6 Bell inequality

In problem 4 we encountered the concurrence $\mathscr{C}$ as a measure of the degree of entanglement of two qubits A, B in the pure state

$$|\Psi\rangle = \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B, \;\; \mathscr{C} = 2|\det c|.$$

If you dispose of a large number of pairs A, B in the same state, then you can measure the concurrence from the mean correlation of spin A and spin B. The Bell inequality is a way to distinguish classical from quantum mechanical correlations.

The recipe is as follows: Choose two unit vectors $\boldsymbol{a}, \boldsymbol{a}'$ along which to measure spin A, and two more $\boldsymbol{b}, \boldsymbol{b}'$ for spin B. Measure for each combination of vectors the spin correlator

$$C_{nm} = \langle\Psi|(\boldsymbol{n}\cdot\boldsymbol{\sigma}_A)(\boldsymbol{m}\cdot\boldsymbol{\sigma}_B)|\Psi\rangle,$$

where $\boldsymbol{\sigma}_A$ acts on spin A and $\boldsymbol{\sigma}_B$ on spin B. Combine the results to obtain the Bell parameter

$$\mathscr{B} = C_{ab} + C_{a'b} + C_{ab'} - C_{a'b'}.$$

Maximize $\mathscr{B}$ by varying $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}', \boldsymbol{b}'$. The maximum $\mathscr{B}_{\max}$ gives the concurrence via

$$\mathscr{B}_{\max} = 2\sqrt{1+\mathscr{C}^2}.$$

Let us prove this.

a) Why can you restrict yourself, without loss of generality, to states of the form $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ (with real α and β) and vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}', \boldsymbol{b}'$ in the $x-z$ plane?

Denote the length of $\boldsymbol{a}+\boldsymbol{a}'$ as $2\cos\theta$ with $0 \leqslant \theta \leqslant \pi/2$. Define two new vectors $\boldsymbol{c}, \boldsymbol{c}'$ by

$$\boldsymbol{a}+\boldsymbol{a}' = 2\boldsymbol{c}\cos\theta, \;\; \boldsymbol{a}-\boldsymbol{a}' = 2\boldsymbol{c}'\sin\theta.$$

b) Explain why $\boldsymbol{c}$ and $\boldsymbol{c}'$ have length 1 and why they are orthogonal.

Now you may immediately maximize

$$\mathscr{B} = 2\cos\theta\,\langle\Psi|(\boldsymbol{c}\cdot\boldsymbol{\sigma}_A)(\boldsymbol{b}\cdot\boldsymbol{\sigma}_B)|\Psi\rangle + 2\sin\theta\,\langle\Psi|(\boldsymbol{c}'\cdot\boldsymbol{\sigma}_A)(\boldsymbol{b}'\cdot\boldsymbol{\sigma}_B)|\Psi\rangle$$

over θ.

c) Derive that

$$\max_\theta \mathscr{B} = 2\left[\langle\Psi|(\boldsymbol{c}\cdot\boldsymbol{\sigma}_A)(\boldsymbol{b}\cdot\boldsymbol{\sigma}_B)|\Psi\rangle^2 + \langle\Psi|(\boldsymbol{c}'\cdot\boldsymbol{\sigma}_A)(\boldsymbol{b}'\cdot\boldsymbol{\sigma}_B)|\Psi\rangle^2\right]^{1/2}.$$

Write $\boldsymbol{b} = (\sin\phi, 0, \cos\phi)$, $\boldsymbol{b}' = (\sin\phi', 0, \cos\phi')$, $\boldsymbol{c} = (\sin\gamma, 0, \cos\gamma)$, $\boldsymbol{c}' = (-\cos\gamma, 0, \sin\gamma)$.

d) Maximize over ϕ, ϕ', γ (in that order) to reach the desired result,

$$\mathscr{B}_{\max} = 2\sqrt{1 + 4(\alpha\beta)^2}.$$

A non-entangled state has $\mathscr{B}_{\max} = 2$. The Bell inequality

$$\mathscr{B} \leqslant 2$$

holds for all classical correlations. John Bell constructed this inequality as a way to demonstrate experimentally that quantum mechanical correlations can not be described in classical terms.

The argument goes as follows. In a classical description the Bell parameter can be found by measuring the spin polarisation P_n along the axes $\boldsymbol{n} = \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}'$. The measurement of spin A is assumed to leave spin B undisturbed (locality), so we can consider each measurement separately. The spin polarisation P_n is $+1$ if the spin is $\uparrow$ along $\boldsymbol{n}$ and -1 if it is $\downarrow$.

e) Show that

$$P_a P_b + P_{a'} P_b + P_a P_{b'} - P_{a'} P_{b'} = \pm 2.$$

f) Explain why $\mathscr{B} \leqslant 2$.

g) problem 5.5 of Preskill.