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1. **Violation of causality in $1 + 1$ dimensions**

   In the lecture notes it is shown that in $3 + 1$ dimensions the Hamiltonian $H = \sqrt{m^2c^4 + \vec{p}^2c^2}$, where $\vec{p} = -i\hbar\vec{\nabla}$, gives rise to violation of causality. In this exercise we will conclude that this is not a special property of this dimension, by considering the $1 + 1$ dimensional case.

a. Give the exact plane wave solutions of Schrödinger's equation for the Hamiltonian $H = \sqrt{m^2c^4 + p^2c^4}$.

b. Let $\psi_0(x, t)$ be the solution of Schrödinger's equation with initial condition $\psi_0(x, 0) = \delta(x)$. It follows that $\psi(x, t) = \int dy f(y)\psi_0(x - y, t)$ is also a solution (for which initial condition?). Therefore it is sufficient to study the time evolution of $\psi_0$.

   Fourier expand $\psi_0(x, t)$ and rewrite this expression as

$$\psi_0(x,t) = \frac{i}{c\pi}\partial_t K_0(z), \quad \text{with } K_0(z) = \int_0^\infty dy \cos(z\sinh(y)), \quad z^2 = \frac{m^2c^2}{\hbar^2}(x^2 - c^2t^2).$$

   ($K_0$ is a modified Bessel function, whence the above expression can be rewritten in terms of ordinary Bessel functions. See for example Abromowitz & Stegun's handbook for details.)

c. Show that for $m \neq 0$ the solution violates causality.

d. Prove that $\mathrm{Re}(\psi_0) = (\psi_0 + \psi_0^*)/2$ respects causality, but does not satisfy Schrödinger's equation. Show that $\psi_0^*$ is a solution of the time inverted Schrödinger equation, or equivalently Schrödinger's equation with opposite (negative) Hamiltonian.

   Prove that both $\psi_0$ and $\psi_0^*$ satisfy the Klein-Gordon equation, and that $\mathrm{Re}(\psi_0)$ is the <u>unique</u> solution that respects causality.

2. **Casimir effect**

   In quantum field theory the vacuum energy depends on the spatial volume. In the lecture notes it has been derived that a free scalar massless field which is spatially contained between two infinite parallel planes with separation $x$, has an energy per unit area

$$E(x) = \frac{1}{2(2\pi\hbar c)^2}\sum_{m=1}^{\infty}\int d^2k\sqrt{k^2 + (\frac{\pi\hbar cm}{x})^2}.$$

   This expression is divergent, but can be made finite in a sensible way by subtraction of a corresponding slice in infinite volume, i.e. without boundary conditions in the $x$ direction. An alternative way of getting rid of the unphysical infinite part, is so-called dimensional regularisation. The above integral (the sum will be attacked analogously later) falls into a class of integrals that is parametrised by (a.o.) the dimension. The method then consists of computing the convergent integrals within this class, and redefine the divergent ones by analytic continuation (in the set of parameters) of the convergent outcomes.

   For the case at hand the following class of integrals is useful

$$I_{n,\lambda,\mu}(\alpha) \equiv \int d^n k \frac{k^{2\lambda}}{(k^2 + \alpha^2)^\mu} \qquad (n \in \mathbb{N};\ \lambda, \mu \in \mathbb{C};\ \ \alpha^2 > 0).$$

For $2\mathrm{Re}(\lambda-\mu)+n<0$ this is convergent (and analytic). For $n=2, \lambda=0, \mu=-1/2$ it reduces to the integral in $E(x)$. Assume for the time being that $2\mathrm{Re}(\lambda-\mu)+n<0$.

a. Change to spherical coordinates and derive

$$I_{n,\lambda,\mu}(\alpha)=\pi B(1,\frac{1}{2})B(\frac{3}{2},\frac{1}{2})\cdots B(\frac{n-1}{2},\frac{1}{2})B(\lambda+\frac{n}{2},\mu-\lambda-\frac{n}{2})\alpha^{n+2\lambda-2\mu},$$

where $B$ is the so-called beta function:

$$B(x,y)\equiv 2\int_0^\infty dt\; t^{2x-1}(1+t^2)^{-x-y}.$$

b. Let the gamma function be defined as

$$\Gamma(x)=\int_0^\infty dt\; t^{x-1}e^{-t} \quad (\mathrm{Re}(x)>0).$$

Show that $\Gamma(x+1)=x\Gamma(x)$ and $\Gamma(1/2)=\sqrt{\pi}$. Also prove

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

c. Now write $I_{n,\lambda,\mu}(\alpha)$ in terms of gamma functions. Note that $\Gamma(x)$ can be continued analytically to $\mathrm{Re}(x)\leq 0$ using $\Gamma(x+1)=x\Gamma(x)$. Therefore we can also continue $I$ analytically to parameter values for which the original integral was divergent. Also notice that the dimension $n$ can now be given arbitrary complex values without difficulties. Show that after having done these regularisations we obtain

$$E(x)=E_2(x), \quad E_n(x)=\sqrt{\pi}\hbar c\sum_{m=1}^{\infty}(\frac{\sqrt{\pi}m}{2x})^{n+1}\frac{\Gamma(-(n+1)/2)}{\Gamma(-1/2)}.$$

d. Only the summation over $m$ remains to be regularised. Define the zeta function

$$\zeta(x)=\sum_{m=1}^{\infty}m^{-x} \quad (\mathrm{Re}(x)>1).$$

For $x=-3$ this coincides with the relevant divergent summation. Like for the integrals, we would like to replace this expression by the analytic continuation of $\zeta(x)|_{\mathrm{Re}(x)>1}$ to $\mathrm{Re}(x)\leq 1$. This continuation satisfies

$$\zeta(1-n)=\frac{(-1)^{n+1}B_n}{n}, \quad n\in\mathbb{N}, \tag{1}$$

where $B_n$ are the Bernouilli numbers:

$$\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\equiv\frac{t}{e^t-1}.$$

Derive eq. (1) by expanding $t/(e^t-1)$ in $e^{-t}$, and $e^{-t}$ in t (be careful with the $t^0$ term).
Hint: introduce new parameters that enable change of summation order. In the end continue back to the relevant parameter values.

Finally compute the fully regularised energy per area $E(x)$ and pressure $F(x)=-dE(x)/dx$. Given the Bernouilli number $B_4=-1/30$, evaluate this pressure for $x=1\mu m$.

3. **Euler-Lagrange equation**

   Let $\phi(x) = \phi(\vec{x}, t)$ be a complex scalar field with action functional $S = \int d^4x\mathcal{L}(x)$. $\mathcal{L}$ is the so-called Lagrange density (the Lagrangian is $L(t) = \int d^3\vec{x}\mathcal{L}(\vec{x}, t)$):

$$\mathcal{L}(x) = \partial_\mu\phi^*(x)\partial^\mu\phi(x) - m^2\phi^*(x)\phi(x),$$

   with metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and units such that $\hbar = 1$, $c = 1$.

a. Prove by variational calculus the Euler-Lagrange equation (equation of motion) $\frac{\delta S}{\delta\phi(x)} = \partial_\mu \frac{\delta S}{\delta(\partial_\mu\phi(x))}$ and show that this gives the Klein-Gordon equation.

b. Given the energy-momentum tensor

$$T_{\mu\nu}(x) = \partial_\mu\phi^*(x)\partial_\nu\phi(x) + \partial_\nu\phi^*(x)\partial_\mu\phi(x) - g_{\mu\nu}\mathcal{L}(x),$$

   show that $\partial_\mu T^{\mu\nu} = 0$.

c. Given the current density

$$J_\mu(x) = i(\phi^*(x)\partial_\mu\phi(x) - (\partial_\mu\phi^*(x))\phi(x)),$$

   show that $\partial_\mu J^\mu(x) = 0$.

d. Prove that the total energy $E$, momentum $P_i$ and charge $Q$, given by $E(t) = \int d^3\vec{x}T_{00}(\vec{x}, t)$, $P_i(t) = \int d^3\vec{x}T_{0i}(\vec{x}, t)$ and $Q(t) = \int d^3\vec{x}J_0(\vec{x}, t)$, are conserved.

4. **Creation and annihilation operators**

   We start from operators $p$ and $q$ satisfying canonical commutation relations $[p, q] = -i\hbar$. Define

$$a = \frac{1}{\sqrt{2\hbar\omega}}(\omega q + ip), \quad a^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega q - ip), \quad N = a^\dagger a.$$

a. Show that $[a, a^\dagger] = 1$. Also calculate the commutators $[a, N]$, $[a^\dagger, N]$ and $[(a^\dagger)^n, N]$.

b. Define $|n\rangle$ by $N|n\rangle = n|n\rangle$, $\langle n|n\rangle = 1$. Show that

$$a|n\rangle = c_n^-|n-1\rangle \quad , \quad a^\dagger|n\rangle = c_n^+|n+1\rangle \quad , \quad |n\rangle = c_n(a^\dagger)^n|0\rangle.$$

   Compute $c_n^-$, $c_n^+$, $c_n$ and show that they can be chosen real.

   Given an algebra of operators and commutation relations, we mean by the associated Hilbert space the (smallest) Hilbert space that may be used to incorporate the algebra. What is the associated Hilbert space in the present case?

c. Derive a matrix representation for the operators $a$, $a^\dagger$ and $N$.

d. Now consider operators with <u>anti</u>commutation relations

$$\{b_r, b_s^\dagger\} = \delta_{r,s}, \quad \{b_r, b_s\} = \{b_r^\dagger, b_s^\dagger\} = 0,$$

   where $\{X, Y\} \equiv XY + YX$. What is the corresponding Hilbert space?

   Define $N_r = b_r^\dagger b_r$. What are the possible eigenvalues of $N_r$? Construct a matrix representation for the operators $b_r$, $b_r^\dagger$ and $N_r$. Why is the algebra generated by $b_r$ and $b_r^\dagger$, with the above anticommutation relations, suitable for describing fermions?

   Prove that exchanging $b_r$ and $b_r^\dagger$ can be described by a unitary transformation.

e. The BCS theory of superconductivity uses the following operators that describe annihilation and creation of electron <u>pairs</u>.

$$c_{\vec{k}} = b_{-\vec{k}\downarrow} b_{\vec{k}\uparrow}, \quad c^\dagger_{\vec{k}} = b^\dagger_{\vec{k}\uparrow} b^\dagger_{-\vec{k}\downarrow}.$$

Prove that $[c_{\vec{k}}, c_{\vec{k}'}] = [c^\dagger_{\vec{k}}, c^\dagger_{\vec{k}'}] = 0$ and calculate $[c_{\vec{k}}, c^\dagger_{\vec{k}'}]$. Also determine the Hilbert space and the action of the operators on this Hilbert space. Would you call the electron pairs fermions or rather bosons?

5. **Real and complex fields**

Let us consider a real scalar field $\varphi(x)$ and a Hamiltonian

$$H = \int d^3\vec{x}\{\frac{1}{2}\pi^2(x) + \frac{1}{2}(\partial_i\varphi(x))^2 + \frac{1}{2}m^2\varphi^2(x)\},$$

where $\pi(x) = \partial_t\varphi(x)$ is the canonical momentum. For quantisation we postulate the following commutators at some time $t$, say $t = 0$. (Argue briefly why these relations are compatible with causality.)

$$[\pi(x), \pi(y)] \mid_{x_0=y_0=0} = [\varphi(x), \varphi(y)] \mid_{x_0=y_0=0} = 0; \quad [\pi(x), \varphi(y)] \mid_{x_0=y_0=0} = -i\delta_3(\vec{x} - \vec{y}).$$

Write the Fourier decomposition of $\varphi(x)$ as follows:

$$\varphi(x) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{1}{\sqrt{2k_0}}(a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}),$$

where $kx = k_0 t - \vec{k}\cdot\vec{x}$ and $k_0 = +\sqrt{\vec{k}^2 + m^2}$.

<u>Remark</u>: $\varphi(\vec{x}, t)$ is the Heisenberg representation of $\varphi(\vec{x}, 0)$. This can be verified explicitly at the end of part d.

a. Give the Fourier decomposition of $\pi(x)$. Why can we (formally) set $\pi(\vec{x}, x_0 = 0) = -i\partial/\partial\varphi(\vec{x}, x_0 = 0)$?

b. Derive the commutation relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$.

c. What is the associated Hilbert space?

d. Write the Hamiltonian $H$ in terms of the occupation number (density) operators $N(\vec{k}) = a^\dagger(\vec{k})a(\vec{k})$. Note that $H$ is time independent.

It is impossible to define a total charge $Q$ for a real field $\varphi(x)$ (in a nontrivial way). Basically this is because a real field describes particles that are their own antiparticles. Therefore let us introduce a <u>complex</u> field $\varphi \neq \varphi^\dagger$ with Hamiltonian

$$H = \int d^3x\{\pi^\dagger(x)\pi(x) + \partial_i\varphi^\dagger(x)\partial_i\varphi(x) + m^2\varphi^\dagger(x)\varphi(x)\},$$

where $\pi(x) = \partial_t\varphi^\dagger(x)$, $\pi^\dagger(x) = \partial_t\varphi(x)$.

e. Show that $H = \int d^3\vec{x}\, T_{00}$ (see exercise 3 for the definition of $T_{\mu\nu}$, in which classical fields now become operator fields).

The nontrivial commutators are postulated to be

$$[\pi(x), \varphi(y)]\,|_{x_0=y_0=0} = [\pi^\dagger(x), \varphi^\dagger(y)]\,|_{x_0=y_0=0} = -i\delta_3(\vec{x}-\vec{y}).$$

Let us write $\varphi(x) = (\varphi_1(x) + i\varphi_2(x))/\sqrt{2}$ and substitute for the real fields $\varphi_i(x)$ the Fourier decompositions in terms of $a_i(\vec{k})$ and $a_i^\dagger(\vec{k})$.

f. Give $a(\vec{k})$ and $b(\vec{k})$ in terms of $a_i(\vec{k})$ such that

$$\varphi(x) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{1}{\sqrt{2k_0}}(a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}).$$

Also derive the Fourier decompositions of $\varphi^\dagger(x)$, $\pi(x)$ and $\pi^\dagger(x)$.

g. Give the mutual commutation relations for the operators $a(\vec{k})$, $a^\dagger(\vec{k})$, $b(\vec{k})$ and $b^\dagger(\vec{k})$.

h. Write $H$ in terms of $N^a(\vec{k}) = a^\dagger(\vec{k})a(\vec{k})$ and $N^b(\vec{k}) = b^\dagger(\vec{k})b(\vec{k})$.

