1 Majorana fermions in a chiral *p*-wave superconductor

The Hamiltonian of a two-dimensional superconductor with chiral p-wave pairing has the form

$$H = \begin{pmatrix} U + p^2/2m & \Delta(p_x - ip_y) \\ \Delta(p_x + ip_y) & -U - p^2/2m \end{pmatrix}. \tag{1}$$

The upper-left block gives the electrostatic potential energy $U(x,y)$ and kinetic energy $(p_x^2 + p_y^2)/2m$ of electrons and the lower-right block the corresponding energy for holes. Electrons and holes are coupled by the superconducting pair potential $\Delta(p_x \pm ip_y)$, with $\boldsymbol{p} = -i\hbar\nabla$ and a real parameter Δ having the dimension of velocity.

Volovik (1999) showed that a magnetic vortex in this superconductor binds a zero mode, which is a Majorana fermion. Here we investigate, following Kitaev (2000), a purely electrostatic way to create a pair of Majoranas at the end points of a wire,[1] defined by

$$U(x,y) = \begin{cases} \infty & \text{if } |y| > W/2, \\ -U_0 & \text{if } |y| < W/2. \end{cases} \tag{2}$$

1.1 Gap closing transition

The appearance of a Majorana fermion is signaled by a closing and reopening of the excitation gap. This transition can be studied for an infinitely long wire, where we can assume translational invariance in the x-direction. We then look for eigenstates of H of the form $\psi(x,y) = \phi(y)e^{ikx}$, with eigenvalue $E(k)$ dependent on the wave number k along the line defect.

(a) Check that the Hamiltonian (1) is Hermitian and satisfies the electron-hole symmetry condition

$$\begin{pmatrix} H_{ee} & H_{eh} \\ H_{he} & H_{hh} \end{pmatrix} = -\begin{pmatrix} H_{hh}^* & H_{he}^* \\ H_{eh}^* & H_{ee}^* \end{pmatrix}. \tag{3}$$

(b) Prove the electron-hole symmetry $E(k) = -E(-k)$. This implies that if the gap closes at a single value k_0 of k, then necessarily $k_0 = 0$. From now on we can restrict ourselves to $k = 0$, so $\psi(x,y) = \phi(y)$.

(c) Construct a solution of $H\phi(y) = 0$ for $|y| < W/2$. Show that there are four independent solutions, labeled by $s, s' \in \{+1, -1\}$, of the form

$$\phi_{ss'} = e^{ik_{ss'}y}\begin{pmatrix} 1 \\ s \end{pmatrix}, \;\; k_{ss'} = (m/\hbar)\big(is\Delta + s'\sqrt{2U_0/m - \Delta^2}\big). \tag{4}$$

[1] M. Wimmer et al., arXiv:1002.3570.

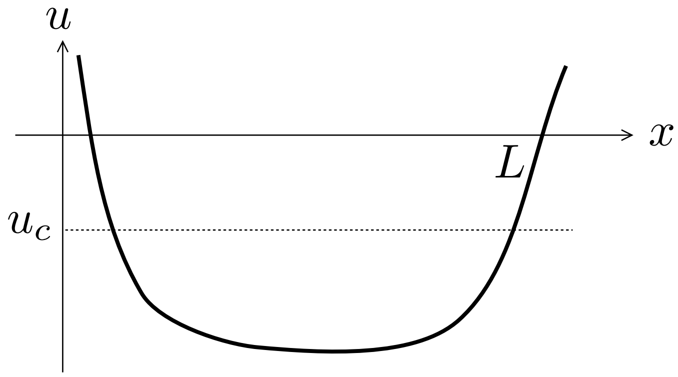
(d) Write the most general solution as a linear superposition of these four functions, and apply the boundary condition $\phi(\pm W/2) = 0$. This gives 4 equations with 4 unknowns.

(e) Derive the gap closing condition

$$U_0 = \tfrac{1}{2} m\Delta^2 + \frac{\hbar^2}{2m}\left(\frac{n\pi}{W}\right)^2, \quad n = 0, 1, 2, 3, \ldots \tag{5}$$

In a wire of finite length, every even gap closing transition ($n = 0, 2, 4, \ldots$) signals the appearance of a zero mode at the end points, and every odd transition the disappearance.

