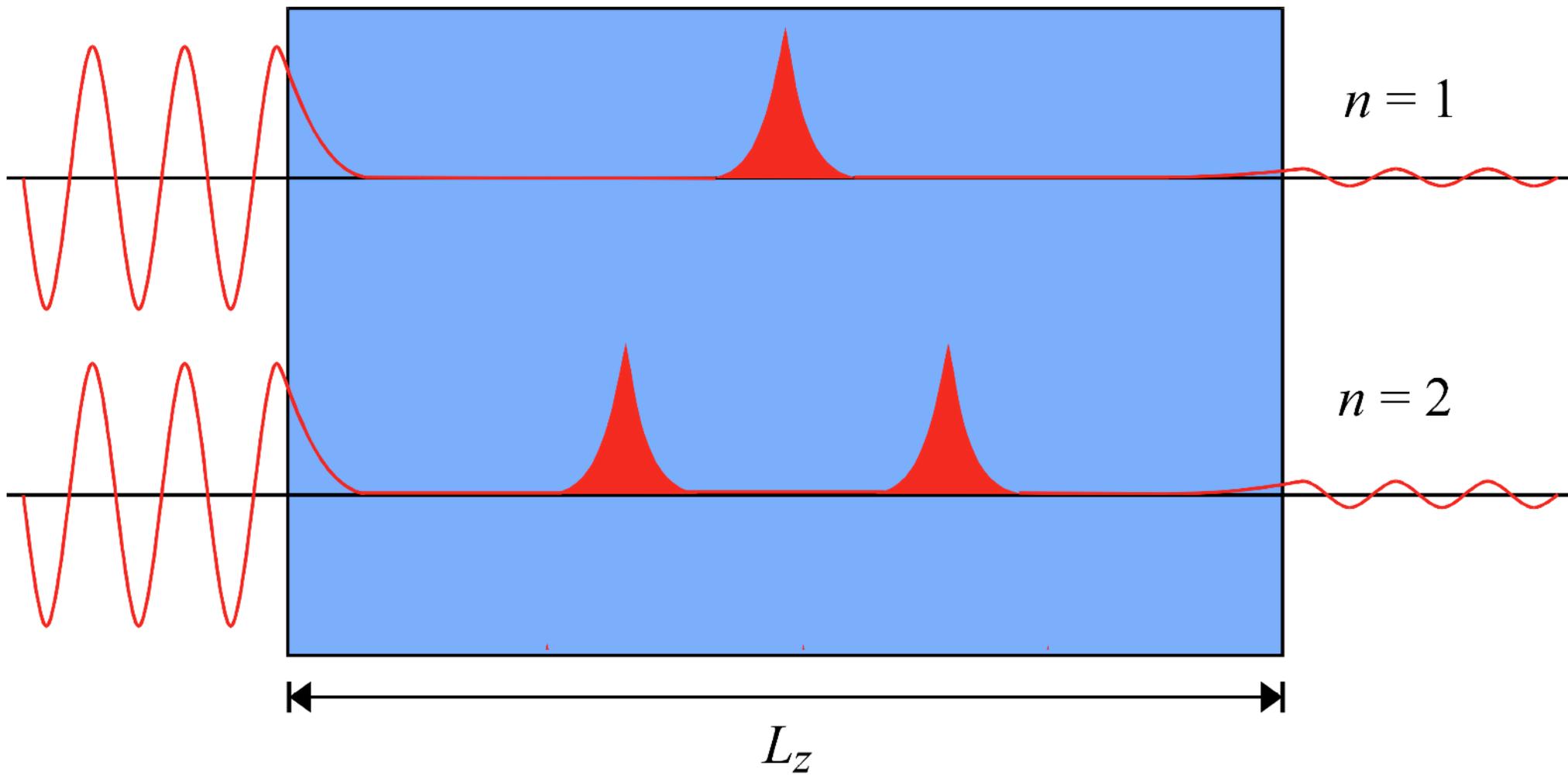
$$\tau_0^{-1} \propto \exp(-L/\ell_0)$$

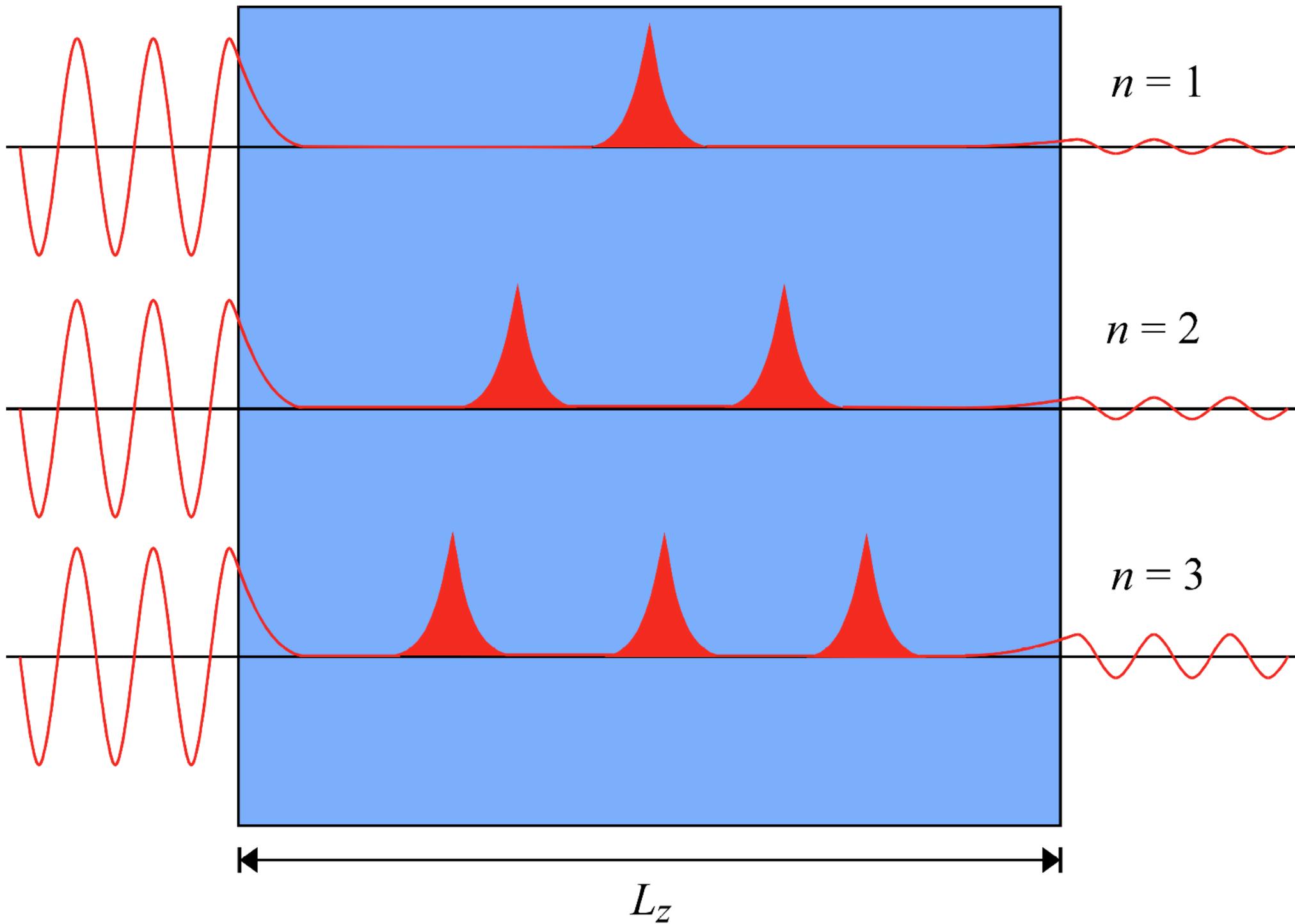
For efficient tunnelling via a resonance, the resonance should lie half way across the specimen: it nearer to the entrance side the wave I deposit back where it came from, too far to the other side and the wave never reaches the resonance. The tunnelling rate for a resonance in the centre is,

$$\tau_1^{-1} \propto \exp(-L/2\ell_0)$$

Thus tunnelling via a resonance beats direct tunnelling hands down.

why stop at one?