We would like to interpret the particles created by $b^\dagger$ as the antiparticles of the ones created by $a^\dagger$. This allows us to define the total charge

$$Q = \text{const.}(\#\text{particles} - \#\text{antiparticles}) = \frac{e}{(2\pi)^3}\int d^3\vec{k}(N^a(\vec{k}) - N^b(\vec{k})).$$

(at $t = 0$).

i. Prove that $Q$ is conserved. Also show that $Q$ can be written as

$$Q = \int d^3\vec{x}\rho(x) + \text{constant},$$

where $\rho(x) = -ie\{(\partial_t\varphi^\dagger)(x)\varphi(x) - \varphi^\dagger(x)(\partial_t\varphi)(x)\}$. Note that $\rho(x) = eJ_0(x)$ (see exercise 3).

6. **Commutation relations and causality**

We reconsider the Hermitian operator field

$$\varphi(x) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{1}{\sqrt{2k_0}}(a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}).$$

In exercise 5 commutation relations ($[\varphi(x), \varphi(y)]\,|_{x_0=y_0=0} = 0$ etc.) and a Hamiltonian have been introduced. Use these for deriving an integral representation for

$$\Delta(x-y) \equiv [\varphi(x), \varphi(y)];$$

$x$, $y$ arbitrary.

Show that $\Delta(x-y) = 0$ whenever $x_0 = y_0$ ($x_0$ arbitrary). Also show that $\Delta(x-y)$ is Lorentz invariant and use this for generalising the result to $x$, $y$ with $(x-y)^2 < 0$ (i.e. spatially separated).

<u>Hint</u>: Prove that $\int d^3\vec{k} = \int d^4k\delta(k^2 - m^2)\theta(k_0)2k_0$, where $\theta$ is the step function.

7. **Feynman rules for a classical field**

   Consider real fields $\varphi_1$ and $\varphi_2$ as described by the Lagrangian

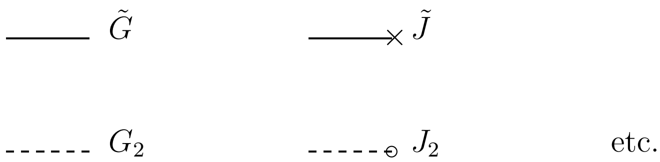
$$\mathcal{L}[\varphi_1, \partial_\mu\varphi_1, \varphi_2, \partial_\mu\varphi_2] = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - g_0 \log[1 + \frac{1}{2}g_1(\varphi_1 - F)^2] - \frac{1}{2}g_2\varphi_1\varphi_2^2.$$

a. Determine the dimension of the fields $\varphi_i$ and the constants $g_i$, $F$. (Remember that for $\hbar = 1,\ c = 1$ all dimensions are powers of $[l] = [m]^{-1}$; also the action $S = \int d^4x\mathcal{L}(x)$ is dimensionless.)

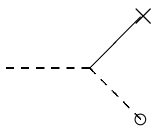
b. In a perturbative calculation $\tilde{\varphi} = \varphi_1 - F$ and $\varphi_2$ are chosen as fundamental fields. Explain why.

   Expand $\mathcal{L}$ in $\tilde{\varphi}$ and $\varphi_2$ (up to $4^{\text{th}}$ order terms). Write the result as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, where $\mathcal{L}_0$ are the quadratic terms and $\mathcal{L}_{\text{int}}$ contains the interaction terms. What are the masses of the fields $\tilde{\varphi}$ and $\varphi_2$?

c. We now introduce source terms $-\tilde{J}\cdot\tilde{\varphi}$ and $-J_2\cdot\varphi_2$. Derive the Feynman rules for the perturbative expansion using the (classical) method in the lecture notes (pp. 13–15). Use the following notation:




d. Which expression is associated to the diagram below?



8. **Photon propagator**

   Gauge invariance complicates the derivation of the photon propagator (see lecture notes p. 16). In this exercise we will fix the gauge by using the following Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \lambda(x)\partial_\mu A^\mu(x) - J_\mu(x)A^\mu(x).$$

   This describes a photon field $A_\mu$ and a Lagrange multiplier $\lambda$ in the presence of an external (i.e. not dynamical) source $J_\mu$.

a. Use partial integration to write the quadratic part of the action as

$$\frac{1}{2}((A_\mu), \lambda)\cdot\hat{M}\begin{pmatrix}(A_\nu)\\ \lambda\end{pmatrix},$$

   where $\hat{M}$ is a <u>hermitian</u> $5\times 5$ matrix operator. The inner product '$\cdot$' includes an integration over space-time.

b. Show that $\hat{M}$ is invertible and that the corresponding photon propagator is the same as in the so-called Landau gauge $\alpha \to \infty$ (lecture notes p. 17).
Hint: work in Fourier space.

9. **Coulomb gauge and temporal gauge**

The gauge freedom of the photon field can be eliminated through an extra constraint besides the equations of motion (imposing the constraint is usually called 'choosing a gauge'). Examples:

(1) Lorentz gauge $\partial_\mu A^\mu = 0$
(2) Coulomb gauge $\partial_i A_i \equiv \vec{\nabla} \cdot \vec{A} = 0$
(3) temporal gauge $A_0 = 0$.

Here we will analyze the conditions (2) and (3). These are not Lorentz invariant, but expose the photon's degrees of freedom nicely.

a. Show that (2) or (3) can always be realised after an appropriate gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$. Furthermore show that (2) and (3) can be imposed simultaneously in the absence of sources (i.e. $J_\mu = 0$).

b. The transversal (T) and longitudinal (L) components of an arbitrary vector field $\vec{v}$ are defined as follows:

$$\vec{v} = \vec{v}_T + \vec{v}_L, \quad \vec{\nabla} \cdot \vec{v}_T = 0, \ \vec{\nabla} \times \vec{v}_L = \vec{0}$$

Write down the relations between $\vec{A}$ and $\vec{E}$, $\vec{B}$ in terms of their T and L components. Also express Maxwell's equations in these components. Here $\vec{A}$, $\vec{E}$ and $\vec{B}$ stand for the vector potential, electric field and magnetic field, respectively.

c. Coulomb gauge

Show that $A_0(\vec{x}, t)$ is completely determined by $\rho(\vec{x}', t)$ and the spatial boundary conditions at time $t$ (hence the name 'instantaneous Coulomb potential'). It follows that the longitudinal component of the physical field $\vec{E}$ is not a degree of freedom in the radiation field. (Why? What do we mean exactly by a degree of freedom in a classical system?)

temporal gauge

Show that $\vec{A}_L$ is completely determined by the charge distribution $\rho$ and the spatial boundary conditions, together with an initial condition $\vec{A}_L(\vec{x}, t \to -\infty)$. Show (again) that $\vec{E}_L$ is not a degree of freedom in the radiation field.

10. **Preparation for the path integral**

Consider a one dimensional harmonic oscillator with the Hamiltonian $H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{q}^2$. Here $\hat{p} = \frac{1}{i} \frac{\partial}{\partial q}$ and $\hat{q}$ is the position operator, so that $< q|H|p >= \frac{1}{\sqrt{2\pi}} e^{ipq} h(p,q)$ with $h(p,q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$ .

a. Prove the following **exact** identity ($\delta t \equiv T/n$):

$$\begin{aligned} K_n(q_n, q_o, T) &\equiv \int \frac{dp_n}{2\pi} \prod_{i=1}^{n-1} \frac{dq_i dp_i}{2\pi} \exp[i \sum_{j=1}^{n} \{p_j(q_j - q_{j-1}) - h(p_j, q_j)\delta t\}] \\ &= < q_n | \{\exp(-i\frac{m\omega^2}{2}\hat{q}^2 \delta t) \exp(-i\frac{\hat{p}^2 \delta t}{2m})\}^n | q_0 > \quad . \end{aligned}$$

b. Show that $K_n(q_n, q_0, T) =< q_n | \exp(-i\frac{m\omega^2}{4}\hat{q}^2\delta t)\mathcal{T}^n \exp(i\frac{m\omega^2}{4}\hat{q}^2\delta t)|q_0 >$, where $\mathcal{T} = \exp(-i\frac{m\omega^2}{4}\hat{q}^2\delta t)\exp(-i\frac{\hat{p}^2\delta t}{2m})\exp(-i\frac{m\omega^2}{4}\hat{q}^2\delta t)$. Prove that $\mathcal{T}$ is a unitary operator.

c. We are going to prove that $\mathcal{T} = \exp(-i\tilde{H}\delta t)$, where $\tilde{H}$ is also a harmonic oscillator Hamiltonian: $\tilde{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\Omega^2\hat{q}^2$.
<u>Note</u>: until that is proven, one should of course use $\mathcal{T}$ as defined in part b.

1. Show that $[\hat{p}, \hat{q}] = -i$ implies

$$[e^{\alpha\hat{p}^2}, \hat{q}] = -2i\alpha\hat{p}e^{\alpha\hat{p}^2} \quad \text{and} \quad [e^{\alpha\hat{q}^2}, \hat{p}] = 2i\alpha\hat{q}e^{\alpha\hat{q}^2}$$

for any $\alpha \in \mathbb{C}$. Now solve the 'eigenvalue equation'

$$\mathcal{T}(\kappa_\pm\hat{q} + \lambda_\pm\hat{p}) = \mu_\pm(\kappa_\pm\hat{q} + \lambda_\pm\hat{p})\mathcal{T}$$

($\kappa_\pm, \lambda_\pm, \mu_\pm \in \mathbb{C}$). Show that $\mu_\pm = \exp(\pm i\Omega\delta t)$, with $\Omega$ defined by $\sin(\frac{1}{2}\Omega\delta t) = \frac{1}{2}\omega\delta t$.

2. Determine the commutation relations between $\kappa_+\hat{q} + \lambda_+\hat{p}$ and $\kappa_-\hat{q} + \lambda_-\hat{p}$. For which normalization are we dealing with creation and annihilation operators $\hat{a}^\dagger$, $\hat{a}$? Show that the corresponding Hamiltonian is given by $\tilde{H}$, with $M = m\sin(\Omega\delta t)/(\Omega\delta t)$.

3. Now that $(\kappa, \lambda, \mu)_\pm$ are known, the eigenvalue equation determines $\mathcal{T}$ uniquely up to a $\hat{p}, \hat{q}$ independent factor. Prove that $\mathcal{T} = C\exp(-i\hat{a}^\dagger\hat{a}\Omega\delta t)$ satisfies the equation ($C \in \mathbb{C}$). Use the definition of $\mathcal{T}$ to show that $C = \exp(-\frac{1}{2}i\Omega\delta t)$.

   <u>Hint</u>: Since $C$ is independent of $\hat{p}, \hat{q}$, it can be determined by calculating $\langle 0|\mathcal{T}|0\rangle$ with $|0\rangle$ the $\tilde{H}$-vacuum ($\hat{a}|0\rangle \equiv 0$). First evaluate $\langle p|\exp(-i\frac{m\omega^2}{4}\hat{q}^2\delta t)|0\rangle = A\exp(-Bp^2/2)$, with $A$ and $B$ defined appropriately.

d. Use the above result to show that $\lim_{n\to\infty} K_n(q_n, q_0, T) =< q_n|e^{-iHT}|q_0 > \quad .$

11. **Path integral for a free particle**

We start from the path integral for the evolution operator associated to Schrödinger's equation (lecture notes p. 21). As Lagrangian we take $L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2$, and problems from integrating rapidly oscillating functions are avoided by choosing so-called Euclidean time $\tau = iT$. The path integral then becomes (with $d\tau \equiv \tau/n$ and for $n \to \infty$):

$$\langle q'|U(\tau)|q\rangle = \left[\frac{m}{2\pi\delta\tau}\right]^{n/2} (\prod_{j=1}^{n-1} \int dq_j) e^{-S(q_0, q_1, \cdots, q_n)}, \tag{1}$$

where $q_0 \equiv q$ and $q_n \equiv q'$, and with action

$$S(q_0, q_1, \cdots, q_n) = \sum_{j=1}^{n} \frac{m}{2} \left[\frac{q_j - q_{j-1}}{\delta\tau}\right]^2 \delta\tau.$$

The Euclidean evolution operator is $U(\tau) = \exp(-H\tau)$, $H$ being the usual quantum mechanical Hamiltonian associated to $L$.

Upon defining

$$U(\tilde{q}, \tau) \equiv \langle \tilde{q}|U(\tau)|0\rangle \stackrel{\text{transl. inv.}}{=} \langle q'|U(\tau)|q\rangle, \quad \tilde{q} \equiv q' - q,$$

$U(\tilde{q}, \tau)$ satisfies the Euclidean Schrödinger equation by construction. Due to the Euclidean time this is a diffusion equation:

$$U(\tilde{q}, 0) = \delta(\tilde{q}) \quad , \quad \frac{\partial}{\partial \tau} U(\tilde{q}, \tau) = \frac{1}{2m} \frac{\partial^2}{\partial \tilde{q}^2} U(\tilde{q}, \tau) \tag{2}$$

a. Determine $U(\tilde{q}, \tau)$ by solving eq. (2) (use a Fourier transform).

b. In this simple case the path integral in eq. (1) can be calculated explicitly. We will do this by changing variables

$$y_j = q_j - q_{j-1}, \quad j = 1, 2, \cdots, n.$$

Show that $\prod_{j=1}^{n-1} dq_j = (\prod_{j=1}^{n} dy_j)\delta(\tilde{q} - \sum_{j'=1}^{n} y_{j'})$.

The $\delta$-function can be written as

$$\delta(\tilde{q} - \sum_{j=1}^{n} y_j) = \frac{1}{2\pi} \int d\omega \exp\Big(i\omega(\sum_{j=1}^{n} y_j - \tilde{q})\Big).$$

These steps reduce the path integral to a product of Gaussian integrals. Perform the integrations and verify that the outcome equals the result in a.

12. **Massive vector fields**

The following Lagrangian (mass $m \neq 0$) describes a massive vector field,

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A_\mu A^\mu.$$

a. Show that this Lagrangian is <u>not</u> gauge invariant.

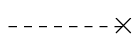
b. Determine the equations of motion for the field $A_\mu(x)$. Show that these are equivalent to

$$\partial_\mu A^\mu = 0 \quad (*) \quad , \quad (\partial^2 + m^2) A_\mu = 0.$$

<u>Remark</u>: the condition $(*)$, being a gauge choice in the massless case (see exercise 8), is now imposed by the equations of motion!

c. Bring $\mathcal{L}$ to the form $\frac{1}{2} A_\mu M^{\mu\nu} A_\nu$ (more precisely, use partial integration to find an $M$ such that this gives the same action—and therefore the same equations of motion). Construct the inverse of the operator $M$ (use a Fourier transform).

d. Now add a source term: $-J_\mu A^\mu$ with $\partial_\mu J^\mu = 0$. Which expression for $A_\mu(k)$ is associated to the following Feynman diagram?



