Introduction to General Relativity

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Preface

This course aims to provide some understanding of general relativity as a theory of gravity in terms of the geometric properties of spacetime. We proceed along the general line of thought formulated by Einstein in his original publications of the general theory of relativity. Only a few parts, including the treatment of the stress-energy tensor are adapted in accordance with later reformulations of the theory, and contravariant coordinates are consistently labeled by superscripts.

In comparison with the special theory of relativity, which applies in flat spacetime, the general theory is quite complicated. Whereas the essential building block of the special theory, namely the Lorentz transformation, can be quickly derived from simple physical principles, the general theory requires the introduction of curved spacetime and an extensive use of differential geometry and tensor calculus. For this reason, this course is not recommended to those who don't have the ambition to work their time-consuming way through these long and perhaps tedious derivations.

While general relativity stand out as a splendid and logic theory, these qualifications apply more in retrospect than during the development of the theory. The path followed by Einstein was, at some times, as if he was trying to find his way in a labyrinth. The creation of this theory was an extremely difficult problem and may still be counted as one of the greatest achievements of mankind. Readers interested in the actual genesis of the theory are advised to read the Einstein biography 'Subtle is the Lord ...' by A. Pais (Oxford University Press, Oxford 1982). This book offers a firsthand view on the laborious path followed by Einstein, but it is not a substitute for a textbook on the subject.

One may ask what is the use of general relativity in the context of requirements of usefulness for industry and technology. Except for its application in the GPS system, one may in principle think of the design of warp engines as known from Star Trek. Such engines would enable travel at extreme speeds, using deformation of the fabric of spacetime. The geometry of spacetime is indeed the subject described by general relativity. However, the feasibility of this type of engine is, to put it mildly, unproved. But even with limited technological applications, there should be some room for the study of fundamental physics. Those who have got the right stuff will find it a fascinating tour of discovery.

The following text is an extension of lecture notes of a course originally given at Delft University of Technology in the period 2006 to 2008.

Henk W.J. Blöte (March 2016)

Chapter 1

Introduction

This chapter displays some unsatisfactory aspects of Newtonian dynamics, even after introducing special relativity, which thus demonstrates the necessity of a theory of gravity and inertia. It provides a summary of some basic concepts and elements that play a role in the development of the theory, and gives some information on possibilities for further reading.

1.1 Basic concepts

The special theory of relativity offers a vast improvement over the older theories based on Galilean transformations. The new 'classical' mechanics which includes kinematics according to special relativity, satisfactorily explains many observed phenomena, including the Michelson-Morley experiment that indicated that the speed of light as measured in an inertial frame does not depend on its state of motion. Nevertheless, some puzzles remained. The orbit of Mercury was known to display a shift in the position of its perihelion. This shift was small but significant, and could not be explained by the known laws of mechanics. Also, the logical foundations of the Newtonian dynamics could be challenged. Let us focus on the so-called Mach's Principle. Imagine a universe, empty but for two liquid bodies. They are sufficiently distant such as to not gravitationally attract one another significantly, and each of them is kept together by its own gravity. One of the bodies does not rotate, and thus assumes a spherical shape, according to Newtonian mechanics. The other body is rotating about an axis pointing to the other body, and, for that reason, takes the shape of an ellipsoid.

But is this entirely logical? If there is no 'aether' that provides a means in which light, fields of gravity, etc. can propagate, and in which one can define an inertial frame of reference, how can one then decide that one of the bodies is rotating, and the other at rest? It would seem that the only observable fact of motion is that the two bodies are rotating with respect to *one another*. This fact does not explain the different shapes of the two bodies.

When we perform the same thought experiment in our present universe (sufficiently far from the Earth and other celestial objects so that their gravitational fields can be ignored) the situation is somewhat different. We have no problem to recognize a non-rotating frame of reference as the one in which the apparent positions of distant galaxies are constant. The distant galaxies provide the concept of 'absolute orientation'. These galaxies may be far away, but are extremely massive in comparison with what we have in our neighborhood. But in an empty universe, there would not be such a concept.

A similar line of thought applies to accelerated motion. According to Newton as well as special relativity, a free and undisturbed particle moves at a constant speed along a straight line. But what does this mean in a universe that is otherwise devoid of matter?

Another thought experiment concerns Einstein's elevator. Imagine an observer in a closed elevator, in the absence of any information from outside. He determines the paths of free-falling objects in his elevator and finds that they undergo an acceleration in the direction of the bottom of the elevator. He will thus ask the question about the origin of the acceleration, *i.e.*, is the elevator at rest, or moving uniformly, in a gravitational field of a massive body such as the Earth, or is it in free space while being accelerated in the upward direction? Since it has been determined (by extremely accurate measurements due to Eötvös, Dicke, Braginsky and others) that all free-falling objects accelerate at the same rate in a gravitational field, the question on the origin of the acceleration is difficult to answer by the observer in the elevator. The 'principle of equivalence' says that the laws of physics are the same in both scenarios (here we neglect the slight but unavoidable inhomogeneity of the gravitational field).

Then, suppose that, at a certain moment, the acceleration observed in the elevator reduces to zero. Thus, any unperturbed objects in the elevator appear to remain at rest or in a state of uniform motion. At that time, the observer will experience weightlessness, but he will be unable to determine whether he is falling freely in a gravitational field or is moving uniformly in space. Given the apparent impossibility to decide this question, Einstein felt that the theory should treat accelerations and gravitational fields on the same footing. The failure of Newtons theory to do so should be seen as a shortcoming.

These considerations led Einstein to explore the possibility that spacetime possesses geometrical properties that, on the one hand, are determined by the mass-energy distribution, and on the other hand, determine the motion of free-falling test particles. In this picture, gravitational forces do not exist; accelerated motion in a gravitational field is just a consequence of the geometry of spacetime.

The development of this theory proved to be a major effort, and it requires considerable effort to study it thoroughly. The bulk of the work involves manipulation of long expressions containing tensors and other quantities with several indices. As already mentioned in the preface, this course is not suitable for those who wish to know more about exotic things as wormholes and warp engines, while lacking the ambition to work their way through long and perhaps difficult and tedious derivations.

1.2 Further Literature

The general theory of relativity is not sufficiently simple to allow an introduction by means of a short text. A book or article on this subject may thus be subject to two dangers: first, the text may be too short so that it becomes incomprehensible; and second, it may become too long so that most potential readers are deterred in an early stage. Another problem is that differences in notation between different books can be confusing. Such differences occur in the order of the indices of the Riemann curvature tensor, and also in the signs of the 'coefficients of the affine connection' when expressed in the derivatives of the metric tensor.

There are many books on general relativity, and it is not feasible to present a complete review. Some of the books avoid mathematical details and are thus only meant for easy reading. We leave these books aside. The following publications will be briefly reviewed here:

1. *Die Grundlage der allgemeinen Relativitätstheorie*, A. Einstein, in *Das Relativitätsprinzip*, issued by O. Blumenthal (Teubner Verlag, Leipzig 1923)

2. *Introduction to the Theory of Relativity*, P.G. Bergmann, (Prentice-Hall, New York 1942).

3. *The Classical Theory of Fields*, L.D. Landau and E.M. Lifshitz, (Addison-Wesley, Reading, Massachusetts 1971).

4. *An Introduction to General Relativity*, S.K. Bose, (Wiley Eastern Limited, New Delhi 1980).

5. *Gravitation*, C.W. Misner, K.S. Thorne, J.A. Wheeler (Freeman and Company, San Francisco 1973).

6. *Tijd en Ruimte, Traagheid en Zwaartekracht* , A.D. Fokker (de Haan's Academische Bibliotheek, Zeist 1960).

7. *Gravity*, J.B. Hartle, (Addison-Wesley, San Francisco 2003).

8. http://en.wikipedia.org/wiki/General_relativity

If one has some reading knowledge of the German language, the first entry in the above list, which is essentially the original work of Einstein, can still be considered as very readable, although some part of his work was executed in a way that was not completely correct, and other parts can be presented in a simpler way. It must be emphasized however that Einstein's work is essentially correct in the sense that it leads to the correct answers, in particular the equation describing the geometry of spacetime – the equation that is now called the Einstein equation. The original work of Einstein may be found in scientific libraries and in special copies issued in later years. It is quite compact, only 67 pages.

The book by Bergmann is very readable, although is contains only few illustrations. It may still be obtainable in scientific libraries.

As usual, the Landau-Lifshitz text provides a compact and excellent overview, but only part of it deals with general relativity, and without the many examples and illustrations provided by newer textbooks.

Also the book by S.K. Bose is rather compact (120 pages). It provides a description of the theory, including an explanation of the Schwarzschild and Kerr metric, and a section on cosmology. It places some emphasis on the description of stellar structures. But it does not always include complete explanations and sufficient background information. The reader may thus feel the necessity to read additional literature.

The book by Fokker is one of the few texts available in Dutch. It is quite compact (162 pp), but not always easy to understand. The author invented new Dutch words like 'tegelijkte' and 'vierschaar', (which describes a set of orthonormal 4-vectors for use as coordinate axes) but these are not very helpful as a substitute of the English language that dominates today's science. The book might be of some use for additional reading. Modern developments are missing, and the book is full of what may now be called non-standard notation.

The book by Misner, Thorne, and Wheeler has as many as 1280 pages. It gives a very extensive description of the subject, and the text contains many examples and exercises. The authors make a strong effort to explain everything as clearly as possible. It is illustrated with many examples and figures, sometimes going to such elementary levels that the reader may feel that certain paragraphs are unnecessary. The book contains so many side tracks that a complete reading is a quite time consuming. This being said, there is much that can be learned from this excellent book, even without reading all of it.

Hartle's book is more modest in size (less than 600 pp.) and is rather easy to read. It contains many illustrations and examples. A disadvantage is that, at least to some extent, the author wrote the book in reverse order – many applications of general relativity are presented before the adequate mathematical description is explained. The derivation of the Einstein equations is given only near the very end of the book, after extensive considerations that actually depend on it.

The entry in the 'wikipedia' online encyclopedia has no 'guarantee' of a publisher or editor, and in principle it can be modified or even withdrawn at any moment. The latest version (March 2016) gives only a short summary of some main points, it fails to describe the logic and mathematics leading to Einstein's field equations. But the literature list may be useful.

1.3 Note on the physical units

By means of a special choice of the units of time and mass, both the speed of light c and the gravitational constant $\hat{G}$ assume the value 1, so that all physical quantities that are normally expressed in units of time, mass, and length, now assume dimensions that are powers of the length unit. Some equations simplify because the fundamental constants c and $\hat{G}$ disappear. These 'geometrical units' are thus convenient for compactness of notation, but have the disadvantage that it becomes less obvious how to separate the 'relativistic effects' from Newtonian mechanics. For this reason we choose the standard mks-units, in which the speed of light is

$$c = 2.99792458 \times 10^8 \text{m/sec} \tag{1.1}$$

and the gravitational constant is

$$\hat{G} = 6.67428 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{sec}^{-2} \,. \tag{1.2}$$

Chapter 2

Special relativity

2.1 The Lorentz transformation

For the time being, we are dealing with Euclidean space plus one time direction. It is called flat spacetime, or Lorentzian spacetime. The extension of the Cartesian coordinate system with a time direction is also called Minkowski space. Furthermore we shall, in general, restrict ourselves to inertial frames. In these frames, unperturbed objects are moving at a constant speed. Any space curvature is thus ignored or supposed to be absent. As we now know, this is only approximately true on the Earth, but it is a good enough approximation in many cases.

Galileo already noted that the laws of mechanics appear to be the same in a moving frame of reference as in a rest frame. Consider a rest frame with spacetime coordinates (x, y, z, t). This set of four numbers describes an 'event'. Then, consider another frame with coordinates (x', y', z', t') moving with velocity v along the x axis. It seemed plausible that the relations between the primed and unprimed coordinates are given by the Galilean transformation

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t\,. \end{aligned} \tag{2.1}$$

For simplicity, the origins $(0, 0, 0, 0)$ of both frames were made to coincide. In general, there may also be a 'shift' between the two coordinate systems. But even then Eq. (2.1) can still be applied to *coordinate intervals*. The equivalence between both frames is expressed by the term Galilean invariance. The principle of Galilean invariance is, however, untenable.

Light that is moving with a speed $\pm c$ in the rest frame should, according to this transformation, have a speed $-v \pm c$ in the moving frame. According to Maxwell's theory of electromagnetism, the speed of light is determined by the permittivity of the vacuum or 'aether'. The speed of light would apply to a special frame of reference that is linked with this 'aether'. In other frames of reference that have nonzero velocities, the Maxwell equations would thus have to be modified if Galilean invariance holds. However, the experiment of Michelson and Morley did not show any movement of the

Earth with respect to the 'aether', irrespective of its state of motion as dependent on the time of the day and the season of the year. The Maxwell equations and the speed of light are the same, independent of the state of motion of the observer, as long as the motion is uniform.

There would be a way out if c were infinite. However, experiments show that c is finite. From the experiment of Michelson and Morley, we may conclude that the speed of light, as measured in mutually moving frames is equal. Thus, the Michelson-Morley result is inconsistent with the principle of Galilean invariance.

In order to explain the invariance of the speed of light, Lorentz maintained the notion of 'rest frame' and formulated a hypothesis that all objects moving at sufficiently high speeds would undergo a contraction in the direction of their motion. He did not explain the physical mechanism of this contraction, but merely remarked that the forces in matter are electromagnetic in origin, and are thus liable to modification in moving frames. That might eventually explain the Lorentz contraction. To describe physics in moving frames, he derived a transformation formula – the Lorentz transformation – that linked observations in the moving frame to those by an observer at rest. He introduced new time and space coordinates to describe observations in the moving frame, but considered the new coordinates as artificial, auxiliary quantities. Remarkably, the laws of physics, expressed by the moving observer in his artificial coordinates, appear to be the same as those in the rest frame using ordinary coordinates.

This is indeed remarkable. If the equations describing the laws of physics are the same in all uniformly moving frames of reference, it is impossible to find out which one is at rest. It would thus be more satisfactory to treat all those frames on the same footing. Einstein succeeded to find a way out of this unsatisfactory situation, by *postulating* that the laws of physics are the same in inertial frames, and by giving up the concept of absolute simultaneity.

Consider the following situation. A train is moving with velocity v in the x-direction. Somewhere in the middle of the train, an apparatus emits a light signal. Detectors, moving with the train, are placed in the front and rear parts of the train, both at a distance d from the device, as measured by the observer in the train. He can verify that both distances are equal, by placing mirrors on the detectors, and requiring that both reflected signals return simultaneously to the apparatus in the middle. In the same way, clocks on the detectors and the apparatus can be synchronized, by requiring that the light (all of this with respect to the observer in the train) travels equally fast in both directions (as measured in the frame moving with the train). If the light signal is emitted by the apparatus at time t_0, it will return at $t_1 = t_0 + 2d/c$ since it travels with the light speed c. Thus, the detectors can synchronize their clocks at the detection time $t_d = t_0 + d/c$. This procedure is, of course, the same as an observer at rest would follow. The moving observer has the right to do so if the laws of physics are the same as in the rest frame. The moving observer will naturally say that the light signal arrives simultaneously at both detectors.

Suppose that the observer at rest is able to follow in detail what happens in the train. He can also read the clocks and will thus agree that the clocks on the detectors will display equal 'times' $t_d = t_0 + d/c$ at the arrival times of the light signals. He will also agree that both reflected light signals will return simultaneously at the apparatus. However, he will not agree that the signals arrive simultaneously at the detectors. According to him, the speed differences between the light signals and the detectors are $c \pm v$. According to the same laws of physics, but now applied in the rest frame, the two detectors are thus triggered at *different times*. Events that are simultaneous in one inertial frame are not in another one. He will also remark that the moving clocks are not running synchronously, and that they are running slow.

2.1.1 Derivation from first principles

In order to formulate coordinate transformations between uniformly moving inertial frames of reference, Einstein gave up the notion of absolute simultaneity. However, he maintained this notion for observations done *within the context of a given inertial frame*. The Galilean transformations (2.1) have to be modified. How should this be done? Consider a frame (x', y', z', t') moving parallel with (x, y, z, t), at a constant speed v in the x direction. We again make origins coincide, noting that we are mainly interested in coordinate intervals and that the choice of the origin is unimportant. One may expect a linear transformation

$$\begin{aligned} x' &= \alpha x - \beta t \\ y' &= y \\ z' &= z \\ t' &= -\gamma x + \delta t\,, \end{aligned} \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta$ depend on v and remain to be determined. A nonlinear transformation would not be consistent with a homogeneous spacetime. The transformation laws for y and z are a consequence of the equivalence of both frames. Next, we describe the same transformation, but by means of coordinates that are mirror inverted in the x direction, *i.e.*, we introduce new coordinates $\tilde{x} \equiv -x$ and $\tilde{x}' \equiv -x'$. After this substitution, the first and last equations of Eqs. (2.2) become

$$\begin{aligned} \tilde{x}' &= \alpha \tilde{x} + \beta t \\ t' &= \gamma \tilde{x} + \delta t\,. \end{aligned} \tag{2.3}$$

If we are willing to accept that spacetime has the property of inversion symmetry, then these equations must describe the transformation between an inertial frame $(\tilde{x}, y, z, t)$ and another such frame $(\tilde{x}', y', z', t')$ that is moving with a speed $-v$ in the $\tilde{x}$ direction. But this is precisely the same relation as between the frames (x', y', z', t') and (x, y, z, t), in that order. Thus we invert Eqs. (2.2), which yields

$$\begin{aligned} x &= (\delta x' + \beta t')/(\alpha\delta - \beta\gamma) \\ t &= (\gamma x' + \alpha t')/(\alpha\delta - \beta\gamma)\,. \end{aligned} \tag{2.4}$$

On physical grounds, the coefficients in Eqs. (2.3) are equal to those in Eqs. (2.4). This yields 4 equations, but only 2 are independent. These are

$$\alpha/\delta = \delta/\alpha = 1 \quad \text{or} \quad \alpha = \delta \tag{2.5}$$

and

$$\alpha\delta - \beta\gamma = \alpha^2 - \beta\gamma = 1 \tag{2.6}$$

A third equation follows from the fact that the relative speed of the two frames is v. For $x' = 0$ we have, according to Eqs. (2.2), $x = \beta t/\alpha$ or $\beta = \alpha v$. Substitution in Eq. (2.6) yields $\alpha^2 - \alpha\gamma v = 1$ or

$$\gamma = (\alpha - \alpha^{-1})/v \tag{2.7}$$

The fourth equation needed to determine all four unknowns $\alpha, \beta, \gamma, \delta$ follows from the requirement that the speed of light is the same in both frames. A light signal emitted at the origin $x = 0$ at time $t = 0$ will reach $x = ct$ at time t. Let us now calculate the corresponding primed coordinates according to Eqs. (2.2). We thus substitute $x = ct$ in Eqs. (2.2), making use of $\beta = \alpha v$:

$$\begin{aligned} x' &= \alpha(ct - vt) = \alpha t(c - v) \\ t' &= \alpha(t - \gamma ct/\alpha) = \alpha t(1 - \gamma c/\alpha)\,. \end{aligned} \tag{2.8}$$

The speed of light in the primed frame is

$$x'/t' = \frac{c - v}{1 - \gamma c/\alpha} = c \quad \text{or} \quad \alpha v = \gamma c^2 \tag{2.9}$$

Elimination of γ according to Eq. (2.7) then yields

$$\alpha = \frac{c^2}{v^2}(\alpha - \alpha^{-1}) \tag{2.10}$$

from which α follows by

$$\alpha^2 = \frac{1}{1 - v^2/c^2} \quad \text{or} \quad \alpha = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.11}$$

where we have, of course, taken the positive root. The other three unknowns now follow as

$$\beta = \alpha v\,, \;\; \gamma = \alpha v/c^2\,, \;\; \delta = \alpha \tag{2.12}$$

so that Eqs. (2.2) become

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{-vx/c^2 + t}{\sqrt{1 - v^2/c^2}}\,. \end{aligned} \tag{2.13}$$

This is the Lorentz transformation for frames of reference moving with respect to one another in the x direction. We observe that at $t = 0$ one has $x < x'$. This is the Lorentz contraction. The coordinates perpendicular to the direction of motion are unchanged.

2.2 General Lorentz transformations

More generally, Lorentz transformations include arbitrary spatial rotations. Therefore Lorentz transformations also include motion in arbitrary directions. They form a group called the Lorentz group. Application of such a Lorentz transformation to an inertial frame or 'Lorentz frame' leads to another inertial frame. The speed of light is invariant under all Lorentz transformations. This means that property $\sqrt{x^2+y^2+z^2} = ct$ that a point (x, y, z, t) may have, is invariant under Lorentz transformations. It follows also that

$$s^2 \equiv c^2t^2 - x^2 - y^2 - z^2 \tag{2.14}$$

is an invariant, as may be checked by explicit calculation of $s'^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$ and substitution of Eqs. (2.13). It is straightforward to enlarge the Lorentz group to include translations (a shift of origin). Then, the invariance of s^2 applies only to coordinate intervals.

2.3 Product of two vectors

For a shorter notation we may use vector notation x^μ for (x, y, z, t), where the superscript μ assumes the values 0 to 3:

$$\begin{aligned} x^0 &= t \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned} \tag{2.15}$$

To distinguish from ordinary (space-like) vectors, we may call x^μ a 4-vector. We use Greek indices for 4-vectors; whenever we wish to restrict to the space-like components we use ordinary indices, e.g., x^k with $k = 1, 2, 3$. Next we define the matrix η by

$$\begin{bmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} = \begin{bmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.16}$$

so that

$$s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu \tag{2.17}$$

The matrix $\eta_{\mu\nu}$ as defined in Eq. (2.16) and used in Eq. (2.17) is characteristic for Lorentz frames, *i.e.*, linear and orthogonal coordinates in a flat spacetime. Eq. (2.17) may still seem a rather complicated way to represent Eq. (2.14), but later we shall see that, in general coordinate systems, the *form* of Eq. (2.17) is adequate while Eq. (2.14) is not, because it will appear that in a more general context η has to be replaced by a nondiagonal matrix.

Eq. (2.17) can be written more compactly when we use the dummy index summation convention, which says that by default sums are executed on pairs of identical indices, one of which is a superscript and one a subscript. Thus

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu \tag{2.18}$$

This expression is a spacetime analog of the scalar product of a vector with itself in Cartesian coordinates. If the time coordinate is 0, it differs only in sign. In pre-relativistic physics, time and distance are separately conserved under coordinate transformations. This does no longer hold in relativistic physics.

Using the same conventions we can form the product of two different 4-vectors x^μ and y^ν:

$$\eta_{\mu\nu} x^\mu y^\nu \tag{2.19}$$

One can easily verify that this product is also invariant under Lorentz transformations. Such a transformation can now be expressed by means of a transformation matrix $\lambda^\mu{}_\nu$. For a velocity v in the x^1 direction, its elements are defined by

$$\begin{bmatrix} \lambda^0{}_0 & \lambda^0{}_1 & \lambda^0{}_2 & \lambda^0{}_3 \\ \lambda^1{}_0 & \lambda^1{}_1 & \lambda^1{}_2 & \lambda^1{}_3 \\ \lambda^2{}_0 & \lambda^2{}_1 & \lambda^2{}_2 & \lambda^2{}_3 \\ \lambda^3{}_0 & \lambda^3{}_1 & \lambda^3{}_2 & \lambda^3{}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c^2}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.20}$$

and the transformation takes the simple form

$$x'^\mu = \lambda^\mu{}_\nu x^\nu \tag{2.21}$$

The above Lorentz transformation maps a vector x^ν, defined as the spacetime interval between two 'events' $(0,0,0,0)$ and (x^0, x^1, x^2, x^3), by means of multiplication by the tensor $\lambda^\mu{}_\nu$, on a vector x'^μ. It is obvious that, in a flat spacetime, the same transformation applies to any interval Δx^μ between two events.

2.4 Relativistic mechanics

The relativistic reformulation of classical mechanics makes extensive use of objects as vectors and tensors. Consider a particle moving with velocity $\vec{v} = (v_x, v_y, v_z)$ with respect to an inertial frame. The 4-vector describing its path covered in time t is thus tv^μ where $v^\mu \equiv (1, v_x, v_y, v_z)$. The velocity of the particle is given by the spatial components divided by the time component. While the interval tv^μ transforms as a 4-vector, v^μ does not. Therefore we define the proper time of the particle as

$$\tau = t\sqrt{1 - (v_x^2 + v_y^2 + v_z^2)/c^2} \tag{2.22}$$

which is just the time measured in the frame in which the particle is at rest, as can be verified by means of a Lorentz transformation. Then define

$$u^\mu \equiv v^\mu \frac{t}{\tau} \tag{2.23}$$

which transforms as a 4-vector because that is the way tv^μ transforms, while τ transforms as a scalar, *i.e.*, is invariant under Lorentz transformations.

Let us now consider a particle with rest mass m. That is the mass as measured by an observer at rest with respect to the particle. On the basis of Lorentz transformations and thought experiments involving collisions, it can now be derived that the 4-momentum of the particle, defined as the 4-vector

$$p^\mu \equiv m u^\mu \tag{2.24}$$

is subject to a conservation law. While the components of this vector can change under collisions with other particles, the sum of the momenta is conserved in a given inertial frame. Naturally the vector changes under Lorentz transformations. The spatial components of the vector p^μ are to be compared with the classical momentum vector; the component p^0 describes the total energy E of the particle as

$$E = p^0 c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq m(c^2 + \frac{1}{2}v^2 + \cdots) \tag{2.25}$$

where the rightmost side is obtained by Taylor expansion in v/c, and is thus only valid for non-relativistic speeds v. The form of these conservation laws provides the physical picture that mass is a form of energy. The inertial mass of a moving object is $p^0 = m/\sqrt{1 - v^2/c^2}$. The difference with respect to the rest mass m is just the kinetic energy of the particle divided by c^2. If two particles collide in an inelastic way and merge, there must be some amount ΔT of kinetic energy that is transformed into other forms of energy carried by the merged particles. This translates into an increase of the inertial mass by an amount $\Delta T/c^2$.

The energy and momentum of zero-mass particles like photons are also described by a 4-vector p^μ. This can be understood in terms of the equations given above, in the limit $m \to 0$, $|v| \to c$ such that $p^0 = m/\sqrt{1 - v^2/c^2}$ is constant.

2.5 Contravariant and covariant vectors

Thus far we have considered vectors that are defined in terms of coordinates or their differences. These carry superscripts. They are called *contravariant* vectors. The product of two such vectors is defined by Eq. (2.19). Their behavior under a linear transformation is

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = \lambda^\mu{}_\nu x^\nu \tag{2.26}$$

where the last step, with $\lambda^\mu{}_\nu$ e.g. as given by Eq. (2.20) applies if the transformation is a Lorentz transformation.

