7 Quantum circuit

Preskill: *chapter 6.2*

An arbitrary quantum mechanical operation on n qubits is a $2^n \times 2^n$ unitary matrix U. The elementary operations (gates) of a quantum circuit act only on 1 or 2 qubits. We wish to show that you can always construct U out of 1 and 2-qubit gates.

The first step is to construct a $d \times d$ unitary matrix U from 2×2 unitary matrices V. We suppose that such a 2×2 matrix is embedded in a $d \times d$ matrix by adding zeroes and ones. To explain the construction we consider the simplest case $d = 3$. The matrix U has the form

$$U = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix}.$$

The first 2×2 unitary matrix that we will need is

$$V_1 = (|a|^2 + |b|^2)^{-1/2} \begin{pmatrix} a^* & b^* & 0 \\ b & -a & 0 \\ 0 & 0 & (|a|^2 + |b|^2)^{1/2} \end{pmatrix}.$$

a) Show that $(V_1 U)_{21} = 0$.

b) Construct another 2×2 unitary matrix V_2, such that

$$V_2 V_1 U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e' & h' \\ 0 & f' & j' \end{pmatrix}.$$

c) Finally, construct a 2×2 unitary matrix V_3, such that $V_3 V_2 V_1 U$ gives the unit matrix.

Inversion of the matrix yields $U = V_1^\dagger V_2^\dagger V_3^\dagger$, so U has been written as the product of three 2×2 unitary matrices.

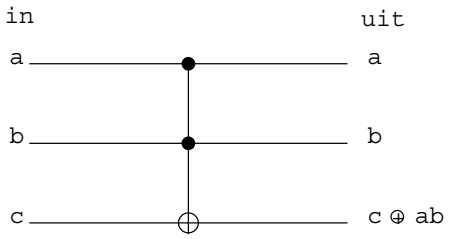
d) Explain that in the more general case of a $d \times d$ unitary matrix U at most $\frac{1}{2} d(d-1)$ building blocks of 2×2 unitary matrices V_i are needed.

The second step in the construction of universal quantum gates requires that we represent an arbitrary 2×2 unitary matrix acting n qubits by gates that act on 1 and 2 qubits only. We omit the proof that this is always possible.

Note 1: The maximal number of 1 and 2-qubit gates that is needed, is of order n^2; so in total you need of order $n^2 2^{2n}$ building blocks, maximally.

Note 2: It is sufficient to use the CNOT as 2-qubit gate.

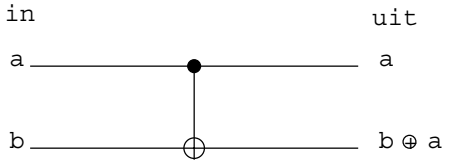
Example: The Toffoli gate is a 3-qubit gate that flips the third qubit c ($0 \leftrightarrow 1$), if and only if the first two qubits a, b are both 1. The notation is



The encircled + means addition modulo 2.

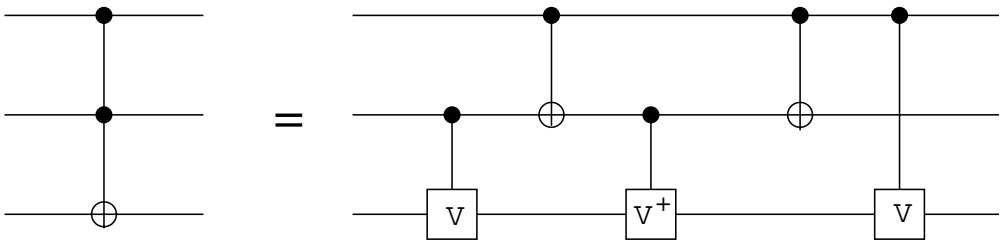
e) Explain that this diagram does what it should do.

Note that the CNOT gate can be represented in the same way by



In a classical computer it is not possible to construct the Toffoli gate out of gates that act only on 1 or 2 qubits. (At least not if the gates should be reversible.) In a quantum computer that is possible.

f) Show that



where $V = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$, $V^2 = \sigma_x$.