Are there other diagrams in this model contributing to $A_\mu(k)$?

$(M^{-1})_{\mu\nu}$ consists of two terms. Show that one of them drops out of $(M^{-1})_{\mu\nu}(k)J^\nu(k)$, and that $\lim_{m\to 0}(M^{-1})_{\mu\nu}(k)J^\nu(k)$ exists. Compare this limit to the Maxwell propagator (lecture notes p. 17) for $\alpha = 1$, the so-called Feynman gauge.

13. **Perturbative approach to the path integral**

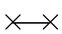
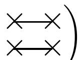
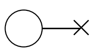
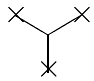
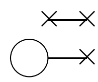
In this exercise we will treat perturbatively the generating function $Z(J)$ for a real scalar $\varphi^3$ theory. The Lagrangian reads

$$\mathcal{L} = \mathcal{L}_2 - V_{\text{int}} - J\varphi, \text{ with } V_{\text{int}} = \frac{g}{3!}\varphi^3, \; \mathcal{L}_2 = \tfrac{1}{2}\varphi G^{-1}\varphi, \; G^{-1} = -(\partial_\mu\partial^\mu + m^2).$$

In the lecture notes (p. 34) it has been shown that the path integral can be reduced to

$$Z(J) = e^{-i\int d^4x V_{\text{int}}(i\frac{\delta}{\delta J(x)})} e^{-\frac{i}{2}\int d^4y \int d^4z J(y)G(y,z)J(z)}.$$

a. Show that

$Z(J) = 1 \; + \; \Big(-\frac{i}{2}$  $-\frac{1}{8}$  $\Big) + \Big(\frac{1}{2}$  $-\frac{i}{3!}$  $-\frac{i}{4}$  $\Big)$

$\quad + \; \mathcal{O}(J^5) + \mathcal{O}(g^2).$

Here  $= \int d^4x \int d^4y J(x)G(x,y)J(y)$ etc. (do not work in Fourier space).

<u>N.B.</u> In this exercise you are not supposed to work out the analytical expressions associated to the Feynman diagrams.

b. Read the page copied from 'Diagrammar' carefully. Verify that the combinatorial factors in the above expression are correctly given by the Diagrammar prescription. <u>Remark</u>: In this prescription the sources $J$ should be considered as 1-vertices.

c. Show up to first order in $g$ and fourth order in $J$ that $Z(J) = \exp(G(J))$, $G(J)$ being the sum of <u>connected</u> diagrams.

d. The 'n-point function' can be expressed in the following way:

$$\langle\varphi_n\cdots\varphi_1\rangle \equiv \langle 0|\varphi_n\cdots\varphi_1|0\rangle = \frac{1}{Z(0)}\left(i\frac{\delta}{\delta J_1}\cdots i\frac{\delta}{\delta J_n}Z(J)\right)_{J=0}$$

($\varphi_i \equiv \varphi(\vec{x}_i, t_i)$, $t_{i+1} > t_i$, $|0\rangle$ = ground state in absence of $J$). This is why $Z(J)$ is called the generating function.

Substitute the result of part a. to obtain the 1,2,3 and 4-point functions up to order $g$. Argue that the product rule guarantees that the Diagrammar prescriptions for diagrams in $Z(J)$ and $\langle\varphi_n\cdots\varphi_1\rangle$ are consistent, and verify explicitly the correctness of the Diagrammar prescription for the 1,2,3 and 4-point functions to the given order in $g$.

e. Show non-perturbatively that

$$\left(i\frac{\delta}{\delta J_1}i\frac{\delta}{\delta J_2}G(J)\right)_{J=0} = \langle\varphi_2\varphi_1\rangle - \langle\varphi_2\rangle\langle\varphi_1\rangle.$$

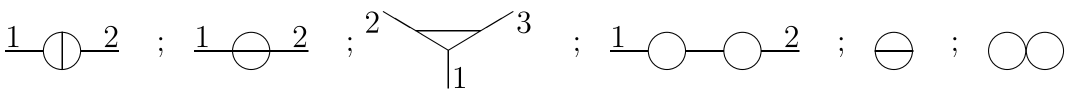
For $n > 2$ similar expressions hold. This means that $G(J)$ is the generator of quantum fluctuations.

14. **Combinatorial factors**

a. Given a real scalar field $\varphi$ with interaction

$$V_{\text{int}} = \frac{\alpha}{3!}\varphi^3 + \frac{\beta}{4!}\varphi^4,$$

determine the combinatorial factors of the following diagrams (see the discussion on pg. 36 and 37 or the section on combinatorial factors in "Diagrammar", CERN Yellow report 73-9, by G. 't Hooft and M. Veltman, reprinted in "Under the spell of the gauge principle" by G. 't Hooft (World Scientific, Singapore, 1994))




b. Consider the following models:

**I** Scalar field $A$,

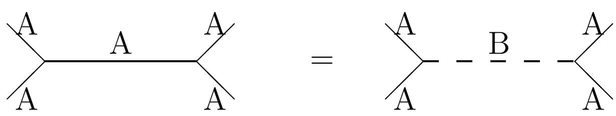
$$\mathcal{L} = \tfrac{1}{2}\partial_\mu A \partial^\mu A - \tfrac{1}{2}m^2 A^2 - \frac{\lambda}{3!}A^3 - JA.$$

**II** Scalar fields $A$ and $B$ with equal mass,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu A \partial^\mu A + \tfrac{1}{2}\partial_\mu B \partial^\mu B - \tfrac{1}{2}m^2 A^2 - \tfrac{1}{2}m^2 B^2 - \frac{\mu}{2!}A^2 B - JA.$$

We limit ourselves to diagrams with an even number of external $A$-lines (and no external $B$'s). Let us pose the question whether we can make a distinction between the above models from knowledge of the amplitudes for its diagrams.

1. Let us first consider tree diagrams.
   - Show that $\lambda$ and $\mu$ can be chosen such that the models I and II give an identical 4-point function:




   - Show that the 6-point function is different for the models I, II.
2. Proceed to show that at 1-loop level even the 2-point function, which at tree level is trivially the same, is different for the models I, II.

15. **Quantum corrections**

In this exercise we set out to prove that the expansion of Feynman diagrams in the number of loops amounts to an expansion in powers of $\hbar$. We consider a real scalar field theory with an arbitrary interaction potential:

$$V_{\text{int}} = \sum_{n\geq 3} \frac{g_n}{n!}\varphi^n.$$

To each Feynman diagram we associate the following quantities: $E$, $I$, $L$ and $V_n$ (number of external lines, internal lines, loops, and vertices with $n$ lines, respectively).

a. Prove that for any connected diagram the following relations hold:

$$\begin{cases} L = I + 1 - \sum_{n\geq 3} V_n \\ \sum_{n\geq 3} nV_n = E + 2I. \end{cases}$$

Hint: Any diagram can be reduced to a tree diagram (i.e. a diagram with no loops) by cutting $L$ times appropriate internal lines (this is the precise definition of $L$). Determine how $E$, $I$, $L$, $\sum V_n$ and $\sum nV_n$ have changed after one such cut. Another operation is the amputation of an external leg. Find the change in the above quantities for this operation too. Finally determine these quantities for a simple diagram in order to obtain the 'initial condition'.

b. Since we are looking for quantum effects, we do not take $\hbar = 1$ (for convenience we keep $c = 1$, though). Powers of $\hbar$ can now pop up at several places in the Lagrangian. We can limit the number of such places by conveniently choosing the dimensions of $\varphi$, $J$ and $\{g_n\}$. Show that this can be done in such a way that

$$\mathcal{L}_J \overset{\hbar=1}{\equiv} \mathcal{L} - J\varphi \overset{\hbar\neq 1}{=} \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}\frac{m^2}{\hbar^2}\varphi^2 - \sum_{n\geq 3}\frac{g_n}{n!}\varphi^n - J\varphi,$$

but that the $\hbar$-dependence in the quadratic part cannot be removed.
Note: for $\hbar \neq 1$, mass and 1/length have independent dimensions.

Remark: It is natural to require that the classical theory (i.e. the Euler-Lagrange equations) is independent of $\hbar$. The above result then implies that $m \sim \hbar$, so that even in the classical theory the mass is an effective parameter of quantum mechanical origin.

Show that the path integral now reads

$$Z(J) = C\int \mathcal{D}\varphi e^{\frac{i}{\hbar}\int d^4x[\mathcal{L}(\varphi(x)) - J(x)\varphi(x)]}$$

($C$ independent of $J$).

c. We absorb the factor $\hbar$ in exponent of $Z(J)$ into the quadratic part of $\mathcal{L}$ by defining

$$\tilde{\varphi} = \hbar^{-1/2}\varphi, \quad \tilde{J} = \hbar^{-1/2}J.$$

This gives

$$Z(J) = \tilde{C}\int \mathcal{D}\tilde{\varphi} e^{i\int d^4x[\tilde{\mathcal{L}}(\tilde{\varphi}(x)) - \tilde{J}(x)\tilde{\varphi}(x)]}.$$

What is the expression for $\tilde{\mathcal{L}}(\tilde{\varphi})$? Show that the propagator for the field $\tilde{\varphi}$ does not have any $\hbar$-dependence. This means that all factors of $\hbar$ in a diagram come from the vertices (and external lines). Express the total power of $\hbar$ in terms of $\{V_n\}$ (and $E$). Finally make use of the results in a. to prove that this power equals $L$, up to a function of $E$ alone.

d. Show that for a model with only four point interactions ($g_n = 0$ for $n \neq 4$) the expansion in the number of loops $L$ can be interpreted as an expansion in powers of $g_4$.

16. **Legendre transformation and classical limit**

In this exercise we will consider the connection between quantum field theory and classical field theory once more. Therefore take $\hbar \neq 1$ again. As explained in the previous exercise, we then have

$$Z[J] = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}\int d^4x(\mathcal{L}-J\varphi)} = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}S[J,\varphi]}, \tag{1}$$

where $S[J,\varphi]$ is independent of $\hbar$. Furthermore, $Z[J] = \exp(G[J]/\hbar)$, $\frac{G[J]}{\hbar}$ being the sum of <u>connected</u> diagrams. The overall factor of $\hbar$ has been conveniently chosen $1/\hbar$ so that

$$G[J] = \sum_{L=0}^{\infty} \hbar^L G_{(L)}[J], \qquad (L = \#\text{loops})$$

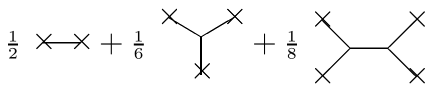
with $\hbar$-independent $G_{(L)}[J]$'s. A saddle point, or stationary phase approximation ($\hbar \to 0$) of eq. (1) then immediately gives

$$G_{(0)}[J] = iS[J, \varphi_{\rm cl}[J]]. \tag{2}$$

Here $\varphi_{\rm cl}[J]$ is the solution of the Euler-Lagrange equations $\delta S[J,\varphi]/\delta\varphi = 0$ (hence the subscript 'cl', for 'classical'). This saddle point $\varphi_{\rm cl}[J]$ is unique under the assumption that in eq. (1) only fields vanishing (sufficiently fast) at infinity are integrated over.

Let us inspect if eq. (2) is reproduced by perturbation theory. For convenience we limit ourselves to $\varphi^3$-theory, whose Lagrangian has already been introduced in exercise 13.

a. 1. Substitute the expansion of $\varphi_{\rm cl}[J]$, as given on p. 15 of the lecture notes, in the action in order to obtain

$-S[J, \varphi_{\rm cl}[J]] =$ 

$+ \mathcal{O}(J^5).$

Verify explicitly that eq. (2) holds to this order in $J$.

2. It follows from the path integral that (cf. exercise 13d.)

$$\langle\varphi\rangle[J](x) = i\frac{\delta G[J]}{\delta J(x)}. \tag{3}$$

Here $\langle\varphi\rangle[J]$ stands for the expectation value of the Heisenberg operator $\hat\varphi(x) = \hat\varphi(\vec{x}, t)$ in the groundstate $|0\rangle[J]$ of the Hamiltonian $\hat H[J]$, i.e. in the presence of a source $J$. Note that $\langle\varphi\rangle[J]$ is an ordinary real valued field, and not an operator field. Also note that in each point $x$ it is a functional of $J$.

Show up to third order in $J$ that

$$\langle\varphi\rangle[J] = \varphi_{\rm cl}[J] + \mathcal{O}(\hbar). \tag{4}$$

b. Now let us see if we can generalise these results to arbitrary order in $J$. Brute force as used in a. is of no use here because this method generates the combinatorial factors for $S[J,\varphi_{\rm cl}[J]]$ in an almost intractable way. The proper framework for the proof is the formalism of Legendre transformations (see Itzykson & Zuber for more details).

We assume that eq. (3) is invertible to $J(x) = J[\langle\varphi\rangle](x)$. This allows us to define a functional $\Gamma$ on $\langle\varphi\rangle$ via a Legendre transform:

$$i\Gamma[\langle\varphi\rangle] \equiv G[J[\langle\varphi\rangle]] + i(J[\langle\varphi\rangle], \langle\varphi\rangle), \tag{5}$$

with $(f, g) \equiv \int d^4x f(x)g(x)$. Derive from eq. (3) that

$$\frac{\delta\Gamma[\langle\varphi\rangle]}{\delta\langle\varphi\rangle(x)} = J[\langle\varphi\rangle](x). \tag{6}$$

Hint: The chain rule for functional derivatives reads:

$$\frac{\delta f[g[h]]}{\delta h(x)} = \int d^4y \left.\frac{\delta f[g]}{\delta g(y)}\right|_{g=g[h]} \frac{\delta g[h](y)}{\delta h(x)}.$$

Remark: an important example of a Legendre transform is the relation between a Lagrangian and its Hamiltonian: $H(q, p) = p\dot{q}(p) - L(q, \dot{q}(p))$ with $p = \partial L(q, \dot{q})/\partial\dot{q}$. (The position $q$ plays no role in this transformation.)

c. 1. It is useful to Taylor expand $G[J]$ around $J = 0$:

$$G[J] = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n G^{(n)}(x_1, \cdots, x_n) J(x_1) \cdots J(x_n).$$

Why can we disregard the $n = 0$ contribution?

Note: $G^{(n)}$ is precisely the connected $n$-point function as defined in exercise 13.