Necklace States

What is the optimum number of resonances in a necklace?

Exact theory for the 1D case can calculate the bandwidth of conducting channels and shows that,

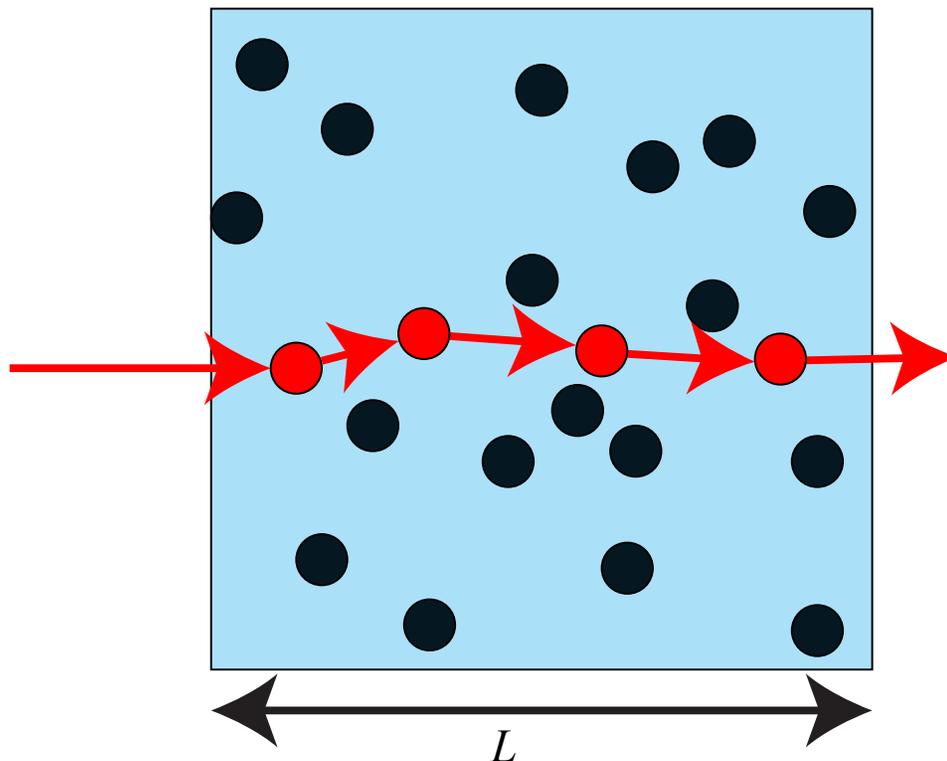
$$\delta\omega \approx \exp\left(-\frac{\pi}{1.816} \left(\overline{\delta_n^2} L\right)^{\frac{1}{2}}\right)$$

which means that a typical hopping distance is $\approx \sqrt{L}$. As the sample gets longer there are more resonances in the necklace, but they are spaced further apart:

the necklaces have fractal dimension of $1/2$.

Necklace States in 3D

Necklaces are important in 3D localised systems. Waves cross the sample through a series of hops. To maximise the overlap with the incident frequency, hops must be as short as possible. As in 1D this requires that they be of approximately equal length, but in addition the path across the sample must be as short as possible and therefore as straight as possible.



In 3D localised systems conduction is through a few 'holes' in the sample that occur exponentially rarely across the surface.

Conclusions

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Transfer matrices have shown great mathematical power in solving difficult problems in 1D systems. They have given results which other methods have only hinted at or not revealed at all and often given them in very general terms.

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More than half of a century's work on the transition between localised and delocalised behaviour has not solved the problem, despite great ingenuity and application of the most powerful mathematical techniques available to us; we do not understand this transition in any real sense of the word.

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More than half of a century's work on the transition between localised and delocalised behaviour has not solved the problem, despite great ingenuity and application of the most powerful mathematical techniques available to us; we do not understand this transition in any real sense of the word.

Perhaps transfer matrices can make an impact. Their mathematical structure is alien to that which has been tried before; they have a record of success in one dimension; and they can already reproduce the major known results in three dimensions.

Thank you for inviting me!

