4 Entanglement

☞ PRESKILL: *chapter 2.4*

The state $|\Psi\rangle$ of two qubits A and B can be constructed out of the building blocks $|0\rangle_A$, $|1\rangle_A$, $|0\rangle_B$, $|1\rangle_B$ of the individual qubits,

$$|\Psi\rangle = c_{00}|0\rangle_A|0\rangle_B + c_{01}|0\rangle_A|1\rangle_B + c_{10}|1\rangle_A|0\rangle_B + c_{11}|1\rangle_A|1\rangle_B.$$

a) Show that the condition of normalisation can be written as $\operatorname{tr} cc^\dagger = 1$, with c the 2×2 matrix with coefficients c_{ij}.

b) Why is it always possible to choose a basis such that

$$|\Psi\rangle = c'_{00}|0\rangle_A|0\rangle_B + c'_{11}|1\rangle_A|1\rangle_B.$$

(This is called the Schmidt decomposition.)

If I only do measurements on qubit A, then it suffices to know the reduced density matrix

$$\rho_A = \operatorname{tr}_B|\Psi\rangle\langle\Psi|.$$

c) Derive that

$$(\rho_A)_{ij} = \sum_k c_{ik}c^*_{jk}, \text{ hence } \rho_A = cc^\dagger.$$

The qubits A and B are called "entangled" if ρ_A describes a mixed (not pure) state.

d) Show that $\det\rho_A \neq 0$ (det = determinant) is a necessary and sufficient condition for entanglement.

e) Why is it equivalent to determine the entanglement from ρ_A or from $\rho_B = \operatorname{tr}_A|\Psi\rangle\langle\Psi|$?

A widely used measure of entanglement is the *concurrence*

$$\mathscr{C} = 2\sqrt{\det\rho_A}.$$

f) Derive that $0 \leqslant \mathscr{C} \leqslant 1$.

g) problem 2.3 of PRESKILL.

h) problem 2.5 of PRESKILL.