We also expand $\Gamma[\langle\varphi\rangle]$ around $\langle\varphi\rangle[J = 0]$. For simplicity we limit ourselves to the case $\langle\varphi\rangle[0] = 0$.

$$\Gamma[\langle\varphi\rangle] = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1, \cdots, x_n) \langle\varphi\rangle(x_1) \cdots \langle\varphi\rangle(x_n).$$

Why do the $n = 0, 1$ terms vanish?

2. $\Gamma^{(n)}$ can be obtained from $\{G^{(m)|m\leq n}\}$ by differentiating eq. (3) $n-1$ times with respect to $\langle\varphi\rangle$ and then setting $\langle\varphi\rangle = 0$ (corresponding to $J[\langle\varphi\rangle] = 0$). Show that

$$\Gamma^{(2)} = i(G^{(2)})^{-1}, \quad \Gamma^{(3)} = -iG^{(3)\ \mathrm{amp}},$$

where ‘amp’ means amputation:

$$G^{(n)\mathrm{amp}}(x_1, \cdots, x_n) \equiv \int \prod_{i=1}^{n} \left(d^4y_i (G^{(2)})^{-1}(x_i, y_i)\right) G^{(n)}(y_1, \cdots, y_n).$$

Argue that $G^{(3)\mathrm{amp}} = G^{(3)\mathrm{1PI}}$. The latter stands for the sum of ‘1 Particle Irreducible’ diagrams, i.e. amputated diagrams that are still connected after cutting one arbitrary internal line. In general the following holds:

$$\Gamma^{(n)} = -iG^{(n)\mathrm{1PI}}. \quad (n \geq 3)$$

You are not asked to prove this, but it might be enlightening to check it for $n = 4$.

3. Use the above to show that, to order $\hbar^0$,

$$\Gamma^{(2)}_{(0)}(x,y) = -\delta_4(x-y)(\partial_\mu\partial^\mu + m^2), \quad \Gamma^{(3)}_{(0)}(x_1,x_2,x_3) = -g\delta_4(x_1-x_2)\delta_4(x_2-x_3),$$

whereas $\Gamma^{(n)}_{(0)} = 0$ for $n \geq 4$. Also show that

$$\Gamma_{(0)}[\langle\varphi\rangle] = S[J=0,\langle\varphi\rangle].$$

d. Show that, to $0^{\text{th}}$ order in $\hbar$, eq. (6) is just the Euler-Lagrange equation (for $\langle\varphi\rangle$). Under what boundary conditions can you now prove eq. (4)? Finally prove eq. (2).

Remark: The above shows that $\Gamma[\langle\varphi\rangle]$ may be viewed as a quantum mechanical generalisation of the classical action (without source term). The physical relevance of this particular generalisation comes from eq. (6). Apparently the observable $\langle\varphi\rangle[J]$ is governed by this generalised Euler-Lagrange equation. The quantum corrections usually cause $\langle\varphi\rangle[J] \neq \varphi_{\text{cl}}[J]$. For $J=0$ a symmetry often prohibits such a shift, but for $J \to 0$ the shift may still be possible. In such a case $\langle\varphi\rangle[J \to 0]$, and therefore $|0\rangle[J \to 0]$, is less symmetric than $\varphi_{\text{cl}}[J \to 0]$. This means that quantum fluctuations can (spontaneously) break a symmetry.

17. **Feynman rules for complex fields**

If two real scalar fields, $\varphi_1$ and $\varphi_2$, are governed by the Lagrangian

$$\mathcal{L}(\varphi_1,\varphi_2) = \tfrac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \tfrac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \tfrac{1}{2}m^2(\varphi_1^2+\varphi_2^2) - V(\varphi_1^2+\varphi_2^2) - J_1\varphi_1 - J_2\varphi_2$$

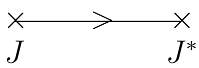
then it is possible to give an equivalent formulation using the complex fields

$$\varphi = \tfrac{\varphi_1+i\varphi_2}{\sqrt{2}}; \quad \varphi^* = \tfrac{\varphi_1-i\varphi_2}{\sqrt{2}}; \quad J = \tfrac{J_1+iJ_2}{\sqrt{2}}; \quad J^* = \tfrac{J_1-iJ_2}{\sqrt{2}}$$

which transforms to the Lagrangian

$$\mathcal{L}(\varphi,\varphi^*) = \partial_\mu\varphi\partial^\mu\varphi^* - m^2\varphi^*\varphi - V(2\varphi^*\varphi) - J^*\varphi - J\varphi^*$$

(see exercise 5 for the interpretation of $\varphi$ and $\varphi^*$ in terms of particles and antiparticles). Among the Feynman rules we now find oriented lines:

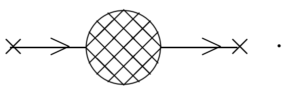



a. Which two processes are described by this diagram?

b. Give all Feynman rules for the model with

$$V(2\varphi^*\varphi) = \tfrac{1}{4}g(\varphi^*\varphi)^2.$$

c. For this potential write down all connected diagrams with at most two loops contributing to

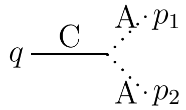


18. **Elementary scalar processes**

Consider three real scalar fields ($A$, $B$, $C$) described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + \partial_\mu C \partial^\mu C - m_A^2 A^2 - m_B^2 B^2 - m_C^2 C^2 - g_A A^2 C - g_B B^2 C).$$

a. If $m_C > 2m_A$ a $C$-particle can decay into two $A$-particles. To lowest order (in the couplings) this process is associated to the Feynman diagram



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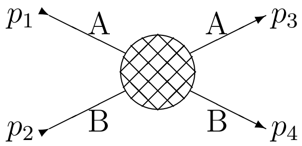
Determine the $S$-matrix element ${}_{\text{out}}\langle p_1 p_2 | q \rangle_{\text{in}}$ to this order (lecture notes p. 40). Also give an expression for the decay width $\Gamma(C \to 2A)$. Work out this expression for a $C$-particle at rest ($\vec{q} = \vec{0}$).

Determine the behaviour of $\Gamma(C \to 2A)$ for $m_C \gg 2m_A$, and for $\frac{m_C - 2m_A}{m_C} \ll 1$.

Give the total width $\Gamma(C)$ if $m_C > 2m_B$ too. Also express the expected $C$-lifetime $\langle \tau \rangle$ in terms of $\Gamma(C)$.

b. Another possible process is the elastic scattering of two particles $A$ and $B$, schematically



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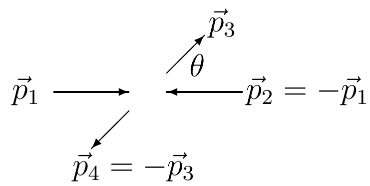
Write down the single diagram that contributes to this process to lowest order. Derive expressions both for the matrix element ${}_{\text{out}}\langle p_3 p_4 | p_1 p_2 \rangle_{\text{in}}$ and for the differential cross section $d\sigma(AB \to AB)$.

In the center of mass (CM) frame the process looks like this:



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In this frame the differential cross section only depends on the external momenta through $p = |\vec{p}_1|$ and $\theta = \angle(\vec{p}_1, \vec{p}_3)$. Work out $\frac{d\sigma}{d\Omega}(p, \theta)$ in case of equal masses $m_A = m_B \equiv m$. To this purpose, first prove these intermediate steps:

$$(i) \quad \int \frac{d^3\vec{p}_3}{E_3} \frac{d^3\vec{p}_4}{E_4} \delta_4(p_3 + p_4 - p_1 - p_2) \overset{\text{CM}}{=} \frac{p}{2\sqrt{p^2+m^2}} \int d\Omega.$$

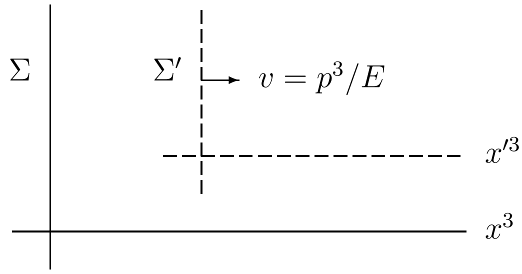
$$(ii) \quad [(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2} \overset{\text{CM}}{=} 2p\sqrt{p^2 + m^2}.$$

Calculate the total cross section $\sigma(AB \to AB)$ by integrating over all directions $\Omega$.

Show that in the limit $m_C \to \infty$, while keeping $\lambda \equiv g_A g_B / m_C^2$ constant, $\sigma(AB \to AB)$ is the same as for a model with only $A$ and $B$ particles and interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4} A^2 B^2.$$

19. **Lorentz transformation for spinors**




An electron is observed in a frame $\Sigma$, where it has velocity $v$ along the 3-axis. It's rest frame is called $\Sigma'$. In $\Sigma'$ the electron's wave function is given by

$$\psi'(x') = \frac{1}{\sqrt{2V'}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-imt'}. \quad \text{(Weyl representation)}$$

(Due to the volume factor $V'$, $\psi'$ has a volume independent norm and one can take $V' \to \infty$.)

a. Verify that this is indeed a positive energy, zero momentum solution of the Dirac equation. What is its spin? Transform the solution to the 'conventional representation' of the lecture notes p. 48.

b. The wave function $\psi$ in the $\Sigma$-frame can be determined via a Lorentz transformation. Show that the transformation $K = (K^\mu{}_\nu)$ from coordinates on $\Sigma$ to coordinates on $\Sigma'$ (i.e. $(x')^\mu = K^\mu{}_\nu x^\nu$) can be written as

$$K = e^{-\alpha L^{03}} \quad \text{with}$$

$$L^{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sinh(\alpha) = \tfrac{p^3}{m}, \quad \cosh(\alpha) = \tfrac{E}{m}.$$

c. Show that the induced transformation of spinors is given by

$$S = \cosh(\frac{\alpha}{2})\mathbf{1} + i\sinh(\frac{\alpha}{2})\sigma^{03}$$

with $\sigma^{\mu\nu}$ as in the lecture notes (p. 50).

d. Determine $\psi(x) = S^{-1}\psi'(Kx)$ in the Weyl representation. Verify explicitly in the $\Sigma$-frame that this is a solution of the Dirac equation with the correct momentum. Finally transform $\psi$ to the conventional representation.

20. **Lorentz algebra vs. su(2)×su(2)**

Using the property that the (Euclidean) Lorentz algebra is isomorphic to su(2)×su(2) one can easily classify all its finite dimensional representations (su(2) is the Lie algebra of SU(2)). We will analyse this situation in the present exercise.

a. Show that the matrices $L^{\mu\nu}$ defined by

$$(L^{\mu\nu})^{\alpha}{}_{\beta} = g^{\mu\alpha} g^{\nu}_{\beta} - g^{\nu\alpha} g^{\mu}_{\beta},$$

generate the Lorentz group (cf. part b. of the previous exercise). Furthermore prove that

$$[L^{\mu\nu}, L^{\rho\sigma}] = g^{\nu\rho} L^{\mu\sigma} + g^{\mu\sigma} L^{\nu\rho} - g^{\mu\rho} L^{\nu\sigma} - g^{\nu\sigma} L^{\mu\rho}.$$

b. Define

$$J^{\pm}_{\ell} = \tfrac{1}{2}(\tfrac{1}{2}\varepsilon_{\ell jk} L^{jk} \pm i L_{\ell 0}). \qquad (\varepsilon_{123} \equiv +1)$$

Determine all commutators $[J^{\pm}_i, J^{\pm}_j]$ and conclude that the Euclidean Lorentz algebra is isomorphic to su(2)×su(2).

c. It is well-known that the set of all finite dimensional representations of su(2)≅so(3) is given by $\{\rho_l | l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$, where

$$\begin{array}{rcl} \rho_0(J_i) & = & 0 \\ \rho_{\frac{1}{2}}(J_i) & = & -\frac{i}{2}\sigma_i \in \mathrm{su}(2) \\ \rho_1(J_i) & = & L_i \in \mathrm{so}(3) \\ & \cdots & \end{array}$$

For each pair $(a, b)$ with $a, b \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$ an irreducible representation of the Euclidean Lorentz algebra can now be defined:

$$\rho_{(a,b)} \equiv \rho_a \otimes \rho_b.$$

In particular $\rho \equiv \rho_{(\frac{1}{2},0)}$ and $\bar{\rho} \equiv \rho_{(0,\frac{1}{2})}$ are defined through

$$\begin{array}{l} \rho(J^-_i)\psi_1 = -\frac{i}{2}\sigma_i \psi_1 \\ \rho(J^+_i)\psi_1 = 0 \\ \bar{\rho}(J^-_i)\psi_2 = 0 \\ \bar{\rho}(J^+_i)\psi_2 = -\frac{i}{2}\sigma_i \psi_2. \end{array}$$

Subsequently we can construct the (reducible) representation $\rho \oplus \bar{\rho}$ acting on pairs $(\psi_1, \psi_2)$.

Give the action of $(\rho \oplus \bar{\rho})(J^{\pm}_i)$ on $(\psi_1, \psi_2)$.

d. Now derive the action of $(\rho \oplus \bar{\rho})(L^{\mu\nu})$ on $(\psi_1, \psi_2)$ and note that these objects are precisely the generators $-\frac{i}{2}\sigma^{\mu\nu}$ in the Weyl representation (lecture notes p. 50).

21. $\gamma$ **algebra**

The defining property of the $\gamma$-matrices $\gamma^1 \cdots \gamma^4$ is

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1} \qquad \mu, \nu = 0, 1, 2, 3.$$

Furthermore one defines $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

a. Show that

$$\{\gamma^\mu, \gamma^5\} = \mathbf{0}, \qquad (\gamma^5)^2 = \mathbf{1}.$$

b. Let $\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n})$ denote the trace over $n$ $\gamma$-matrices (take $\mu_i \in \{0,1,2,3\}$).

