# 5 Teleportation

☞ PRESKILL: *chapter 4*

The basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ of two qubits from problem 4 is not entangled. Sometimes it is more convenient to use another basis,

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},\ |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},\ |\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},\ |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

which *is* entangled. This basis is called the Bell basis.

*a)* Show that each of the four Bell states is maximally entangled (has concurrence = 1).

You can produce these states starting from a non-entangled basis, by using two gates: The Hadamard gate and the CNOT gate. The Hadamard gate rotates the first qubit, $|x\rangle \to 2^{-1/2}(|0\rangle + (-1)^x|1\rangle)$. The CNOT gate (controlled NOT) exchanges $0 \leftrightarrow 1$ at the second qubit, but only if the first qubit is 1.

*b)* Show that $|ij\rangle \to |\beta_{ij}\rangle$.

Suppose that Alice and Bob have met to produce the state $|\beta_{00}\rangle$ and then have separated, each taking 1 qubit. Alice wants to use this entangled pair to transmit to Bob the unknown state $\alpha|0\rangle + \beta|1\rangle$ of a second qubit in her possession.

*c)* Why can't Alice just measure $\alpha$ and $\beta$ and send the result to Bob?

The state of the 3 qubits (2 with Alice, 1 with Bob) reads

$$|\Psi\rangle = 2^{-1/2}\big[\alpha(|00\rangle + |11\rangle)|0\rangle + \beta(|00\rangle + |11\rangle)|1\rangle\big].$$

Alice sends both her qubits through a CNOT gate and then sends her unknown qubit through a Hadamard gate. She finally measures both her qubits.

*d)* Specify for each of the four outcomes 00, 01, 10, 11 of the measurement of Alice, wat is the resulting state of Bob's qubit.

Alice communicates to Bob the result of her measurement.

*e)* Indicate how Bob can use that knowledge to bring his qubit in the state $\alpha|0\rangle + \beta|1\rangle$ (without actually knowing that state!). This completes the "teleportation".

To communicate the result of her measurement to Bob, Alice must send a message. This takes time and ensures that teleportation does not exceed the speed of light. Suppose that

Bob tries to extract some information from his qubit *after* Alice's measurement but *before* her message has arrived.

*f)* Calculate the reduced density matrix of Bob, $\rho_B = \mathrm{tr}_A \rho$, and show that $\rho_B$ contains no information at all about the unknown qubit $\alpha|0\rangle + \beta|1\rangle$.

Alice must destroy her qubit to transmit its state to Bob. This is an example of the general "no-cloning theorem".

*g)* Show that it is not possible to construct a gate that can clone an arbitrary unknown qubit $|\Psi\rangle$, that is to say

$$|\Psi\rangle|0\rangle \rightarrow |\Psi\rangle|\Psi\rangle$$

is not possible.