We shall now introduce a different sort of vectors, the so-called *covariant* vectors. Such vectors carry subscripts, e.g., a_ν. Their behavior under a linear transformation is

$$a'_\mu = a_\nu \frac{\partial x^\nu}{\partial x'^\mu} \tag{2.27}$$

For instance, covariant vectors serve to describe the gradient of a scalar field. A field $a(x^\mu)$ is called scalar if it is invariant under a transformation $x^\mu \to x'^\mu$, *i.e.*, we have $a(x^\mu) \to a'(x'^\mu) = a(x^\mu)$. The gradient of a is defined as

$$a_\mu \equiv \frac{\partial a}{\partial x^\mu} \tag{2.28}$$

The difference between the values of a in two infinitesimally separated spacetime points x^μ and $x^\mu + dx^\mu$ is thus

$$da = a_\mu dx^\mu \tag{2.29}$$

This may be interpreted as the scalar product of the covariant 4-vector a_μ and the contravariant 4-vector dx^μ. Since da is a scalar, it is invariant under coordinate transformations. But its form is different from the product of two contravariant 4-vectors as given by Eq. (2.19). That is because a_μ, being a covariant vector, has different transformation properties.

Let us now illustrate the invariance of da by means of the transformation properties of a_μ and dx^μ. Since dx^μ is contravariant, it transforms as

$$dx'^\mu = \lambda^\mu{}_\nu(v) dx^\nu \tag{2.30}$$

where $\lambda^\mu{}_\nu(v)$ is defined by Eq. (2.20) for relative motion with velocity v along the common x axes. The transformation of the gradient a_μ is

$$a'{}_\mu = \frac{\partial a}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = a_\nu \lambda^\nu{}_\mu(-v) \tag{2.31}$$

The last step follows because the unprimed frame moves with speed $-v$ with respect to the primed frame. Thus v changes sign, and the primed and unprimed variables are interchanged with respect to Eq. (2.21). If we now define

$$\lambda_\mu{}^\nu(v) \equiv \lambda^\nu{}_\mu(-v) \tag{2.32}$$

we can write

$$a'{}_\mu = \lambda_\mu{}^\nu(v) a_\nu \tag{2.33}$$

where $\lambda_\mu{}^\nu(v)$ is given by

$$\begin{bmatrix} \lambda_0{}^0 & \lambda_0{}^1 & \lambda_0{}^2 & \lambda_0{}^3 \\ \lambda_1{}^0 & \lambda_1{}^1 & \lambda_1{}^2 & \lambda_1{}^3 \\ \lambda_2{}^0 & \lambda_2{}^1 & \lambda_2{}^2 & \lambda_2{}^3 \\ \lambda_3{}^0 & \lambda_3{}^1 & \lambda_3{}^2 & \lambda_3{}^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c^2}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.34}$$

It is now readily verified that indeed we have

$$da' = a'{}_\mu dx'^\mu = \lambda_\mu{}^\nu \lambda^\mu{}_\tau \frac{\partial a}{\partial x^\nu} dx^\tau = \frac{\partial a}{\partial x^\tau} dx^\tau = da \tag{2.35}$$

because $\lambda_\mu{}^\nu$ and $\lambda^\mu{}_\tau(v) = \lambda_\tau{}^\mu(-v)$ are each others inverse.

2.6 Tensors

Tensors are defined as quantities with indices that obey the same transformation laws as those of products of vectors. A tensor of rank n is one with n indices. In general, the order of the indices is important. As with vectors, the upper or lower position of an index indicates contravariance or covariance with respect of that index. A tensor can thus be fully contravariant or fully covariant, or of a mixed type. For instance, the mixed tensor ${A_\mu}^\nu$ transforms as

$$ {A'_\mu}^\nu = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\tau}{A_\sigma}^\tau \tag{2.36} $$

This *transformation behavior* is called *covariant* which just means that it behaves as a tensor. This introduces a second meaning of the word 'covariant'; which one applies will be clear from the context. A tensor may also be *invariant* under transformations, which means that its elements keep the same values. Under Lorentz transformations, let the matrix η transform as a tensor:

$$ {\eta'}_{\mu\nu} = {\lambda_\mu}^\tau {\lambda_\nu}^\sigma \eta_{\tau\sigma} \tag{2.37} $$

Using the explicit forms given by Eqs. (2.16) and (2.34) one finds ${\eta'}_{\mu\nu} = \eta_{\mu\nu}$ which means that η is invariant. This invariance confirms that the scalar product, as defined in Eq. (2.18) is invariant under Lorentz transformations.

The fully contravariant form of η is defined as the inverse of $\eta_{\mu\nu}$ and is given by

$$ \begin{bmatrix} \eta^{00} & \eta^{01} & \eta^{02} & \eta^{03} \\ \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} \\ \eta^{20} & \eta^{21} & \eta^{22} & \eta^{23} \\ \eta^{30} & \eta^{31} & \eta^{32} & \eta^{33} \end{bmatrix} = \begin{bmatrix} c^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.38} $$

It is, of course, invariant under Lorentz transformations:

$$ \eta'^{\mu\nu} = {\lambda^\mu}_\tau {\lambda^\nu}_\sigma \eta^{\tau\sigma} \tag{2.39} $$

Partial contraction of η and its inverse yields the identity

$$ \eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu \tag{2.40} $$

The matrix form of the unit matrix δ^μ_ν is

$$ \begin{bmatrix} \delta^0_0 & \delta^0_1 & \delta^0_2 & \delta^0_3 \\ \delta^1_0 & \delta^1_1 & \delta^1_2 & \delta^1_3 \\ \delta^2_0 & \delta^2_1 & \delta^2_2 & \delta^2_3 \\ \delta^3_0 & \delta^3_1 & \delta^3_2 & \delta^3_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.41} $$

This matrix, also called the identity, or the Kronecker delta, is invariant not only under Lorentz transformations, but also under general coordinate transformations:

$$ {\delta'}^\mu_\nu = \frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\sigma}\delta^\sigma_\tau = \frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\tau} = \delta^\mu_\nu \tag{2.42} $$

because $\partial x^\tau / \partial x'^\nu$ and $\partial x'^\mu / \partial x^\tau$ are one another's inverse.

2.6.1 Index raising and lowering

The two forms of η can be used to toggle between the covariant and contravariant form of a vector, or even between these two forms of a tensor with respect to a given index. For instance, the definition

$$x_\mu \equiv \eta_{\mu\nu} x^\nu \tag{2.43}$$

describes index lowering, and the inverse operation is

$$x^\mu \equiv \eta^{\mu\nu} x_\nu \tag{2.44}$$

so that the Lorentz invariant scalar product (2.18) can written

$$s^2 = x^\mu x^\nu \eta_{\mu\nu} = x^\mu x_\mu \tag{2.45}$$

We can now also verify that the definition of $\lambda_\mu{}^\nu(v)$, as explicitly written in Eq. (2.34), satisfies the index raising and lowering rules

$$\lambda_\mu{}^\nu(v) = \eta_{\mu\tau} \eta^{\nu\sigma} \lambda^\tau{}_\sigma \tag{2.46}$$

From these definitions, and from Eq. (2.40), it is clear that the unit matrix δ^μ_ν can be considered to be the mixed form of η. Or, in other words, η describes the fully covariant and contravariant forms of the unit matrix.

2.7 Mechanics of continuous matter

At this stage of the course, we do not yet know the precise equations describing how matter generates gravity. But it is certain that these equations shall contain the mass distribution, and they must be consistent with the principles of relativity formulated in the preceding pages. Naturally, under normal circumstances, the main source of gravity is ordinary matter. But, as we have seen, mass and energy are equivalent, and the energy shall thus also contribute to the source of gravity. And there are further contributions, as we shall see. It is necessary to combine all contributions and to bring them in a relativistically invariant form such as vectors or tensors. Only then can we attempt to couple this information with the gravitational field.

The number zero component of the momentum-energy vector determines the mass, which includes the rest mass as well as other forms of energy, of a particle. But an equation including only that component, and not the three remaining components, is not relativistically invariant. In order to find a proper special-relativistic formulation, we should describe the mass distribution in an invariant form. We should therefore not restrict our attention to the mass-energy distribution, but also include the momentum distribution. This information is expressed by the 4-momentum density field (the dependence on the coordinates x^μ is not explicitly shown):

$$\rho u^\mu \quad \text{where} \quad u^\mu = v^\mu \frac{\partial t}{\partial \tau} \tag{2.47}$$

where the density field $\rho = \rho(x^\mu)$ is defined as the density in the rest frame of the matter at the pertinent position. The quantity (2.47) does however not have the right transformation properties. These are displayed by the tensor

$$\rho u^\mu u^\nu \tag{2.48}$$

of which $\rho u^0 u^0$ represents the mass-energy density. This is obvious in the rest frame. Furthermore, the transformation law of contravariant tensors shows that in another Lorentz frame with speed v, the mass-energy density becomes

$$\rho u'^0 u'^0 = \frac{\rho u^0 u^0}{1 - v^2/c^2} \tag{2.49}$$

The denominator consists of two factors $\sqrt{1 - v^2/c^2}$, one of which can be ascribed to the increase of mass because of the kinetic contribution to the energy, and the other to the volume contraction of the volume element in the moving frame.

The elements $\rho u^0 u^i$ with $i = 1, 2, 3$ represent the momentum density, and the elements $\rho u^i u^j$ with $j = 1, 2, 3$ the flow of i-momentum density in the j-direction.

But there is still another contribution to the flow of momentum. The atoms that constitute matter are subject to mutual forces. In the language of continuous matter, there is a pressure and perhaps also a shear tension. These forces are described by the stress tensor t^{ij} with indices taking the values 1 to 3. The element t^{ij} denotes the ith component of the force per unit of area perpendicular to the jth Cartesian direction. The forces applied by the opposite sides of the surface are, however, of opposite signs. The ambiguity in the sign of t^{ij} is resolved by the rule that, along any closed surface, the forces are those applied by the inside matter on the outside. The matrix t^{ij} is symmetric, and in many cases one expects that it is dominated by the isotropic pressure, which appears as contributions the diagonal elements of t^{ij}.

Note that t^{ij} has the dimension of energy per unit of volume. Indeed a change of volume changes the energy correspondingly. Furthermore, the forces acting on a unit of area can be interpreted as a flow of momentum density. The latter elements thus have a similar meaning as, and are supplementary to, the $\rho u^\mu u^\nu$ mentioned above. We may now define the contravariant symmetric tensor $P^{\mu\nu}$ as

$$P^{\mu\nu} \equiv \rho u^\mu u^\nu + t^{\mu\nu} \tag{2.50}$$

where the contravariant tensor $t^{\mu\nu}$ is defined via its element in the rest frame, as

$$t^{\mu\nu} \equiv \begin{bmatrix} 0 & 0 \\ 0 & t^{mn} \end{bmatrix} \tag{2.51}$$

while the known Lorentz transformation properties allow one to obtain the elements in any other inertial frame. The tensor $P^{\mu\nu}$ is usually called the stress-energy tensor. Without going into further details, we mention that its 4-divergence satisfies the equations of motion

$$\frac{\partial P^{\mu\nu}}{\partial x^\nu} = f^\mu\,, \tag{2.52}$$

where f^μ is a 4-vector field defined as follows. Its space components are, in the frame that is moving with the matter at position x^ν, equal to the external forces (such as electromagnetic or gravitational forces) per unit of volume, and its time-like component is $f^0 = 0$. It satisfies the covariant equation $\eta_{\mu\nu} f^\mu u^\nu = 0$, as one can check in the local rest frame.

Let us reflect on the meaning of Eq. (2.52). If external forces are absent, the 4-divergence of $P^{\mu\nu}$ vanishes. Thus $dP^{\mu 0}/dx^0 = -dP^{\mu n}/dx^n$ which says that the time derivative of $P^{\mu 0}$ equals minus the spatial divergence of the space-like components. In other words, the momentum-energy density is subject to a conservation law. Any local change is compensated by changes elsewhere. The time derivative of $P^{\mu 0}$ contains several contributions, including the acceleration of the matter due to the forces acting on a volume element. But it also includes the fact that the density field and the velocity field may be inhomogeneous so that the value of $P^{\mu 0}$ is also affected by motion of the matter with respect to the frame of reference. The acceleration of the matter is due to forces that are determined by the inhomogeneity of $t^{\mu\nu}$.

The tensor $P^{\mu\nu}$ describes the stress-energy-momentum density and flow due to sources such as matter. In general there are additional contributions to $P^{\mu\nu}$ due to radiation and the electromagnetic field. The latter contributions can be brought in a relativistically invariant form and be included in $P^{\mu\nu}$. Further contributions might arise from unknown or partly unknown sources. For instance, from the viewpoint of quantum field theory and the standard model of elementary-particle physics it is plausible that, even in the absence of matter or radiation, there may exist a nonzero vacuum energy that contributes to $P^{\mu\nu}$. Observational constraints imply that such a contribution must be very small; such a small contribution might explain the existence of the so-called dark energy that is postulated by cosmologists. The main problem here is that any reasonable estimate of the dark energy density according to the standard model is many orders of magnitude larger than that suggested by cosmology. This indicates that the standard model of elementary particle physics is still insufficient or incomplete.

Including all possible contributions, the tensor $P^{\mu\nu}$ serves as a candidate for the source field in the relativistic theory of gravity that remains to be formulated.

Chapter 3

General coordinate transformations

3.1 Non-Euclidean geometry in non-inertial frames

The laws of mechanics remain relatively simple in inertial frames, but we shall see that complications arise in non-inertial frames. The following example, due to Ehrenfest, considers a frame that is uniformly rotating with an angular speed ω about the z axis of an inertial frame. The z' and z axes coincide. An observer in the rotating frame constructs a circle described by ${x'}^2 + {y'}^2 = r^2$ in the (x', y') plane. The speed of the points on the rotating circle is $v = \omega r$, according to an observer in the inertial frame (x, y, z, t). Both observers will agree that the radius is r, because each radius is everywhere moving perpendicular to its own orientation, and is therefore not Lorentz contracted.

According to the observer in the inertial frame, the circumference of the circle is equal to $2\pi r$. Let the observer in the rotating frame also measure the circumference. He is using a measurement stick that is, as may be observed in the rest frame, Lorentz contracted with a factor $\sqrt{1 - v^2/c^2}$. The outcome of his measurement is that the circumference of the circle is $2\pi r/\sqrt{1 - v^2/c^2}$.

Thus, the ratio between the circumference and the radius of the circle in the rotating frame is not 2π. Euclidean geometry does not apply in the rotating frame. This conclusion already follows without considering the time variable. More generally we should however compare the 'Lorentzian geometry' of flat spacetime with curved spacetime.

Furthermore, the observer in the rotating frame finds that free-falling objects are subject to an acceleration away from the origin. He may ascribe this to a 'centrifugal force'. One may argue that such accelerations, as well as the curvature in the rotating frame, are artificial and introduced only by our choice of coordinates. While this is true, it is precisely the point we wish to investigate in order to be able to describe gravity. We shall thus work with general (nonlinear) coordinates and transformations. In general, curvature and accelerations are introduced at the same time by nonlinear transformations (the above transformation between the inertial and the rotating frames contains $\sin \omega t$ etc.). The general theory should be formulated such that it has a large degree of independence with regard to the choice of the frame of reference. We shall however use only analytic transformations (except in special limiting cases). We shall also make the assumption that, even in a gravitational field, it is possible to apply a coordinate transformation to a frame in which the laws of physics appear, at least locally, to be the same as in an inertial frame. Such a frame is called a 'local Lorentz frame'. This expresses the principle of equivalence. In other words, we may perform a coordinate transformation to Einstein's elevator. Since gravitational fields are inhomogeneous, it is natural that the laws of the inertial frame apply only locally. But, within the range of applicability, (within Einstein's elevator) we can use such useful tools as uniformity of motion, Lorentz transformations and relativistic invariance.

In this course we shall focus on the algebraic formulation of these transformations. The need for the use of nonlinear transformations suggests the use of the tools of differential geometry. It is thus, in general, necessary to bring coordinate transformations in differential form. Eq. (2.18) has to be replaced by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{3.1}$$

where ds^2 is Lorentz invariant and denotes the square of the infinitesimal 'line element' ds. Furthermore, Eq. (2.26) is replaced by

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \tag{3.2}$$

The covariant analog uses the inverse transformation:

$$dx'_\mu = dx_\nu \frac{\partial x^\nu}{\partial x'^\mu} \tag{3.3}$$

In the case of a gradient vector, Eq. 2.27 need not be modified, because the gradient itself is already composed of infinitesimals. The application of Lorentz transformations in the local inertial frame should, in principle, also be formulated using infinitesimal coordinate intervals.

3.2 The metric tensor

Under general coordinate transformations, we still wish to keep track of the value of the 'invariant' ds^2 which has the value $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ in the local inertial frame, but its invariance is only valid under local Lorentz transformations, at least if we keep using the definition (2.16). We can however redefine ds^2 such that it preserves its invariance under general transformations. Consider

$$dx^\tau = \frac{\partial x^\tau}{\partial x'^\mu} dx'^\mu \quad \text{and} \quad dx^\sigma = \frac{\partial x^\sigma}{\partial x'^\nu} dx'^\nu \tag{3.4}$$

so that

$$ds^2 = \eta_{\tau\sigma} dx^\tau dx^\sigma = \eta_{\tau\sigma} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} dx'^\mu dx'^\nu \tag{3.5}$$

This suggests the use of the notation

$$g'_{\mu\nu} \equiv \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\tau\sigma} \tag{3.6}$$

to write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.7}$$

where we have dropped the primes, for the line element ds as expressed in general coordinates. From its definition it is clear that the matrix $g_{\mu\nu}$ is, like η, symmetric. But it depends on the coordinate system. It takes the values $\eta_{\mu\nu}$ in a Lorentz frame. It transforms as a covariant tensor and is called the *metric tensor*. Its covariant character is already obvious from transformations to a Lorentz frame, where the metric reduces to the Minkowski metric $\eta_{\mu\nu}$. Between two non-Lorentz frames, the transformation reads

$$g'_{\mu\nu} = \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\tau\sigma} \tag{3.8}$$

The fully contravariant form of g is, analogous to η, equal to the inverse of $g_{\mu\nu}$:

$$g^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\tau}\frac{\partial x^\nu}{\partial x'^\sigma}\eta^{\tau\sigma} = [(g_{\mu\nu})^{-1}]^{\mu\nu} \quad \text{and} \quad g^{\mu\tau}g_{\tau\nu} = \delta^\mu_\nu \tag{3.9}$$

where the primed coordinates apply to the Minkowski metric $\eta^{\tau\sigma}$. The metric tensor serves for index raising and lowering, and to define the distance ds between two nearby points in spacetime. The sign of ds^2 tells whether the interval is spacelike or timelike. If we know all distances between nearby points, we know, in principle, the complete geometry. Thus, g is a quantity that is of fundamental importance. While we know g in flat spacetime, we do not yet know how g changes in the presence of a gravitational field. We may try to find this out by means of coordinate transformations to the local inertial frame.

3.3 Differentiation of vectors

3.3.1 Contravariant vectors

We have already seen how to differentiate a scalar field and thus to define a vector field. One may differentiate a vector field but this does not, in general, lead to a tensor field. That is because, in addition to the vector field, also the metric may be position dependent. Let us investigate the transformation properties of the derivative of a contravariant vector field A^τ:

$$\begin{aligned}
\frac{\partial A^\tau}{\partial x^\mu} \quad \rightarrow \quad \frac{dA'^\tau}{dx'^\mu} &= \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\tau}{\partial x^\nu}A^\nu\right) \\
&= \frac{\partial x'^\tau}{\partial x^\nu}\frac{\partial A^\nu}{\partial x'^\mu} + \frac{\partial^2 x'^\tau}{\partial x'^\mu \partial x^\nu}A^\nu \\
&= \frac{\partial x'^\tau}{\partial x^\nu}\frac{\partial A^\nu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\mu} + \frac{\partial^2 x'^\tau}{\partial x^\sigma \partial x^\alpha}\frac{\partial x^\sigma}{\partial x'^\mu}A^\alpha
\end{aligned} \tag{3.10}$$

The first term is the one expected for the transformation of a mixed tensor of rank 2. The second term is nonzero if the transformation is nonlinear, and shows that, in general, the derivative $\partial A^\tau/\partial x^\mu$ does not transform as a tensor. But we wish to formulate covariant equations involving derivatives, so we must find a trick to construct a covariant derivative. To this purpose we define a $4\times 4\times 4$ matrix $\Gamma^\tau_{\mu\nu}$, also called 'affine connection', 'affinity' or 'Christoffel symbol'. It is a measure of the nonlinearity of the coordinate system, which may still be due to the choice of the coordinates, and/or deviations from Lorentzian geometry. The definition is

$$\Gamma^\tau_{\mu\nu} \equiv \frac{1}{2} g^{\tau\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) \tag{3.11}$$

which is symmetric in the subscripts μ and ν. To determine the transformation behavior of $\Gamma^\tau_{\mu\nu}$, we first transform the first term between parentheses:

$$\begin{aligned}
\frac{\partial g'_{\sigma\nu}}{\partial x'^\mu} &= \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\nu} g_{\gamma\rho}\right) \\
&= \frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial g_{\gamma\rho}}{\partial x'^\mu} + g_{\gamma\rho}\left(\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\sigma}\right) \\
&= \frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} + g_{\gamma\rho}\left(\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\sigma}\right)
\end{aligned}$$

Similarly, transformation of the second term between parentheses in Eq. (3.11) yields

$$\frac{\partial g'_{\sigma\mu}}{\partial x'^\nu} = \frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} + g_{\gamma\rho}\left(\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x^\gamma}{\partial x'^\nu \partial x'^\sigma}\right)$$

and the third term yields

$$-\frac{\partial g'_{\mu\nu}}{\partial x'^\sigma} = -\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\sigma}\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} - g_{\gamma\rho}\left(\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial^2 x^\rho}{\partial x'^\sigma \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\sigma}\right)$$

The sum of the three terms thus becomes

$$\frac{\partial g'_{\sigma\nu}}{\partial x'^\mu} + \frac{\partial g'_{\sigma\mu}}{\partial x'^\nu} - \frac{\partial g'_{\mu\nu}}{\partial x'^\sigma} = \frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial x^\rho} - \frac{\partial g_{\alpha\rho}}{\partial x^\gamma}\right) + 2g_{\gamma\rho}\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}$$

where we have rewritten some terms using the symmetry of $g_{\gamma\rho}$ and swapped dummy indices α, ρ and α, γ. Thus the affine connection transforms as

$$\begin{aligned}
\Gamma'^\tau_{\mu\nu} &= \frac{1}{2}\frac{\partial x'^\tau}{\partial x^\beta}\frac{\partial x'^\sigma}{\partial x^\delta} g^{\beta\delta}\left[\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial x^\rho} - \frac{\partial g_{\alpha\rho}}{\partial x^\gamma}\right) + 2g_{\gamma\rho}\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right] = \\
&\frac{1}{2}\frac{\partial x'^\tau}{\partial x^\beta}\frac{\partial x'^\sigma}{\partial x^\delta}\frac{\partial x^\gamma}{\partial x'^\sigma} g^{\beta\delta}\left[\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial x^\rho} - \frac{\partial g_{\alpha\rho}}{\partial x^\gamma}\right) + 2g_{\gamma\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right] = \\
&\frac{1}{2}\frac{\partial x'^\tau}{\partial x^\beta} g^{\beta\gamma}\left[\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial g_{\gamma\rho}}{\partial x^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial x^\rho} - \frac{\partial g_{\alpha\rho}}{\partial x^\gamma}\right) + 2g_{\gamma\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right]
\end{aligned} \tag{3.12}$$

which can be rewritten as

$$\Gamma'^{\tau}_{\mu\nu} = \frac{\partial x'^{\tau}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\Gamma^{\beta}_{\alpha\rho} + \frac{\partial x'^{\tau}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}} \tag{3.13}$$

The first term is how Γ would transform if it were a tensor; the second one shows that Γ is not a tensor. It is nonzero for nonlinear transformations. A second derivative appears also in Eq. (3.10), but there the prime is in the numerator. We can however relate such two second derivatives by means of

$$\frac{\partial x'^{\tau}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}} + \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\tau}}{\partial x^{\sigma}\partial x^{\alpha}} = \frac{d}{\partial x'^{\mu}}\left(\frac{\partial x'^{\tau}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\right) = \frac{\partial}{\partial x'^{\mu}}\delta^{\tau}_{\nu} = 0 \tag{3.14}$$

so that we can rewrite the transformation of Γ as

$$\Gamma'^{\tau}_{\mu\nu} = \frac{\partial x'^{\tau}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\Gamma^{\beta}_{\alpha\rho} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\tau}}{\partial x^{\sigma}\partial x^{\alpha}} \tag{3.15}$$

Using this formula we write the transformation behavior of $\Gamma^{\tau}_{\mu\nu}A^{\nu}$ as

$$\begin{aligned}\Gamma'^{\tau}_{\mu\nu}A'^{\nu} &= \left(\frac{\partial x'^{\tau}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\Gamma^{\beta}_{\alpha\rho} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\tau}}{\partial x^{\sigma}\partial x^{\alpha}}\right)\frac{\partial x'^{\nu}}{\partial x^{\gamma}}A^{\gamma} \\ &= \frac{\partial x'^{\tau}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\Gamma^{\beta}_{\alpha\gamma}A^{\gamma} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\tau}}{\partial x^{\sigma}\partial x^{\alpha}}A^{\alpha}\end{aligned} \tag{3.16}$$

We make use of Eq. (3.10), which displays the transformation of the derivative of A^{α}, to form the sum

$$\begin{aligned}&\frac{\partial A'^{\tau}}{\partial x'^{\mu}} + \Gamma'^{\tau}_{\mu\alpha}A'^{\alpha} = \frac{\partial x'^{\tau}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial A^{\nu}}{\partial x^{\alpha}} + \frac{\partial^2 x'^{\tau}}{\partial x^{\sigma}\partial x^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}A^{\alpha} \\ &+\frac{\partial x'^{\tau}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\Gamma^{\beta}_{\alpha\gamma}A^{\gamma} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\tau}}{\partial x^{\sigma}\partial x^{\alpha}}A^{\alpha} = \frac{\partial x'^{\tau}}{\partial x^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\gamma}A^{\gamma}\right)\end{aligned} \tag{3.17}$$

from which one observes that the 'covariant derivative'

$$\frac{\partial A^{\tau}}{\partial x^{\mu}} + \Gamma^{\tau}_{\mu\nu}A^{\nu} \tag{3.18}$$

transforms as a mixed tensor. The significance of this fact is that we can now construct covariant formulas containing derivatives of contravariant vector fields. If such a formula is valid in one frame of reference (which may be a local Lorentz frame), then it is valid in all coordinate systems.

3.3.2 Covariant vectors

We shall also need to find an analogous expression for the covariant derivative of a covariant vector field. Ordinary differentiation yields

$$\frac{\partial A'_{\tau}}{\partial x'^{\mu}} = \frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial x^{\nu}}{\partial x'^{\tau}}A_{\nu}\right) = \frac{\partial x^{\nu}}{\partial x'^{\tau}}\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial A_{\nu}}{\partial x^{\gamma}} + \frac{\partial^2 x^{\nu}}{\partial x'^{\tau}\partial x'^{\mu}}A_{\nu} \tag{3.19}$$

From Eq. (3.13) one finds

$$\Gamma'^{\nu}_{\tau\mu}A'_{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\tau}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\Gamma^{\gamma}_{\alpha\rho}A_{\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\tau}\partial x'^{\mu}}A_{\alpha} \tag{3.20}$$

From Eqs. (3.19) and (3.20) we observe that

$$\frac{\partial A'_\tau}{\partial x'^\mu} - \Gamma'^\nu_{\tau\mu} A'_\nu = \frac{\partial x^\nu}{\partial x'^\tau}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial A_\nu}{\partial x^\gamma} + \frac{\partial^2 x^\nu}{\partial x'^\tau \partial x'^\mu} A_\nu - \frac{\partial x^\alpha}{\partial x'^\tau}\frac{\partial x^\rho}{\partial x'^\mu}\Gamma^\gamma_{\alpha\rho} A_\gamma$$

$$- \frac{\partial^2 x^\alpha}{\partial x'^\tau \partial x'^\mu} A_\alpha = \frac{\partial x^\nu}{\partial x'^\tau}\frac{\partial x^\gamma}{\partial x'^\mu}\left(\frac{\partial A_\nu}{\partial x^\gamma} - \Gamma^\alpha_{\nu\gamma} A_\alpha\right) \tag{3.21}$$

transforms as a covariant tensor, and thus serves as the covariant derivative of a covariant vector.