1. Prove that such a trace equals zero for $n$ odd. Also prove that $\mathrm{Tr}\gamma^5=0$.
2. Compute

$$\begin{array}{l}\mathrm{Tr}(\gamma^\mu\gamma^\nu)\\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5)\\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5)\end{array}$$

c. Prove the following identities:

$$\begin{aligned}\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha &= -2\gamma^\rho\gamma^\nu\gamma^\mu\\ \gamma^\mu\gamma^\nu\gamma^\rho+\gamma^\rho\gamma^\nu\gamma^\mu &= \tfrac{1}{2}\mathrm{Tr}(\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho)\gamma_\alpha.\end{aligned}$$

22. **Majorana and Weyl fermions**

a. Given any set of gamma matrices $\{\gamma^\mu\}$ another set is defined by

$$\gamma'^\mu = U^\dagger\gamma^\mu U, \qquad U^\dagger U = 1.$$

1. Take $\{\gamma^\mu\}$ to be

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

(the ‘conventional representation’ on pp. 48, 51 of the lecture notes) and choose

$$U = \sigma_1 \otimes \tfrac{1}{2}(\sigma_1 - i\sigma_3) + \sigma_3 \otimes \tfrac{1}{2}(\sigma_1 + i\sigma_3) = \tfrac{1}{2}\begin{pmatrix} \sigma_1 + i\sigma_3 & \sigma_1 - i\sigma_3 \\ \sigma_1 - i\sigma_3 & -(\sigma_1 + i\sigma_3) \end{pmatrix}.$$

Show that $U$ is unitary and that the set $\{\gamma'^\mu\}$ is given by

$$\gamma'^0 = \sigma_3 \otimes \sigma_2, \quad \gamma'^1 = -i \otimes \sigma_3, \quad \gamma'^2 = i\sigma_2 \otimes \sigma_2, \quad \gamma'^3 = -i \otimes \sigma_1.$$

Note that all $\gamma'$-matrices are purely imaginary.

2. This so-called Majorana representation $\{\gamma'^\mu\}$ allows us to impose the following:

$$\psi^* = \psi \qquad \text{(Majorana condition)}$$

(with $\psi = \psi(x)$). Show that this is consistent with the Dirac equation.

3. Prove that the condition implies $\bar\psi\psi = 0$.

<u>Remark</u>: This result is no longer valid for anticommuting $\psi$, $\bar\psi$.

4. How can we interpret Majorana fermions? (charge; antiparticles?)

b. Another possible condition on fermions is

$$\gamma^5\psi = \psi. \qquad \text{(Weyl condition)}$$

1. Use the Dirac equation to prove that necessarily $m = 0$.
2. Prove that Weyl fermions $\omega$ and $\psi$ satisfy $\bar\omega\psi = 0$.

3. The helicity operator $\vec{\Sigma}\cdot\hat{k}$ (with $\hat{k}=\vec{k}/|\vec{k}|$) is defined via $\Sigma^i \equiv \frac{1}{2}\varepsilon_{ijk}\sigma^{jk}$. Show that in the original $\gamma$-representation of part a.1. this reads

$$\Sigma^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

Also prove that $(\vec{\Sigma}\cdot\hat{k})^2 = \mathbf{1}$.

4. Show that for $m=0$ the plane wave solutions of the Dirac equation can be written as

$$k_0 = +E: \qquad U_+(\vec{k}) = \begin{pmatrix} \chi \\ (\vec{\sigma}\cdot\hat{k})\chi \end{pmatrix}$$
$$k_0 = -E: \qquad U_-(\vec{k}) = \begin{pmatrix} -(\vec{\sigma}\cdot\hat{k})\chi \\ \chi \end{pmatrix}.$$

Determine the action on $U_\pm$ of the chirality operator $\gamma^5$ and the helicity operator $\vec{\Sigma}\cdot\hat{k}$. In particular show that their action is the same, up to signs.

5. Conclude that a massless spinor satisfying the Weyl condition can either describe right-handed particles or left-handed antiparticles. Here right-(left-)handed means having positive (negative) helicity.

Remark: from b.3. it is clear that helicity equals the (anti-)particle's spin component in its direction of movement. Therefore the helicity operator commutes with the Dirac equation for any mass $m$, this equation being Lorentz (hence rotation) covariant. Chirality, however, is only a good quantum number in the massless case.

c. Is it possible to realise the Majorana and Weyl conditions simultaneously?

23. **Dirac equation**

We start from the Dirac action

$$S_{\text{Dirac}} = \int d^4x \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x).$$

a. In the lecture notes (p. 11) the energy-momentum tensor $T^{\mu\nu}$ has been constructed for the (scalar) bosonic case. As this construction uses general coordinate invariance, its generalisation to fermions is complicated (the formulation of spinors in general relativity is involved). For this we refer to section 10 of "The spacetime approach to quantum field theory", by B.S. DeWitt in Relativity, groups and topology II, ed. B.S. DeWitt and R. Stora (Norht-Holland, Amsterdam, 1984). The generalisation of eq. (3.20) yields an energy-momentum tensor that is no longer symmetric:

$$T^{\mu\nu} = \bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma^\alpha\partial_\alpha - m)\psi + \frac{1}{4}\partial_\lambda\Big(\bar{\psi}(\gamma^\lambda\sigma^{\mu\nu} - \gamma^\mu\sigma^{\lambda\nu} + \gamma^\nu\sigma^{\mu\lambda})\psi\Big).$$

Use the equations of motion to show that $\partial_\mu T^{\mu\nu}=0$ and that the energy-momentum tensor is equivalent to the following symmetric result:

$$T^{\mu\nu} = \frac{1}{4}\Big(\bar{\psi}i\gamma^\mu\partial^\nu\psi + \bar{\psi}i\gamma^\nu\partial^\mu\psi - (\partial^\nu\bar{\psi})i\gamma^\mu\psi - (\partial^\mu\bar{\psi})i\gamma^\nu\psi\Big).$$

It is possible, however, to use only translational invariance (and of course, as always for Noether currents, the equations of motion). This derivation is very close in spirit to the discussion on pg. 10 and to eq. (3.20). Show that translational invariance can be formulated in the following way:

$$\mathcal{L}[\psi_\Lambda, \bar{\psi}_\Lambda, \partial_\mu \psi_\Lambda, \partial_\mu \bar{\psi}_\Lambda](x - \Lambda) = \mathcal{L}[\psi, \bar{\psi}, \partial_\mu \psi, \partial_\mu \bar{\psi}](x), \tag{1}$$

where $\psi_\Lambda(x) \equiv \psi(x + \Lambda)$ ($\Lambda$ independent of $x$) etc.

Expand the left-hand side of eq. (1) to first order in $\Lambda$. Now use the equations of motion, and eq. (1), to prove that $\partial_\mu T^{\mu\nu} = 0$ for

$$T^{\mu\nu} = \bar{\psi} i \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \bar{\psi}(i\gamma^\alpha \partial_\alpha - m)\psi.$$

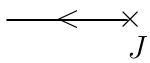
Also verify *explicitly* from the Dirac equation that $\partial_\mu T^{\mu\nu} = 0$ is satisfied.

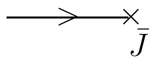
Show that all three definitions of the energy-momentum tensor give the same results for $H = \int d_3\vec{x}\; T_{00}$ and $P_i = \int d_3\vec{x}\; T_{0i}$, that these quantitities are conserved and that for a plane wave solution of the Dirac equation with momentum $\vec{k}$ (see section 13 of the lecture notes) they coincide, as it should be, with the energy and momentum of that solution, i.e. $H = k_0(\vec{k})$ and $P_i = k_i$.

b. Now add interactions and (external) sources:

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} - V_{\text{int}} + \mathcal{L}_{\text{source}}, \quad V_{\text{int}} = \tfrac{1}{4} g (\bar{\psi}\psi)^2, \quad \mathcal{L}_{\text{source}} = -(\bar{J}\psi + J\bar{\psi}).$$

What are the corresponding Euler-Lagrange equations for $\psi$ and $\bar{\psi}$? Solve these equations in a perturbative way, like in the scalar case (exercise 7; pp. 13–15 of the lecture notes). In particular give the Feynman rules for the equivalent diagrammatic expansion. For the lowest order result use the following notation:

$\psi =$  ,

$\bar{\psi} =$  .

24. **Canonical formalism for spinors**

a. 1. On p. 54 of the lecture notes creation and annihilation operators for the Dirac field are introduced:

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k_0}} \sum_{a=1}^{2} \left( b_a(\vec{k}) u^{(a)}(\vec{k}) e^{-ikx} + d_a^\dagger(\vec{k}) v^{(a)}(\vec{k}) e^{ikx} \right),$$

with $k_0 = +\sqrt{\vec{k}^2 + m^2}$. Give the corresponding expression for $\bar{\psi}$.

2. Postulate anticommutation relations as on p. 55:

$$\{b_a(\vec{k}), b_b^\dagger(\vec{k}')\} = \{d_a(\vec{k}), d_b^\dagger(\vec{k}')\} = \delta_{ab}\delta^3(\vec{k} - \vec{k}');$$

the remaining anticommutators are zero. Show that

$$\begin{aligned}
\{\psi_\alpha(x), \psi_\beta(y)\} &= \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)\} = 0 \quad \text{and} \\
\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} &= (i\gamma^\mu \partial_\mu + m)_{\alpha\beta} \Delta(x - y), \quad \text{with} \\
\Delta(x - y) &= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2k_0} \left( e^{-ik(x-y)} - e^{ik(x-y)} \right)
\end{aligned}$$

(and $\partial_\mu = \partial/\partial x^\mu$). Compare with exercise 6 and conclude that causality is respected.

3. Alternatively, postulate commutation relations (substitute $\{\ ,\ \} \to [\ ,\ ]$ in above anticommutation relations). Show that

$$\begin{aligned}
[\psi_\alpha(x), \psi_\beta(y)] &= [\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)] = 0 \quad \text{and} \\
[\psi_\alpha(x), \bar{\psi}_\beta(y)] &= (i\gamma^\mu\partial_\mu + m)_{\alpha\beta}\tilde{\Delta}(x-y), \quad \text{with} \\
\tilde{\Delta}(x-y) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2k_0}\left(e^{-ik(x-y)} + e^{ik(x-y)}\right).
\end{aligned}$$

Conclude that causality is violated.

Remark: This result can be generalised to a theorem stating that any local quantum field theory that respects causality admits only fermions with half integer spin and bosons with integer spin.

b. Add a source term $\bar{J}\psi + \bar{\psi}J$ to the Dirac Hamiltonian density $\bar{\psi}(-i\gamma^i\partial_i + m)\psi$, $J_\alpha(x)$ and $\bar{J}_\alpha(x)$ being anticommuting external fields. Expand $\langle 0|\exp(-iHt)|0\rangle$ up to second order in the sources. To this purpose use the Hamiltonian perturbation formalism (lecture notes pp. 17–20); use the properties of $u$, $v$ (p. 54) and the gamma matrices to simplify the spinor structure. Your final result should be

$$\begin{aligned}
\langle 0|e^{-iHt}|0\rangle e^{iE_0 t} &= 1 - i\int d^4x d^4y \bar{J}(x) G_{\rm F}(x-y) J(y) + \cdots \quad \text{with} \\
G_{\rm F}(x-y) &= \int \frac{d^4k}{(2\pi)^4}\frac{k_\mu\gamma^\mu + m}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}.
\end{aligned}$$

Note that $G_{\rm F}$ is precisely the classical fermion propagator of the previous exercise.

25. **Anticommuting variables**

In this exercise Greek letters denote anticommuting variables, ordinary letters commuting ones.

a. Compute the following integrals:

$$\int d\theta\, e^{\theta a}, \quad \int d\theta \frac{1}{1 - a\theta}, \quad \int d\theta \ln(1+\theta).$$

b. Given the following linear relation between two sets of $n$ independent anticommuting variables,

$$\eta_i = \sum_{j=1}^n B_{ij}\theta_j,$$

show that (for invertible $B$)

$$d\eta_1 d\eta_2 \cdots d\eta_n = \frac{1}{\det B} d\theta_1 d\theta_2 \cdots d\theta_n.$$

Compare this to the case of commuting variables.

Hint: Consider the most general function of $n$ anticommuting variables, which is a polynomial of degree $n$. Analyse its behaviour under integrations and linear transformations on the variables.

c. Prove that, for independent $\eta_i, \bar{\eta}_j$,

$$\int d\eta_1 d\bar{\eta}_1 \cdots d\eta_n d\bar{\eta}_n e^{\bar{\eta}_i A_{ij} \eta_j} = \det A.$$

Use this result to prove the following result, which holds for any antisymmetric matrix $A$:

$$\int d\theta_1 \cdots d\theta_n e^{\frac{1}{2}\theta_i A_{ij} \theta_j} = \pm\sqrt{\det A}.$$

Hint: Substitute $\eta_i = \theta_i + i\bar{\theta}_i$, $\bar{\eta}_i = \theta_i - i\bar{\theta}_i$.

d. Given a smooth function $f$ satisfying $\lim_{y\to\infty} f(y) = 0$, prove that

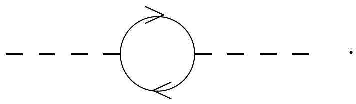
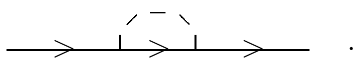
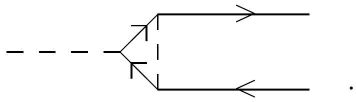
$$\int dx_1 dx_2 d\theta d\bar{\theta} f(x_1^2 + x_2^2 + \bar{\theta}\theta) = -\pi f(0).$$

26. **One loop Feynman diagrams**

Consider a model consisting of fermions $\psi$ and real scalar particles $\varphi$ with interaction

$$V_{\text{int}} = g\bar{\psi}\varphi\psi.$$

Determine the reduced matrix elements corresponding to the following diagrams (do not work out the analytical expressions):

1. $\varphi$ self-energy 
2. $\psi$ self-energy 
3. vertex correction 

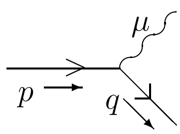
27. **Compton scattering for pions**

At not too high energies the pion-photon interaction is well approximated by scalar QED:

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\varphi^*(\partial^\mu + ieA^\mu)\varphi - m^2\varphi^*\varphi.$$

The pion ($\pi^-$) is described by the complex scalar field $\varphi$, the photon ($\gamma$) by the vector field $A_\mu$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$).

a. Show that the 3 point vertex is given by




$$= e(p_\mu + q_\mu),$$

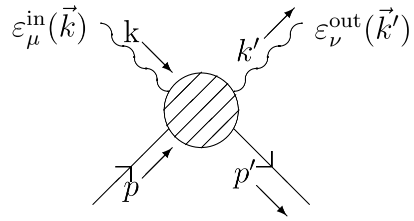
and give the other Feynman rules in the Lorentz gauge.

b. Which Feynman diagrams contribute, to order $e^2$, to $\pi^-\gamma \to \pi^-\gamma$ elastic scattering?

c. The initial and final photon states are plane waves:

$$\frac{\varepsilon_\mu^{\text{in}}(\vec{k})e^{-ik^i x_i}}{\sqrt{(2\pi)^3 2k_0}} \quad \text{resp.} \quad \frac{\varepsilon_\mu^{\text{out}}(\vec{k}')e^{-ik'^i x_i}}{\sqrt{(2\pi)^3 2k'_0}}.$$

Express the reduced matrix element for the scattering process, to order $e^2$, in terms of the polarisation vectors $\varepsilon^{\text{in,out}}$ and the external momenta. Use the following notation:




d. Using the result of part c. prove (to order $e^2$) that the $S$-matrix vanishes whenever the initial or final photon is longitudinal (i.e. $\varepsilon_\mu(\vec{k}) \sim k_\mu$). Explain which property of the model is responsible for this.

