3 Density matrix

Preskill: *chapter 2.3*

The expectation value of an operator M in the state $|\Psi\rangle$ is $\langle M\rangle = \langle\Psi|M|\Psi\rangle$. We can write this as the trace of the product of the operator M and an operator ρ,

$$\langle M\rangle = \operatorname{tr} M\rho,\ \ \rho = |\Psi\rangle\langle\Psi|.$$

The operator ρ is the density matrix corresponding to the state $|\Psi\rangle$.

a) Derive that $\operatorname{tr}\rho = 1$, $\rho = \rho^\dagger$, $\rho^2 = \rho$. What are the eigenvalues of ρ?

More generally, a system can consist of a mixture of states. If the state $|\Psi_n\rangle$ occurs with probability p_n, then the density matrix is

$$\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|.$$

b) What is now the expectation value of M?

c) Derive that it still holds that $\operatorname{tr}\rho = 1$, $\rho = \rho^\dagger$. However, unlike for a pure (not mixed) state, it no longer holds that $\rho^2 = \rho$.

d) Derive that $\langle\Psi|\rho|\Psi\rangle \geqslant 0$ for each $|\Psi\rangle$. What restriction does this inequality impose on the eigenvalues of ρ?

e) For a single qubit ρ is a 2×2 matrix. Derive that ρ can be written in terms of the Pauli matrices in the form

$$\rho = \frac{1}{2}\left(\sigma_0 + \boldsymbol{a}\cdot\boldsymbol{\sigma}\right),$$

with a vector of real coefficients $\boldsymbol{a} = (a_x, a_y, a_z)$. Why is $|\boldsymbol{a}| \leqslant 1$? The vectors $\boldsymbol{a}$ which satisfy $|\boldsymbol{a}| \leqslant 1$ form a sphere, the Bloch sphere.

f) Where on the Bloch sphere lies the density matrix $\rho = |\Psi\rangle\langle\Psi|$ of a pure state? Write ρ for this case in terms of the angles θ, ϕ from problem 1.

g) Derive that the expectation value of the spin operator $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ along the direction $\boldsymbol{n}$ is given by the inner product $\boldsymbol{a}\cdot\boldsymbol{n}$. Explain how you can use this property to measure the density matrix. Why is this not possible if you have only a single system at your disposal?

h) problem 2.2 of Preskill.