Looking back at the derivation of Eqs. (3.17) and (3.21) we see that it works because of the cancellation of two terms associated with the nonlinearity of the transformation. One term appears in the transformation of the ordinary derivative of the vector field, and the other in the transformation of the Christoffel symbol.

The geometric picture of covariant differentiation of a vector field is as follows. Taking the derivative of A^α means that we have to shift the vector $A^\alpha(x^\mu)$ to position $x^\mu + dx^\mu$ and then subtract it from the local value $A^\alpha(x^\mu + dx^\mu)$. The shift should be done such that the orientation of A^α remains the same: A^α and its shifted replica should be parallel. But how to shift A^α to another position such that it remains the same? That is where the contribution with Γ comes in, the Christoffel symbol enables the parallel shift of a vector. More precisely, $\Gamma^\alpha_{\beta\gamma}$ represents the change of the α component of a contravariant unit vector in the β direction when shifted in the γ direction.

3.3.3 Differentiation of tensors

Having observed how to covariantly differentiate vectors, it is now also clear how to form covariant derivatives of tensors with several subscripts and/or superscripts. For each contravariant index, we have to add a term with Γ as we did in Eq. (3.17), and for each covariant index we have to subtract one like in Eq. (3.21). For instance,

$$\frac{\partial A^{\sigma\tau}_{\mu\nu}}{\partial x^\gamma} + \Gamma^\sigma_{\alpha\gamma} A^{\alpha\tau}_{\mu\nu} + \Gamma^\tau_{\alpha\gamma} A^{\sigma\alpha}_{\mu\nu} - \Gamma^\alpha_{\mu\gamma} A^{\sigma\tau}_{\alpha\nu} - \Gamma^\alpha_{\nu\gamma} A^{\sigma\tau}_{\mu\alpha} \tag{3.22}$$

transforms covariantly because, for each index of A, the term due to the nonlinearity of the transformation cancels with a term coming from the transformation of a Γ.

Application of this procedure to the metric tensor leads to

$$\frac{\partial g_{\mu\nu}}{\partial x^\gamma} - \Gamma^\alpha_{\mu\gamma} g_{\alpha\nu} - \Gamma^\alpha_{\nu\gamma} g_{\mu\alpha} \tag{3.23}$$

Substitution of the definition of the affine connection leads to

$$\frac{\partial g_{\mu\nu}}{\partial x^\gamma} - \frac{1}{2} g_{\alpha\nu} g^{\alpha\sigma}\left(\frac{\partial g_{\sigma\gamma}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\gamma} - \frac{\partial g_{\mu\gamma}}{\partial x^\sigma}\right) - \frac{1}{2} g_{\alpha\mu} g^{\alpha\sigma}\left(\frac{\partial g_{\sigma\gamma}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\gamma} - \frac{\partial g_{\nu\gamma}}{\partial x^\sigma}\right) =$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\gamma} + \frac{1}{2}\left(-\frac{\partial g_{\nu\gamma}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\nu} - \frac{\partial g_{\mu\gamma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\gamma} + \frac{\partial g_{\nu\gamma}}{\partial x^\mu}\right) = 0 \tag{3.24}$$

which shows that the covariant derivative of the covariant metric tensor vanishes. We could write a similar derivation for the contravariant metric tensor, but we give a faster proof. To this purpose, we first note that the covariant derivative of a product of vectors is expanded in the same way as an ordinary derivative of such a product. Thus the covariant derivative of $A^{\cdots}_{\cdots\mu}B^{\cdots\mu}_{\cdots}$ to x^α is equal to $A^{\cdots}_{\cdots\mu}$ times the covariant derivative of $B^{\cdots\mu}_{\cdots}$, plus $B^{\cdots\mu}_{\cdots}$ times the covariant derivative of $A^{\cdots}_{\cdots\mu}$. The latter expression is:

$$(\frac{\partial}{\partial x^\alpha}A^{\cdots}_{\cdots\mu} + \cdots - \Gamma^\nu_{\mu\alpha}A^{\cdots}_{\cdots\nu})B^{\cdots\mu}_{\cdots} + A^{\cdots}_{\cdots\mu}(\frac{\partial}{\partial x^\alpha}B^{\cdots\mu}_{\cdots} + \cdots + \Gamma^\mu_{\alpha\nu}B^{\cdots\nu}_{\cdots}) \tag{3.25}$$

where the dots in the superscripts and subscripts denote possible further tensor indices, and the other dots denote the terms with Γ's generated by these indices due to the covariant differentiation. It is obvious that the two terms with Γ that are shown explicitly cancel; the other terms with Γ remain and are precisely those generated by covariant differentiation of a tensor that combines all indices of A and B, except the ones summed out in the product.

This differentiation rule can now be applied to the covariant derivative of the identity $\delta^\mu_\nu = g^{\mu\sigma}g_{\nu\sigma}$, which is equal to zero. The differentiation rule thus produces two terms, one involving the covariant derivative of $g^{\mu\sigma}$ and the other one of $g_{\nu\sigma}$, that add up to 0. Since the covariant derivative of $g_{\nu\sigma}$ vanishes, that of $g^{\mu\sigma}$ must also vanish.

3.4 The equations of motion of a test particle

3.4.1 Mapping on an inertial frame

In spacetime, the metric does not only specify distances, it also determines the path of a test particle that is not subject to external forces. If a mapping exists to an inertial frame, we can specify the equation of motion very easily. In such a frame, a particle will move uniformly, *i.e.*, the 4-velocity v^μ and the associated 4-vector u^μ are constant in time, but the constant is frame-dependent (but still subject to the condition $g_{\mu\nu}u^\mu u^\nu = 1$). In a local Lorentz frame, the equation of motion is, in differential form,

$$\frac{dv^\mu}{ds} = \frac{du^\mu}{ds} = 0 \tag{3.26}$$

with $ds = cd\tau$ where τ is the proper time of the particle, *i.e.*, measured in its rest frame. But this equation, while valid in inertial frames, is not covariant under general transformations because it involves the differentiation of a vector. The relation with other frames is

$$\begin{aligned}\left(\frac{du^\mu}{ds}\right)' = \frac{d}{ds}\left(\frac{\partial x'^\mu}{\partial x^\nu}u^\nu\right) &= \frac{\partial x'^\mu}{\partial x^\nu}\frac{du^\nu}{ds} + \frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu}\frac{dx^\sigma}{ds}u^\nu \\ &= \frac{\partial x'^\mu}{\partial x^\nu}\frac{du^\nu}{ds} + \frac{1}{c}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu}u^\sigma u^\nu\end{aligned} \tag{3.27}$$

The second derivative can be eliminated by adding a term with the affine connection, and using its transformation behavior according to Eq. (3.15), as follows

$$\left(\frac{du^\mu}{ds} + \frac{1}{c}\,\Gamma^\mu_{\sigma\nu} u^\sigma u^\nu\right)' = \frac{\partial x'^\mu}{\partial x^\nu}\frac{du^\nu}{ds} + \frac{1}{c}\,\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu} u^\sigma u^\nu$$

$$+\frac{1}{c}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\tau}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x'^\sigma}{\partial x^\beta}\frac{\partial x'^\nu}{\partial x^\gamma}\Gamma^\alpha_{\tau\rho} u^\beta u^\gamma - \frac{1}{c}\frac{\partial x^\beta}{\partial x'^\sigma}\frac{\partial x^\gamma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma}\frac{\partial x'^\sigma}{\partial x^\delta}\frac{\partial x'^\nu}{\partial x^\zeta} u^\delta u^\zeta$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\frac{du^\nu}{ds} + \frac{1}{c}\,\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu} u^\sigma u^\nu + \frac{1}{c}\,\frac{\partial x'^\mu}{\partial x^\alpha}\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma - \frac{1}{c}\,\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma} u^\beta u^\gamma$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\left(\frac{du^\nu}{ds} + \frac{1}{c}\,\Gamma^\nu_{\alpha\beta} u^\alpha u^\beta\right) \qquad (3.28)$$

from which it follows that $du^\mu/ds + \frac{1}{c}\,\Gamma^\mu_{\sigma\nu} u^\sigma u^\nu$ transforms as a contravariant vector. Thus

$$\frac{du^\mu}{ds} + \frac{1}{c}\,\Gamma^\mu_{\sigma\nu} u^\sigma u^\nu = 0 \qquad (3.29)$$

is the covariant equation of motion that describes the motion of a free-falling object.

3.4.2 Geodesics

More generally, one can find the equation of motion of a test particle that is not subject to external forces without making use of a transformation to an inertial frame, by requiring that the particle follows a *geodesic*, also called a geodesic line or geodetic line. This is a straight line in a flat spacetime. A geodesic will be seen to remain a geodesic under general coordinate transformations; thus a geodesic is the closest thing to a straight line in a curved geometry. On the surface of a sphere, a geodesic is a large circle. In space it is defined as the path between two points that minimizes the distance. In spacetime, the definition is modified such that the integrated line element is extremal. Let us analyze the length w (the integrated line element ds) of a line connecting two fixed events with time-like separation. Let the line be parametrized by a parameter λ in the interval $\lambda_1 < \lambda < \lambda_2$. The coordinates x^μ along the line are thus functions of λ. The invariant length of the line is

$$L \equiv \int_{\lambda_1}^{\lambda_2} dw\,, \qquad dw = \sqrt{ds^2} = \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda \qquad (3.30)$$

We write

$$w^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \qquad (3.31)$$

so that

$$L = \int_{\lambda_1}^{\lambda_2} w(\lambda) d\lambda \qquad (3.32)$$

We wish to find the functions $x^\mu(\lambda)$ that describe the geodesic, *i.e.*, that extremize Eq. (3.32). Under a variation δx^μ with respect to the geodesic, the variation of w is $\delta w = \delta(w^2)/(2w)$ or

$$\delta w = \frac{1}{2w}\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,\delta x^\sigma + g_{\mu\nu}\frac{dx^\nu}{d\lambda}\,\delta\left(\frac{dx^\mu}{d\lambda}\right) + g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\,\delta\left(\frac{dx^\sigma}{d\lambda}\right)\right] =$$

$$\frac{1}{2w}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,\delta x^\sigma + 2g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\,\frac{d\delta x^\sigma}{d\lambda}\right) \qquad (3.33)$$

so that the equation of the geodesic becomes

$$\delta L = \int_{\lambda_1}^{\lambda_2} \delta w(\lambda) d\lambda = \int_{\lambda_1}^{\lambda_2} \frac{1}{2w}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,\delta x^\sigma + 2g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\,\frac{d\delta x^\sigma}{d\lambda}\right) d\lambda = 0 \qquad (3.34)$$

Partial integration yields

$$\int_{\lambda_1}^{\lambda_2} \frac{1}{2w}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,\delta x^\sigma\right) d\lambda + \frac{g_{\mu\sigma}}{w}\frac{dx^\mu}{d\lambda}\,\delta x^\sigma\bigg|_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda}\left(\frac{g_{\mu\sigma}}{w}\frac{dx^\mu}{d\lambda}\right)\delta x^\sigma d\lambda = 0 \qquad (3.35)$$

The midmost term vanishes since we choose $\delta x^\sigma = 0$ at λ_1 and λ_2. Thus

$$\int_{\lambda_1}^{\lambda_2} \left[\frac{d}{d\lambda}\left(\frac{g_{\mu\sigma}}{w}\frac{dx^\mu}{d\lambda}\right) - \frac{1}{2w}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]\delta x^\sigma d\lambda = 0 \qquad (3.36)$$

Since $\delta x^\mu(\lambda)$ is arbitrary for $\lambda_1 < \lambda < \lambda_2$, the geodesic should satisfy

$$\frac{d}{d\lambda}\left(\frac{g_{\mu\sigma}}{w}\frac{dx^\mu}{d\lambda}\right) - \frac{1}{2w}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \qquad (3.37)$$

The substitution $ds = wd\lambda$, and dividing out w leads to

$$\frac{d}{ds}\left(g_{\mu\sigma}\frac{dx^\mu}{ds}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = g_{\mu\sigma}\frac{d^2x^\mu}{ds^2} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$= g_{\mu\sigma}\frac{d^2x^\mu}{ds^2} + \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (3.38)$$

Multiplication by $g^{\tau\sigma}$ then yields

$$\frac{d^2x^\tau}{ds^2} + \frac{1}{2}g^{\tau\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{d^2x^\tau}{ds^2} + \Gamma^\tau_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (3.39)$$

The substitution of the velocity-like 4-vector $u^\mu = (1/c)dx^\mu/ds$ reproduces, as expected, the earlier result for the equation of motion, Eq. (3.29).

3.4.3 Motion of massless particles

Equation (3.29) is not directly applicable to the path of a photon, because ds vanishes, and the velocity-like 4-vector u^α is thus divergent. Part of the problem is solved by the substitution of u^α by the 4-momentum $p^\alpha = mu^\alpha$ where m is the rest mass of the particle. This leads to

$$m\frac{dp^\mu}{ds} + \frac{1}{c}\Gamma^\mu_{\sigma\nu}p^\sigma p^\nu = 0 \tag{3.40}$$

There are still one m and one ds that disable the direct application of this equation to a photon which has $m = 0$. However the ratio ds/m is well behaved in the limit $m \to 0$ while p^0 is kept constant. We thus substitute

$$\frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau} = \frac{1}{c}\frac{dx^0}{d\tau}\frac{d}{dx^0} = \frac{u^0}{c}\frac{d}{dx^0} = \frac{p^0}{mc}\frac{d}{dx^0} \tag{3.41}$$

in Eq. (3.40) which leads to

$$\frac{dp^\mu}{dx^0}p^0 + \Gamma^\mu_{\sigma\nu}p^\sigma p^\nu = 0\,, \tag{3.42}$$

in which the explicit dependences on s and m are removed, so that Eq. (3.42) remains valid in the limit $m \to 0$.

3.5 Conserved quantities

Multiplication of Eq. (3.29) by $g_{\alpha\mu}$, and substitution of the definition of the affinity, Eq. (3.11) yields

$$g_{\alpha\mu}\frac{du^\mu}{ds} + \frac{1}{2c}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\alpha}\right)u^\sigma u^\nu = 0 \tag{3.43}$$

Since

$$\frac{du_\alpha}{ds} = \frac{d}{ds}g_{\alpha\mu}u^\mu = \frac{dg_{\alpha\mu}}{ds}u^\mu + g_{\alpha\mu}\frac{du^\mu}{ds} \tag{3.44}$$

Eq. (3.43) can be rewritten as

$$\frac{du_\alpha}{ds} - \frac{dg_{\alpha\mu}}{ds}u^\mu + \frac{1}{2c}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\alpha}\right)u^\sigma u^\nu = 0 \tag{3.45}$$

Note that, after multiplication with $u^\sigma u^\nu$, the first and the second term between brackets produce equal results. Furthermore, together they cancel the second term in Eq. (3.45) because

$$\frac{dg_{\alpha\mu}}{ds} = \frac{\partial g_{\alpha\mu}}{\partial x^\sigma}\frac{dx^\sigma}{ds} = \frac{1}{c}\frac{\partial g_{\alpha\mu}}{\partial x^\sigma}u^\sigma \tag{3.46}$$

Thus one obtains

$$\frac{du_\alpha}{ds} = \frac{1}{2c}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}u^\mu u^\nu \tag{3.47}$$

The physical meaning of this equation is as follows. If the metric does not depend on the coordinate x^α, then both sides of Eq. (3.47) vanish, so that the corresponding component of the *covariant* vector u_α is constant along the path of the particle, *i.e.*, conserved.

Chapter 4

Curved spacetime

4.1 Metric and gravity

Let us try to illustrate the role of the metric in the context of a weak and uniform gravitational field, such as exists, in a good approximation, near the surface of the Earth. Weak means that the space curvature is small and that the metric can be chosen as $g_{\mu\nu} \approx \eta_{\mu\nu}$. There is some freedom of choice, for instance, we have the freedom to choose spherical coordinates. But here we try to remain as closely to a flat frame, at rest with respect to the surface of the Earth, as possible. Let us try to describe the equations of motion in a frame that is fixed with respect to the surface of the Earth by the following metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.1}$$

where the small parameters $h_{\mu\nu}$ account in lowest order for the deviations from the Minkowski metric due to the gravitational field. The affine connection is

$$\Gamma^{\mu}_{\sigma\nu} \equiv \frac{1}{2} g^{\mu\alpha} \left(\frac{\partial g_{\alpha\nu}}{\partial x^{\sigma}} + \frac{\partial g_{\alpha\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\sigma\nu}}{\partial x^{\alpha}} \right) = \frac{1}{2} \eta^{\mu\alpha} \left(\frac{\partial h_{\alpha\nu}}{\partial x^{\sigma}} + \frac{\partial h_{\alpha\sigma}}{\partial x^{\nu}} - \frac{\partial h_{\sigma\nu}}{\partial x^{\alpha}} \right) \tag{4.2}$$

since $\eta^{\mu\nu}$ is independent of x^{α}. We have neglected higher orders of $h_{\mu\nu}$. The geodesic equation is thus

$$\frac{du^{\mu}}{ds} = -\frac{1}{c} \Gamma^{\mu}_{\sigma\nu} u^{\sigma} u^{\nu} = -\frac{1}{2c} \eta^{\mu\alpha} \left(\frac{\partial h_{\alpha\nu}}{\partial x^{\sigma}} + \frac{\partial h_{\alpha\sigma}}{\partial x^{\nu}} - \frac{\partial h_{\sigma\nu}}{\partial x^{\alpha}} \right) u^{\sigma} u^{\nu} \tag{4.3}$$

Let the gravitational acceleration be $-\hat{g}$ in the $z = x^3$ direction. In the coordinates that we would normally use, we then have, since $ds = cdt$ in the rest frame,

$$\frac{du^{\mu}}{ds} = \frac{1}{c} \frac{du^{\mu}}{d\tau} = (0, 0, 0, -\hat{g}/c) \tag{4.4}$$

where $\hat{g}$ is the acceleration of the gravitational field; this notation with a 'hat' is used to distinguish it from the metric tensor. These equations are, for small speeds, independent of the speed. Thus we put $u^{\sigma} = u^{\nu} = (1, 0, 0, 0)$ which leads to

$$\frac{du^{\mu}}{ds} = -\frac{1}{2c} \eta^{\mu\alpha} \left(\frac{\partial h_{\alpha 0}}{\partial x^{0}} + \frac{\partial h_{\alpha 0}}{\partial x^{0}} - \frac{\partial h_{00}}{\partial x^{\alpha}} \right) \tag{4.5}$$

We are looking for a time-independent metric so that the derivatives to x^0 are zero. Furthermore, consistency with Eq. (4.4) requires that only the term with $\eta^{33} = -1$ is nonzero, so that $\partial g_{00}/\partial x^3 = \partial h_{00}/\partial x^3 = 2\hat{g}$. The other derivatives of $g_{\mu\nu}$ and $h_{\mu\nu}$ vanish in first order.

Obviously, the element g_{00} assumes the role of a constant plus 2 times the gravitational potential V_{grav}. Since in the Lorentzian spacetime we have $V_{\text{grav}} = 0$ and $g_{00} = c^2$, the constant is c^2. Thus, for weak gravitational fields we have

$$g_{00} = c^2 + 2V_{\text{grav}} \,. \tag{4.6}$$

In lowest order, one can 'transform away' the gravitational field by

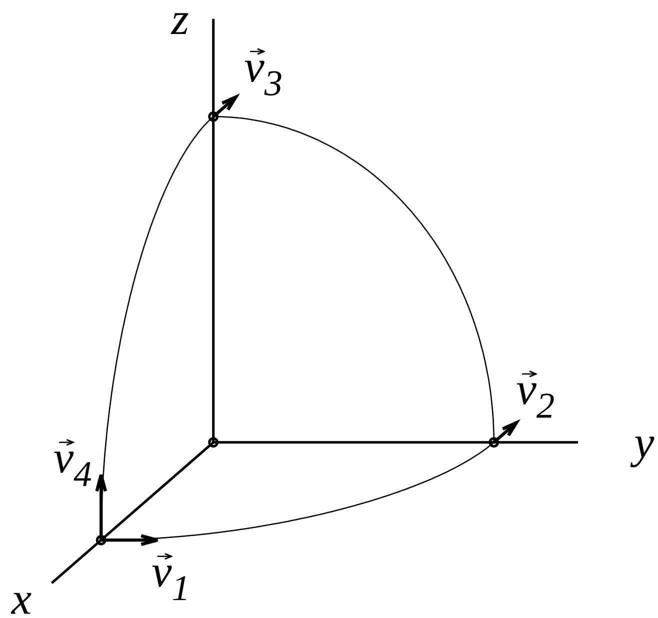
$$\begin{aligned} x'^0 &= x^0 + \hat{g}x^0x^3/c^2 \\ x'^1 &= x^1 \\ x'^2 &= x^2 \\ x'^3 &= x^3 + (\hat{g}/2)(x^0)^2 \,. \end{aligned} \tag{4.7}$$

The last term in the first line is recognized as due to a Lorentz transformation with a velocity $-\hat{g}x^0$. Because of the presence of higher-order terms, and also because of the inhomogeneity of the gravitational field of the Earth and of other bodies, Eq. (4.7) can only transform away the gravitational field locally. In terms of the metric $g_{\mu\nu}$ this means that one can transform it locally into $\eta_{\mu\nu}$, and also get rid of the first derivative. Because of this restriction, the general theory has to be formulated such as to include curved spacetime. However, one can still transform to a local Lorentz frame, apply the laws of physics as we know them in inertial frames, then transform back covariantly to general coordinates, and thus find the general equations of motion.

4.2 Parallel transport of a vector

In the case described above, the metric can be transformed locally to that of a flat spacetime, but not globally. In that case, the metric is called a *Riemann metric*. We illustrate this situation using a low-dimensional example.

Imagine a two-dimensional universe shaped as the surface of a unit sphere $x^2 + y^2 + z^2 = 1$. A two-dimensional physicist, living in this world, wishes to check if his world is flat. While he could check whether the angles in a triangle add up to π, he performs a somewhat different experiment. He transports a 2-component vector, of course confined to the surface, along a closed path, while maintaining its original orientation. After completing the 'parallel transport' along his path, he checks whether the vector is indeed unchanged. Let us quantify this scenario, using the language of an observer in our three-dimensional world. The closed path consists of three parts of large circles (*i.e.*, geodesics) in the xy, the yz, and the zx plane respectively. The vector first travels from position $\vec{p}_1 = (1, 0, 0)$ to $\vec{p}_2 = (0, 1, 0)$, then to $\vec{p}_3 = (0, 0, 1)$, and finally back to $\vec{p}_0 = (1, 0, 0)$. The original orientation of the vector is in the y direction: $\vec{v}_1 = (0, v, 0)$. The two-dimensional physicist does not know anything about this three-dimensional picture, but he has a good sense of orientation, and therefore keeps, during the first leg of his round trip, his vector in the same direction as the path he is following.



When he arrives in $\vec{p}_2 = (0, 1, 0)$, the vector has thus changed into $\vec{v}_2 = (-v, 0, 0)$. He makes a turn to the left and travels to $\vec{p}_3 = (0, 0, 1)$, while keeping the vector perpendicular to his path, thus $\vec{v}_3 = (-v, 0, 0)$. He makes another turn to the left and travels back to the starting point $\vec{p}_4 = \vec{p}_0 = (1, 0, 0)$. The vector is kept oriented along his path, pointing backwards. After completion of his trip, the vector has evolved into $\vec{v}_4 = (0, 0, v)$, perpendicular to the original orientation, as illustrated in the figure. The outcome of this experiment shows that this two-dimensional space is curved. In a flat space the observer would have found that the orientation is unchanged after completing a closed path.