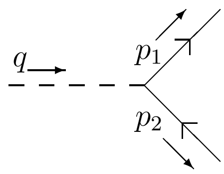
e. Give all Feynman diagrams that are needed for an order $e^6$ calculation of the *cross section*. Which of them are *UV* divergent, i.e. which give rise to expressions that diverge due to integrations over large momenta?

28. **Elementary fermionic processes**

Let us reconsider the situation in exercise 18, with the bosonic fields $A$ and $B$ replaced by fermionic fields $\psi_A$ and $\psi_B$. The Lagrangian now is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu C \partial^\mu C - \tfrac{1}{2}m_C^2 C^2 + \bar{\psi}_A(i\not{\partial} - m_A)\psi_A + \bar{\psi}_B(i\not{\partial} - m_B)\psi_B - g_A\bar{\psi}_A C\psi_A - g_B\bar{\psi}_B C\psi_B.$$

a. For $m_C > 2m_A$, $C$ can decay into $A$ and anti-$A$ according to the diagram

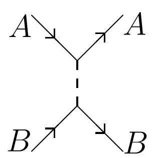



.

Determine like in exercise 18a. the $S$ matrix element and the decay width $\Gamma$, which now are functions of the fermion spins. Perform a summation over all possible spins to obtain the expected lifetime of $C$.

<u>Hint</u>: some properties of the $u$ and $v$ spinors (lecture notes p. 54) are very useful here.

b. Scattering of $A$ and $B$ is described by




.

Write down the analytic expression for the corresponding $S$ matrix element.

Now assume that $m_A = m_B$ and work in the CM frame. Determine the differential cross section $d\sigma(AB \to AB)$. *Average* over the incoming spins and *sum* over the outgoing spins. For which experimental situation is this justified?

Work out your result as a function of the CM variables $|\vec{p}|$ and $\theta$. As a check it is given that $d\sigma/d\Omega$ is spherically symmetric for $m_C = 2m_A$.

29. $e^-e^+$ **collisions in QED**

The QED Lagrangian with two flavours, electrons and muons, reads

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\alpha_{\text{gauge}}(\partial_\mu A^\mu)^2 + \sum_{f=e,\mu} \bar{\psi}_f(i\gamma^\mu D_\mu - m_f)\psi_f,$$

where $D_\mu = \partial_\mu - ieA_\mu$ (lecture notes p. 74).

a. $e^-e^+ \to e^-e^+$ (Bhabha scattering).

Which two diagrams contribute to lowest order?

In the lecture notes the Møller ($e^-e^- \to e^-e^-$) differential cross section is calculated. Perform an analogous calculation to obtain (in the CM frame)

$$\frac{d\sigma}{d\Omega}(e^-e^+ \to e^-e^+) = \frac{\alpha^2}{16E^2}\left\{\frac{(2E^2-m^2)^2}{(E^2-m^2)^2\sin^4(\theta/2)} + \frac{-8E^4+m^4}{E^2(E^2-m^2)\sin^2(\theta/2)}\right.$$
$$\left. + \frac{12E^4+m^4}{E^4} - \frac{4(2E^2-m^2)(E^2-m^2)\sin^2(\theta/2)}{E^4} + \frac{4(E^2-m^2)^2\sin^4(\theta/2)}{E^4}\right\} ,$$

where $\alpha = e^2/4\pi$, $m = m_e$; $E$ and $\theta$ are the CM variables for the energy of the incoming electron resp. the angle between the in and outgoing electrons.

Note that unlike in the Møller case there is no divergence at $\theta = \pi$. What is the reason for this?

b. $e^-e^+ \to \mu^-\mu^+$.

How many diagrams contribute to lowest order?

Show that in the CM frame, and in the limit $m_e/E, m_\mu/E \to 0$,

$$\frac{d\sigma}{d\Omega}(e^-e^+ \to \mu^-\mu^+) = \frac{\alpha^2}{16E^2}(1+\cos^2\theta).$$

Calculate the total cross section. Use dimensional analysis to express your result in units $\hbar, c \neq 1$.

30. **Weak interaction in the standard model**

The Lagrangian given below describes a simplified version of the standard model. This simplification, which only contains fermions $\psi$ and massive vector bosons $W_\mu$, captures the mechanism through which the standard model gives rise to an effective 4-fermion interaction.

$$\mathcal{L}(W_\mu, \psi) = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}M^2 W_\mu W^\mu + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + gW_\mu(\bar{\psi}\gamma^\mu\psi)$$

with $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$.

a. Give the Feynman rules.

b. We restrict ourselves to tree diagrams which satisfy two conditions: 1. all external lines are fermionic; 2. $p^2 \ll M^2$ for all momenta $p_\mu$. Show that such diagrams can effectively be described by

$$\mathcal{L}_{\text{effective}}(\psi) = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{\lambda}{(2!)^2}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$$

and express the parameter $\lambda$ in terms of $g$ and $M$.

31. **Gauge fields**

In this exercise we use the following gauge field conventions (cf. lecture notes):

$$\begin{array}{ccc} A_\mu = qA_\mu^a T^a & F_{\mu\nu} = qF_{\mu\nu}^a T^a & [T^a, T^b] = f_{abc}T^c \\ \mathrm{Tr}(T^a T^b) = -\frac{1}{2}\delta_{ab} & D_\mu = \partial_\mu + A_\mu & F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \end{array}$$

where $\{T^a\}$ spans a matrix representation $\rho(L_G)$ of the Lie algebra. Note that we absorb the coupling $q$ in $A_\mu$.

a. Notation: $X, Y, Z$ stand for arbitrary elements of the Lie algebra $\rho(L_G)$.

In the lecture notes the generators of the adjoint representation are defined as

$$(\mathrm{ad}T^a)_{bc} = -f_{abc}.$$

Show that this representation can be thought of as acting on the Lie algebra itself, in the following way:

$$(\mathrm{ad}T^a)Y = [T^a, Y].$$

Also prove from this formula that $X \to \mathrm{ad}X$ indeed is a representation of the Lie algebra, i.e. prove that it is a linear map, satisfying

$$(\mathrm{ad}[X, Y]) = [(\mathrm{ad}X), (\mathrm{ad}Y)].$$

Hint: work out $(\mathrm{ad}[X, Y])Z$, using the Jacobi identity $[X, [Y, Z]]$+cyclic$= 0$ (which can be seen to hold trivially by expanding the commutators).

Finally prove that

$$e^{\mathrm{ad}X}Y = e^X Y e^{-X}.$$

This means that the adjoint representation of the *group* $G$ is a conjugation. Therefore gauge transformations act on the field strength through the adjoint group representation (as $F_{\mu\nu} \to gF_{\mu\nu}g^{-1}$, lecture notes p. 87).

b. Define $D_\mu^{(\mathrm{ad})}X \equiv (\mathrm{ad}D_\mu)X = [\partial_\mu + A_\mu, X] = (\partial_\mu X) + [A_\mu, X]$. Prove that

$$D_\mu^{(\mathrm{ad})} D_\nu^{(\mathrm{ad})} F^{\mu\nu} = 0.$$

Hint: What is $[D_\mu^{(\mathrm{ad})}, D_\nu^{(\mathrm{ad})}]$?

c. The gauge invariant Lagrangian for a fermion field coupled to a dynamical SU(N) gauge field is (lecture notes pp. 86, 87)

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + \frac{1}{2q^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2q^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{2}{q^2}\mathrm{Tr}(J^\mu A_\mu) \quad \text{with} \\ J^\mu &= qJ^{\mu a}T^a, \quad J^{\mu a} = -iq\bar{\psi}\gamma^\mu T^a\psi. \end{aligned}$$

Derive the Euler-Lagrange equations:

$$D_\mu^{(\mathrm{ad})}F^{\mu\nu} = J^\nu \tag{1}$$

$$(i\gamma^\mu D_\mu - m)\psi = 0 \tag{2}$$

Show from eq. (1) that

$$D_\mu^{(\mathrm{ad})}J^\mu = 0.$$

Show that this equation follows from eq. (2) as well.

d. Use the Jacobi identity to prove the Bianchi identity:

$$D_\mu^{(\mathrm{ad})}F_{\nu\rho} + \text{cyclic} = 0.$$

Show that for electromagnetism ($G$ =U(1)) this gives the homogeneous Maxwell equations.

32. **Dirac equation with gauge fields**

a. By construction the Klein-Gordon equation is obtained from the free Dirac equation in the following way:

$$0 = -(i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi = (\partial^2 + m^2)\psi.$$

Analogously prove from the Yang-Mills Dirac equation (problem 31c. eq. (2)) that

$$(D^2 + m^2 - \tfrac{i}{2}\sigma^{\mu\nu}F_{\mu\nu})\psi = 0. \tag{1}$$

b. Now specify to electromagnetism,

$$T = i, \quad q = -e, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{etc.}$$

Choose the gauge $A_0 = 0$ and turn off the electric field by assuming that $\partial_t\vec{A} = \vec{0}$. Show that eq. (1) reduces to

$$(D^2 + m^2)\psi + e\begin{pmatrix} \vec{\sigma}\cdot\vec{B} & 0 \\ 0 & \vec{\sigma}\cdot\vec{B} \end{pmatrix}\psi = 0.$$

Write $\psi = \binom{\psi_G}{\psi_S}$ and define the 2-spinor $\psi_{\mathrm{sch}}$ to be

$$\psi_{\mathrm{sch}} = e^{imt}\psi_G$$

(subtraction of the rest energy from the Hamiltonian). Show that[1]

$$-\frac{1}{2m}\frac{\partial^2}{\partial t^2}\psi_{\text{sch}} + i\frac{\partial}{\partial t}\psi_{\text{sch}} = \left[\frac{(\vec{p}+e\vec{A})^2}{2m} + \frac{e}{2m}(\vec{\sigma}\cdot\vec{B})\right]\psi_{\text{sch}}.$$

Show that in the non-relativistic limit this equation simplifies to the well known Schrödinger equation for an electron in a magnetic field, i.e.

$$\begin{cases} i\dfrac{\partial}{\partial t}\psi_{\text{sch}} = H\psi_{\text{sch}} \\ H = \dfrac{1}{2m}(\vec{p}+e\vec{A})^2 + \dfrac{e}{2m}(\vec{\sigma}\cdot\vec{B}). \end{cases}$$

33. **Linear sigma model**

The Lagrangian for the linear sigma model reads

$$\mathcal{L} = \tfrac{1}{2}[\partial_\mu\vec{\varphi}\cdot\partial^\mu\vec{\varphi} + \partial_\mu\sigma\partial^\mu\sigma] + \tfrac{1}{4}\mu^2[|\vec{\varphi}|^2 + (\sigma+v)^2] - g[|\vec{\varphi}|^2 + (\sigma+v)^2]^2,$$

where $\vec{\varphi}$ is the pion field (3 real components), $\sigma$ is the sigma field (1 real component), $v$ a constant and $\mu$, $g$ are real positive parameters.

a. Show that this Lagrangian contains a linear term $\alpha\sigma$ and express $\alpha$ in terms of $\mu$, $g$ and $v$. What is the Feynman rule for such a term?

For $\alpha \neq 0$, this Feynman rule makes the perturbative approach unnecessarily complicated. Argue that this complication is avoided when $v$ is such that $\vec{\varphi} = \vec{0}$ and $\sigma = 0$ corresponds to the minumum of the potential associated to the Lagrangian.

Determine $v$, and show that the $\vec{\varphi}$ and $\sigma$ masses are 0 resp. $\mu$.

b. Show that the Lagrangian is invariant under the global infinitesimal (isospin) transformations

$$\delta_\Lambda\sigma(x) = 0, \quad \delta_\Lambda\varphi_i(x) = -\varepsilon_{ijk}\Lambda_j\varphi_k(x), \tag{1}$$

and also under the global transformations

$$\delta_\xi\sigma(x) = -\vec{\varphi}(x)\cdot\vec{\xi}, \quad \delta_\xi\vec{\varphi}(x) = (\sigma(x)+v)\vec{\xi}\ . \tag{2}$$

Now write eqs. (1,2) in matrix notation w.r.t. the 4-component vector $\varphi_\mu(x)$ by defining $\varphi_4(x) \equiv \sigma(x) + v$ (i.e. write $\delta_\Lambda\varphi_\mu(x) = \Lambda_i L^i_{\mu\nu}\varphi_\nu(x)$, resp. $\delta_\xi\varphi_\mu(x) = \xi_i K^i_{\mu\nu}\varphi_\nu(x)$ for suitably defined $4\times4$ matrices $L^i$ and $K^i$). Prove that $L^i$ and $K^i$ span the space of real antisymmetric $4\times4$ matrices.

Conclude that eqs. (1,2) are in fact infinitesimal SO(4) transformations. Verify this by showing that the Lagrangian, written in terms of the 4-component vector $\varphi_\mu$, is manifestly SO(4) invariant.

c. Give the Noether currents associated to eqs. (1,2)

[1]Please note: To respect covariance $p_k = -i\partial/\partial x^k$, whereas $p^k = -i\partial/\partial x^k$ or $\vec{p} = (p^1, p^2, p^3) = -i\vec{\nabla}$ as used in non-relativistic quantum mechanics.

