1 Quantum bit

☞ PRESKILL: *chapter 2.1 and 2.2*

A wave function $|\Psi\rangle$ in a two-dimensional Hilbert space (with basis vectors $|0\rangle$ and $|1\rangle$) has the general form

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The coefficients α and β are two arbitrary complex numbers.

a) To specify α and β one needs 4 real numbers. Argue that 2 of those 4 numbers are superfluous for the specification of a physical state.

b) Show that, without loss of generality, one can write the state $|\Psi\rangle$ in the form

$$|\Psi\rangle = e^{-i\phi/2}\cos(\theta/2)|0\rangle + e^{i\phi/2}\sin(\theta/2)|1\rangle,$$

with real θ and ϕ. These are the spherical coordinates of a point on the unit sphere, called the *Bloch sphere*.

For an electron spin we can interpret the basis vectors $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$ as, respectively, spin up $|\uparrow\rangle$ or down $|\downarrow\rangle$ along the z-axis.

c) Where do these two states lie on the Bloch sphere?

More generally the spin is oriented along the x, y, or z-axis if it is an eigenfunction of the Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For later use we also define the unit matrix

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

d) Where on the Bloch sphere lies a spin oriented along the x or y-axis?

A spin state that points in the direction of the unit vector $\boldsymbol{n} = (n_x, n_y, n_z)$ is an eigenfunction of

$$\boldsymbol{n}\cdot\boldsymbol{\sigma} \equiv n_x\sigma_x + n_y\sigma_y + n_z\sigma_z.$$

e) Derive that $\boldsymbol{n}$ indicates the point on the Bloch sphere corresponding to that spin state.

g) Problem 2.1 of PRESKILL.