In the above example, a rather large change of orientation appears because the scale of the path is of the same order as the inverse curvature, *i.e.*, the radius of the sphere. A 'parallel transport' of a vector over an infinitesimal closed path will naturally lead to a change of orientation proportional to the surface area enclosed by the path. Transport along a path composed of such infinitesimal surface elements will naturally yield a change of orientation equal to the sum of the infinitesimal contributions associated with the surface elements that compose the enclosed area. Let us work this out for a spacetime path surrounding an infinitesimal area spanned by dy^μ and dz^μ. The four legs of the path thus run between the following points respectively:

$$\begin{aligned} x^\mu(0) &= x^\mu \\ x^\mu(1) &= x^\mu + dy^\mu \\ x^\mu(2) &= x^\mu + dy^\mu + dz^\mu \\ x^\mu(3) &= x^\mu + dz^\mu \\ x^\mu(4) &= x^\mu \end{aligned} \tag{4.8}$$

As we have seen, along this path, A^τ changes by an amount determined by the Christoffel symbol: the correction term used in the covariant differentiation, with a minus sign. The changes along the four legs are, in first order,

$$\begin{aligned} A^\tau(1) - A^\tau(0) &= -\Gamma^\tau_{\nu\rho}(0)A^\rho(0)dy^\nu \\ A^\tau(2) - A^\tau(1) &= -\Gamma^\tau_{\nu\rho}(1)A^\rho(1)dz^\nu \\ A^\tau(3) - A^\tau(2) &= +\Gamma^\tau_{\nu\rho}(2)A^\rho(2)dy^\nu \\ A^\tau(4) - A^\tau(3) &= +\Gamma^\tau_{\nu\rho}(3)A^\rho(3)dz^\nu \end{aligned} \tag{4.9}$$

which adds up to

$$A^\tau(4) - A^\tau(0) = \tag{4.10}$$

$$-\Gamma^\tau_{\nu\rho}(0)A^\rho(0)dy^\nu - \Gamma^\tau_{\nu\rho}(1)A^\rho(1)dz^\nu + \Gamma^\tau_{\nu\rho}(2)A^\rho(2)dy^\nu + \Gamma^\tau_{\nu\rho}(3)A^\rho(3)dz^\nu$$

But we need to determine the changes up to second order in dy^ν and dz^ν. We need not bother about contributions that are quadratic in dy^ν and independent of dz^ν, because the net change along the closed path vanishes in all orders of dy^ν when we let $dz^\nu \to 0$. The same holds for quadratic contributions in dz^ν. But we need to keep track of terms proportional to $dy^\mu dz^\nu$: the surface area spanned by the infinitesimal displacements. Such contributions arise because Γ and A in the first-order terms of Eq. (4.10) are position dependent. The first-order changes of A are already given by Eq. (4.10), and obviously cancel in $A^\tau(4) - A^\tau(0)$. The first order changes in the components of Γ are, relative to $\Gamma(0)$,

$$\begin{aligned} \Gamma^\tau_{\nu\rho}(1) &= \Gamma^\tau_{\nu\rho}(0) + (\partial\Gamma^\tau_{\nu\rho}/\partial x^\sigma)\, dy^\sigma \\ \Gamma^\tau_{\nu\rho}(2) &= \Gamma^\tau_{\nu\rho}(0) + (\partial\Gamma^\tau_{\nu\rho}/\partial x^\sigma)(dy^\sigma + dz^\sigma) \\ \Gamma^\tau_{\nu\rho}(3) &= \Gamma^\tau_{\nu\rho}(0) + (\partial\Gamma^\tau_{\nu\rho}/\partial x^\sigma)\, dz^\sigma \\ \Gamma^\tau_{\nu\rho}(4) &= \Gamma^\tau_{\nu\rho}(0) \end{aligned} \tag{4.11}$$

The relevant second-order contributions are contained in

$$A^\tau(4) - A^\tau(0) = [\Gamma^\tau_{\nu\rho}(2) - \Gamma^\tau_{\nu\rho}(0)]A^\rho dy^\nu + [\Gamma^\tau_{\nu\rho}(3) - \Gamma^\tau_{\nu\rho}(1)]A^\rho dz^\nu +$$

$$\Gamma^\tau_{\nu\rho}[A^\rho(2) - A^\rho(0)]dy^\nu + \Gamma^\tau_{\nu\rho}[A^\rho(3) - A^\rho(1)]dz^\nu \tag{4.12}$$

After substituting the relevant parts of this equation we collect the remaining second-order contributions proportional to $dy^\mu dz^\nu$:

$$A^\tau(4) - A^\tau(0) = \frac{\partial\Gamma^\tau_{\nu\rho}}{\partial x^\sigma}A^\rho dy^\nu dz^\sigma - \frac{\partial\Gamma^\tau_{\nu\rho}}{\partial x^\sigma}A^\rho dy^\sigma dz^\nu -$$

$$\Gamma^\tau_{\nu\rho}\Gamma^\rho_{\beta\alpha}A^\alpha dy^\nu dz^\beta + \Gamma^\tau_{\nu\rho}\Gamma^\rho_{\beta\alpha}A^\alpha dy^\beta dz^\nu \tag{4.13}$$

Renaming dummy indices, and using the symmetry of Γ in its two subscripts, one finds

$$A^\tau(4) - A^\tau(0) = \left(\frac{\partial\Gamma^\tau_{\nu\rho}}{\partial x^\mu} - \frac{\partial\Gamma^\tau_{\mu\rho}}{\partial x^\nu} + \Gamma^\tau_{\mu\beta}\Gamma^\beta_{\nu\rho} - \Gamma^\tau_{\nu\beta}\Gamma^\beta_{\mu\rho}\right) A^\rho dy^\nu dz^\mu \tag{4.14}$$

4.3 The Riemann tensor

With the definition of

$$R^{\tau}{}_{\beta\gamma\delta} \equiv \frac{\partial\Gamma^{\tau}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\tau}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\sigma}_{\beta\delta}\Gamma^{\tau}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\tau}_{\sigma\delta} \tag{4.15}$$

we can rewrite Eq. (4.14) as

$$A^{\tau}(4) - A^{\tau}(0) = R^{\tau}{}_{\beta\gamma\delta}A^{\beta}dy^{\delta}dz^{\gamma} \tag{4.16}$$

We simplify our notation by abbreviations of ordinary and covariant differentiations. Ordinary differentiation of any vector or tensor to x^{ν} is denoted by appending a subscript '$,\nu$':

$$A^{\mu}_{,\nu} \equiv \frac{\partial A^{\mu}}{\partial x^{\nu}} \quad \text{and} \quad A_{\mu,\nu} \equiv \frac{\partial A_{\mu}}{\partial x^{\nu}} \tag{4.17}$$

and covariant differentiation by appending a subscript '$;\nu$':

$$A^{\mu}_{;\nu} \equiv \frac{\partial A^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\tau}A^{\tau} \quad \text{and} \quad A_{\mu;\nu} \equiv \frac{\partial A_{\mu}}{\partial x^{\nu}} - \Gamma^{\tau}_{\mu\nu}A_{\tau} \tag{4.18}$$

The definition of $R^{\tau}{}_{\beta\gamma\delta}$ can thus be written more compactly as

$$R^{\tau}{}_{\beta\gamma\delta} \equiv \Gamma^{\tau}_{\beta\delta,\gamma} - \Gamma^{\tau}_{\beta\gamma,\delta} + \Gamma^{\sigma}_{\beta\delta}\Gamma^{\tau}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\tau}_{\sigma\delta} \tag{4.19}$$

Higher derivatives can be denoted similarly, for instance

$$A_{\beta,\gamma\delta} \equiv A_{\beta,\gamma,\delta} \equiv (A_{\beta,\gamma})_{,\delta} = \left(\frac{\partial A_{\beta}}{\partial x^{\gamma}}\right)_{,\delta} = \frac{\partial^2 A_{\beta}}{\partial x^{\gamma}\partial x^{\delta}} \tag{4.20}$$

and

$$A_{\beta;\gamma\delta} \equiv A_{\beta;\gamma;\delta} \equiv (A_{\beta;\gamma})_{;\delta} = A_{\beta;\gamma,\delta} - \Gamma^{\tau}_{\beta\delta}A_{\tau;\gamma} - \Gamma^{\tau}_{\gamma\delta}A_{\beta;\tau} =$$

$$A_{\beta,\gamma,\delta} - \Gamma^{\tau}_{\beta\gamma,\delta}A_{\tau} - \Gamma^{\tau}_{\beta\gamma}A_{\tau,\delta} - \Gamma^{\tau}_{\beta\delta}A_{\tau;\gamma} - \Gamma^{\tau}_{\gamma\delta}A_{\beta;\tau} =$$

$$A_{\beta,\gamma,\delta} - \Gamma^{\tau}_{\beta\gamma,\delta}A_{\tau} - \Gamma^{\tau}_{\beta\gamma}A_{\tau,\delta} - \Gamma^{\tau}_{\beta\delta}A_{\tau,\gamma} + \Gamma^{\tau}_{\beta\delta}\Gamma^{\sigma}_{\tau\gamma}A_{\sigma} - \Gamma^{\tau}_{\gamma\delta}A_{\beta,\tau} + \Gamma^{\tau}_{\gamma\delta}\Gamma^{\sigma}_{\beta\tau}A_{\sigma}$$

This expression is, unlike ordinary second derivatives, not symmetric in the indices γ and δ. This asymmetry is expressed by the antisymmetric form

$$A_{\beta;\gamma;\delta} - A_{\beta;\delta;\gamma} = \left(\Gamma^{\tau}_{\beta\delta,\gamma} - \Gamma^{\tau}_{\beta\gamma,\delta} + \Gamma^{\sigma}_{\beta\delta}\Gamma^{\tau}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\tau}_{\sigma\delta}\right)A_{\tau} = R^{\tau}{}_{\beta\gamma\delta}A_{\tau} \tag{4.21}$$

Here A_{τ} is an arbitrary vector, and its contraction with $R^{\tau}{}_{\beta\gamma\delta}$ yields the tensor $A_{\beta;\gamma;\delta} - A_{\beta;\delta;\gamma}$. Therefore, $R^{\tau}{}_{\beta\gamma\delta}$ must also transform covariantly as a tensor. It is called the Riemann tensor, the Riemann curvature tensor, or the Riemann-Christoffel tensor.

Note that the formation of the antisymmetric second covariant derivative is very close in spirit to the parallel transportation of the vector A_{τ} around an infinitesimal surface element. The antisymmetric form of the *ordinary* second derivative vanishes of course. What remains is the result of the parallel transportation, which thus explains that Eqs. (4.16) and (4.21) look very similar. Actually, if we bring the infinitesimals in Eq. (4.16) to the other side, they become identical, after an application of index raising and lowering and of a symmetry property of the Riemann tensor.

4.3.1 Symmetries of the Riemann tensor

The symmetries of the Riemann tensor are most clearly exposed in the fully covariant form

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\tau} R^{\tau}{}_{\beta\gamma\delta} \tag{4.22}$$

The symmetry relations are

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma} \\ R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} \\ R_{\alpha\beta\gamma\delta} &= +R_{\gamma\delta\alpha\beta} \end{aligned} \tag{4.23}$$

and

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \tag{4.24}$$

In general one expects that a 4-index tensor has $4^4 = 256$ independent elements, but this number is reduced because of these relations. The first relation, which says that the Riemann tensor is antisymmetric in δ and γ, reduces the number of independent combinations of these indices from 16 to 6. From the second equation, we see that the same applies to the indices α and β. Thus only $6^2 = 36$ elements remain which may be thought as to form a 6×6 matrix, which is, according to the third relation, symmetric. Thus, only $(36 + 6)/2 = 21$ remain. Finally, one may check what further restrictions follow from Eq. (4.24). Only the combination with four different indices leads to one additional condition. Thus, 20 'independent' elements remain. However, there must be more relations, because the metric tensor, which is symmetric and has only 10 independent elements, fully determines the Riemann tensor.

The symmetry relations (4.23) and (4.24) can be simply verified directly from the definition of the Riemann tensor. It is even simpler to transform to a local inertial frame, where $g_{\mu\nu}$ reduces to $\eta_{\mu\nu}$ plus terms of second and higher order in the coordinates. Then the Γ's (but not their derivatives) vanish and it becomes relatively easy to check the symmetry relations. Since the relations are tensor relations and thus transform covariantly, they are also valid in other frames. Let us use this procedure to learn something about the covariant derivative of the Riemann tensor:

$$R_{\alpha\beta\gamma\delta;\zeta} = g_{\alpha\mu}\left(\Gamma^{\mu}_{\beta\delta,\gamma} - \Gamma^{\mu}_{\beta\gamma,\delta} + \Gamma^{\nu}_{\beta\delta}\Gamma^{\mu}_{\nu\gamma} - \Gamma^{\nu}_{\beta\gamma}\Gamma^{\mu}_{\nu\delta}\right)_{;\zeta} \tag{4.25}$$

which reduces in an inertial frame to

$$R_{\alpha\beta\gamma\delta;\zeta} = g_{\alpha\mu}(\Gamma^{\mu}_{\beta\delta,\gamma,\zeta} - \Gamma^{\mu}_{\beta\gamma,\delta,\zeta}) \tag{4.26}$$

We sum Eq. (4.26) over the cyclic permutations of the indices γ, δ, ζ, after which the terms cancel on the right-hand side:

$$R_{\alpha\beta\gamma\delta;\zeta} + R_{\alpha\beta\delta\zeta;\gamma} + R_{\alpha\beta\zeta\gamma;\delta} = 0 \tag{4.27}$$

This covariant equation is called the *Bianchi identity*, and of course also valid in general frames of reference. It is important because, as we shall see later, it can guide us in the search for the relation between the metric and the mass-energy-momentum distribution.

The Ricci tensor $R_{\beta\delta}$ is defined as a contracted form of the Riemann tensor:

$$R_{\beta\delta} \equiv g^{\alpha\gamma} R_{\alpha\beta\gamma\delta} \tag{4.28}$$

which is symmetric because of the third symmetry relation (4.23). The *curvature scalar* R is defined as the contracted form of the Ricci tensor:

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} \tag{4.29}$$

Multiplication of the Bianchi identity by $g^{\alpha\gamma} g^{\beta\delta}$ leads to

$$(g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\gamma\delta})_{;\zeta} + (g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\delta\zeta})_{;\gamma} + (g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\zeta\gamma})_{;\delta} = 0 \tag{4.30}$$

which simplifies to

$$R_{;\zeta} - R^{\gamma}_{\zeta;\gamma} - R^{\delta}_{\zeta;\delta} = 0 \tag{4.31}$$

Multiplication by $-\frac{1}{2}$ yields

$$R^{\alpha}_{\beta;\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta} R_{;\alpha} = (R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} R)_{;\alpha} = 0 \tag{4.32}$$

An equivalent version of this contracted form of the Bianchi identity follows after multiplication by $g^{\beta\gamma}$:

$$g^{\beta\gamma}(R^{\alpha}_{\beta;\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta} R_{;\alpha}) = (R^{\alpha\gamma} - \frac{1}{2} g^{\alpha\gamma} R)_{;\alpha} = 0 \tag{4.33}$$

which means that the covariant 4-divergence of the tensor $R^{\alpha\gamma} - \frac{1}{2} g^{\alpha\gamma} R$ vanishes.

Chapter 5

The Einstein equation

In this chapter we shall focus on the relation between the metric and the mass-energy-momentum distribution. Unfortunately, it is not possible to derive the desired equations without making new assumptions. This is by no means strange, for instance because also the Newtonian theory of gravity is based on an assumption for the equation for the gravitational attraction. According to Newtonian mechanics, the attractive force F between two spherical masses m and M whose centers are separated by a distance r, is

$$F = \hat{G}\frac{mM}{r^2} \tag{5.1}$$

where $\hat{G}$ is the gravitational constant (see section 1.3). The gravitational potential of a spherical mass M is therefore

$$V_{\text{grav}} = -\hat{G}\frac{M}{r} \tag{5.2}$$

More generally one can describe the relation between the mass density distribution $\rho(x, y, z)$ and the gravitational potential by means of the differential equation

$$\nabla^2 V_{\text{grav}}(x, y, z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V_{\text{grav}}(x, y, z) = 4\pi\hat{G}\rho(x, y, z) \tag{5.3}$$

which is fully analogous with the relation between the electrostatic field and the charge density distribution.

This equation is not covariant under coordinate transformations, but it provides some information on how its covariant generalization must look like. The relation between the metric and the gravitational potential was, in a weak-field approximation, already given in Sec. 4.1. There we found that $g_{00} = c^2 + 2V_{\text{grav}}$. Thus we must require that, in the weak-field limit and in the rest frame,

$$\nabla^2 g_{00}(x, y, z) = 8\pi\hat{G}\rho(x, y, z) \tag{5.4}$$

Our task is thus to find a covariant equation that reduces to Eq. (5.4) in the pertinent limit. It would be natural to covariantly differentiate the metric tensor twice and to assume that this form is equal to some tensor involving the mass-momentum distribution. However, this does not work because the covariant derivative of $g_{\mu\nu}$ vanishes according to Eq. (3.24). Fortunately, the Riemann curvature tensor seems to have the desired properties: it is a tensor that contains the second derivatives of the metric to the coordinates. Its contracted form, the Ricci tensor $R^{\mu\nu}$, has two indices, just as the stress-energy tensor $P^{\mu\nu}$. But the properties of $R^{\mu\nu}$ and $P^{\mu\nu}$ under differentiation are not well compatible. According to Eq. (2.52), in te absence of external forces, the 4-divergence of $P^{\mu\nu}$ vanishes in a local Lorentz frame. With general coordinates, we may substitute the covariant form $P^{\mu\nu}{}_{;\nu} = 0$. This does not hold for $R^{\mu\nu}$. But Eq. (4.33) shows that $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ has precisely the property that makes it compatible with $P^{\mu\nu}$. We may thus assume that both tensors are proportional. We shall verify later that the Newtonian limit determines the proportionality constant as $8\pi c^{-4}\hat{G}$. The resulting equation is thus

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi c^{-4}\hat{G}P^{\mu\nu} \tag{5.5}$$

and is called the *Einstein field equation.* With the definition of the Einstein tensor $G^{\mu\nu}$ as

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{5.6}$$

the Einstein equation becomes

$$G^{\mu\nu} = 8\pi c^{-4}\hat{G}P^{\mu\nu} \tag{5.7}$$

This is the fundamental equation in Einstein's theory describing the geometry of spacetime. It is a postulate that has to be verified by measurements. It is the simplest way to formulate the coupling of spacetime with matter, subject to conditions of covariance and logic. Other theories have been formulated in which the coupling is formulated in a less direct way. As far as the differences of these theories with Einstein's theory are measurable experimentally, these theories have been rejected.

An alternative form to write the Einstein equation follows after substitution of

$$G \equiv g_{\mu\nu}G^{\mu\nu} = g_{\mu\nu}R^{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\mu\nu}R = R - \frac{1}{2}\delta^{\mu}_{\mu}R = -R \tag{5.8}$$

and of Eq. (5.7) into Eq. (5.6) which leads to

$$R^{\mu\nu} = G^{\mu\nu} - \frac{1}{2}g^{\mu\nu}G = 8\pi c^{-4}\hat{G}(P^{\mu\nu} - \frac{1}{2}g^{\mu\nu}P) \tag{5.9}$$

where $P \equiv g_{\mu\nu}P^{\mu\nu}$.

5.1 The Newtonian limit

Let us see if the Einstein equation is consistent with Newtonian gravity in the limit of weak fields and small forces and speeds. Thus the stress-energy tensor is dominated by the element $P^{00} = \rho$, and we neglect the other elements. Then, Eq. (5.6) implies that the space-like elements of the Ricci tensor satisfy

$$R^{ij} = \frac{1}{2}g^{ij}R \simeq \frac{1}{2}\eta^{ij}R \tag{5.10}$$

The last part of this equation holds in first order because R is already first order in the deviations $h^{\mu\nu}$ with respect to Minkowski metric:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{with} \quad h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} \tag{5.11}$$

so that $h^{ij}R$ can be neglected in the lowest order approximation. Note that the minus sign of $h^{\mu\nu}$ is for compatibility with Eq. (4.1): $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$ up to terms of second order in the deviations $h^{\mu\nu}$. The curvature scalar thus satisfies

$$R \simeq \eta_{\mu\nu} R^{\mu\nu} = c^2 R^{00} + \frac{3}{2} R \quad \text{or} \quad R = -2c^2 R^{00} = -\frac{2}{c^2} R_{00} \tag{5.12}$$

Thus the only surviving element of the Einstein tensor is

$$G^{00} = R^{00} - \frac{1}{2} g^{00} R = R^{00} - \frac{1}{2c^2} R = R^{00} - \frac{1}{2c^2}(-2c^2 R^{00}) = 2R^{00} \tag{5.13}$$

so that the Einstein equation (5.5) reduces to

$$R^{00} = 4\pi c^{-4} \hat{G} P^{00} = 4\pi c^{-4} \hat{G} \rho \quad \text{or} \quad R_{00} = (\eta_{00})^2 R^{00} = 4\pi \hat{G} \rho \tag{5.14}$$

We want to determine the metric tensor, so we have to have to express R_{00} in terms of the $g_{\mu\nu}$. As a first step we express the Riemann tensor in the affine connection according to Eqs. (4.19) and (4.22):

$$R_{\alpha\beta\gamma\delta} \simeq g_{\alpha\tau}(\Gamma^\tau_{\beta\delta,\gamma} - \Gamma^\tau_{\beta\gamma,\delta}) \simeq \eta_{\alpha\tau}(\Gamma^\tau_{\beta\delta,\gamma} - \Gamma^\tau_{\beta\gamma,\delta}) \tag{5.15}$$

where we have neglected the terms quadratic in the Γ's which are also quadratic in the $h_{\mu\nu}$. The Γ's are expressed in terms of the metric as

$$\Gamma^\tau_{\beta\delta} = \frac{1}{2} g^{\tau\sigma}(g_{\sigma\delta,\beta} + g_{\sigma\beta,\delta} - g_{\beta\delta,\sigma}) \tag{5.16}$$

and

$$\Gamma^\tau_{\beta\gamma} = \frac{1}{2} g^{\tau\sigma}(g_{\sigma\gamma,\beta} + g_{\sigma\beta,\gamma} - g_{\beta\gamma,\sigma}) \tag{5.17}$$

Differentiation yields

$$\Gamma^\tau_{\beta\delta,\gamma} = \frac{1}{2} \eta^{\tau\sigma}(g_{\sigma\delta,\beta\gamma} + g_{\sigma\beta,\gamma\delta} - g_{\beta\delta,\sigma\gamma}) \tag{5.18}$$

and

$$\Gamma^\tau_{\beta\gamma,\delta} = \frac{1}{2} \eta^{\tau\sigma}(g_{\sigma\gamma,\beta\delta} + g_{\sigma\beta,\delta\gamma} - g_{\beta\gamma,\sigma\delta}) \tag{5.19}$$

where we have again neglected terms of second order. Thus

$$R_{\alpha\beta\gamma\delta} \simeq \frac{1}{2}(g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta}) \tag{5.20}$$

and

$$R_{00} \simeq \eta^{\alpha\gamma} R_{\alpha 0 \gamma 0} = \eta^{ij} R_{i0j0} \tag{5.21}$$

because the term with $\alpha = \gamma = 0$ involves differentiations to time, and we are looking for a static solution. Using Eqs. (5.20) and (5.14) we thus obtain

$$R_{00} \simeq -\frac{1}{2} \eta^{ij} g_{00,ij} = \frac{1}{2} \nabla^2 g_{00} = 4\pi \hat{G} \rho \tag{5.22}$$

which reproduces Eq. (5.4), and thereby confirms the proportionality constant in Eqs. (5.5), (5.7) and (5.9).

5.2 Classes of solutions

The Einstein field equation is actually 10 equations. But these cannot be independent in view of the differential identities that exist for both members of the equation: the contracted Bianchi identities Eq. (4.33), in line with the vanishing of the 4-divergence of the energy-momentum tensor, whose covariant formulation reads $P^{\mu\nu}{}_{;\nu} = 0$, specify 4 relations that are already implicit in both sides of the Einstein equation. Thus, there are only 6 independent equations. Since the metric tensor has 10 independent elements, we do not have enough equations to find a unique solution for the metric.

That is fortunate, otherwise we would be in trouble, because of the following reason. We can apply a general coordinate transformation, such that the 4 new coordinates are arbitrary analytic functions of the original ones. That means that, if a solution of the Einstein equations exists, that there exists a 4-parameter family of such solutions. These solutions have the same physical content: although the metric is different, every line element between two infinitesimally separated events is invariant under general transformations.

Thus, in effect, there are 4 gauge fields or coordinate conditions that can be arbitrarily chosen. They have to be chosen in order to determine a unique solution of the field equations. In practice one can use this freedom of choice in order to simplify the equations and their solution.

This does by no means imply that we relax the condition of general covariance of the theory. But the choice of the gauge fields is necessary in any explicit calculation. It is in fact the same sort of choice that one naturally makes when defining a Cartesian coordinate system in Euclidean space. In typical cases, the gauge fields are determined by the symmetry of the problem, or the independence of the coordinates if such independence is expected or possible. For instance, in the Newtonian limit in the preceding section 5.1 we made the choice that the metric does not depend on time x^0.

For spherically symmetric problems, such as the gravitational field of a spherical object, it is natural to choose a frame of reference that explicitly reflects this symmetry. That means that we describe orientations by means of the angles θ and ϕ, in the same way as usual. But the components of the metric have to be more general than for Euclidean space. The most general form of the metric that is consistent with spherical symmetry, can be expressed as follows

$$\begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} \gamma & -\alpha/2 & 0 & 0 \\ -\alpha/2 & -\beta & 0 & 0 \\ 0 & 0 & -\delta & 0 \\ 0 & 0 & 0 & -\delta\sin^2\theta \end{bmatrix} \tag{5.23}$$

where we identify, as usual, x^0 with the time t. The second coordinate x^1 parametrizes the distance to the origin by a radius-like quantity r, and $x^2 \equiv \theta$ and $x^3 \equiv \phi$ in the usual notation. The parameters α, β, γ and δ are analytic functions of r and t. These functions still contain some freedom of choice resulting from the coordinate conditions for t and r. It is easy to check that ordinary rotations $\theta \to \theta'(\theta, \phi)$, $\phi \to \phi'(\theta, \phi)$ do not change the form of the line element

$$ds^2 = \gamma(r,t)dt^2 - \beta(r,t)dr^2 - \alpha(r,t)dtdr - \delta(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.24}$$

but there are other coordinate transformations that do change this expression. First, we can introduce new coordinates $r'(r,t)$ and $t'(r,t)$ such that two conditions are satisfied: first, that $\alpha(r',t')$ vanishes; and second, that $\delta(r',t') = (r')^2$. Forgetting the primes we thus have

$$ds^2 = \gamma(r,t)dt^2 - \beta(r,t)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{5.25}$$

This form shows that the circumference of a large circle is $2\pi r$, but r is not necessarily the same as the distance between the origin and the circle; that quantity has to be found by integration of the line element, which involves the function β which remains to be found.

The nonzero elements of the metric are

$$g_{00} = \gamma(r,t), \quad g_{11} = -\beta(r,t), \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta \tag{5.26}$$

which form a diagonal matrix which is therefore easy to invert:

$$g^{00} = \gamma^{-1}, \quad g^{11} = -\beta^{-1}, \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2}\sin^{-2}\theta \tag{5.27}$$

5.3 The Schwarzschild solution

We proceed to find a rotationally invariant solution for the metric. Thus we express the affinities and the Einstein tensor in $g^{\mu\nu}$. To reduce the writing, we abbreviate differentiations by appending a subscript, such as $\alpha_t \equiv \partial\alpha/\partial t$ and $\alpha_r \equiv \partial\alpha/\partial r$. With this notation the Christoffel symbols as defined in Eq. (3.11) become

$$\begin{array}{lll}
\Gamma^0_{00} = \gamma_t/2\gamma & \Gamma^0_{01} = \gamma_r/2\gamma & \Gamma^0_{11} = \beta_t/2\gamma \\
\Gamma^1_{00} = \gamma_r/2\beta & \Gamma^1_{01} = \beta_t/2\beta & \Gamma^1_{11} = \beta_r/2\beta \\
\Gamma^1_{22} = -r/\beta & \Gamma^1_{33} = -r\sin^2\theta/\beta & \Gamma^2_{12} = 1/r \\
\Gamma^2_{33} = -\sin\theta\cos\theta & \Gamma^3_{13} = 1/r & \Gamma^3_{23} = \cos\theta/\sin\theta
\end{array} \tag{5.28}$$

A few other nonzero elements follow from the symmetry in the two subscripts, *i.e.*, $\Gamma^\tau_{\alpha\beta} = \Gamma^\tau_{\beta\alpha}$. The remaining elements vanish. These elements next serve to calculate the Ricci tensor which is, according to Eqs. (4.19), (4.22) and (4.28)

$$R_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\gamma,\beta} + \Gamma^\sigma_{\alpha\beta}\Gamma^\gamma_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\gamma_{\sigma\beta} \tag{5.29}$$

The elements of $R_{\alpha\beta}$ can be computed by straightforward substitution of the affinities. It is left as a (somewhat time consuming) exercise to the reader to verify that the nonzero elements of the Ricci tensor are

$$\begin{aligned}
R_{00} &= \frac{\gamma_{rr}}{2\beta} - \frac{\gamma_r}{4\beta}\left(\frac{\beta_r}{\beta} + \frac{\gamma_r}{\gamma}\right) + \frac{\gamma_r}{\beta r} - \frac{\beta_{tt}}{2\beta} + \frac{\beta_t}{4\beta}\left(\frac{\beta_t}{\beta} + \frac{\gamma_t}{\gamma}\right) \\
R_{11} &= -\frac{\gamma_{rr}}{2\gamma} + \frac{\gamma_r}{4\gamma}\left(\frac{\beta_r}{\beta} + \frac{\gamma_r}{\gamma}\right) + \frac{\beta_r}{\beta r} + \frac{\beta_{tt}}{2\gamma} - \frac{\beta_t}{4\gamma}\left(\frac{\beta_t}{\beta} + \frac{\gamma_t}{\gamma}\right) \\
R_{22} &= 1 - \frac{1}{\beta} + \frac{\beta_r r}{2\beta^2} - \frac{\gamma_r r}{2\beta\gamma}, \quad R_{33} = \sin^2\theta R_{22}\,, \quad R_{01} = R_{10} = \frac{\beta_t}{\beta r}
\end{aligned} \tag{5.30}$$

where 2 subscripts t or r indicate a second derivative.