34. **Higgs mechanism**

We consider a model with real scalar fields $\varphi^i$ and vector fields $A^i_\mu$, $i = 1, 2, 3$. These fields transform in the fundamental representation of an internal group SO(3), with generators

$$(T^i)_{jk} = -\epsilon_{ijk} \qquad i, j, k = 1, 2, 3.$$

In particular the covariant derivative and field tensor read

$$\begin{aligned} (D_\mu\varphi)^i &= \partial_\mu\varphi^i - g\epsilon_{kij}A^k_\mu\varphi^j, \\ F^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon_{ijk}A^j_\mu A^k_\nu. \end{aligned}$$

The Lagrangian is taken to be

$$\begin{aligned} \mathcal{L} = -\tfrac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} + \tfrac{1}{2}(D_\mu\varphi)^i(D^\mu\varphi)^i - V(|\vec{\varphi}|^2) & \quad \text{with potential} \\ V(|\vec{\varphi}|^2) = \tfrac{\lambda}{4}[|\vec{\varphi}|^2 - F^2]^2, \qquad\qquad & \quad \lambda, F > 0. \end{aligned}$$

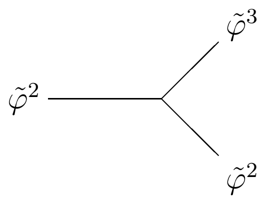
The (0-loop) vacuum expectation value (vev) of the scalar fields is chosen to be

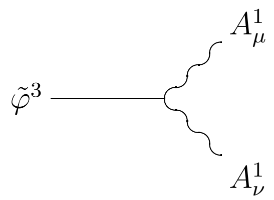
$$\langle\vec{\varphi}(x)\rangle = \vec{F} \equiv \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} \qquad (F \text{ constant}).$$

a. Explain why this is a valid choice for the vev. Show that this vev is invariant under a 1-dimensional subgroup of SO(3).

b. Define $\tilde{\varphi}^i = \varphi^i - F^i$, and expand the Lagrangian in terms of $\tilde{\varphi}^i$ and $A^i_\mu$. Note that the quadratic part of the Lagrangian contains off-diagonal elements (mixing $A$ and $\tilde{\varphi}$). In general such terms can be handled by diagonalisation (redefining $A$ and $\tilde{\varphi}$ in terms of each other in an appropriate way), but anticipating the gauge choice in part c. you may neglect them here.

Interpret the various terms (mass terms, kinetic terms, interaction terms). Give the masses for the fields $A^i_\mu$ and $\tilde{\varphi}^i$ (i=1,2,3), and also the couplings for the following 3-vertices:




c. Show that we can choose the gauge such that

$$\tilde{\varphi}^1 = \tilde{\varphi}^2 = 0.$$

Does this completely fix the gauge?

This model contains 9 physical degrees of freedom (dof). Read off from the quadratic terms how in the above gauge these physical dof are distributed over the fields.

d. Reconsider the situation for a different potential, $V(|\vec{\varphi}|^2) = \frac{1}{2}m^2|\vec{\varphi}|^2$. What is the $\vec{\varphi}$ vev in this case? Read off the number of physical dof again ($\varphi^1 = \varphi^2 = 0$ is <u>not</u> a convenient gauge choice now. Why?).

35. **Higgs effect and ghosts**

Take the model in the previous exercise and add the following gauge fixing term to the Lagrangian:

$$\mathcal{L}^{\text{gauge}} = -\frac{\alpha}{2}\mathcal{F}_a^2 \quad , \quad \mathcal{F}_a = \partial_\mu A^{\mu a} - \frac{g}{\alpha}\epsilon_{abc}F^b\varphi^c.$$

a. Expand $\mathcal{L} + \mathcal{L}^{\text{gauge}}$ up to quadratic terms in $A_\mu^a$ and $\tilde{\varphi}^a \equiv \varphi^a - F^a$. For convenience choose $F^a = F\delta_{a3}$, as in the previous exercise. Show that due to the special gauge choice, the quadratic terms mixing $A$ and $\tilde{\varphi}$ cancel among $\mathcal{L}$ and $\mathcal{L}^{\text{gauge}}$.

b. Determine how $\mathcal{F}_a$ transforms under infinitesimal local gauge transformations $\varphi \to \Omega\varphi$; $A_\mu \equiv A_\mu^a T^a \to \Omega A_\mu \Omega^{-1} + g^{-1}\Omega\partial_\mu\Omega^{-1}$ with $\Omega_{ij} = (\exp(\Lambda^a T^a))_{ij} = \delta_{ij} - \Lambda^a\epsilon_{aij}$, and write the result as $\delta\mathcal{F}_a = M_{ab}\Lambda^b$ (cf. lecture notes pp. 90, 91). Read off the ghost masses.

c. Determine the vector, scalar and ghost propagators as a function of $\alpha$. Which limits correspond to the transversal gauge ($\partial_\mu A^{\mu a} = 0$) and the unitary gauge ($\tilde{\varphi}^1 = \tilde{\varphi}^2 = 0$, as in part c. of the previous exercise)?

d. Which poles in the propagators correspond to *physical* masses? Check that unphysical poles always coincide mutually, and argue why this is necessary.

36. **Elektroweak interactions in the standard model**

If $\psi$ is an SU(2) doublet and has U(1) hypercharge $-\frac{1}{2}Yg'$, its covariant derivative reads (for SU(2) generators $-i\sigma_a/2$ and U(1) generator $i$)

$$D_\mu\psi = \partial_\mu\psi - \tfrac{1}{2}ig\sigma_a W_\mu^a\psi - \tfrac{1}{2}iYg'B_\mu\psi.$$

It is given that the fermion fields $e^L, e^R$ (electron), $\nu$ (neutrino), $u^L, u^R, d^L, d^R$ (up and down quarks) have the following SU(2)$\times$U(1) properties:

$$\begin{array}{rll}
\psi_e^L = \binom{\nu}{e^L} & \text{doublet} & Y = -1 \\
\psi_e^R = e^R & \text{singlet} & Y = -2 \\
\psi_q^L = \binom{u^L}{d^L} & \text{doublet} & Y = +\frac{1}{3} \\
\psi_u^R = u^R & \text{singlet} & Y = +\frac{4}{3} \\
\psi_d^R = d^R & \text{singlet} & Y = -\frac{2}{3}
\end{array}$$

Furthermore, we reformulate the gauge fields:

$$\begin{aligned}
W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \\
Z_\mu &= -W_\mu^3\cos\theta_W + B_\mu\sin\theta_W, \quad A_\mu^{\text{em}} = W_\mu^3\sin\theta_W + B_\mu\cos\theta_W,
\end{aligned}$$

and $\sigma_\pm = \sigma_1 \pm i\sigma_2$.

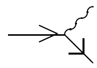
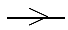
a. Write the covariant derivative of $\psi_e^L$ and $\psi_e^R$ in terms of $W_\mu^\pm, Z_\mu, A_\mu^{\text{em}}$ and $\sigma_+, \sigma_-, \sigma_3$.

b. We require the interaction between the electron fields $e^L, e^R$ and the photon field $A_\mu^{\text{em}}$ to be the same as in quantum electrodynamics (QED). Derive from this requirement two relations between $g, g', \theta_W$ and the electron charge $-e$.

c. Work out the relevant covariant derivatives to determine the electromagnetic charge of the neutrino and the up and down quarks. Also analyse the electromagnetic properties of the fields $W_\mu^\pm$ and $Z_\mu$. Discuss the particle interpretation of the complex fields $W_\mu^\pm$.

d. The Lagrangian of the standard model contains a.o. the following terms:

$$i\bar{\psi}_e^L \gamma^\mu D_\mu \psi_e^L + i\bar{\psi}_e^R \gamma^\mu D_\mu \psi_e^R.$$

Determine from this <u>all</u> possible 3-vertices of type  , where  stands for $W_\mu^+, W_\mu^-$ or $Z_\mu$, and  for $e^L, e^R$ or $\nu$.

e. The standard model allows for the process $W^- \to e^L + \bar{\nu}$. It is given that the decay rate $\Gamma_0$ equals

$$\Gamma_0 \equiv \Gamma(W^- \to e^L \bar{\nu}) = \frac{1}{48\pi} g^2 M_W$$

($M_W$ is the mass of $W^-$, the masses of $e^L, e^R$ and $\nu$ are neglected). Use your results from part d. to express the decay rates $\Gamma_1 \equiv \Gamma(Z \to e^R \bar{e}^R)$ and $\Gamma_2 \equiv \Gamma(Z \to \nu\bar{\nu})$ in terms of $\Gamma_0$ and $\theta_W$.

f. Show that

$$\Gamma(Z \to e^L \bar{e}^L) \neq \Gamma(Z \to e^R \bar{e}^R)$$

and interpret this result.

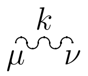
37. **LEP experiment**

Since 1989 CERN has been operating the "Large Electron-Positron" (LEP) collider, a ring with a circumference of 27 km. Electrons ($e^-$) and positrons ($e^+$) are accelerated in opposite directions, each reaching an energy ($E$) of about 45 GeV. The CM energy ($2E$) in a collision is comparable to the mass of the neutral vector boson Z that was encountered in the previous exercise. From Heisenberg's uncertainty relation it is then clear that a Z-boson created in the collision can exist for a relatively long time. This gives rise to a so-called resonance in the electron-positron cross section. In the present exercise we will analyse this phenomenon for the process $e^+e^- \to \mu^+\mu^-$.

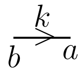
The following part of the standard model Lagrangian (in the unitary gauge) suffices for a leading order calculation:

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu) + \tfrac{1}{2} M_Z^2 Z_\mu Z^\mu + \\ &\quad + \sum_{f=e,\mu} \left\{ \bar{\psi}_f (i\gamma^\mu \partial_\mu - m_f)\psi_f - \frac{e}{\sqrt{3}} \bar{\psi}_f Z_\mu \gamma^\mu \gamma_5 \psi_f \right\}.\end{aligned}$$

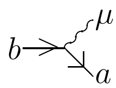
In this Lagrangian the spinor field $\psi_{e(\mu)}$ describes the electron (muon) with mass $m_{e(\mu)}$ =0.511 MeV (105.7 MeV), while the Z-particle (with mass $M_Z$ =91.2 GeV) is described by the vector field $Z_\mu$. For convenience the Weinberg angle $\theta_W$ has been approximated ($\sin^2\theta_W = 0.25$ instead of $\sin^2\theta_W = 0.23$). As can be seen from part d. of the previous exercise this considerably simplifies the interaction between the Z-particle and the fermions. The Feynman rules now are




$$= \frac{-(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_Z^2})}{k^2 - M_Z^2 + i\epsilon} \quad ,$$




$$= \frac{(m_f + \gamma^\mu k_\mu)_{ab}}{k^2 - m_f^2 + i\epsilon} \quad ,$$




$$= \frac{e}{\sqrt{3}}(\gamma^\mu \gamma_5)_{ab}.$$

a. Give all Feynman diagrams and the S-matrix for $e^+e^- \to \mu^+\mu^-$ via a Z-particle. Also give the Feynman diagrams for $e^+e^- \to e^+e^-$ via a Z-particle.

b. Prove from the Dirac equation that $\bar{u}^{s_1}(p)\gamma^\mu v^{s_2}(q)(p_\mu + q_\mu) = 0$ and the same for $u \leftrightarrow v$. Here $u^s(p)$ ($v^s(p)$) is a Dirac spinor describing a particle (anti-particle) with arbitrary spin $s$. For QED, explain why these equalities are related to U(1) gauge invariance.

c. Show like in part b. that $\bar{u}^{s_1}(p)\gamma^\mu\gamma_5 v^{s_2}(q)(p_\mu + q_\mu) = 2m_f \bar{u}^{s_1}(p)\gamma_5 v^{s_2}(q)$ and derive an analogous formula for $u \leftrightarrow v$. Here $m_f$ stands for the fermion mass.

d. The typical energy scale in the LEP experiment is $M_Z$. This means that $m_e$ and $m_\mu$ can be neglected. Show that this implies that in part a. the Z-propagator can be replaced by

$$\frac{-g_{\mu\nu}}{k^2 - M_Z^2 + i\epsilon}.$$

e. In the lecture notes (pp. 39, 46) it is explained that quantum corrections modify the propagator. Show for the present case that the Z-propagator will be modified to

$$\frac{-Z_Z g_{\mu\nu}}{k^2 - M_Z^2 + iM_Z\Gamma_Z + i\epsilon}$$

($Z_Z$ =Z's wavefunction renormalisation; $\Gamma_Z$ =Z's total decay rate).

To a good approximation the $k-$dependence of $\Gamma_Z$ may be neglected near the resonance. Furthermore, $\mathcal{O}(e^2)$ corrections to $Z_Z$ will be neglected too, i.e. we take $Z_Z = 1$. Why is $\Gamma_Z \neq 0$?

f. In your calculation below you may (or rather should) use the result from exercise 29b, namely that in QED the total cross section for $e^+e^- \to \mu^+\mu^-$ (i.e. via a photon) equals $\frac{\pi\alpha^2}{3E^2}$. Here $\alpha$ is the fine structure constant, the fermion masses are neglected, and the incoming particles are *not* polarised.

Show that the total cross section $\sigma$ for $e^+e^- \to \mu^+\mu^-$ via a Z-particle equals, for unpolarised electron-positron bundles and in the approximations discussed before,

$$\sigma = \frac{\frac{1}{3}\pi\alpha^2(4E/3)^2}{((2E)^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}.$$

Remark Since $\Gamma_Z \ll M_Z$ (see below) it is clear from the above formulas that near the resonance photon "exchange" can be neglected. Also Higgs "exchange", possible in the standard model, is negligible, as the coupling between the Higgs particle and fermions is proportional to $m_f/M_Z$ ($\ll e$).

g. The figure on the first page of the lecture notes shows the LEP data (in the figure Energy=$2E$). Explain this plot qualitatively from your calculations, and extract $\Gamma_Z$.

Note that each fermion anti-fermion pair, into which the Z-particle can decay will give a positive contribution to $\Gamma_Z$. As also neutrinos contribute, one has been able to determine the existence of precisely three (light) neutrino types.