1.2 Appearance of a Majorana fermion



To establish the appearance of a zero mode (Majorana fermion) at a gap closing transition, we investigate the effective one-dimensional Hamiltonian

$$H_{\text{eff}} = \begin{pmatrix} \alpha p_x + u(x) - u_c & (\beta + i\gamma) p_x \\ (\beta - i\gamma) p_x & \alpha p_x - u(x) + u_c \end{pmatrix} + \text{terms of order } p_x^2. \tag{6}$$

The parameters α, β, γ are real numbers. A wire of length L is constructed through the potential profile $u(x)$ shown in the figure. We seek a zero mode near one end of the wire, say near $x = L$. The other end is assumed to be far away.

(a) Convince yourself that H_{eff} has the most general form and that the gap closes at $u = u_c$.

(b) The velocity operator $v_x = \partial H/\partial p_x$ should have a positive and a negative eigenvalue, to allow for bidirectional propagation along the wire. Show that this requires $\beta^2 + \gamma^2 > \alpha^2$.

(c) Derive a first order differential equation for the zero mode,

$$\frac{d}{dx}\psi_0(x) = -\frac{u(x) - u_c}{\beta^2 + \gamma^2 - \alpha^2} M\psi_0(x), \text{ with matrix } M = \begin{pmatrix} -i\alpha & \gamma - i\beta \\ \gamma + i\beta & i\alpha \end{pmatrix}. \tag{7}$$

(d) Diagonalize M and obtain a solution localized near $x = L$.

2 Topological quantum computation with Majorana fermions

2.1 Topological qubits

Majorana operators γ_n are Hermitian operators ($\gamma_n = \gamma_n^\dagger$) that satisfy the anticommutation relation

$$\{\gamma_n, \gamma_m\} \equiv \gamma_n \gamma_m + \gamma_m \gamma_n = 2\delta_{nm}. \tag{8}$$

From four Majorana operators $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ we can construct a pair of Dirac operators

$$a_1 = \frac{\gamma_1 + i\gamma_2}{2}, \;\; a_2 = \frac{\gamma_3 + i\gamma_4}{2}. \tag{9}$$

(a) Check that these operators satisfy the usual fermion anticommutation relations

$$\{a_n, a_m^\dagger\} = \delta_{nm}. \tag{10}$$

(b) Explain why the *most general* Hermitian operator that couples Majoranas n and m equals a constant times $i\gamma_n\gamma_m$.
(c) Check that the operator

$$\mathscr{P}_{nm} = \tfrac{1}{2}(1 + i\gamma_n\gamma_m) \tag{11}$$

has eigenvalues 0 and 1. We denote by $|s, s\rangle$, $s, s' \in \{0, 1\}$, an eigenstate of $\mathscr{P}_{12}$ and $\mathscr{P}_{34}$ with eigenvalues s and s', respectively.
(d) Explain why fermion parity conservation prohibits transitions between states with different $s + s'$ modulo 2.
(e) The two states $|0\rangle \equiv |0, 0\rangle$ and $|1\rangle \equiv |1, 1\rangle$ define a qubit. Show that the NOT operation σ_x (being the Pauli matrix that interchanges $|0\rangle$ and $|1\rangle$) is obtained by

$$\sigma_x = -i\gamma_2\gamma_3. \tag{12}$$

The other Pauli matrices can be obtained similarly by coupling other pairs of Majoranas,

$\sigma_y = i\gamma_1\gamma_3$, $\sigma_z = -i\gamma_1\gamma_2$.

2.2 Braiding

The exchange of Majoranas 1 and 2 transforms $\gamma_1 \mapsto e^{i\alpha}\gamma_2$, $\gamma_2 \mapsto e^{i\beta}\gamma_1$. To maintain Hermitian operators the phases α, β are restricted by $\alpha, \beta \in \{0, \pi\}$.

(a) Conservation of fermion parity requires that the operator $\mathscr{P}_{12}$ remains unchanged. Argue that this requires either $\alpha = 0, \beta = \pi$ or $\alpha = \pi, \beta = 0$.

(b) Assume $\alpha = \pi$, $\beta = 0$. Show that the exchange is a unitary transformation $\gamma_n \mapsto U^\dagger \gamma_n U$ with

$$U = \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2). \tag{13}$$

(c) Check that U can also be written as

$$U = e^{i(\pi/4)\sigma_z}, \tag{14}$$

which has the form of a $\pi/2$ rotation of the qubit around the z-axis on the Bloch sphere. Rotations by $\pi/2$ around other axes can be obtained by exchanging other pairs of Majoranas. These operations are called *topological* because the angle of rotation is exactly $\pi/2$, irrespective of the details of the exchange operation (which is called *braiding*).

(d) Show that if one repeats the entire braiding operation, the Majoranas 1 and 2 have returned to their original positions but the final state differs from the initial state by a unitary operator and not just by a phase factor. That is called *non-Abelian statistics.*