Here we consider the case that all gravitating matter is concentrated within a finite distance from the origin, and we shall solve the equations in the empty outside region, where $P^{\mu\nu} = 0$ so that also $G^{\mu\nu}$ vanishes. Since $g_{01} = 0$ and, according to the Einstein equation in empty space,

$$R_{01} - \frac{1}{2} g_{01} R = 0 \tag{5.31}$$

it follows that also $R_{01} = 0$ and, in view of Eqs. (5.30), that $\beta_t = 0$. Therefore all time derivatives disappear from Eqs. (5.30), even if γ_t is nonzero. We may thus try to eliminate all time dependences from the problem, as may already be suggested by intuition. To achieve this formally we first note that the multiplication of the Einstein tensor in Eq. (5.6) by $g_{\nu\sigma}$

$$G^{\mu\nu} g_{\nu\sigma} = R^{\mu\nu} g_{\nu\sigma} - \frac{1}{2} g^{\mu\nu} g_{\nu\sigma} R = R^{\mu}{}_{\sigma} - \frac{1}{2} \delta^{\mu}_{\sigma} R = 0 \tag{5.32}$$

shows that $R^{\mu}{}_{\sigma}$ is diagonal and that the 4 diagonal elements have the same value. Thus $\beta r (R^0_0 - R^1_1) = 0$. Substitution of the elements of the Ricci tensor yields

$$\beta r (g^{00} R_{00} - g^{11} R_{11}) = \beta r \left(\frac{R_{00}}{\gamma} + \frac{R_{11}}{\beta} \right) = \frac{\beta_r}{\beta} + \frac{\gamma_r}{\gamma} = \frac{\partial \ln(\beta\gamma)}{\partial r} = 0 \tag{5.33}$$

from which we conclude that $\beta(r)\gamma(r,t)$ does not depend on r and can be written as

$$\beta(r)\gamma(r,t) = \psi(t) \tag{5.34}$$

where $\psi(t)$ is a function of time. If there exist functions $\beta(r), \gamma(r,t)$ that satisfy the Einstein equation with the $R_{\mu\nu}$ as given by Eqs. (5.30), then the same holds for the pair $\beta(r), \gamma(r,t')$ where t' is a function of t. This change of metric corresponds to choosing a coordinate condition or gauge function. We use this freedom to choose t' such that

$$\gamma/\gamma' = \psi(t)/c^2 \tag{5.35}$$

or, since $g_{00} dt^2 = g'_{00} dt'^2$ and $g_{00} = \gamma$,

$$\frac{\partial t'}{\partial t} = \sqrt{\psi(t)}/c \tag{5.36}$$

so that the time dependence of $\gamma(r,t)$ is neutralized. This implies that the coordinate conditions can be chosen such that the solution for the metric is independent of time. The physical interpretation is that the gravitational field is time independent. Making use of $\beta\gamma = c^2$, we can simplify the equation for R_{22}. Differentiation of $\beta\gamma$ to r furthermore yields $\beta_r/\beta = -\gamma_r/\gamma$. Substitution in the equation for R_{22} yields

$$R_{22} = 1 - \frac{1}{\beta} \left(1 - \frac{\beta_r r}{2\beta} + \frac{\gamma_r r}{2\gamma} \right) = 1 - \frac{\gamma}{c^2} \left(1 + \frac{\gamma_r r}{\gamma} \right) = 1 - (\gamma + \gamma_r r)/c^2 \tag{5.37}$$

In order to show that the vanishing of the Einstein tensor in vacuum implies $R_{22} = 0$, we write the definition of the curvature scalar as

$$R = \delta^\mu_\sigma R^\sigma{}_\mu \tag{5.38}$$

In combination with Eq. (5.32) it follows that the diagonal elements, for instance $R^2{}_2$ satisfy

$$R = 4R^2{}_2 \quad \text{and} \quad R = 2R^2{}_2 \tag{5.39}$$

so that $R_{22} = R = 0$ and Eq. (5.37) becomes

$$\frac{\partial}{\partial r}[\gamma(r)r] = c^2 \tag{5.40}$$

Integration over r yields

$$\gamma(r) = c^2\left(1 + \frac{A}{r}\right) \tag{5.41}$$

where $c^2 A$ is the integration constant and remains to be determined. The result for the line element is

$$ds^2 = \left(1 + \frac{A}{r}\right)c^2 dt^2 - \left(1 + \frac{A}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{5.42}$$

If mass is absent, the metric extends to $r = 0$ and the absence of a divergence implies $A = 0$. In that case the expression reduces, as expected to that of flat space, which follows immediately by transforming the Minkowski metric to spherical coordinates. This result holds also inside a spherical cavity in a spherically symmetric mass distribution. In the case that this solution applies to the metric outside a spherical mass distribution, we can determine the integration constant A by requiring that the solution asymptotically reproduces Newton's theory. Let us therefore have a closer look at the 4 terms in the expression (5.42) for the line element. The third and fourth term look normal in the sense that they are the same as in flat space. They determine the circumference of circles with radial parameter r as $2\pi r$, just as in flat space. However, the second term represents a deviation from flat space, *i.e.*, the 'radius' r is not equal to the distance in the radial direction as specified by Eq. (5.42). The line element in the radial direction is $|ds| = (1 + A/r)^{-1/2} dr$. If A is nonzero, the distance between two circles with radial parameters r and $r + dr$ is not dr. Euclidean geometry is not valid if $A \neq 0$.

The first term in Eq. (5.42) depends on r, which means that there is a gravitational field. In Sec. 4.1 we have seen that in the Newtonian limit $g_{00} = c^2 + 2V_{\text{grav}}$. Since then $V_{\text{grav}} = -\hat{G}M/r$, we may identify

$$\frac{Ac^2}{r} = -\frac{2\hat{G}M}{r} \quad \text{or} \quad A = -\frac{2\hat{G}M}{c^2} \tag{5.43}$$

and the line element is now fully specified as

$$ds^2 = \left(1 - \frac{2\hat{G}M}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2\hat{G}M}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{5.44}$$

This is the Schwarzschild solution for the metric describing a spherically symmetric field of gravity.

5.4 Gravitational redshift

It will be useful to reflect on the meaning of the expressions that we have derived and used in the preceding section. The Schwarzschild metric given implicitly in Eq. (5.44) tells us, besides the distance between spatially separated points, *i.e.*, the outcome of geometric measurements, also about the relative speed of clocks at different locations. Let us work this out in some detail by means of a thought experiment involving two observers having identical clocks and other physical equipment that emits light signals at intervals of ΔT seconds, as determined by the pertinent observer's clock. The observers detect one another's signals and are thus able to check whether the clocks are running equal.

Let observer 1 sit at rest in the Schwarzschild frame, but at such a large distance $r >> A$ from the center that the metric implied by Eq. (5.44) is indistinguishable from flat, *i.e.*, $g^{\mu\nu} \simeq \eta^{\mu\nu}$. His clock ticks at the rate specified by the Schwarzschild metric which is $\eta^{\mu\nu}$ at his location. His clock shows the Schwarzschild time. This time applies everywhere in the Schwarzschild frame, but it is not necessarily the same time as shown by other clocks.

Let the second observer sit at a finite value of the radius parameter r. He is also at rest in the Schwarzschild frame. His flashing light emits signals with period ΔT *as measured by his own clock*. For a careful analysis of this process, we should realize that the second observer is subject to a gravitational field, so he does not sit in an inertial frame. However, he may independently check that the rate of his clock and flashing light is not influenced by the gravitational field, for instance by jumping from a table, holding a third clock and comparing the time of his free-falling clock with the clock on the table. He may do the comparison in an arbitrarily short time after jumping, so that his speed is kept arbitrarily small. Clocks of a sufficient quality will, under these circumstances, not be influenced by accelerations. Furthermore, we note that general coordinate transformations between the frames of the two observers do in first order not depend on accelerations.

The invariant line element of the interval between two flashes of the equipment of observer 2 is, as measured in his own inertial frame, is given by $ds^2 = \eta_{00}(\Delta T)^2$. But these flashes are also observed by the first observer who is using the Schwarzschild metric. Let, according to the measurement of observer 1, the flashes occur at intervals of ΔT_2 seconds. The invariant line element is thus $ds^2 = g_{00}(\Delta T_2)^2$, where g_{00} is the Schwarzschild metric at the position of observer 2. Its invariance under a transformation from the Schwarzschild metric to the local inertial frame implies that

$$\eta_{00}(\Delta T)^2 = g_{00}(\Delta T_2)^2 \tag{5.45}$$

Thus

$$\frac{\Delta T_2}{\Delta T} = \sqrt{\frac{\eta_{00}}{g_{00}}} = \left(1 - \frac{2\hat{G}M}{c^2 r}\right)^{-1/2} > 1 \tag{5.46}$$

The clock of observer 2 runs slower. The frequency ν_2 of the signals emitted by him will be redshifted by a factor

$$\frac{\nu_2}{\nu} = \left(1 - \frac{2\hat{G}M}{c^2 r}\right)^{1/2} \tag{5.47}$$

with respect to the observer far away.

The existence of the gravitational redshift in the Earth's gravity field has already been amply confirmed, for instance from a frequency shift of γ rays as observed using the Mössbauer effect, and from data taken during a high-altitude rocket flight of an atomic clock in the context of the Gravity Probe A project. More recently the gravitational redshift effect is (as a matter of necessity) routinely taken into account in the operation of the GPS (Global Positioning System) navigation system. Height differences of about 10 m are already observable from the frequency shift of very accurate atomic clocks.

Equation (5.47) shows that something catastrophic happens at $2\hat{G}M/c^2r = 1$, where $g_{00} = 0$ and the redshift becomes infinite. This means that nothing, not even light, can escape from a point within the 'event horizon' located at the Schwarzschild radius

$$r_{\rm S} = \frac{2\hat{G}M}{c^2} \tag{5.48}$$

so that one may speak of a 'black hole'. Note that g_{11} diverges at the Schwarzschild radius. However, the radial distance s_{12} between two points $x^\mu(1) = (0, r_1, 0, 0)$ and $x^\mu(2) = (0, r_2, 0, 0)$ as determined by integration of the line element $|ds|$

$$s_{12} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - r_{\rm S}/r}} = r\sqrt{1 - r_{\rm S}/r}\Big|_{r_1}^{r_2} + r_{\rm S}\ln(\sqrt{r} + \sqrt{r - r_{\rm S}})\Big|_{r_1}^{r_2} \tag{5.49}$$

remains finite when one of the radial parameters r_1 or r_2 approaches $r_{\rm S}$. This does not mean that the Schwarzschild time (the time expressed in Schwarzschild coordinates) needed to travel the distance between r_1 and r_2 remains finite. The velocity of light in the radial direction is determined by the vanishing of the line element in Eq. (5.44)

$$\frac{dr}{dt} = \pm c(1 - r_{\rm S}/r) \tag{5.50}$$

Remarkably this goes to zero for $r \to r_{\rm S}$. The time needed by a light ray is, for $r_2 > r_1$, given by

$$t_{12} = \int_{r_1}^{r_2} \frac{dr}{c(1 - r_{\rm S}/r)} = \frac{r_2 - r_1}{c} + \frac{r_{\rm S}}{c}\ln\left(\frac{r_2 - r_{\rm S}}{r_1 - r_{\rm S}}\right) \tag{5.51}$$

which diverges for $r_1 \to r_{\rm S}$. The Schwarzschild time for a light ray to travel the same path in the opposite direction is the same. Also a free-falling object takes an infinite length of coordinate time to enter the black hole ($r = r_{\rm S}$). An observer falling into a black hole may thus seem to posses eternal life according to distant observers, but his own clock will measure only a finite time to cross the horizon and a short additional time to hit the central singularity at $r = 0$.

For common celestial bodies such as the known objects in our solar system, the Schwarzschild radius is very small in comparison with the actual size of the body. The Schwarzschild metric applies only outside their actual radius (and subject to some perturbation if the mass distribution deviates from sphericity). The presence of mass inside their radius implies $P^{\mu\nu} \neq 0$ so that the Schwarzschild solution does not apply there. Also within these objects there is no event horizon, the metric remains nonsingular if we may assume a reasonable mass distribution. For an object with the mass of the Earth, the Schwarzschild radius is nearly 1 cm, for the mass of the Sun it is about 3 km.

5.5 Orbital mechanics

According to Eq. (5.28) and the results for β and γ in Sec. 5.3, the nonzero elements of the metric and the affine connection are given by

$$\begin{array}{ll} g_{00} = c^2(1+A/r) & g_{11} = -(1+A/r)^{-1} \\ g_{22} = -r^2 & g_{33} = -r^2\sin^2\theta \\ \Gamma^0_{01} = -(A/2r^2)(1+A/r)^{-1} & \Gamma^1_{00} = (Ac^2/2r^2)(1+A/r) \\ \Gamma^1_{11} = -(A/2r^2)(1+A/r)^{-2} & \Gamma^1_{22} = r(1+A/r) \\ \Gamma^1_{33} = -r(1+A/r)\sin^2\theta & \Gamma^2_{12} = 1/r \\ \Gamma^2_{33} = -\sin\theta\cos\theta & \Gamma^3_{13} = 1/r \\ \Gamma^3_{23} = \cos\theta/\sin\theta & A = -2\hat{G}M/c^2 \end{array} \tag{5.52}$$

with $t = x^0$, $r = x^1$, $\theta = x^2$, and $\phi = x^3$.

To facilitate the solution of the equation of motion of a test particle in this metric, we choose the orientation of our coordinate system such that both the initial position and velocity lie in the $\theta = \pi/2$ plane, *i.e.*, $x^2 = \pi/2$ and $u^2 = cd\theta/ds = 0$. Then, the substitution of Eqs. (5.52) into the geodesic equation (3.29) shows that

$$\frac{du^2}{ds} = -\frac{1}{c}\,\Gamma^2_{\mu\nu}u^\mu u^\nu = 0\,. \tag{5.53}$$

The solution

$$x^2(s) = \pi/2\,, \quad u^2(s) = 0 \tag{5.54}$$

shows that the whole orbit of the test particle lies in the $\theta = \pi/2$ plane. For these initial conditions $\theta = \pi/2$ is conserved, in line with angular momentum conservation.

For the solution of the remaining coordinates along the geodesic, we make use of the conservation laws described in Sec. 3.5, rather than the equations of motion (3.29). Since the Schwarzschild metric does not explicitly depend on $x^0 = t$ and $x^3 = \phi$, there are two more conserved quantities u_0 and u_3. Using

$$u_0 = cg_{00}\frac{dx^0}{ds} \quad \text{or} \quad \frac{dx^0}{ds} = \frac{u_0}{cg_{00}}\,, \tag{5.55}$$

which expresses energy conservation, and

$$u_3 = cg_{33}\frac{dx^3}{ds} \quad \text{or} \quad \frac{dx^3}{ds} = \frac{u_3}{cg_{33}}\,, \tag{5.56}$$

also in line with angular momentum conservation, one finds that

$$\frac{d\phi}{dt} = \frac{dx^3}{dx^0} = \frac{g_{00}u_3}{g_{33}u_0}\,. \tag{5.57}$$

The definition of the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ can, in the present context, be rewritten as

$$g_{00}\left(\frac{dt}{ds}\right)^2 + g_{11}\left(\frac{dr}{ds}\right)^2 + g_{33}\left(\frac{d\phi}{ds}\right)^2 = 1 \tag{5.58}$$

or

$$\left(\frac{dr}{ds}\right)^2 = \frac{1}{g_{11}} - \frac{(u_0)^2}{c^2 g_{00} g_{11}} - \frac{(u_3)^2}{c^2 g_{33} g_{11}} = c^{-4}[(u_0)^2 - c^2 g_{00} - g_{00}(u_3/r)^2] \tag{5.59}$$

The right-hand side of this equation is a function of the coordinate r. The range of possible values of r is obviously restricted, because this function must be non-negative. Depending on the conserved quantities, r may be restricted to a bounded interval, which corresponds with a bound orbit (an ellipse in the Newtonian limit). It is also clear that, by tuning u_0 and u_3, one can make the interval shrink to zero. This limiting case corresponds with a circular orbit. For large enough u_0, one of the interval bounds vanishes to infinity, and the orbit becomes unbound.

The square root of Eq. (5.59) is

$$\frac{dr}{ds} = \pm\frac{1}{c^2}\sqrt{(u_0)^2 - c^2 g_{00} - g_{00}(u_3/r)^2} \tag{5.60}$$

By integration of ds/dr, and the substitution $ds = c d\tau$, one obtains the proper time τ of the moving object as a function of the radial parameter r:

$$\tau = \tau_0 \pm c \int dr\, [(u_0)^2 - c^2 g_{00} - g_{00}(u_3/r)^2]^{-1/2} \tag{5.61}$$

Division of the right-hand version of Eq. (5.56) by Eq. (5.60) leads to

$$\frac{d\phi}{dr} = \pm c u_3 r^{-2}[(u_0)^2 - c^2 g_{00} - g_{00}(u_3/r)^2]^{-1/2} \tag{5.62}$$

and integration provides $\phi(r)$

$$\phi = \phi_0 \pm c u_3 \int dr\, r^{-2}[(u_0)^2 - c^2 g_{00} - g_{00}(u_3/r)^2]^{-1/2} \tag{5.63}$$

which describes the orbit of the test particle. Evaluation of Eq. (5.63) for bound orbits shows that orbits resembling an ellipse still occur, but the long axis of the orbit is found to precess in the direction of the orbital motion. This result explained the long-standing problem of the perihelion shift of the planet Mercury, thus providing an early confirmation of Einstein's theory.

Finally, combination of Eq. (5.55) with Eq. (5.60) and subsequent integration yields

$$t = t_0 \pm c u_0 \int dr\, (g_{00})^{-1}[(u_0)^2 - c^2 g_{00} - g_{00}(u_3/r)^2]^{-1/2} \tag{5.64}$$

which describes the Schwarzschild time as a function of the coordinate r. Since g_{00} vanishes linearly as a function of r for an orbit crossing the event horizon, the integrand diverges as $1/(r - r_S)$, and the coordinate time needed to enter the horizon is infinite. In contrast, the *proper time* shown by a falling clock, given by Eq. (5.61), remains finite when it crosses the horizon. Unfortunately, the part of the orbit inside the horizon can only be investigated directly by an observer moving with the clock, and he will be unable to report his findings to the external world.

5.6 Gravitational waves

We are looking for solutions of the Einstein equation that correspond with small plane-wave-like perturbations of the flat metric in empty space. On the basis of the analogy with electromagnetism, one expects that such perturbations will originate from accelerated masses. One would also expect that, on a large scale, these perturbations will propagate with a more or less spherical wave pattern. With a sufficiently large extent, such a wave can be well approximated by a plane wave, which thus justifies the use of plane waves.

As before, we describe small perturbations of the flat metric by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{5.65}$$

It is readily verified that $g_{\mu\alpha}g^{\mu\beta} = \delta^\beta_\alpha$ in first order of the perturbations $h_{\mu\nu}$. Next, we calculate the Ricci tensor in first order of the perturbations. On the basis of Eq. (5.20), we immediately find

$$R_{\beta\delta} = \eta^{\alpha\gamma}R_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta^{\alpha\gamma}(h_{\alpha\delta,\beta\gamma} - h_{\beta\delta,\alpha\gamma} - h_{\alpha\gamma,\beta\delta} + h_{\beta\gamma,\alpha\delta}) =$$
$$-\frac{1}{2}(\eta^{\alpha\gamma}h_{\beta\delta,\alpha\gamma} + h^\alpha_{\alpha,\beta\delta} - h^\alpha_{\delta,\alpha\beta} - h^\alpha_{\beta,\alpha\delta}) \tag{5.66}$$

We shall now attempt to choose the coordinates such that the last three terms between brackets in the last part of Eq. (5.66) vanish. For this purpose, it would be sufficient to choose the coordinates such that

$$h^\alpha_{\delta,\alpha} - \frac{1}{2}h^\alpha_{\alpha,\delta} = 0 \tag{5.67}$$

After differentiating this to x_β, we add the same form with δ and β interchanged. The result is:

$$h^\alpha_{\delta,\alpha\beta} - \frac{1}{2}h^\alpha_{\alpha,\delta\beta} + h^\alpha_{\beta,\alpha\delta} - \frac{1}{2}h^\alpha_{\alpha,\beta\delta} = h^\alpha_{\delta,\alpha\beta} + h^\alpha_{\beta,\alpha\delta} - h^\alpha_{\alpha,\beta\delta} = 0 \tag{5.68}$$

so that the last three terms Eq. (5.66) do indeed vanish if Eq. (5.67) is satisfied. To satisfy hat equation, we apply a coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu \tag{5.69}$$

where ξ^μ is a small quantity, expected to be of the same order as the deviations from the flat metric. Under this transformation, the metric transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta} \tag{5.70}$$

or

$$\eta_{\mu\nu} + h'_{\mu\nu} = \left(\delta^\alpha_\mu - \frac{\partial \xi^\alpha}{\partial x'^\mu}\right)\left(\delta^\beta_\nu - \frac{\partial \xi^\beta}{\partial x'^\nu}\right)(\eta_{\alpha\beta} + h_{\alpha\beta}) =$$
$$\eta_{\mu\nu} + h_{\mu\nu} - \frac{\partial \xi_\nu}{\partial x'^\mu} - \frac{\partial \xi_\mu}{\partial x'^\nu} + \cdots \tag{5.71}$$

We keep only terms up to first order in the small quantities. After subtracting $\eta_{\mu\nu}$ and raising one index, one finds

$$h'^{\alpha}_{\beta} = h^{\alpha}_{\beta} - \eta^{\alpha\mu}\frac{\partial \xi_\beta}{\partial x'^\mu} - \eta^{\alpha\mu}\frac{\partial \xi_\mu}{\partial x'^\beta} \tag{5.72}$$

from which it follows that

$$h'^{\sigma}_{\sigma} = h^{\sigma}_{\sigma} - \eta^{\sigma\mu}\frac{\partial \xi_\sigma}{\partial x'^\mu} - \eta^{\sigma\mu}\frac{\partial \xi_\mu}{\partial x'^\sigma} \tag{5.73}$$

Combining these two equations we can form

$$h'^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}h'^{\sigma}_{\sigma} =$$

$$h^{\alpha}_{\beta} - \eta^{\alpha\mu}\frac{\partial \xi_\beta}{\partial x'^\mu} - \eta^{\alpha\mu}\frac{\partial \xi_\mu}{\partial x'^\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}h^{\sigma}_{\sigma} + \frac{1}{2}\delta^{\alpha}_{\beta}\eta^{\sigma\mu}\frac{\partial \xi_\sigma}{\partial x'^\mu} + \frac{1}{2}\delta^{\alpha}_{\beta}\eta^{\sigma\mu}\frac{\partial \xi_\mu}{\partial x'^\sigma} \tag{5.74}$$

After differentiation to x^α and using the abbreviated form of the derivatives one finds

$$(h'^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}h'^{\sigma}_{\sigma})_{,\alpha} =$$

$$(h^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}h^{\sigma}_{\sigma})_{,\alpha} - \eta^{\alpha\mu}\xi_{\beta,\alpha\mu} - \eta^{\alpha\mu}\xi_{\mu,\alpha\beta} + \frac{1}{2}\eta^{\sigma\mu}\xi_{\sigma,\mu\beta} + \frac{1}{2}\eta^{\sigma\mu}\xi_{\mu,\sigma\beta} \tag{5.75}$$

The last three terms add up to zero. It follows that both sides of Eq. (5.75) vanish when we choose

$$\eta^{\alpha\mu}\xi_{\beta,\alpha\mu} = (h^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}h^{\sigma}_{\sigma})_{,\alpha} \tag{5.76}$$

This differential equation, which can also be written

$$\eta^{\alpha\mu}\xi_{\beta,\alpha\mu} = \Box\,\xi_\beta = c^{-2}\frac{\partial^2 \xi_\beta}{\partial t^2} - \frac{\partial^2 \xi_\beta}{\partial x^2} - \frac{\partial^2 \xi_\beta}{\partial y^2} - \frac{\partial^2 \xi_\beta}{\partial z^2} = (h^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}h^{\sigma}_{\sigma})_{,\alpha} \tag{5.77}$$

is of a type that has in principle always a solution. This means that we can choose our frame of reference such that Eq. (5.67) is satisfied. After taking notice that we have chosen new coordinates, we forget the primes and Eq. (5.66) does reduce to

$$R_{\beta\delta} = -\frac{1}{2}\eta^{\alpha\gamma}h_{\beta\delta,\alpha\gamma} \tag{5.78}$$

The Einstein equation Eq. (5.9) implies that, in the absence of a source term $P_{\mu\nu}$, the Ricci curvature tensor vanishes. Thus

$$\eta^{\alpha\gamma}h_{\beta\delta,\alpha\gamma} = \Box\,h_{\beta\delta} = c^{-2}\frac{\partial^2 h_{\beta\delta}}{\partial t^2} - \frac{\partial^2 h_{\beta\delta}}{\partial x^2} - \frac{\partial^2 h_{\beta\delta}}{\partial y^2} - \frac{\partial^2 h_{\beta\delta}}{\partial z^2} = 0 \tag{5.79}$$

This equation has solutions in the form of running waves such as $h_{\beta\delta} \propto f(x - ct)$ and rotated versions $h_{\beta\delta} \propto f(e_i x^i - ct)$ where e_i is some covariant unit 3-vector. The linear approximation that we used allows the use of a Fourier decomposition so that we may restrict ourselves, without real loss of generality, to linear combinations of sine and cosine waves.