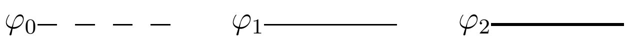
38. **1 loop calculation with scalar fields**

A model with scalar fields $\varphi_0$, $\varphi_1$ and $\varphi_2$ is described by the Lagrangian

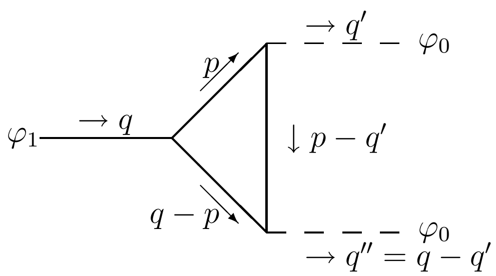
$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi_0\partial^\mu\varphi_0 + \partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 - m_0^2\varphi_0^2 - m_1^2\varphi_1^2 - M^2\varphi_2^2 - \lambda_0\varphi_0\varphi_2^2 - \lambda_1\varphi_1\varphi_2^2),$$

with $M \gg m_1 > 3m_0$.

a. Even though there is no direct interaction between $\varphi_0$ and $\varphi_1$, the model gives rise to diagrams with only external $\varphi_0$ and $\varphi_1$ lines. Clarify this statement by drawing some diagrams contributing to the processes $\varphi_1 \to \varphi_0\varphi_0\varphi_0$ and $\varphi_1\varphi_0 \to \varphi_1\varphi_0$. Use the following notation for the propagators:




b. Consider the diagram




Give the associated S-matrix element without working out the $d^4p$ integration yet.

c. The S-matrix element contains the following expression:

$$g(q, q') = \int \frac{d^4p}{(2\pi)^4} \frac{i\lambda_1\lambda_0^2}{(p^2 - M^2 + i\varepsilon)((p-q)^2 - M^2 + i\varepsilon)((p-q')^2 - M^2 + i\varepsilon)}.$$

Argue that $g(q, q')$ can be viewed as the effective coupling constant for the leading order contribution to the process $\varphi_1 \to \varphi_0\varphi_0$. Do you expect $g(q, q')$ to be real or complex?

d. Compute $g(q, q')$ with the techniques introduced in the lecture notes pp. 101–103: write $g(q, q')$ as an integral over a function of the form $I_{n,\alpha,\beta}(\tilde{m})$. To this purpose use a Wick rotation, the Feynman trick and a shift of the integration variable $p$. Write the relevant function $I_{n,\alpha,\beta}(\tilde{m})$ in terms of Gamma functions.

e. Assuming $q$, $q'$, $q''$ on-shell, compute $q^2$, $q'^2$ and $q \cdot q'$, and observe that these scalars are much smaller than $M^2$. Use this observation to expand your result from part d. in terms of $1/M^2$. In this way show that

$$g(q, q') = \frac{g_0}{M^2} + \frac{c_0 m_0^2}{M^4} + \frac{c_1 m_1^2}{M^4} + \mathcal{O}\left(\frac{m^4}{M^6}\right)$$

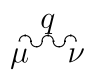
and calculate $g_0, c_0, c_1$.

39. **Vacuum polarisation and Pauli-Villars regularisation**

QED, quantum electrodynamics, is the field theory of minimally coupled photons and electrons. Their fields are a U(1) vector field $A_\mu$ and a spinor field $\psi(x)$, governed by the Lagrangian

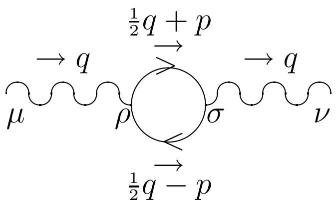
$$\begin{aligned}\mathcal{L}_{\text{QED}}[A,\psi] &= \mathcal{L}_{\text{photon}} + \mathcal{L}_{\text{electron}} + \mathcal{L}_{\text{int}} = \\ &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + eA_\mu(\bar{\psi}\gamma^\mu\psi).\end{aligned}$$

Choosing the Landau gauge ($\alpha \to \infty$, lecture notes p. 17) the 2-point function reads to $0^{\text{th}}$ order




$$\equiv \Pi^{(0)}_{\mu\nu}(q) = (-i)\left(\frac{-1}{(q^2+i\varepsilon)^2}[q^2 g_{\mu\nu} - q_\mu q_\nu]\right).$$

(the propagator equals $i\Pi^{(0)}_{\mu\nu}(q)$, cf. exercise 13). We are interested in the leading correction to $i\Pi^{(0)}_{\mu\nu}(q)$, the so-called vacuum polarisation:




$$\equiv \Pi^{(1)}_{\mu\nu}(q).$$

a. Show that

$$\Pi^{(1)}_{\mu\nu}(q) = \Pi^{(0)}_{\mu\rho}(q)\omega^{\rho\sigma}(q)\Pi^{(0)}_{\sigma\nu}(q),$$

with

$$\omega^{\rho\sigma}(q) = -\frac{e^2}{(2\pi)^4}\int d^4p\,\mathrm{Tr}\left(\gamma^\sigma\frac{m+\gamma\cdot(\frac{1}{2}q+p)}{(\frac{1}{2}q+p)^2 - m^2 + i\varepsilon}\gamma^\rho\frac{m+\gamma\cdot(p-\frac{1}{2}q)}{(p-\frac{1}{2}q)^2 - m^2 + i\varepsilon}\right).$$

Note that this expression is divergent.

b. Compute the trace in the above expression using your results from exercise 21.

<u>N.B.</u> Throughout the present exercise do <u>not</u> assume $q$ on-shell.

c. Show that $q_\rho\omega^{\rho\sigma}(q) = 0$ (strictly speaking this is only valid after regularisation as in part e.). From this conclude that $\omega^{\rho\sigma}(q)$ takes the form

$$\omega^{\rho\sigma}(q) = \omega(q^2)(q^2 g^{\rho\sigma} - q^\rho q^\sigma).$$

d. Determine the scalar function $\omega(q^2)$ by contracting the above expression with $g_{\rho\sigma}$.

e. The result in part d. contains a divergent $d^4p$ integration. This divergence will now be regularised by the method of Pauli-Villars. In this method attention shifts from $\omega(q^2, m) \equiv \omega(q^2)$ to the sum

$$\bar{\omega}(q^2) \equiv \sum_{s\geq 0} C_s\omega(q^2, m_s),$$

where $C_0 = 1$ and $m_0 = m$. Here the sum should be performed before doing the $d^4p$ integration.

Show that $\bar{\omega}(q^2)$ is of the form

$$\bar{\omega}(q^2) = \int d^4p \sum_s C_s \frac{P_2 + m_s^2 P_0}{\tilde{P}_4 + m_s^2 \tilde{P}_2 + m_s^4 \tilde{P}_0 + i\varepsilon},$$

where $P_n$ and $\tilde{P}_n$ are polynomials of degree $n$ in $p$ and of arbitrary degree in $q$; furthermore they are independent of $m_s$.

Expand the quotient $I_s$ (appearing within the sum and integration in $\bar{\omega}(q^2)$) for large values of $p^2$:

$$I_s = C_s \frac{P_2}{\tilde{P}_4} + C_s m_s^2 \left[ \frac{P_0}{\tilde{P}_4} - \frac{P_2 \tilde{P}_2}{\tilde{P}_4^2} \right] + \mathcal{O}(p^{-6}).$$

Show that the conditions $\sum_s C_s = 0, \sum_s C_s m_s^2 = 0$ guarantee that $\bar{\omega}(q^2)$ is given by a convergent integral.

Remark: Closer inspection shows that the second term in $I_s$ is only of order $p^{-6}$. The naive conclusion that the condition $\sum_s C_s m_s^2 = 0$ is superfluous is wrong, as the cancellation of the $p^{-4}$ contribution does not take place at the level of $\omega^{\rho\sigma}$. Hence leaving out the second condition makes the derivation in part c. invalid.

f. A solution to the conditions is

$$\begin{array}{ccc} C_0 = 1 & C_1 = 1 & C_2 = -2 \\ m_0^2 = m^2 & m_1^2 = m^2 + 2\Lambda^2 & m_2^2 = m^2 + \Lambda^2 \end{array}$$

for arbitrary $\Lambda^2$. Show that for this choice $\bar{\omega}(q^2)$ gives the same vacuum polarisation as a model consisting of the photon field and the following fields:

| | (mass)$^2$ | statistics |
|---|---|---|
| $\psi_0$ | $m^2$ | Fermi |
| $\psi_1$ | $m^2 + 2\Lambda^2$ | Fermi |
| $\psi_2$ | $m^2 + \Lambda^2$ | Bose |
| $\psi_3$ | $m^2 + \Lambda^2$ | Bose |

with a Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{photon}} + \sum_{s=0}^{3} \bar{\psi}_s \left[ i\gamma^\mu (\partial_\mu - ieA_\mu) + m_s \right] \psi_s.$$

g. Remark
The situation in part f. describes the regularised theory. To return to the true theory we would like to eliminate the fields $\psi_{1,2,3}$ by pushing their masses to infinity, i.e. by taking the limit $\Lambda^2 \to \infty$. However, careful inspection shows that $\bar{\omega}(q^2)$ diverges as a function of $\Lambda^2$:

$$\bar{\omega}(q^2, \Lambda^2) \overset{\Lambda^2 \to \infty}{\sim} \log\left(\frac{\Lambda^2}{m^2}\right).$$

This divergence can be absorbed in a wavefunction renormalisation, which will never appear in physical quantities.

40. **Beta decay of the neutron**

Through the weak interactions a neutron ($N$) can decay in a proton ($P$), an electron ($e$) and an anti-neutrino ($\bar{\nu}^e$). At quark-level this so-called beta decay reads $d \to u + e + \bar{\nu}^e$. The following interaction term in the Lagrangian of the Standard Model is relevant to this decay:

$$\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}}\left(W_\mu^- \bar{\psi}_d^L \gamma^\mu \psi_u^L + W_\mu^- \bar{\psi}_e^L \gamma^\mu \psi_{\nu^e} + h.c.\right)$$

($h.c.$ =hermitian conjugate, $\psi^L = \frac{1-\gamma_5}{2}\psi$).

a. Give the lowest order Feynman diagram for the above process (for quarks).

b. Show that if the external momenta are much smaller than the W-boson mass $M_W$, we can just as well consider the effective interaction

$$\mathcal{L}_{\text{int}}^{\text{eff}} = -\frac{G}{\sqrt{2}}\left(\bar{\psi}_d \gamma^\mu (1-\gamma_5)\psi_u \bar{\psi}_{\nu^e}\gamma_\mu(1-\gamma_5)\psi_e\right) + h.c. \tag{1}$$

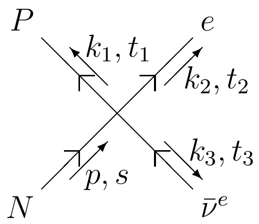
and express the so-called Fermi-constant $G$ in terms of $g$ and $M_W$.

c. Prove that $S_{\text{int}}^{\text{eff}} = \int d^4x \mathcal{L}_{\text{int}}^{\text{eff}}$ is <u>not</u> invariant under parity.

d. Since the proton and neutron are built out of 3 quarks ($N = ddu$, $P = uud$), one can derive from eq.(1) an effective Lagrangian for neutron decay.

$$\tilde{\mathcal{L}}_{\text{int}}^{\text{eff}} = -\frac{G}{\sqrt{2}}\left(\bar{\psi}_N \gamma^\mu (1-\alpha\gamma_5)\psi_P \bar{\psi}_{\nu^e}\gamma_\mu(1-\gamma_5)\psi_e\right) + h.c. \tag{2}$$

Through QCD-effects $\alpha$ will deviate from 1. In good approximation one has $\alpha = 1.22$. Give the reduced matrix element $\mathcal{M}$ for the decay of the neutron. Use the following conventions for the momenta $(p, k_i)$ and spins $(s, t_i)$:




e. During the beta decay of the neutron, which is assumed at rest, only the momentum of the electron ($\vec{k}_2$) is measured. Using a magnetic field the spin of the neutron is aligned along the positive $z$-axis.
Give the expression for the spinor $u_N$ for this polarisation and prove that

$$\begin{aligned}\sum_{t_1,t_2,t_3} |\mathcal{M}|^2 &= \frac{G^2}{2}\bar{u}_N \gamma^\mu (1-\alpha\gamma_5)(\not{k}_1 + m_P)\gamma^\nu(1-\alpha\gamma_5)u_N \cdot \\ &\quad \cdot \text{Tr}\left((\not{k}_2 + m_e)\gamma_\nu(1-\gamma_5)\not{k}_3\gamma_\mu(1-\gamma_5)\right). \end{aligned} \tag{3}$$

Here $m_e, m_P$ and $m_N$ are the masses of resp. the electron, proton and neutron. We work in the limit $m_N \to \infty, m_P \to \infty$, but we keep $\Delta m = m_N - m_P$ fixed.

f. Show that in this limit

$$\bar{u}_N\gamma^\mu(1-\alpha\gamma_5)(\not{k}_1+m_P)\gamma^\nu(1-\alpha\gamma_5)u_N = 4m_P^2\left(c^\mu g^{\mu\nu} - \alpha(\delta_0^\mu\delta_3^\nu + \delta_3^\mu\delta_0^\nu) - i\alpha^2\varepsilon^{0\mu\nu3}\right),$$

with $c^\mu = 1$ for $\mu = 0$ and $c^\mu = -\alpha^2$ for $\mu = 1,2,3$; $\varepsilon^{0123} = -1$, $\varepsilon^{\mu\nu\rho\sigma}$ completely antisymmetric; $g$ is the metric diag$(1,-1,-1,-1)$.
Prove that the 'partial' decay width is given by

$$d\Gamma_\uparrow = f(|\vec{k}_2|)\left(1 - \frac{2\alpha(\alpha-1)}{1+3\alpha^2}\frac{|\vec{k}_2|}{(k_2)^0}\cos\theta\right)d^3k_2, \tag{4}$$

where $\theta$ is the angle with the positive $z$-axis, along which the electron is detected.
Hint: Prove first that in the limit $m_N, m_P \to \infty$ conservation of energy implies that

$$\Delta m = |\vec{k}_3| + \sqrt{m_e^2 + |\vec{k}_2|^2}.$$

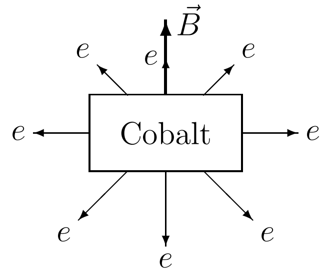
g. Explain why the unpolarised partial decay width is given by

$$d\bar{\Gamma} = f(|\vec{k}_2|)d^3k_2 \quad ?$$

Compute from this the life-time of the neutron in the approximation that $m_e/\Delta m = 0$ (in reality $m_e/\Delta m \approx 0.4$, which leads roughly to a correction with a factor 2). In units where $\hbar = c = 1$, you may use that

$$\begin{aligned} \Delta m &= 2.0\cdot 10^{21}s^{-1}, \\ \frac{m_P}{\Delta m} &= 7.3\cdot 10^2, \\ G &= 1.0\cdot 10^{-5}m_P^{-2}. \end{aligned}$$

h. Already in 1957 (breaking of) invariance under parity in the weak interactions was tested. Free neutrons are experimentally hard to handle. This is why a piece of Cobalt ($Co^{60}$) was used, whose nucleus changes under beta decay into Nickel ($Ni^{60}$). Schematically the following result was obtained:




(that is a bigger electron flux in the direction of $-\vec{B}$ than in the direction of $\vec{B}$, where $\vec{B}$ is the applied magnetic field).
Argue why this experiment demonstrated the violation of invariance under parity transformations.
Note: Nuclear complications make the precise computations for $Co^{60}$ rather more difficult than those for a free neutron. The result is nevertheless given by eq.(4), but with appropriately modified $\alpha$. For the above question this is not relevant; To conclude that parity invariance (mirror symmetry) is broken, the details of the underling theory are not required.