2 Quantum gate

The elementary operation (gate) of a quantum computer acts on a single qubit. It is a linear operation, which can be represented by a matrix multiplication,

$$|\Psi\rangle_{\text{out}} = U|\Psi\rangle_{\text{in}} \Leftrightarrow \begin{pmatrix} \alpha^{\text{out}} \\ \beta^{\text{out}} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha^{\text{in}} \\ \beta^{\text{in}} \end{pmatrix}.$$

a) Show that normalization of initial state $|\Psi_{\text{in}}\rangle$ and final state $|\Psi_{\text{out}}\rangle$ requires that U is a unitary matrix,

$$UU^\dagger = U^\dagger U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) Are the Pauli matrices allowed as quantum gates? Explain what each Pauli matrix does to a point on the Bloch sphere.

c) Are these three gates independent, or can you construct one out of the other two?

The Hadamard gate has the form

$$H = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

d) What does this gate do to a point on the Bloch sphere?

e) Show that the combination of H and a single Pauli matrix (e.g. σ_x) can be used to construct the other two Pauli matrices.

The most general gate rotates a point on the Bloch sphere by an angle θ along the unit vector $\boldsymbol{n}$.

f) Demonstrate that this gate is given by

$$R_{\boldsymbol{n}}(\theta) = \cos(\theta/2)\sigma_0 - i\sin(\theta/2)\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$