5.6.1 Transverse character of gravitational waves

We still have to consider the question about the number and the nature of the independent components of such a wave, as associated with the free indices β and δ. It is now convenient to describe the deviations from flatness of the metric by

$$f^{\alpha}_{\beta} \equiv h^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} h^{\sigma}_{\sigma} \tag{5.80}$$

Multiplication with δ^{β}_{α} shows that $f^{\sigma}_{\sigma} = -h^{\sigma}_{\sigma}$ so that the reverse relation is

$$h^{\alpha}_{\beta} \equiv f^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} f^{\sigma}_{\sigma} \tag{5.81}$$

The coordinate condition Eq. (5.67) is thus simply

$$\eta^{\beta\delta} f_{\alpha\beta,\delta} = 0 \tag{5.82}$$

We consider a plane wave $f_{\alpha\beta}(x^1 - cx^0)$, for instance a cosine wave running in the x direction. Its components must satisfy the coordinate condition:

$$\begin{array}{rcl} -(f_{00,0}/c^2 - f_{01,1}) &=& f'_{00}/c + f'_{01} = 0 \\ -(f_{01,0}/c^2 - f_{11,1}) &=& f'_{01}/c + f'_{11} = 0 \\ -(f_{02,0}/c^2 - f_{12,1}) &=& f'_{02}/c + f'_{12} = 0 \\ -(f_{03,0}/c^2 - f_{13,1}) &=& f'_{03}/c + f'_{13} = 0 \end{array} \tag{5.83}$$

where $f'_{\alpha\beta} \equiv df_{\alpha\beta}(y)/dy$. Since $f_{\alpha\beta}(y)$ depends only on y, Eq. (5.83) can be integrated over x^0 and thus yields direct relations between the *unprimed* $f_{\alpha\beta}$. We need not worry about any infinitesimal integration constants, since they can simply be transformed away by an infinitesimal shift of coordinates. Thus

$$\begin{array}{rcl} f_{00} &=& -cf_{01} \\ f_{01} &=& -cf_{11} \\ f_{02} &=& -cf_{12} \\ f_{03} &=& -cf_{13} \end{array} \tag{5.84}$$

These relations eliminate 4 of the 10 elements of $f_{\alpha\beta}$ as independent parameters. A further reduction of this number is still possible because we have not yet fully exploited the freedom to choose the most general coordinate condition. Given a solution $\xi_{\alpha}(x^{\mu})$ of Eq. (5.76), we have the freedom to substitute ξ^{α} by $\xi^{\alpha} + \omega^{\alpha}$ where $\omega^{\alpha}(x^{\mu})$ is a solution of

$$\eta^{\alpha\mu}\omega_{\beta,\alpha\mu} = 0 \tag{5.85}$$

because $\xi^{\alpha} + \omega^{\alpha}$ qualifies equally as a solution of Eq. (5.76). Let the effect of this transformation be that $f_{\alpha\beta}$ changes into $\hat{f}_{\alpha\beta}$ (we use this notation because f' is already defined with a different meaning). The result is read directly from Eq. (5.74):

$$\hat{f}^{\alpha}_{\beta} = f^{\alpha}_{\beta} - \eta^{\alpha\mu}(\omega_{\beta,\mu} + \omega_{\mu,\beta}) + \frac{1}{2}\delta^{\alpha}_{\beta}\eta^{\sigma\mu}(\omega_{\mu,\sigma} + \omega_{\sigma,\mu}) \tag{5.86}$$

Multiplication by $\eta_{\alpha\gamma}$ leads to

$$\hat{f}_{\gamma\beta} = f_{\gamma\beta} - \omega_{\beta,\gamma} - \omega_{\gamma,\beta} + \eta_{\gamma\beta}\omega^{\sigma}{}_{,\sigma} \tag{5.87}$$

In line with the $\hat{f}_{\mu\nu}$, we choose the $\omega_\beta(x^\mu)$ as running waves depending on $x^1 - cx^0 = x - ct$, in agreement with Eq. (5.85). We denote the derivative to this argument as ${\omega'}_\beta$. Substitution in Eq. (5.87) leads to

$$\begin{aligned}
\hat{f}_{00} &= f_{00} + c{\omega'}_0 - c^2{\omega'}_1 \\
\hat{f}_{01} &= f_{01} - {\omega'}_0 + c{\omega'}_1 \\
\hat{f}_{02} &= f_{02} + c{\omega'}_2 \\
\hat{f}_{03} &= f_{03} + c{\omega'}_3 \\
\hat{f}_{11} &= f_{11} + c^{-1}{\omega'}_0 - {\omega'}_1 \\
\hat{f}_{12} &= f_{12} - {\omega'}_2 \\
\hat{f}_{13} &= f_{13} - {\omega'}_3 \\
\hat{f}_{22} &= f_{22} + c^{-1}{\omega'}_0 + {\omega'}_1 \\
\hat{f}_{23} &= f_{23} \\
\hat{f}_{33} &= f_{33} + c^{-1}{\omega'}_0 + {\omega'}_1
\end{aligned} \tag{5.88}$$

As expected, the ratios of the elements of $\hat{f}_{0\mu}$ and $\hat{f}_{1\mu}$ are still consistent with Eq. (5.84). A few other elements are determined by the symmetry of $f_{\alpha\beta}$ in its indices. It is now clear that we may choose the 4 elements of ω_μ such as to eliminate 4 more elements of $\hat{f}_{\mu,\nu}$. First, we choose ω_0 and ω_1 such that $\omega_0 - c\omega_1 = f_{00}/c$ so that $\hat{f}_{00}$ vanishes. Then, in view of the coordinate relations as expressed by Eq. (5.84), $\hat{f}_{11}$ and $\hat{f}_{10}$ also vanish. Similarly, ω_2 and ω_3 can be chosen such that $\hat{f}_{12}$ and $\hat{f}_{13}$ vanish. Then also $\hat{f}_{20}$ and $\hat{f}_{30}$ vanish. Finally, we may treat $c^{-1}{\omega'}_0 + {\omega'}_1$ as an independent variable, so that we can arrange it such that $\hat{f}_{22} + \hat{f}_{33} = 0$.

Thus, with this choice of coordinates, only two independent components of the gravitational waves remain. We forget the 'hats' and denote them as f_{23} and $f_{22} = -f_{33}$. In view of Eq. (5.80) the same holds for the components of $h_{\alpha\beta}$: the deviations from the flat metric are $h_{23} = h_{32}$ and $h_{22} = -h_{33}$. For this wave running in the x direction, the deviations from flat metric apply to the y and z directions, *i.e.*, the waves are transversely polarized. Furthermore, a rotation of $\pi/4$ about the x axis transforms the two components into one another. To demonstrate this we write out the rotation, which takes the form ${h'}_{\alpha\beta} = T_\alpha{}^\mu h_{\mu\nu}(T^{-1})^\nu{}_\beta$, in matrix form:

$${h'}_{\alpha\beta} = \begin{bmatrix} {h'}_{00} & {h'}_{01} & {h'}_{02} & {h'}_{03} \\ {h'}_{10} & {h'}_{11} & {h'}_{12} & {h'}_{13} \\ {h'}_{20} & {h'}_{21} & {h'}_{22} & {h'}_{23} \\ {h'}_{30} & {h'}_{31} & {h'}_{32} & {h'}_{33} \end{bmatrix} = T_\alpha{}^\mu h_{\mu\nu}(T^{-1})^\nu{}_\beta =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1/2} & -\sqrt{1/2} \\ 0 & 0 & \sqrt{1/2} & \sqrt{1/2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & -\sqrt{1/2} & \sqrt{1/2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -h_{23} & h_{22} \\ 0 & 0 & h_{22} & h_{23} \end{bmatrix} \tag{5.89}$$

Since $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and $\eta'_{\alpha\beta} = T_\alpha{}^\mu \eta_{\mu\nu} (T^{-1})^\nu{}_\beta = \eta_{\alpha\beta}$, Eq. (5.89) is sufficient to describe the change of the metric under a rotation of $\pi/4$ radians.

It is now easy to see that another $\pi/4$ rotation leads to a change of sign with respect to the original perturbations of the metric. Thus the transformation behavior is the same as that of electromagnetic quadrupoles. One may therefore say that gravitational waves have a quadrupole character. The strength of the waves emitted by a source with a time-dependent mass distribution, for instance a pair of binary stars circling the common center of gravity, can be found by small-field expansion of the Einstein equation and integration of the retarded potential, similar to the procedure used in electrodynamics. At distances r that are large in comparison with the source of the waves, the result is the Einstein quadrupole formula

$$h^{ij}(r,t) \equiv \frac{2\hat{G}}{c^4 r} \ddot{Q}^{ij}(t - r/c) \tag{5.90}$$

where $\ddot{Q}^{ij}(t)$ is the second time derivative of the traceless quadrupole moment $Q^{ij}(t)$ of the source. This quantity is determined by the mass density distribution $\rho(\vec{x}, t)$ as

$$Q^{ij}(t) \equiv \int d^3x \rho(\vec{x}, t) \left(x^i x^j - \frac{1}{3} \eta^{ij} \eta_{kl} x^k x^l \right) . \tag{5.91}$$

The term with η^{ij} ensures that the trace of $Q^{ij}(t)$ vanishes. As a consequence of the linearization, the result for $h^{ij}(r,t)$ is valid only for low densities ρ, *i.e.*, the size Δr of the mass distribution should be large in comparison with the Schwarzschild radius of its total mass. Furthermore the derivation assumes that the variation of the mass distribution in time takes place on a time scale $\Delta t >> \Delta r/c$.

5.6.2 Observation of gravitational waves

Observable gravitational waves may be expected from various astronomical objects whose mass distributions have rapidly changing quadrupole moments, such as asymmetric supernova explosions and compact objects orbiting one another. The latter type of systems are expected to lose energy due to the gravitational radiation, so that their orbits will shrink in time. In the case that at least one of the compact objects is a pulsar, which emits radio signals at an extremely constant rate, it is possible to observe this decrease quantitatively. Such decreases have been observed, and appear to be in a perfect agreement with the theoretical predictions, and can thus be seen as an indirect observation of gravitational waves.

Efforts are under way to observe gravitational waves directly, using laser interferometry, pulsar timing arrays, and cryogenic detectors. The problem is here that, according to Eq. (5.90), all plausible sources produce extremely weak gravitational waves in our neighborhood. Direct observations of gravitational waves have been achieved since 2015, by means of large earth-based optical interferometers. The first observations exposed the final revolutions and the merging of two black holes, with masses up to about 30 solar masses, at cosmological distances of one or a few times 10^9 light years.

Chapter 6

Cosmology

One appealing application of general relativity is the description of the geometry of the universe as a whole. It is however necessary to make further assumptions concerning its mass distribution. In modern times it is considered natural to assume that on a sufficiently large scale, the universe is homogeneous and isotropic. This assumption is called the *cosmological principle*. While it is known that considerable deviations from homogeneity exist up to the size of galactic superclusters in the order of 10^8 light years or 10^{24} [m], the size of the overseeable universe is in the order of 10^{10} light years or 10^{26} [m]. The available observations indicate that the assumption of homogeneity and isotropy is approximately satisfied. We treat the matter in the universe as a homogeneous fluid with the appropriate average density.

Even before attempting a detailed calculation of the metric of the universe, it is already possible to make an order-of-magnitude estimate of the scale of curvature of the universe. Actually we do not yet know whether such a universal curvature exists, but if so, we expect that we can guess its value on the basis of the Einstein equation and an average matter density estimate provided by astronomical observations as $P^{00} \approx 10^{-27}$ [kg m^{-3}]. Combined with the Einstein equation and the values of c and $\hat{G}$ given in Sec. 1.3 this leads to $R^{00} \approx 2 \times 10^{-70}$ [sec^2 m^{-4}]. The curvature scalar, which has the dimension of square inverse length, follows as $R \approx g_{00} R^{00} \approx 2 \times 10^{-53}$ [m^{-2}]. This indicates a 'size', or length scale of the curvature of the universe, as $R^{-1/2} \approx 2 \times 10^{26}$ [m]. The result of this admittedly crude calculation is still remarkably close to the above-mentioned value of 10^{26} [m]. This is sufficiently promising to proceed with the analysis of the metric of the universe.

6.1 Symmetry properties

The statement that te universe is homogeneous and isotropic does not imply that the metric is homogeneous and isotropic, but it means that it is possible to specify a frame of reference such that the metric, which is a function of the coordinates, does not depend on the location of the origin and the orientation of the axes. So our task will be to choose the coordinate conditions accordingly. It is natural that we choose the coordinates such that, at every position in the universe, the matter (on a sufficiently large scale) is locally at rest. We denote the time $t = x^0$ in this co-moving frame; it is equal to the proper time. We choose our units of time such that the line element satisfies, as in a local Lorentz frame, $ds = cdt$, which leads to $g_{00} = c^2$. The other elements of the metric remain to be determined, and the squared line element is thus written as

$$ds^2 = c^2dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^idx^j \tag{6.1}$$

As before, the summation over dummy Roman indices excludes the time components. The choice of a frame of reference includes the choice of an origin somewhere. This does not seem in line with the symmetry of the problem, and makes it somewhat tricky to further determine the coordinate conditions. For instance, at this point it would not be correct to assume that the elements of the metric are independent of the space- and/or time coordinates. We thus have to proceed carefully, and note that a particle at rest in this frame must satisfy the condition of free fall expressed by the geodesic equation Eq. (3.29). Multiplication by $g_{\tau\mu}$ yields

$$g_{\tau\mu}\frac{du^\mu}{ds} = -\frac{1}{c}\,g_{\tau\mu}\Gamma^\mu_{\sigma\nu}u^\sigma u^\nu = -\frac{1}{2c}\left(\frac{\partial g_{\tau\nu}}{\partial x^\sigma} + \frac{\partial g_{\tau\sigma}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\tau}\right)u^\sigma u^\nu \tag{6.2}$$

Here we can make a substitution according to

$$\frac{du_\tau}{ds} = g_{\tau\mu}\frac{du^\mu}{ds} + \frac{dg_{\tau\mu}}{ds}u^\mu = g_{\tau\mu}\frac{du^\mu}{ds} + \frac{1}{c}\frac{\partial g_{\tau\mu}}{\partial x^\sigma}u^\mu u^\sigma \tag{6.3}$$

which leads to

$$\frac{du_\tau}{ds} = \frac{1}{2c}\frac{\partial g_{\sigma\nu}}{\partial x^\tau}u^\sigma u^\nu \tag{6.4}$$

The choice of the co-moving frame implies that the spacelike components of the 4-velocities are zero. The timelike components are nonzero, but as implied by Eq. (6.1), g_{00} does not depend on the coordinates. Thus the right-hand side of Eq. (6.4), and thereby also its left-hand side, is equal to 0 and we have

$$\frac{d}{dt}u_\tau = \frac{d}{dt}(g_{\tau\mu}u^\mu) = \frac{d}{dt}g_{\tau0}u^0 = 0 \tag{6.5}$$

so that we have made another small step towards the determination of the metric, namely the result that $g_{\tau0}$ is time-independent:

$$g_{\tau0,0} = 0 \tag{6.6}$$

This result does, of course, rely on our special choice of coordinates.

But we have not yet fully exploited the symmetries. The condition of isotropy is most clearly displayed by spherical coordinates $x^1 = r$, $x_2 = \theta$, $x_3 = \phi$:

$$ds^2 = c^2dt^2 - \beta(r,t)dr^2 - \gamma(r)dtdr - \alpha(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.7}$$

It is possible to transform away $\gamma(r)$ by means of a 'shift of time' that depends only on the spatial coordinates:

$$t' = t - \frac{1}{2c^2}\int dr\gamma(r) \;\text{ or }\; dt' = dt - \frac{1}{2c^2}\gamma(r)dr \tag{6.8}$$

Substitution in Eq. (6.7) leaves g_{00} unchanged. However, the cross term in $dtdr$ vanishes. We do not care about transforming the other elements of the metric which involve (presently) unknown functions anyway. We simply append primes to denote the result of the transformation and, as usual, then drop the primes. We thus rewrite Eq. (6.7) as

$$ds^2 = c^2dt^2 - \beta(r,t)dr^2 - \alpha(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.9}$$

This form of the metric imposes isotropy, but not homogeneity. The latter condition requires the invariance of the metric under translations. It is sufficient to consider infinitesimal spatial translations:

$$x'^i = x^i + \xi^i(\vec{x}) \tag{6.10}$$

where ξ^i is an infinitesimal function of the space coordinates (x^1, x^2, x^3). It has to be chosen such that the actual shift, which depends not only on $\xi^i(\vec{x})$ but also on the metric, is independent of the spatial coordinates. But it is chosen independent of the time coordinate. This already enables us, as we shall see, to further restrict the form of the metric. The condition of homogeneity implies that, under such a translation, the metric satisfies

$$g'_{ij}(\vec{x}\,') = g_{ij}(\vec{x}\,') \tag{6.11}$$

Note the precise meaning of this equation. The prime on the right hand side is essential: without it, the equation would express that the metric at a given location is invariant under a coordinate transformation. which is not what we require. We wish to compare the metric in different locations. Our aim is that, when we apply a translation, the original metric at a given set of coordinates is equal to the transformed metric at the same coordinates in the new frame. In order to explore the consequences of Eq. (6.11) we use two steps to relate both sides. The first step transforms the metric in the usual way according to

$$g'_{ij}(\vec{x}\,') = g_{ij}(\vec{x}) - \frac{\partial \xi^k}{\partial x^i} g_{jk} - \frac{\partial \xi^k}{\partial x^j} g_{ik} \tag{6.12}$$

and next we relate the metric in two neighboring points by linearizing in the ξ^k:

$$g_{ij}(\vec{x}\,') = g_{ij}(\vec{x}) + \xi^k \frac{\partial g_{ij}}{\partial x^k} \tag{6.13}$$

Substitution in Eq. (6.11) yields

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial \xi^k}{\partial x^i} g_{jk} + \frac{\partial \xi^k}{\partial x^j} g_{ik} = 0 \tag{6.14}$$

This is actually 9 equations, of which we select 3:

$$\begin{aligned} 2\frac{\partial \xi^k}{\partial x^1} g_{1k} + \xi^k \frac{\partial g_{11}}{\partial x^k} &= 0 \\ \frac{\partial \xi^k}{\partial x^1} g_{2k} + \frac{\partial \xi^k}{\partial x^2} g_{1k} &= 0 \\ 2\frac{\partial \xi^k}{\partial x^2} g_{2k} + \xi^k \frac{\partial g_{22}}{\partial x^k} &= 0 \end{aligned} \tag{6.15}$$

After substitution of the nonzero elements of the metric as given in Eq. (6.9) we find

$$\begin{aligned} 2\frac{\partial \xi^1}{\partial r}\beta + \xi^1\frac{\partial \beta}{\partial r} &= 0 \\ \frac{\partial \xi^2}{\partial r}\alpha + \frac{\partial \xi^1}{\partial \theta}\beta &= 0 \\ 2\frac{\partial \xi^2}{\partial \theta}\alpha + \xi^1\frac{\partial \alpha}{\partial r} &= 0 \end{aligned} \tag{6.16}$$

Since ξ^1 does not depend on the time t, the first of these three equations implies that $(\partial\beta(r,t)/\partial r)/\beta(r,t) = \partial \ln \beta(r,t)/\partial r$ is independent of time. This can only hold if $\beta(r,t)$ factorizes in an r-dependent part and a t-dependent part. We denote these parts as $\beta_r(r)$ and $\tilde{R}^2(t)$ respectively, where we write $\tilde{R}(t)$ and not $R(t)$ to avoid confusion with the curvature scalar R. Thus

$$\beta(r,t) = \beta_r(r)\tilde{R}^2(t)\,. \tag{6.17}$$

Substitution in the second line of Eqs. (6.16) shows that this kind of factorization applies also to $\alpha(r,t)$:

$$\alpha(r,t) = \alpha_r(r)\tilde{R}^2(t) \tag{6.18}$$

so that $\tilde{R}(t)$ describes the evolution in time of the scale factor of the universe. The foregoing considerations have not yet fixed the scale of the radius variable r. We may thus define r' by

$$r'(r) \equiv \sqrt{\alpha_r(r)} \tag{6.19}$$

so that

$$\alpha'_r(r') = \alpha_r(r) = r'^2 \tag{6.20}$$

The substitution $r(r')$ in $\beta_r(r)dr^2$ leads to a result similarly denoted $\beta'_r(r')dr'^2$. After making the proper substitutions in Eq. (6.9) and forgetting the primes one obtains

$$ds^2 = c^2dt^2 - \tilde{R}^2(t)[\beta_r(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{6.21}$$

Substitution of the factorization of $\beta(r,t)$ in the first line of Eq. (6.16) and dividing out $\tilde{R}^2(t)$ yields

$$2\frac{\partial \xi^1}{\partial r}\beta_r(r) + \xi^1\frac{d\beta_r(r)}{dr} = 0 \tag{6.22}$$

or

$$\frac{\partial \ln \xi^1}{\partial r} = -\frac{1}{2}\frac{d\ln\beta_r(r)}{dr} \tag{6.23}$$

The 'shift' ξ^1 is naturally dependent on r, θ and ϕ. Integration yields that $\ln \xi^1(r,\theta,\phi)$ is equal to $-\frac{1}{2}\ln\beta_r(r)$ plus an integration constant that is some function ζ of θ and ϕ. Thus

$$\xi^1(r,\theta,\phi) = \zeta(\theta,\phi)/\sqrt{\beta_r(r)} \tag{6.24}$$

We are now able to further specify the terms appearing in the second and third line of Eqs. (6.16), by substitution of $\xi^1(r,\theta,\phi)$ and the factorization of $\alpha(r,t)$ and $\beta(r,t)$, and by dividing out $\tilde{R}^2(t)$:

$$\begin{aligned} \frac{\partial \xi^2}{\partial r} r^2 + \frac{\partial \zeta(\theta,\phi)}{\partial \theta}\sqrt{\beta_r(r)} &= 0 \\ \frac{\partial \xi^2}{\partial \theta} r^2 + \zeta(\theta,\phi)\frac{r}{\sqrt{\beta_r(r)}} &= 0 \end{aligned} \tag{6.25}$$

Differentiation of the first equation to θ, and of the second one to r yields

$$\begin{aligned} \frac{\partial^2 \xi^2}{\partial r \partial \theta} + \frac{\partial^2 \zeta(\theta,\phi)}{\partial \theta^2}\frac{\sqrt{\beta_r(r)}}{r^2} &= 0 \\ \frac{\partial^2 \xi^2}{\partial r \partial \theta} + \zeta(\theta,\phi)\frac{d}{dr}\frac{1}{r\sqrt{\beta_r(r)}} &= 0 \end{aligned} \tag{6.26}$$

Elimination of the term with ξ^2 leads to

$$\frac{1}{\zeta}\frac{\partial^2 \zeta(\theta,\phi)}{\partial \theta^2} = \frac{r^2}{\sqrt{\beta_r(r)}}\frac{d}{dr}\frac{1}{r\sqrt{\beta_r(r)}} \tag{6.27}$$

Note that the left-hand side may depend only on θ and ϕ, and the right-hand side only on r. Both sides are therefore a constant independent of the coordinates. We can determine this constant by considering a region sufficiently close to the origin (r small) where the metric can be well approximated by spherical coordinates as in Euclidean space. We choose the translation in Eq. (6.10) parallel to the z axis (the $\theta = 0$ direction). Thus $\xi^1 = |\xi| \cos\theta$ and therefore, according to Eq. (6.24), $\zeta \propto \cos\theta$ so that

$$\frac{1}{\zeta}\frac{\partial^2 \zeta(\theta,\phi)}{\partial \theta^2} = -1 \tag{6.28}$$

The consequence for the right-hand side of Eq. (6.27) follows as

$$\frac{d}{dr}\frac{1}{r\sqrt{\beta_r(r)}} = -\frac{\sqrt{\beta_r(r)}}{r^2} \tag{6.29}$$

or

$$\frac{1}{r\sqrt{\beta_r(r)}}\frac{d}{dr}\frac{1}{r\sqrt{\beta_r(r)}} = -\frac{1}{r^3} \tag{6.30}$$

or

$$\frac{1}{r\sqrt{\beta_r(r)}}\, d\,\frac{1}{r\sqrt{\beta_r(r)}} = \frac{1}{2} d\left(\frac{1}{r\sqrt{\beta_r(r)}}\right)^2 = -\frac{dr}{r^3} \tag{6.31}$$

Integration yields

$$\frac{1}{r^2\beta_r(r)} = \frac{1}{r^2} - k \tag{6.32}$$

where $-k$ denotes the integration constant. The solution for $\beta_r(r)$ is

$$\beta_r(r) = \frac{1}{1 - kr^2} \tag{6.33}$$

Substitution of this result in Eq. (6.21) yields the Robertson-Walker metric as expressed by the squared line element

$$ds^2 = c^2dt^2 - \tilde{R}^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.34}$$

The constant k and the function $\tilde{R}(t)$, which plays the role of the spatial scale factor of the universe, are not determined by the constraints of symmetry that we have imposed.

6.2 The Robertson-Walker metric

Before we attempt to obtain further information on the free parameters k and $\tilde{R}(t)$ in Eq. (6.34) by application of the Einstein equation, we shall explore some general properties of that metric, and determine its spatial curvature. For the time being we leave the time dependences, which are determined by the unknown function $\tilde{R}(t)$, out of consideration. We thus determine the curvature of the metric at a fixed time t on the basis of the space-like line element dl given by

$$dl^2 = \tilde{R}^2\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.35}$$

where, as noted before, we write $\tilde{R}$ instead of $R(t)$ to avoid confusion with the curvature scalar R. The form of the line element shows immediately that there are three qualitatively different cases: $k > 0$, $k = 0$, and $k < 0$. For $k > 0$ the range of the radial parameter r is restricted to $0 \leq r \leq 1/\sqrt{k}$. Universes in this class are finite and closed. In the second case $k = 0$ the Robertson-Walker metric reduces to flat infinite space as described by ordinary spherical coordinates. In the third case $k < 0$, we again have an infinite and open space.

Further information follows from the ratio between the radius and the circumference of a circle about the origin. From Eq. (6.35) we observe that a large circle at $\theta = \pi/2$ with radial parameter r has a circumference $2\pi r\tilde{R}$, but the integrated line element along the radial direction is, for $k \neq 0$, not $r\tilde{R}$ but

$$\int dl = \tilde{R}\int \frac{dr}{\sqrt{1 - kr^2}} \tag{6.36}$$

For $k > 0$ we may substitute $z \equiv \arcsin(r\sqrt{k})$:

$$\int dl = \frac{\tilde{R}}{\sqrt{k}}\int_0^{\arcsin r\sqrt{k}} \frac{dz\cos z}{\cos z} = \frac{\tilde{R}\arcsin r\sqrt{k}}{\sqrt{k}} \tag{6.37}$$

Since $\arcsin(r\sqrt{k})/\sqrt{k} > r$ for $r > 0$, the circumference is less than 2π times the integrated line element in the radial direction.

For $k < 0$ we instead substitute $z \equiv \text{arcsinh}\sqrt{-kr^2} = \ln(\sqrt{1-kr^2} + \sqrt{-kr^2}\,)$:

$$\int dl = \frac{\tilde{R}}{\sqrt{-k}} \int_0^{\text{arcsinh}\sqrt{-kr^2}} \frac{dz \cosh z}{\cosh z} =$$

$$\frac{\tilde{R}}{\sqrt{-k}} \text{arcsinh}\sqrt{-kr^2} = \frac{\tilde{R}}{\sqrt{-k}} \ln(\sqrt{1-kr^2} + \sqrt{-kr^2}\,) \qquad (6.38)$$

Since $\ln(\sqrt{1-kr^2} + \sqrt{-kr^2})/\sqrt{-k} < r$ for $r > 0$, the circumference exceeds 2π times the integrated line element in the radial direction. This shows already that there exists a space curvature for $k \neq 0$.

