9 Quantum error correction

☞ PRESKILL: *chapter 7.1–7.4*

We wish to protect a single qubit against the occurrence of a single error. The error could be a bit flip (σ_x), a phase shift (σ_z), or the combination of a bit flip and a phase shift ($\sigma_y = i\sigma_x\sigma_z$).

Let us first restrict the error to a bit flip. A classical bit can be protected against a single bit flip by encoding it in three bits. The code is a simple repetition,

$$0 \to 000, \;\; 1 \to 111.$$

a) Explain how parity checking allows you to correct for a flip of one of the three bits.

Similarly, a quantum bit can be protected agains a σ_x error by encoding it in three qubits,

$$|0\rangle \to |0\rangle|0\rangle|0\rangle, \;\; |1\rangle \to |1\rangle|1\rangle|1\rangle.$$

b) Construct a circuit that encodes the state $\alpha|0\rangle + \beta|1\rangle$ as $\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle$.

c) We know that at most one of the three qubits in the encoded state has flipped. Explain how a measurement of the two observables $S_1 = \sigma_z \otimes \sigma_z \otimes 1$ and $S_2 = 1 \otimes \sigma_z \otimes \sigma_z$ allows you to determine which of the qubits has flipped — without disturbing the encoded state. A σ_x operation on the flipped qubit then allows you to recover the original state.

The same code can correct for a phase shift (σ_z) instead of a bit flip error, if we apply a Hadamard gate to each of the three qubits in the encoded state.

d) Explain why this works. What are the two observables that we have to measure in this case to determine the error?

More generally, an error correcting code consists of a 2^M dimensional subspace of a 2^N dimensional Hilbert space, such that

$$\langle\psi_0|\sigma_{n_1}^{\alpha_1}\sigma_{n_2}^{\alpha_2}\ldots\sigma_{n_k}^{\alpha_k}|\psi_0'\rangle = 0, \;\; 1 \leqslant k \leqslant 2K, \tag{1}$$

for any two (possibly identical) states ψ_0, ψ_0' in the code space and any product of up to $2K$ Pauli matrices σ_n^α (acting on different spins $n_1, n_2, \ldots$). The number M is the number of qubits encoded in N spins. The number K is the number of errors that the code can correct.

e) Demonstrate that the bit-flip code discussed in *b* satisfies this condition.

The smallest code that can protect a single qubit against any single error (σ_x, σ_y, or σ_z) requires five qubits. It is rather complicated to analyze. A simpler nine-qubit code (due to Shor) is given by

$$
\begin{aligned}
|0\rangle &\to 2^{-3/2}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle), \\
|1\rangle &\to 2^{-3/2}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).
\end{aligned}
$$

f) Verify that the Shor code satisfies the error correction condition.