6.2.1 The spatial curvature

The calculation of the curvature scalar R follows the usual path: the Christoffel symbols, the Riemann tensor, the Ricci tensor, and finally R. Also the contracted indices are restricted to 1, 2, and 3. The calculated quantities are thus not to be confused with the analogous ones in spacetime. The nonzero elements of the metric, and their nonzero derivatives are

$$\begin{array}{lll} g_{11} = \frac{\tilde{R}^2}{1-kr^2} & g^{11} = \frac{1-kr^2}{\tilde{R}^2} & g_{11,1} = g_{11}\frac{2kr}{1-kr^2} \\ g_{22} = r^2\tilde{R}^2 & g^{22} = \frac{1}{r^2\tilde{R}^2} & g_{22,1} = 2r\tilde{R}^2 \\ g_{33} = r^2\tilde{R}^2\sin^2\theta & g^{33} = \frac{1}{r^2\tilde{R}^2\sin^2\theta} & g_{33,1} = 2r\tilde{R}^2\sin^2\theta \\ & & g_{33,2} = 2r^2\tilde{R}^2\sin\theta\cos\theta \end{array} \qquad (6.39)$$

The Christoffel symbols are evaluated according to Eq. (3.11). The results for the nonvanishing Γ's and their derivatives are

$$\begin{array}{ll} \Gamma^1_{11} = \frac{1}{2}g^{11}g_{11,1} = \frac{kr}{1-kr^2} & \Gamma^1_{11,1} = \frac{k(1+kr^2)}{(1-kr^2)^2} \\ \Gamma^1_{22} = -\frac{1}{2}g^{11}g_{22,1} = -r(1-kr^2) & \Gamma^1_{22,1} = -1+3kr^2 \\ \Gamma^1_{33} = -\frac{1}{2}g^{11}g_{33,1} = -r(1-kr^2)\sin^2\theta & \Gamma^1_{33,1} = (-1+3kr^2)\sin^2\theta \\ & \Gamma^1_{33,2} = (-2r+2kr^3)\sin\theta\cos\theta \\ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}g_{22,1} = \frac{1}{r} & \Gamma^2_{12,1} = \Gamma^2_{21,1} = -\frac{1}{r^2} \\ \Gamma^2_{33} = -\frac{1}{2}g^{22}g_{33,2} = -\sin\theta\cos\theta & \Gamma^2_{33,2} = 2\sin^2\theta - 1 \\ \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}g_{33,1} = \frac{1}{r} & \Gamma^3_{13,1} = \Gamma^3_{31,1} = -\frac{1}{r^2} \\ \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}g_{33,2} = \frac{\cos\theta}{\sin\theta} & \Gamma^3_{23,2} = \Gamma^3_{32,2} = -\frac{1}{\sin^2\theta} \end{array} \qquad (6.40)$$

Three of the nonzero elements of the Riemann tensor now follow from Eq. (4.19) as

$$\begin{array}{lll} R^1{}_{212} & = \Gamma^1_{22,1} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^1_{22} & = kr^2 \\ R^1{}_{313} & = \Gamma^1_{33,1} + \Gamma^1_{33}\Gamma^1_{11} - \Gamma^3_{13}\Gamma^1_{33} & = kr^2\sin^2\theta \\ R^2{}_{323} & = \Gamma^2_{33,2} + \Gamma^1_{33}\Gamma^2_{12} - \Gamma^3_{23}\Gamma^2_{33} & = kr^2\sin^2\theta \end{array} \qquad (6.41)$$

The fully covariant form of these elements follows as

$$\begin{array}{ll} R_{1212} & = \frac{kr^2\tilde{R}^2}{1-kr^2} \\ R_{1313} & = \frac{kr^2\tilde{R}^2}{1-kr^2}\sin^2\theta \\ R_{2323} & = kr^4\tilde{R}^2\sin^2\theta \end{array} \qquad (6.42)$$

The other nonzero elements follow from the symmetries of the Riemann tensor given in Sec. 4.3.1. These result for the elements of the Riemann tensor can be compactly written as

$$R_{ijkl} = \frac{k}{\tilde{R}^2}(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{6.43}$$

Multiplication and contraction with g^{ik} yields the Ricci tensor

$$R_{jl} = g^{ik}R_{ijkl} = \frac{2k}{\tilde{R}^2}g_{jl} \tag{6.44}$$

and one more further contraction with g^{jl} leads to the curvature scalar

$$R = R^j_j = g^{jl}R_{jl} = \frac{6k}{\tilde{R}^2} \tag{6.45}$$

This result tells us that the curvature has the same sign as k, and that it is everywhere the same. As expected, it satisfies the cosmological principle.

6.2.2 Hyperspherical symmetry

While the isotropy of the Robertson-Walker metric is obvious from Eq. (6.35), the full symmetry remains hidden. To expose it, we introduce four new coordinates as

$$\begin{aligned} x &= r\tilde{R}\sin\theta\cos\phi \\ y &= r\tilde{R}\sin\theta\sin\phi \\ z &= r\tilde{R}\cos\theta \\ w^2 &= (1-kr^2)\tilde{R}^2/k \end{aligned} \tag{6.46}$$

The fourth parameter w is not independent and can be expressed in x, y, z:

$$w^2 = \tilde{R}^2/k - r^2\tilde{R}^2 = \tilde{R}^2/k - x^2 - y^2 - z^2 \tag{6.47}$$

This is meaningful only if $w^2 > 0$, *i.e.*, for $k > 0$, the case of a finite universe with a positive curvature. It follows from the last line of Eq. (6.46) that

$$wdw = -\tilde{R}^2 rdr \quad \text{or} \quad dw^2 \equiv (dw)^2 = \frac{k\tilde{R}^2r^2}{1-kr^2}dr^2 \tag{6.48}$$

The substitution of $\frac{1}{1-kr^2} = 1 + \frac{kr^2}{1-kr^2}$ in Eq. (6.35) leads to

$$dl^2 = \tilde{R}^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] + \frac{k\tilde{R}^2r^2}{1-kr^2}dr^2 \tag{6.49}$$

or, after substitution according to Eqs. (6.46) and (6.48)

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{6.50}$$

which is the line element of four-dimensional Euclidean space in Cartesian coordinates. Since the 4 coordinates are subject to the constraint Eq. (6.47), this line element describes the geometry of the three-dimensional hypersurface of a 4-dimensional hypersphere of radius $\tilde{R}/\sqrt{k}$.

For the case $k < 0$, one can rewrite dl^2 a similar way, but with dw^2 replaced by $-dw^2$ so that the 4-dimensional space becomes hyperbolic in the w-dimension.

6.3 Dynamical equations

The next task is to express the Einstein tensor in terms of the elements of the metric given by Eq. (6.34) and to analyze the Einstein equation, which will purportedly yield some relation between the parameters $\tilde{R}(t)$, k and the elements of the stress-energy tensor. Note that the space-like elements of the metric are just those of Eq. (6.39) with an additional minus sign. They have to be supplemented by the element $g_{00} = c^2$, $g^{00} = c^{-2}$ and by the nonvanishing time derivatives which are

$$\begin{aligned} g_{11,0} &= 2g_{11}\dot{R}/\tilde{R} \\ g_{22,0} &= 2g_{22}\dot{R}/\tilde{R} \\ g_{33,0} &= 2g_{33}\dot{R}/\tilde{R} \end{aligned} \tag{6.51}$$

where $\dot{R} \equiv \frac{d\tilde{R}}{dt}$ should not be confused with the time derivative of the curvature scalar. Recalculation of the Christoffel symbols, with summations running over all 4 indices, reproduces those given in Eq. (6.40) (with the same sign) and yields in addition

$$\Gamma^0_{ij} = -\frac{1}{c^2}\frac{\dot{R}}{\tilde{R}}\,g_{ij} \quad \text{and} \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{R}}{\tilde{R}}\,\delta^i_j \tag{6.52}$$

The additional nonzero derivatives of the Christoffel symbols are

$$\begin{array}{ll} \Gamma^0_{11,0} = c^{-2}\frac{\tilde{R}\ddot{R}+\dot{R}^2}{1-kr^2} & \Gamma^0_{11,1} = c^{-2}\frac{2kr\tilde{R}\dot{R}}{(1-kr^2)^2} \\ \Gamma^0_{22,0} = c^{-2}r^2(\tilde{R}\ddot{R}+\dot{R}^2) & \Gamma^0_{22,1} = 2c^{-2}r\tilde{R}\dot{R} \\ \Gamma^0_{33,0} = c^{-2}r^2(\tilde{R}\ddot{R}+\dot{R}^2)\sin^2\theta & \Gamma^0_{33,1} = 2c^{-2}r\tilde{R}\dot{R}\sin^2\theta \\ & \Gamma^0_{33,2} = 2c^{-2}r^2\tilde{R}\dot{R}\sin\theta\cos\theta \\ \Gamma^i_{0j,0} = \Gamma^i_{j0,0} = \frac{\ddot{R}\tilde{R}-\dot{R}^2}{\tilde{R}^2}\,\delta^i_j & \end{array} \tag{6.53}$$

On the basis of these results it is straightforward to calculate the elements of the Riemann tensor from Eq. (4.19). A sufficient subset of these elements is:

$$\begin{aligned} R^0{}_{101} &= \Gamma^0_{11,0} - \Gamma^1_{10}\Gamma^0_{11} && = c^{-2}\frac{\tilde{R}\ddot{R}}{1-kr^2} \\ R^0{}_{202} &= \Gamma^0_{22,0} - \Gamma^2_{20}\Gamma^0_{22} && = c^{-2}\frac{\tilde{R}\ddot{R}}{1-kr^2} \\ R^0{}_{303} &= \Gamma^0_{33,0} - \Gamma^3_{30}\Gamma^0_{33} && = c^{-2}\frac{\tilde{R}\ddot{R}}{1-kr^2} \\ R^1{}_{212} &= \Gamma^1_{22,1} + \Gamma^1_{01}\Gamma^0_{22} + \Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{21} && = r^2(k + c^{-2}\dot{R}^2) \\ R^1{}_{313} &= \Gamma^1_{33,1} - \Gamma^1_{31,3} + \Gamma^1_{01}\Gamma^0_{33} + \Gamma^1_{11}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^3_{13} && = r^2(k + c^{-2}\dot{R}^2)\sin^2\theta \\ R^2{}_{323} &= \Gamma^2_{33,2} + \Gamma^2_{02}\Gamma^0_{33} + \Gamma^2_{12}\Gamma^1_{33} - \Gamma^2_{33}\Gamma^3_{32} && = r^2(k + c^{-2}\dot{R}^2)\sin^2\theta \end{aligned} \tag{6.54}$$

where ‘sufficient’ means that one can obtain all nonzero elements by translating to the fully covariant form of of these elements

$$\begin{aligned} R_{0101} &= \frac{\tilde{R}\ddot{R}}{1-kr^2} \\ R_{0202} &= r^2\tilde{R}\ddot{R} \\ R_{0303} &= r^2\tilde{R}\ddot{R}\sin^2\theta \\ R_{1212} &= -\frac{r^2\tilde{R}^2}{1-kr^2}\,(k + c^{-2}\dot{R}^2) \\ R_{1313} &= -\frac{r^2\tilde{R}^2}{1-kr^2}\,(k + c^{-2}\dot{R}^2)\sin^2\theta \\ R_{2323} &= -r^4\tilde{R}^2(k + c^{-2}\dot{R}^2)\sin^2\theta \end{aligned} \tag{6.55}$$

and using the symmetries of the fully covariant form of the Riemann tensor given in Sec. 4.3.1. After contraction with the metric tensor, and another multiplication, the nonzero elements of the mixed form of the Ricci tensor follow as

$$R^0_0 = -\frac{3}{c^2}\frac{\ddot{R}}{\tilde{R}} \quad \text{and} \quad R^1_1 = R^2_2 = R^3_3 = -c^{-2}\frac{\ddot{R}}{\tilde{R}} - 2\,\frac{k + c^{-2}\dot{R}^2}{\tilde{R}^2} \tag{6.56}$$

and a further contraction yields the curvature scalar as

$$R = -\frac{6}{c^2}\,\frac{c^2 k + \dot{R}^2 + \tilde{R}\ddot{R}}{\tilde{R}^2} \tag{6.57}$$

The mixed form of the Einstein tensor (see Eq. 5.6) is now simply obtained as

$$\begin{aligned} G^0_0 &= R^0_0 - \tfrac{1}{2}R = \tfrac{3}{c^2}\,\tfrac{c^2k+\dot{R}^2}{\tilde{R}^2} \\ G^1_1 &= R^1_1 - \tfrac{1}{2}R = \tfrac{1}{c^2}\,\tfrac{c^2k+\dot{R}^2+2\tilde{R}\ddot{R}}{\tilde{R}^2} \\ G^2_2 &= G^3_3 = G^1_1 \end{aligned} \tag{6.58}$$

This tensor is to be compared with the stress-energy tensor. We use the mixed form of the Einstein equation (cf. Eq. 5.7)

$$G^\mu_\nu = 8\pi c^{-4}\hat{G}P^\mu_\nu \tag{6.59}$$

The assumptions of isotropy and the form of the Robertson-Walker metric already imply that the off-diagonal elements of P^μ_ν vanish. In the simple case that the latter tensor is dominated by the mass of ordinary matter and/or radiation pressure, the diagonal elements satisfy

$$P^\mu_\nu = (\rho c^2 + p)\delta^0_\nu \delta^\mu_0 - p\delta^\mu_\nu \tag{6.60}$$

where ρ is the matter density and p the pressure. The latter parameter may not be neglected in the early stages of the universe.

We are now in a position to apply the Einstein equations. For $\mu = \nu = 0$, Eqs. (6.58), (6.59) and (6.60) yield

$$c^2k + \dot{R}^2 = \frac{8\pi}{3}\rho\hat{G}\tilde{R}^2 \tag{6.61}$$

The $\mu = \nu = 1$, 2 and 3 components of the Einstein equations are identical:

$$c^2k + \dot{R}^2 + 2\tilde{R}\ddot{R} = -\frac{8\pi}{c^2}p\hat{G}\tilde{R}^2 \tag{6.62}$$

Eqs. (6.61) and (6.62) provide a possible basis for the exploration of the dynamics of the universe. The relevant equations can also be cast in other forms. The difference of Eqs. (6.61) and (6.62) is

$$2\tilde{R}\ddot{R} = -\frac{8\pi}{3}\hat{G}\tilde{R}^2(3p/c^2 + \rho) \tag{6.63}$$

Multiplication with $\dot{R}/\tilde{R}$ yields

$$2\dot{R}\ddot{R} = -\frac{8\pi}{3}\hat{G}\tilde{R}(3\dot{R}p/c^2 + \rho\dot{R}) \tag{6.64}$$

Taking the time derivative of Eq. (6.61) yields

$$2\dot{R}\ddot{R} = \frac{8\pi}{3}\hat{G}\tilde{R}(\tilde{R}\dot{\rho} + 2\rho\dot{R}) \tag{6.65}$$

Comparison with Eq. (6.64) shows that

$$\dot{\rho} + 3(p/c^2 + \rho)\frac{\dot{R}}{\tilde{R}} = 0 \tag{6.66}$$

6.4 Time evolution of the universe

Consider equation Eq. (6.63), which can be written more simply as

$$\frac{\ddot{R}}{\tilde{R}} = -\frac{4\pi}{3}\hat{G}(3p/c^2 + \rho) \tag{6.67}$$

This equation indicates that the universe cannot be stationary. There is an inward acceleration or deceleration. Now let us imagine the situation in the year 1917. The general theory of relativity was essentially complete and the theory was first being applied to cosmology. The Hubble expansion, was not yet discovered and the universe was believed to be stationary. Einstein therefore believed that something was wrong in the theory, and he attempted to remedy the problem by adding a term a with the 'cosmological constant' Λ to the Einstein equation

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi c^{-4}\hat{G}P^{\mu\nu} \tag{6.68}$$

which works against the gravitational attraction and thus might compensate the inward acceleration. After the discovery of the Hubble expansion, the belief in a stationary universe collapsed and, as there was no more obvious need for a nonzero cosmological constant, Einstein withdrew the modification of the theory. All observations were perfectly consistent with a decelerated expansion as described be Eq. (6.67). Further astronomical research thus aimed at more accurate determinations of the average density ρ so that it would become possible to determine the future fate of the universe. For sufficiently low densities, the universe would expand forever; at higher densities, it would collapse under its own gravitation.

The situation has again changed in recent years. Newer observations, which include brightness measurements of extremely distant supernovae and small irregularities in the distribution of the microwave background radiation, provide some evidence for an *accelerated* instead of a decelerated expansion. If true, there must be a nonzero cosmological term because an acceleration is inconsistent with Eq. (6.67), which implies $\ddot{R} < 0$. Nowadays it seems plausible that a nonzero 'dark energy' density could be explained by the 'vacuum energy' of relativistic quantum field theories for elementary particle physics. But nobody knows how to calculate the value of this background energy from first principles. Such a cosmological background energy should logically be included in the stress-energy tensor $P^{\mu\nu}$ of Eq. (6.59), which is thus redefined as

$$P^{\mu\nu} = P^{\mu\nu}(\text{matter}) + P^{\mu\nu}(\text{vacuum}) \tag{6.69}$$

with

$$P^{\mu\nu}(\text{vacuum}) = \frac{\Lambda c^4}{8\pi\hat{G}} g^{\mu\nu} \tag{6.70}$$

and $P^{\mu\nu}(\text{matter})$ represents all other contributions, *i.e.*, the stress-energy tensor as defined in Sec. 2.7, including the contributions from the energy and pressure of radiation. Including the cosmological term, Eq. (6.60) changes into

$$P^\mu_\nu = (\rho c^2 + p)\delta^0_\nu \delta^\mu_0 - p\delta^\mu_\nu + \frac{\Lambda c^4}{8\pi\hat{G}}\delta^\mu_\nu \tag{6.71}$$

and Eqs. (6.61) and (6.62) change into

$$c^2 k + \dot{R}^2 = \frac{8\pi c^{-2}\tilde{R}^2\hat{G}}{3}\left(\rho c^2 + \frac{\Lambda c^4}{8\pi\hat{G}}\right) \tag{6.72}$$

and

$$c^2 k + \dot{R}^2 + 2\tilde{R}\ddot{R} = 8\pi c^{-2}\tilde{R}^2\hat{G}\left(-p + \frac{\Lambda c^4}{8\pi\hat{G}}\right) \tag{6.73}$$

Via subtraction one obtains

$$\frac{\ddot{R}}{\tilde{R}} = -\frac{4\pi\hat{G}}{3}\left(\frac{3p}{c^2} + \rho\right) + \frac{1}{3}c^2\Lambda \tag{6.74}$$

which shows that an accelerated expansion is possible if $\Lambda > 0$.

From Eq. (6.72) we see that the parameter k which determines the sign of the curvature and thereby the topology of the universe, satisfies

$$k = \left(\frac{8\pi\hat{G}}{3}\rho + \frac{\Lambda c^2}{3} - \frac{\dot{R}^2}{\tilde{R}^2}\right)\tilde{R}^2 c^{-2} \tag{6.75}$$

This makes it, in principle, possible to determine the sign of k, which determines whether the universe is closed or open. Then, three parameters have to be determined. First, the matter density ρ can be determined from the dynamics of galaxies, including the so called 'dark matter', but excluding the 'dark energy' associated with Λ. Second, $\dot{R}/\tilde{R}$ is the cosmic expansion parameter, *i.e.*, the Hubble constant which is now reasonably well known from a comparison between the recession speeds of far-away galaxies and their distances. Third, the quantity that is most difficult to estimate is Λ, but rather recent observations of extremely distant supernovae, which therefore also probe a distant past, indicate that the expansion is accelerating. On this basis, there is no evidence that k differs significantly from 0, suggesting a flat universe according to Eq. (6.45), which ignores time dependences. The absence of curvature also agrees with the 'inflation' scenario for the infant universe.

The combination $k = 0$, $\Lambda > 0$ determines the eventual fate of the universe. It follows from Eq. (6.72) that

$$\dot{R}^2/\tilde{R}^2 \geq \Lambda \times \text{constant} \tag{6.76}$$

which implies an exponential growth as a function of time.

Chapter 7

Appendices

7.1 Appendix 1: Angular momentum and charge

The Schwarzschild solution describes a centrally symmetric situation, and thus corresponds with the gravitational field of an object with zero angular momentum. For a rotating object, the spherical symmetry is broken and only axial symmetry, *i.e.*, rotational symmetry about for instance the z axis or in the ϕ direction, remains. It is thus interesting to look at possible solutions of the Einstein equations that possess only axial symmetry. Such a solution was found by Kerr. Naturally it is somewhat more complicated than the Schwarzschild solution. Off-diagonal elements appear in the metric tensor due to the angular momentum of the rotating substance that acts as the source of the gravitational field. Furthermore, the Schwarzschild metric is also modified if the gravitating object is electrically charged.

7.1.1 Angular momentum and frame dragging

Before describing the actual Kerr metric we explore the influence of angular momentum on the metric in first order of the strength of the source, and therefore in first order in the deviations from the flat metric $\eta_{\alpha\beta}$. The source term $P^{\mu\nu}$, which was restricted to P^{00} in the Newtonian approximation in Sec. 5.1, will now include P^{j0} and P^{0j}, $j = 1, 2, 3$, due to the motion of the source, expressed by Eq. (2.50).

Making convenient use of the first-order analysis given in Sec. 5.6, we denote the perturbation of the metric tensor as $h_{\alpha\beta}$. Index raising and lowering of $h_{\alpha\beta}$, $R^{\alpha\beta}$ can thus be done with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$. In Sec. 5.6 we have seen that, after the choice of the coordinate conditions such as made following Eq. (5.66), the Ricci curvature tensor reduces to

$$R_{\beta\delta} = -\frac{1}{2}\eta^{\alpha\gamma}h_{\beta\delta,\alpha\gamma} \tag{7.1}$$

The term with $\alpha = \gamma = 0$ vanishes for a time-independent metric, so that

$$R_{\beta\delta} = \frac{1}{2}\nabla^2 h_{\beta\delta} \tag{7.2}$$

which is a generalization of a part of Eq. (5.22). The relevant elements of the Einstein tensor are G_{00} and G_{0k} with $k = 1$, 2, and 3. Since the steps leading to Eq. (5.13) remain valid in the present context, we still have $R^{00} = \frac{1}{2}G^{00}$. The off-diagonal elements satisfy $G_{0k} = R_{0k} - \frac{1}{2}h_{0k}R$, of which the second term is second order in the perturbation, and can therefore be neglected. The Einstein equations thus reduce to

$$R_{00} = \frac{1}{2}\nabla^2 h_{00} = 4\pi c^{-4}\hat{G}\rho u_0 u_0 = 4\pi\hat{G}\rho u^0 u^0 \tag{7.3}$$

and

$$R_{0k} = \frac{1}{2}\nabla^2 h_{0k} = 8\pi c^{-4}\hat{G}\rho u_0 u_k = -8\pi c^{-2}\hat{G}\rho u^0 u^k \tag{7.4}$$

For a given density field $\rho(\vec{r})$, $\vec{r} \equiv (x^1, x^2, x^3)$, and motion described by $u^\alpha(\vec{r})$, the solutions proceed as for the Poisson equation $\nabla^2\phi = -\rho/\epsilon_0$ in electrostatics, which is solved by $\phi(\vec{r}) = (4\pi\epsilon_0)^{-1}\int d\vec{r'}\rho(\vec{r'})/|\vec{r} - \vec{r'}|$. Thus, at a position $\vec{r}$ one finds h_{00} as

$$h_{00}(\vec{r}) = -2\hat{G}\int d\vec{r'}\,\frac{\rho(\vec{r'})u^0(\vec{r'})u^0(\vec{r'})}{|\vec{r} - \vec{r'}\,|}\,. \tag{7.5}$$

For small speeds, where one has $u^0 \approx 1$, and for a spherical mass distribution, Eq. (7.5) reduces to $h_{00}(r) = -2\hat{G}M/r$ where $M = \int d\vec{r}\rho(\vec{r})$ is the gravitating mass. For relativistic speeds, u^0 differs significantly from 1 and the gravitating mass increases due to the contribution of the kinetic energy.

The elements of the metric with one time-like index follow as

$$h_{0k}(\vec{r}) = \frac{4\hat{G}}{c^2}\int d\vec{r'}\,\frac{\rho(\vec{r'})u^0(\vec{r'})u^k(\vec{r'})}{|\vec{r} - \vec{r'}\,|}\,. \tag{7.6}$$

To expose the physical meaning of Eq. (7.6), we consider an object rotating with angular velocity ω about an axis with orientation $\vec{\omega}$. We restrict the speeds to be small in comparison with c, so that the velocity field is $v^k(\vec{r}) = u^k(\vec{r}) = \epsilon_{klm}\omega^l x^m$, where $\epsilon_{klm} = 1$ (-1) if klm is an even (odd) permutation of 123, and $\epsilon_{klm} = 0$ otherwise. Next we expand the inverse numerator of Eq. (7.6) as

$$|\vec{r} - \vec{r'}\,|^{-1} = (\vec{r}.\vec{r} - 2\vec{r}.\vec{r'} + \vec{r'}.\vec{r'})^{-1/2} \approx r^{-1}(1 + \vec{r}.\vec{r'}/r^2 + \ldots) \tag{7.7}$$

where the rightmost form neglects contributions decaying faster than $1/r^2$. Substitution of this expression and of $u^k(\vec{r})$ in Eq. (7.6) yields

$$h_{0k}(\vec{r}) = \frac{4\hat{G}\epsilon_{klm}\omega^l}{c^2}\int d\vec{r'}\,\frac{\rho(\vec{r'})x'^{\,m}[1 + (x_n x'^{\,n})/r^2]}{r}\,. \tag{7.8}$$

For mass distributions $\rho(\vec{r'})$ that are symmetric with respect to the three spatial directions, only even powers of the components of $\vec{r'}$, *i.e.*, only the terms with $n = m$ survive the integration:

$$h_{0k}(\vec{r}) = \sum_{m=1}^{3}\frac{4\hat{G}\epsilon_{klm}\omega^l x^m}{c^2 r^3}\int d\vec{r'}\,\rho(\vec{r'})(x'^{\,m})^2\,, \tag{7.9}$$

where the sum on m is written explicitly. For simplicity we choose the x^3 axis along $\vec{\omega}$, and consider a mass distribution that is rotationally symmetric about the x^3 axis.

Then the term with $m = 3$ in the sum vanishes, and for $m = 1$ the integral is the same as for $m = 2$, so that

$$h_{0k}(\vec{r}) = \frac{2\hat{G}\epsilon_{k3m}\omega^3 x^m}{c^2 r^3} \int d\vec{r'}\, \rho(\vec{r'})[(x'^{\,1})^2 + (x'^{\,2})^2]\,. \tag{7.10}$$

Here one recognizes that the integral is equal to the moment of inertia I_z about the $x^3 = z$ axis, and we identify $J = J^3 \equiv \omega^3 I_z$ as the angular momentum of the rotating mass distribution. Thus

$$h_{0k}(\vec{r}) = 2\hat{G}\epsilon_{k3m} J x^m/(c^2 r^3)\,. \tag{7.11}$$

The line element for the case $x^3 = 0$ reduces to

$$ds^2 = (\text{diagonal contributions}) + \frac{4\hat{G}J}{c^2 r^3}\,(x^2 dx^1 - x^1 dx^2) dt\,, \tag{7.12}$$

where we used that $g_{0k} = g_{k0}$. Instead of Cartesian coordinates we may use polar coordinates, so that the part between parentheses becomes $r^2 d\phi$, which leads to

$$ds^2 = (\text{diagonal contributions}) + \frac{4\hat{G}J}{c^2 r}\, d\phi dt\,. \tag{7.13}$$

The term proportional to $dtd\phi$ represents the 'frame dragging' phenomenon. It means that, in our coordinate system, the speed of light in the $-\phi$ direction differs from that in the $+\phi$ direction. Under terrestrial circumstances, the effect is extremely small. It can still be detected experimentally by means of very precise measurements of a small precession effect of an orbiting gyroscope. The observation of the frame-dragging effect was achieved by the Gravity Probe B project, although the result was less accurate than originally expected. Another verification of the frame-dragging effect has been reported on the basis of the precession of satellite orbits. The numerical analysis of these orbits is complicated by inhomogeneities in the mass distribution of the Earth.

7.1.2 Description of the Kerr metric

We skip the actual solution procedure, which is beyond the scope of this introductory text, and only list the result. Also in the rest of this Appendix we do not aim at complete explanations, but we merely wish to provide some information about the fascinating properties of the metric around rotating and/or charged objects.

The squared line element according to the Kerr metric, which pertains to a rotating, uncharged source, can be written in different forms, by application of coordinate transformations. Here we present it in terms of the so-called Boyer-Lindquist coordinates as

$$\begin{aligned} ds^2 = &\left(1 - \frac{2\hat{G}Mr}{c^2 b^2}\right) c^2 dt^2 \;-\; \frac{b^2}{q^2} dr^2 - b^2 d\theta^2 \\ &- \sin^2\theta \left[r^2 + a^2 + 2\,\frac{a^2 \sin^2\theta}{c^2 b^2}\hat{G}Mr\right] d\phi^2 + 4\,\frac{a \sin^2\theta}{c b^2}\hat{G}Mr dt d\phi \end{aligned} \tag{7.14}$$

where

$$b^2 = r^2 + a^2 \cos^2\theta \quad \text{and} \quad q^2 = r^2 + a^2 - 2\hat{G}Mr/c^2 \tag{7.15}$$

The physical meaning of the constants M and a will be specified later. The nonzero covariant and contravariant elements of the metric are

$$\begin{array}{ll}
g_{00} = (1 - \frac{2\hat{G}Mr}{c^2b^2})c^2 & g^{00} = \frac{1}{b^2q^2}\left[(r^2+a^2)^2 - q^2a^2\sin^2\theta\right]c^{-2} \\
g_{11} = -\frac{b^2}{q^2} & g^{11} = -\frac{q^2}{b^2} \\
g_{22} = -b^2 & g^{22} = -\frac{1}{b^2} \\
g_{33} = -(r^2 + a^2 + 2\,\frac{a^2\sin^2\theta}{c^2b^2}\hat{G}Mr)\sin^2\theta & g^{33} = -\frac{c^2b^2 - 2\hat{G}Mr}{c^2q^2b^2\sin^2\theta} \\
g_{03} = g_{30} = 2\,\frac{a\sin^2\theta}{cb^2}\hat{G}Mr & g^{03} = g^{30} = 2\frac{a\hat{G}Mr}{cq^2b^2}
\end{array} \tag{7.16}$$

The covariant elements are directly taken from Eq. (7.14). The contravariant elements follow by inversion of the matrix $g_{\mu\nu}$. It requires some trivial algebra to check that the elements given in Eq. (7.16) satisfy $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$.

For sufficiently large radial parameters r, contributions with with relative measure a^2/r^2 with respect to the leading ones become negligible. The metric including contributions up to first order in a/r is

$$ds^2 \simeq \left(1 - \frac{2\hat{G}M}{c^2r}\right)c^2dt^2 - \left(1 - \frac{2\hat{G}M}{c^2r}\right)^{-1}dr^2$$

$$-r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{4a\sin^2\theta}{cr}\hat{G}Mdtd\phi \tag{7.17}$$

A comparison with the Schwarzschild metric as given by Eq. (5.44) shows that M is the mass that generates the gravitational field, and that, for large distances away from the gravitating object, the leading deviation from the Schwarzschild metric is the term $2g_{03}dtd\phi$. This term does not have spherical, but only axial symmetry and has the same form as the $dtd\phi$ term in Eq. (7.13). A comparison of the amplitudes of these frame-dragging terms shows that $a = J/Mc$, where J is the angular momentum of the gravitating object.

For a gravitationally collapsed object (black hole), we can identify the locus of infinite redshift, *i.e.*, $g_{00} = 0$, for light emitted by a source at rest in the Kerr coordinates. This occurs in Eq. (7.14) for

$$2r\hat{G}M = c^2b^2 \quad \text{or} \quad r = \frac{\hat{G}M}{c^2} \pm \sqrt{\left(\frac{\hat{G}M}{c^2}\right)^2 - a^2\cos^2\theta} \tag{7.18}$$

This equation describes two different surfaces. The shape of the outer surface (with the + sign) resembles an oblate spheroid. As expected, this surface reduces to a sphere with the Schwarzschild value $r = 2\hat{G}M/c^2$ for $a \to 0$. In this limit, the inner surface reduces to the central singularity of the Schwarzschild metric. In the space between the two surfaces one has, just as in the Schwarzschild metric, $g_{00} < 0$, which means that the time-like coordinate x^0 becomes space-like. However, the proper time continues to exist for an infalling object, which is not at rest, but has a superluminal speed. The existence of the region inside the inner surface, with $g_{00} > 0$ again, is a new phenomenon in comparison with the Schwarzschild metric. Furthermore, b vanishes for $r \to 0$ and $\theta \to \pi/2$, corresponding to a highly singular metric at that point. By means of a singular coordinate transformation, this singularity can be repositioned to a nonzero value of the radial coordinate. The singularity then assumes the form of a circle where the spatial curvature diverges.

For nonzero angular momentum ($a > 0$) the frame-dragging effect becomes so large at the outer surface of infinite redshift that objects cannot be stationary in the ϕ direction. Only light can be standing still in one of the ϕ directions.

The surface of infinite redshift described by Eq. (7.18) apllies only to stationary objects, and does therefore not correspond with the event horizon. Objects that are corotating with the black hole can have a less than infinite redshift even within this surface. But inspection of Eq. (7.14) shows that something special occurs at $q = 0$ or

$$r = \hat{G}M/c^2 \pm \sqrt{\left(\frac{\hat{G}M}{c^2}\right)^2 - a^2} \tag{7.19}$$

where the metric becomes singular. This singularity can be transformed away, but the outer surface described by Eq. (7.19) with the + sign remains special in the sense that, in the Kerr coordinates, a free falling object takes an infinite length of time to reach that surface. This is the same situation as for the event horizon of the Schwarzschild metric described in Sec. 5.3. For this reason, the outer surface of Eq. (7.19) is called the event horizon of the Kerr metric. For $a \neq 0$ and $\cos\theta < 1$ it lies inside the infinite redshift surface of Eq. (7.18). The region between the two surfaces is called the *ergosphere*. It is the stage of special phenomena such as the 'Penrose process' and negative-energy orbits. Negative energy here means the energy *including* the mass term mc^2. The Penrose process, which should in the present stage be considered a thought experiment, makes use of this phenomenon: an incoming object enters into the ergosphere and splits up in two parts with significantly different velocities, such that one part enters a negative-energy orbit. This part falls toward the event horizon and is thus captured by the black hole, and contributes a negative energy to it, and decreases its angular momentum. The remaining part of the object escapes from the ergosphere, with an increased mass-energy content in comparison with the original object.

7.1.3 Other exact solutions

The metric around a spherically symmetric object having an electrical charge Q was exactly solved by Reissner and Nordström, and can be described by

$$ds^2 = \left(1 - \frac{r_\mathrm{S}}{r} + \frac{r_Q^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{r_\mathrm{S}}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{7.20}$$

where $r_\mathrm{S} = 2\hat{G}M/c^2$ is the known formula for the Schwarzschild radius, and

$$r_Q \equiv \frac{Q^2 \hat{G}}{4\pi c^4 \epsilon_0} \tag{7.21}$$

A strange phenomenon is that g_{00} vanishes at *two* values of r, so that there is an inner as well as an outer horizon. However, the amount of electrical charge of existing black holes, and thereby r_Q is likely to be very small.

Even more structure is contained in the Kerr-Newman solution describing the case of a rotating, charged body. This metric can be expressed as

$$ds^2 = \frac{q^2}{b^2}(cdt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{b^2}[(r^2+a^2)d\phi - acdt]^2 - b^2\left(\frac{dr^2}{q^2} + d\theta^2\right) \tag{7.22}$$

with $a \equiv J/Mc$, $b^2 \equiv r^2 + a^2\cos^2\theta$, and $q^2 \equiv r^2 - r_{\mathrm{S}}r + a^2 + r_Q^2$. This metric has the Reissner-Nordström metric, the Kerr metric, and thus also the Schwarzschild metric as appropriate limiting cases. While the relevance of electric charges for the metric in the context of observational astronomy seems limited, rotating and charged black holes play a role in string theory and applications of the AdS-CFT (Anti de Sitter-Conformal Field Theory) correspondence. It was recently shown that this correspondence can be used to describe and predict properties of quantum many-body systems.

7.2 Appendix 2: The equations of motion

7.2.1 Derivation from the Einstein equation

In Section 3.4 we derived the equations of motion from the assumption that a free falling particle follows a geodesic. After the introduction of the Einstein equation such an assumption is no longer necessary. We derive the equation of motion using the language of continuum mechanics in a general frame of reference.

As a consequence of the contracted Bianchi identity Eq. (4.33), the Einstein tensor satisfies $G^{\mu\nu}{}_{;\nu} = 0$. Therefore, the Einstein equation implies that also $P^{\mu\nu}{}_{;\nu} = 0$ which is a generalization of Eq. (2.52) with zero external forces. The latter equation applies only to Lorentz frames. There it expresses the law of conservation of momentum and energy. In general coordinates it becomes less obvious what conservation laws are being expressed. Gravitational forces are absent in general relativity, but acceleration due to deviations from Minkowski metric exist. Thus, in general coordinates, the corresponding external forces are absent: it is the inertial force that accounts for the acceleration. Therefore the vanishing of the covariant divergence of $P^{\mu\nu}$ must be interpreted as the equation of motion.

We demonstrate this for a single particle with mass m and spacetime coordinate y^α which is parametrized by the time-like invariant length s as determined by the metric. We thus have to express the continuum language for $P^{\mu\nu}$ in terms of the rest-mass density distribution $\rho x^\alpha = m\delta(x^\alpha - y^\alpha(s))$. A complication is that the delta function is not generally covariant, but transforms as an inverse volume. It can however be shown that $\delta(x^\alpha - y^\alpha(s))/\sqrt{-|g|}$ is a scalar, where $|g|$ is the determinant of the covariant metric tensor. Thus the energy-momentum tensor at position x^α due to the presence of the particle is

$$P^{\mu\nu}(x^\alpha) = \frac{m}{\sqrt{-|g|}} \int u^\mu u^\nu \delta(x^\alpha - y^\alpha(s)) ds \tag{7.23}$$

where $u^\mu = dy^\mu/ds$, and the integral is taken over the trajectory $y^\alpha(s)$ of the particle. The path is thus parametrized by s but later we shall use the freedom to choose another parametrization. The covariant divergence of this expression, namely

$$P^{\mu\nu}{}_{;\nu} = P^{\mu\nu}{}_{,\nu} + \Gamma^\nu_{\tau\nu} P^{\mu\tau} + \Gamma^\mu_{\tau\nu} P^{\nu\tau} = 0 \tag{7.24}$$

should thus vanish. From the definition of the affine connection one has

$$\Gamma^\nu_{\tau\nu} = \frac{1}{2} g^{\nu\sigma} g_{\nu\sigma,\tau} \tag{7.25}$$

Here we have to apply some elementary linear algebra. Consider a matrix $A_{\mu\nu}$ whose inverse is denoted $A^{\mu\nu}$. The elements of the inverse matrix are

$$A^{\mu\nu} = s(\mu\nu)/|A| \tag{7.26}$$

where $|A|$ is the determinant of $A_{\mu\nu}$, and $s(\mu\nu)$ is the subdeterminant of $A_{\mu\nu}$, *i.e.*, the determinant of the submatrix that remains when the μth row and the νth column of

$A_{\mu\nu}$ are erased, times a factor ± 1 equal to the signature of the permutation $\mu \leftrightarrow \nu$. The expansion of the determinant $|A|$ in column ν of $A_{\mu\nu}$ leads to

$$|A| = \sum_{\mu} A_{\mu\nu} s(\mu\nu) \tag{7.27}$$

where no dummy index summation on ν is implied. It follows that

$$\frac{\partial |A|}{\partial A_{\mu\nu}} = s(\mu\nu) = |A| A^{\mu\nu} \quad \text{or} \quad \frac{\partial}{\partial A_{\mu\nu}} \ln ||A|| = s(\mu\nu) = A^{\mu\nu} \tag{7.28}$$

We added absolute-value signs because the determinant may be negative. Application of these formulas with g instead of A shows that the gradient of $\ln(-|g|)$ is equal to the contracted Christoffel symbol:

$$\frac{\partial}{\partial x^\tau} \ln(-|g|) = \frac{\partial g_{\mu\nu}}{\partial x^\tau} \frac{\partial}{\partial g_{\mu\nu}} \ln(-|g|) = \frac{\partial g_{\mu\nu}}{\partial x^\tau} g^{\mu\nu} = \Gamma^\nu_{\tau\nu} \tag{7.29}$$

so that Eq. (7.25) can be extended to

$$\Gamma^\nu_{\tau\nu} = \frac{1}{2} g^{\nu\sigma} g_{\nu\sigma,\tau} = \frac{1}{2} \frac{\partial}{\partial x^\tau} \ln(-|g|) = \frac{1}{\sqrt{-|g|}} \frac{\partial}{\partial x^\tau} \sqrt{-|g|} \tag{7.30}$$

Combination of this result with Eq. (7.24) yields

$$P^{\mu\nu}_{\ ;\nu} = \Gamma^\mu_{\tau\nu} P^{\nu\tau} + \frac{1}{\sqrt{-|g|}} \frac{\partial}{\partial x^\tau} (\sqrt{-|g|} P^{\mu\tau}) = 0 \tag{7.31}$$

Substitution of Eq. (7.23) in this equation leads, after dividing out a factor $m/\sqrt{-|g|}$, to

$$\int u^\mu u^\nu \frac{\partial}{\partial x^\nu} \delta(x^\alpha - y^\alpha(s)) ds + \Gamma^\mu_{\tau\nu} \int u^\nu u^\tau \delta(x^\alpha - y^\alpha(s)) ds = 0 \tag{7.32}$$

The singular character of this equation is due to the representation of the test particle as a point, *i.e.*, a delta-function contribution to $P^{\mu\nu}$. Due to the singular form of the energy-momentum distribution, we have to integrate over the derivative of the delta function. This is doable if we realize that a delta function is just a function with a large value in a small interval, and its derivative is just the difference of two properly upscaled delta functions separated by an accordingly downscaled interval. Furthermore, the delta function $\delta(x^\alpha - y^\alpha)$ is actually the product of four separate delta functions $\delta(x^0 - y^0)\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3)$. We rewrite the term with $\nu = 0$ in the implied sum in the first integral of Eq. (7.32) as follows

$$\int u^\mu u^0 \delta(x^a - y^a(s)) \frac{\partial}{\partial x^0} \delta(x^0 - y^0(s)) ds = c \int u^\mu \delta(x^a - y^a) \frac{\partial}{\partial x^0} \delta(x^0 - y^0) dy^0 =$$
$$- c \int u^\mu \delta(x^a - y^a) \frac{\partial}{\partial y^0} \delta(x^0 - y^0) dy^0 = c \frac{d}{dx^0} (u^\mu \delta(x^a - y^a)) \tag{7.33}$$

Here we used the abbreviation $\delta(x^a - y^a) \equiv \delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3)$ for the three space-like factors of the delta function $\delta(x^\alpha - y^\alpha)$. The last integral can be executed because the integral of a function times the derivative of the delta function is minus the derivative of the function. Next, we rewrite the term with $\nu = 1$ in the first integral of Eq. (7.32) as

$$
\begin{aligned}
\int u^\mu u^1 \delta(x^0 - y^0(s))\delta(x^2 - y^2(s))\delta(x^3 - y^3(s))\frac{\partial}{\partial x^1}\delta(x^1 - y^1(s))ds = \\
- c\int u^\mu \delta(x^0 - y^0)\delta(x^2 - y^2)\delta(x^3 - y^3)\frac{\partial}{\partial y^1}\delta(x^1 - y^1)dy^1 = \\
- c\int u^\mu \delta(x^0 - y^0)\frac{dy^1}{dy^0}\frac{\partial}{\partial y^1}\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3)dy^0
\end{aligned}
\tag{7.34}
$$

where the path is now understood to be parametrized by y^0. The other space-like values $\nu = 2, 3$ are rewritten similarly. The sum of all three is

$$
\begin{aligned}
-c\int u^\mu \delta(x^0 - y^0)\frac{dy^a}{dy^0}\frac{\partial}{\partial y^a}\delta(x^a - y^a)dy^0 = \\
- c\int u^\mu \delta(x^0 - y^0)\frac{d}{dy^0}\delta(x^a - y^a)dy^0 = -cu^\mu \frac{d}{dx^0}\delta(x^a - y^a)
\end{aligned}
\tag{7.35}
$$

The sum of Eqs. (7.33) and (7.35) is

$$
\begin{aligned}
c\frac{du^\mu}{dx^0}\delta(x^a - y^a(x^0)) = c\int \frac{du^\mu}{dx^0}\delta(x^a - y^a(x^0))\delta(x^0 - y^0(x^0))dx^0 \\
= c\int \frac{du^\mu}{ds}\delta(x^\alpha - y^\alpha(s))ds
\end{aligned}
\tag{7.36}
$$

Substitution in Eq. (7.32) and division by c leads to

$$
\int \left(\frac{du^\mu}{ds} + \frac{1}{c}\Gamma^\mu_{\tau\nu}u^\nu u^\tau \right) \delta(x^\alpha - y^\alpha(s))ds = 0 \tag{7.37}
$$

The equation is, of course, singular at the position of the particle in spacetime. For every point along the trajectory of the particle, the geodesic equation, and thus the equation of motion, Eq. (3.29), are satisfied. The field equations themselves determine the path of the singularity associated with the particle.

7.2.2 Numerical solution of the equation of motion

In many cases, the metric and its derivatives as represented by the Christoffel symbols are too complicated for an analytic solution of Eq. (3.29). In principle it is, however, simple to obtain a numerical solution on the basis of a given metric, and initial conditions x^μ and u^μ. To this purpose we divide the path in sufficiently small intervals Δs and linearize the dependence of u^μ on s. With the help of Eq. (3.29), this yields the recursion for the vector u^μ as

$$
u^\mu(s + \Delta s) \approx u^\mu(s) + \frac{du^\mu}{ds}\Delta s = u^\mu(s) - \frac{\Delta s}{c}\,\Gamma^\mu_{\sigma\nu}u^\sigma u^\nu\,. \tag{7.38}
$$

Similarly, the recursion for the coordinates is

$$
x^\mu(s + \Delta s) \approx x^\mu(s) + \frac{dx^\mu}{ds}\Delta s = x^\mu(s) + \frac{\Delta s}{c}u^\mu \tag{7.39}
$$

Starting from initial conditions $x^\mu(s_0), u^\mu(s_0)$, repeated application of these recursions yields the evolution of u^μ and x^μ along the path parametrized by s. This includes the coordinate time x^0 so that the evolution in time follows as well.

7.2.3 Motion as a function of coordinate time

The path $x^i(x^0)$ of a particle along a geodesic may be obtained by elimination of s from the solution $x^\mu(s)$ of Eq. (3.29), or by solving the equation that is obtained by eliminating ds in Eq. (3.29). To this purpose, we rewrite Eq. (3.7) as

$$ds^2 = g_{\mu\nu}\,\frac{dx^\mu}{dt}\,\frac{dx^\nu}{dt}\,dt^2 \tag{7.40}$$

where $t \equiv x^0$ is the coordinate time. The derivatives to t are taken along the path of the particle. This equation determines the ratio of the infinitesimal increments ds and dt along the path. We may thus replace the derivatives to s according to the chain rule $d/ds = (dt/ds)d/dt$, where

$$\frac{dt}{ds} = \left(g_{\alpha\beta}\,\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1/2} \tag{7.41}$$

so that Eq. (3.29) becomes

$$\frac{du^\mu}{dt}\left(g_{\alpha\beta}\,\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1/2} + \frac{1}{c}\,\Gamma^\mu_{\sigma\nu}u^\sigma u^\nu = 0\,. \tag{7.42}$$

The 4-vector u^μ still depends implicitly on s since $u^\mu = dx^\mu/d\tau = c dx^\mu/ds$. This dependence is eliminated by the substitution

$$u^\mu = c\,\frac{dx^\mu}{dt}\frac{dt}{ds} \tag{7.43}$$

which yields

$$\left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1/2}\frac{d}{dt}\left(\frac{dx^\mu}{dt}\frac{dt}{ds}\right) + \left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1}\Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{dt}\,\frac{dx^\nu}{dt} = 0\,. \tag{7.44}$$

Evaluation of the derivative leads to

$$\frac{d^2x^\mu}{dt^2}\left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1} - \frac{1}{2}\left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-2}\frac{dx^\mu}{dt}\frac{d}{dt}\left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right) +$$
$$\left(g_{\alpha\beta}\,\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1}\Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{dt}\frac{dx^\nu}{dt} = 0\,. \tag{7.45}$$

Dividing out factors and taking into account the implicit dependence of the metric on time t, one finds

$$\frac{d^2x^\mu}{dt^2} - \frac{dx^\mu}{dt}\left(g_{\alpha\beta}\,\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)^{-1}\left(\frac{1}{2}\,g_{\alpha\beta,\gamma}\,\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} + g_{\alpha\beta}\,\frac{d^2x^\alpha}{dt^2}\frac{dx^\beta}{dt}\right)$$
$$+\Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{dt}\frac{dx^\nu}{dt} = 0\,. \tag{7.46}$$

In some simple cases, depending on the metric tensor and the affine connection as a function of the coordinates, Eq. (7.46) may be solved analytically.

7.3 Appendix 3: Wormholes

Consider a hypothetical spacetime metric given by the squared line element

$$ds^2 = c^2 dt^2 - dr^2 - (r^2 + b^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.47}$$

without, for the present, considering the Einstein equation. The parameter b is a positive constant. Far away from the origin ($r = 0$) we have $b^2 + r^2 \simeq (r + b/2r)^2$ and thus the metric becomes, at large distances, asymptotically equal to that of flat space, expressed in spherical coordinates. But this holds for $r >> 0$ as well as for $r << 0$. Apparently there are *two* asymptotically flat spaces, whose metric is approximately described by spherical coordinates. The two parts are connected in a curved region $r^2 \lesssim b^2$. To analyze the metric in more detail, we construct an intersection at t =constant, $\theta = \pi/2$:

$$dl^2 = dr^2 + (r^2 + b^2) d\phi^2 \tag{7.48}$$

This is a two-dimensional metric. We shall construct a visualization of it by a surface in three-dimensional space. The three-dimensional space can as usual be parametrized by Cartesian axes (x, y, z). But here, in line with the axial symmetry of the problem, we use (w, ϕ, z) where w and ϕ are polar coordinates parametrizing the (x, y) plane:

$$\begin{aligned} x &= w\cos\phi \\ y &= w\sin\phi \end{aligned}$$

and we search for the functions $z(r)$ and $w(r)$ that describe the surface. The line element, expressed in (w, ϕ, z), is

$$dl^2 = dz^2 + dw^2 + w^2 d\phi^2 \tag{7.49}$$

Substitution of z and w as functions of r leads to

$$dl^2 = \left[\left(\frac{dz}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2\right] dr^2 + w^2 d\phi^2 \tag{7.50}$$

which is the same as Eq. (7.48) if

$$\left(\frac{dz}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2 = 1 \quad \text{and} \quad w^2 = b^2 + r^2 \tag{7.51}$$

From the second equation one finds that

$$\left(\frac{dw}{dr}\right)^2 = \frac{r^2}{b^2 + r^2} \tag{7.52}$$

and substitution in the first condition of Eq. (7.51) yields

$$\frac{dz}{dr} = \pm \frac{1}{\sqrt{1 + b^2/r^2}} \tag{7.53}$$

which can be integrated to

$$\frac{z}{b} = \pm \operatorname{arcsinh}\left(\frac{r}{b}\right) \quad \text{or} \quad \frac{r}{b} = \pm \sinh\left(\frac{z}{b}\right) \tag{7.54}$$

Substitution in the second part of Eq. (7.51) yields

$$\frac{w}{b} = \pm \cosh\left(\frac{z}{b}\right) \quad \text{or} \quad \frac{z}{b} = \pm \operatorname{arccosh}\left(\frac{w}{b}\right) \tag{7.55}$$

As expected, the shape of the curve does not depend on the third variable ϕ. By means of a rotation of this curve about the z axis, one obtains a curved plane whose form can be described in terms of a wormhole-type connection between two asymptotically flat sheets. This situation, while described for a 2-dimensional intersection of Eq. (7.47), provides a quite general picture in the sense that it applies to all t. Moreover, the spherical symmetry of the metric of Eq. (7.47) in the coordinates (θ, ϕ) implies that the picture does not depend on the orientation $\theta = \pi/2$ of the intersection that we have chosen.

Formally, one can construct a metric in which the wormhole connects a spacetime to itself, but with an additional 'shift' in the space and / or time directions. The existence of such wormholes would lead to severe paradoxes concerning causality, and the question remains if such wormholes can possibly exist.

Therefore, let us finally address the question if such a wormhole geometry is consistent with the Einstein equation. A calculation in reverse direction yields a stress-energy tensor P^{μ}_{ν} with a *negative* energy. This situation is only known in short-lived quantum fluctuations. The prospects of finding wormholes of a size exceeding the Planck length $\sqrt{\hat{G}\hbar/c^3} \approx 10^{-35}$ [m] are therefore not favorable.

7.4 Appendix 4: Dimensions

For actual calculations it is highly desirable to keep track of the physical units of the relevant observables and constants. For some quantities they are given below in mks units: meter, kilogram and second. In agreement with the foregoing treatments of e.g. the Schwarzschild and the Robertson-Walker metric, we use the notation that x^0 is a time variable, and x^1 a spatial variable measuring a distance (not an angle).

quantity	units
x^0	[sec]
x^1	[m]
x_0	[$\mathrm{m}^2\mathrm{sec}^{-1}$]
x_1	[m]
s	[m]
c	[$\mathrm{m\,sec}^{-1}$]
u^0	[1]
u^1	[$\mathrm{m\,sec}^{-1}$]
g^{00}	[$\mathrm{m}^{-2}\mathrm{sec}^2$]
g^{01}	[$\mathrm{m}^{-1}\mathrm{sec}$]
g^{11}	[1]
g_{00}	[$\mathrm{m}^2\mathrm{sec}^{-2}$]
g_{01}	[$\mathrm{m\,sec}^{-1}$]
g_{11}	[1]
Γ^0_{00}	[sec^{-1}]
Γ^1_{00}	[$\mathrm{m\,sec}^{-2}$]
Γ^0_{10}	[m^{-1}]
Γ^0_{11}	[$\mathrm{m}^{-2}\mathrm{sec}$]
Γ^1_{11}	[m^{-1}]
R_{0000}	[$\mathrm{m}^2\mathrm{sec}^{-4}$]
R_{00}	[sec^{-2}]
R	[m^{-2}]
R^{00}	[$\mathrm{m}^{-4}\mathrm{sec}^2$]
P^{00}	[$\mathrm{kg\,m}^{-3}$]
$\hat{G}$	[$\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{sec}^{-2}$]
$\hat{G}P^{00}$	[sec^{-2}]
$c^{-4}\hat{G}P^{00}$	[$\mathrm{m}^{-4}\mathrm{sec}^2$]
$\tilde{R}$	[m]
Λ	[m^{-2}]